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Introduction to focus issue on hydrodynamic quantum analogs

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Hydrodynamic quantum analogs is a nascent field initiated in 2005 by the discovery of a hydrodynamic pilot-wave system [Y. Couder, S. Protière, E. Fort, and A. Boudaoud, Nature **437**, 208 (2005)]. The system consists of a millimetric droplet self-propeling along the surface of a vibrating bath through a resonant interaction with its own wave field [J. W. M. Bush, Annu. Rev. Fluid Mech. **47**, 269–292 (2015)]. There are three critical ingredients for the quantum like-behavior. The first is "path memory" [A. Eddi, E. Sultan, J. Moukhtar, E. Fort, M. Rossi, and Y. Couder, J. Fluid Mech. **675**, 433–463 (2011)], which renders the system non-Markovian: the instantaneous wave force acting on the droplet depends explicitly on its past. The second is the resonance condition between droplet and wave that ensures a highly structured monochromatic pilot wave field that imposes an effective potential on the walking droplet, resulting in preferred, quantized states. The third ingredient is chaos, which in several systems is characterized by unpredictable switching between unstable periodic orbits. This focus issue is devoted to recent studies of and relating to pilot-wave hydrodynamics, a field that attempts to answer the following simple but provocative question: *Might deterministic chaotic pilot-wave dynamics underlie quantum statistics? Published by AIP Publishing.* https://doi.org/10.1063/1.5055383

I. INTRODUCTION

In 2005, Yves Couder discovered that a millimetric droplet may walk on the surface of a vibrating fluid bath through a resonant interaction with its own wave field.[1,4] This hydrodynamic pilot wave system has since generated considerable activity in the physics and applied mathematics communities[2,5] and drawn the attention of philosophers of science.[6] Not only is it a rich new dynamical system, but it extends the range of classical systems to include several features once thought to be exclusive to the microscopic, quantum realm. It is, moreover, strongly reminiscent of the double-solution pilot-wave theory proposed by Louis de Broglie,[7,8] according to which quantum particles have an associated vibration or internal clock responsible for wave generation and move in resonance with the resulting guiding or "pilot" wave. Studies of pilot-wave hydrodynamics seek to assess the potential and limitations of this walking droplet or "walker" system as a quantum analog, while exploring its connections with extant realist models of quantum dynamics,[2,5] including de Broglie's double-solution theory,[7,8] Bohmian mechanics,[9,10] and stochastic electrodynamics.[11]

Pilot-wave hydrodynamics is an example of hereditary mechanics,[12,13] a class of dynamical systems in which initial conditions (even if known precisely) are insufficient to specify the system evolution; one must also have knowledge of the system's past.[75] In the walker system, this non-Markovian feature arises by virtue of the system's "path memory":[3] the wave force acting on the drop depends explicitly on the droplet's past trajectory, along the entirety of which the bouncing droplet generates waves. As the vibrating bath approaches its Faraday threshold (the critical vibrational acceleration above which waves form even in the absence of the droplet), the droplet's wave field becomes more persistent, so the system memory increases. When coupled with the resonance between droplet and bath that results in a highly structured, monochromatic wave field, the path memory may lead to the emergence of stable quantized states. Further increasing the memory may result in the emergence of chaos characterized by unpredictable switching between unstable periodic states and ultimately the emergence of multimodal, quantum-like statistics.

With the walking droplet system, a number of hydrodynamic quantum analogs (HQA) have been discovered in the laboratory and investigated both experimentally and theoretically. Examples include tunneling across barriers[14–16] and refraction from single and double slits.[17–19] A number of static and dynamic bound states of multiple droplets have been discovered, including crystal lattices,[20] orbiting pairs,[21,22] ratcheting pairs,[23] and promenading pairs.[24,25] In the context of orbital pilot-wave dynamics, walker motion in a rotating frame has yielded analogs for quantized Larmor levels,[26,27] Zeeman splitting,[28] and multimodal quantum-like statistics emerging in the chaotic regime.[27,29] The stability of

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hydrodynamic spin states, characterized by a droplet trapped in a circular orbit by its own wave field, has been considered in a number of studies[29,30] and is revisited in this issue.[31] Walker motion in a simple harmonic well is characterized by orbital trajectories that are quantized in both energy and mean angular momentum.[32,33] Coherent quantum-like statistics also emerge in the hydrodynamic analog of the quantum corral,[34,35] where statistical projection effects and an analog of the "quantum mirage" have been reported.[36] The Faraday system has also been explored as an optical analog; for example, a narrow channel has been explored as a walker waveguide,[37] and walker trapping in the vicinity of a linear array of pillars is reminiscent of particle trapping with the Talbot effect.[38] The majority of these hydrodynamic and optical analogs will be revisited in this focus issue.

A hierarchy of theoretical models have been developed to rationalize various walker behaviors.[26,39–43] In this focus issue, Turton *et al.*[44] review these models, providing valuable perspective in detailing their successes and failures. A table summarizes the different theoretical models developed by the HQA community. Stroboscopic models, deduced by averaging the equations of motion over the droplet's bouncing period,[39] prescribe the horizontal motion of walkers assuming perfect synchronization between the droplet's bouncing motion and the resulting wave form. While the stroboscopic models have proven to be relatively successful in describing single-droplet behavior,[29,33,40,42,45] they have limitations in describing the dynamics and stability of orbiting pairs[22] and promenading pairs.[25] A current area of focus in the HQA community is thus resolving the variability in the vertical bouncing dynamics,[43] a feature explicitly neglected in the stroboscopic models. The importance of variability in the bouncing dynamics is highlighted in three separate studies in this special issue.[46–48] Finally, the recent theoretical models of Nachbin *et al.*[16] and Faria[49] allow for robust treatment of walker-boundary interactions[19,50] and will both be called upon in this focus issue.

The manner in which quantum-like behavior emerges in the walker system is well studied in the context of "closed" systems, which arise when the drop is confined spatially through either imposed boundaries or applied forces. Examples include walker motion in corrals[34,36] or in the presence of a central spring force,[32,33] both of which will be revisited in this focus issue, and walker motion in a rotating frame.[26,27,29] In such closed systems, the droplet navigates its monochromatic wave field, which imposes a dynamic constraint on its motion. At relatively low memory, this constraint gives rise to stable quantized orbits. As the memory is increased progressively, the quantized orbital states go unstable via one of the classic routes to chaos.[35,51–53] At high memory, the walker switches between unstable periodic orbits, resulting in multimodal, quantum-like statistics.[27,32] While some of this behavior is captured by the stroboscopic models, discrepancies suggest the significance of variable vertical dynamics. In this issue, Perrard and Labousse[48] revisit walker motion in a harmonic potential and postulate that uncertainty in the vertical dynamics may be responsible for the chaotic switching in closed systems. The routes to chaos in this rich new dynamical system are evidently relatively subtle.

The walker behavior in open systems is no less rich, but the links with quantum systems are not as clear. For example, the extent to which quantum-like behavior arises in the diffraction of walkers from single and double slits remains a point of contention.[17–19] While the experiments of Pucci *et al.*[19] indicate that the walker behavior is predictable over the majority of parameter space considered, they did find unpredictable, evidently chaotic, behavior at the highest memories considered, the origins of which are not well understood. While Nachbin *et al.*[16] elucidated how unpredictability arises in tunneling between cavities, the manner in which it arises in tunneling across a barrier in an effectively open system[14] is similarly unclear. We note that Perrard and Labousse's[48] proposal that noise associated with variability in the bouncing dynamics may be a source of chaos is expected to be equally valid in open and closed walkers and so may have some bearing on this class of open systems. In this issue, Valani *et al.*[54] demonstrate the rich behavior of walker-walker collisions as described with a stroboscopic model. Tadrist *et al.*[47] demonstrate the importance of vertical phase variations as the source of unpredictability and chaos in colliding walkers. Finally, Galeano-Rios *et al.*[46] demonstrate that the perturbation of the vertical motion through the influence of the transient wave generated by their partner is critical. Evidently, variability in the vertical dynamics may prove to be a critical ingredient for chaos in both open and closed hydrodynamic pilot-wave systems.

Bush[2] proposed the extension of existing pilot-wave models to a generalized pilot-wave framework capable of achieving behavior inaccessible to the hydrodynamic pilot-wave system, a theme revisited here in the review of Turton *et al.*[55] While certain features of the walker system are essential for the apparent quantum-like behavior, others are evidently not. The simplest possible generalization is an exploration of a parameter regime beyond that accessible in the laboratory, an approach taken in three studies in this focus issue.[31,56,57] Other possibilities include the consideration of alternative particle-wave couplings or exploration of pilot-wave dynamics in three spatial dimensions. The results reported here indicate the vast range of dynamical behaviors accessible with this generalized pilot-wave framework.

In what follows, we briefly outline the contents of this focus issue and categorize the contributions into three central themes. First, we summarize studies of closed pilot-wave systems, including walker motion in corrals, a circular annulus, and a simple harmonic potential. Second, we summarize studies of open systems, including walker-walker interactions in various guises, walkers interacting with background wave fields, and walkers interacting with submerged topography. Finally, we report explorations of a generalized pilot-wave framework, of dynamical systems inspired by the walker system whose range of behavior extends beyond it. We close with a brief discussion of current perspectives in the field as informed by this special issue.

II. CLOSED SYSTEMS

In their paper, "Transition to chaos in wave memory dynamics in an harmonic well: Deterministic and noise-driven behaviour," Perrard and Labousse[48] present the results of an experimental and theoretical investigation of the chaotic dynamics of a walker in a harmonic well. The authors show chaotic trajectories in which the droplet oscillates between strange attractors that can be associated with the eigenstates of the system. Two distinct transition mechanisms are identified. The first mechanism is an intermittency dynamics typical of low-dimensional chaos. As such transitions have been captured theoretically with stroboscopic models,[42,45,53] one may surmise that variability in the vertical dynamics is not an essential ingredient for them. Conversely, the second mechanism, noise-driven chaos, relies explicitly on the influence of noise in the vertical dynamics. Specifically, it results from the fact that the global Lyapunov exponent of the system, although strictly negative, tends asymptotically to zero when the memory parameter is increased. The walker then becomes acutely sensitive to the discreteness of droplet impacts, which induces chaos. Both mechanisms yield insight into the selection rules for the transitions between eigenstates.

In their contribution, "Walking droplets in a circular corral: Quantization and chaos," Cristea-Platon *et al.*[58] report the results of an experimental investigation of walkers confined to circular cavities. Prior studies of corrals[34,36,51] have focused on the statistics emerging in the fully chaotic regime arising at high memory in relatively large corrals. Here, attention is given to smaller corrals, where periodic and quasi-periodic orbits are prevalent. These orbits exhibit a double quantization in average radius and angular momentum reminiscent of that arising for walker motion in a simple harmonic potential.[32,33] The study thus serves to unify the studies of walker motion in closed systems. In such closed systems, when the memory time exceeds the crossing time of the domain, the walker is always encountering a perturbed wave field, so responding to its environment as it navigates its own wave field.[45] In serving to drastically decrease the dimensionality of the system to a few wave eigenmodes, confinement is responsible for the quantum-like statistics in this and other closed systems.

Durey *et al.*[59] present the results of a theoretical investigation of a walker subject to a central force and confined to one-dimensional motion. The discrete-time numerical model[42] makes use of the spectral decomposition of the wave solution and velocity potential that collapses the dynamics onto fundamental matrices. The authors rationalize a number of regimes, including periodic quantized oscillations, chaotic motion, and the emergence of wave-like statistics. For an unbounded geometry, the authors deduce a simple relationship between the time-averaged (mean) pilot-wave field and the statistics of the droplets. Specifically, the mean wave field is proven to be the convolution of the droplet's histogram and the wave field of a stationary bouncer. They further report that in closed systems such as walker motion in a one-dimensional simple-harmonic potential, the instantaneous pilot-wave field converges to the mean, which thus plays the role of an additional applied potential. The walker dynamics is then cast in terms of a Langevin-like equation wherein the wave force has a component associated with the mean-pilot-wave potential as well as a stochastic element associated with the local perturbation of the instantaneous wave field from its mean. The study represents a substantial breakthrough in linking the dynamics and statistics of walking droplets. Moreover, it makes exciting new connections to both Bohmian mechanics[9,10] and de Broglie's double-solution pilot-wave theory.[7,8]

In "Standard map-like models for single and multiple walkers in an annular cavity," Rahman[60] takes a discrete-dynamical-systems approach in describing single and multiple walkers confined to a circular annulus. Inspired by the kicked rotator, a canonical problem in chaos and quantum chaos, the author develops a simple pilot-wave model in which each impact imparts a discrete kick to the droplet. The dynamical system then takes the form of the standard map (the most well-studied one-dimensional map) for the angular position of the droplet, with the kick strength being the control parameter. The bifurcations and the transition to chaos arising with increasing kick strength are characterized. The model is then extended to the case of multiple droplets. The models are simulated and exhibit several features reported in the experimental studies of strings of walkers in an annular cavity.[61] Using dynamical systems techniques and bifurcation theory, the single droplet model is analyzed to prove results suggested by the numerical simulations.

In "Walking droplets correlated at a distance," Nachbin[62] reports the results of simulations of two droplets confined to separate wells and divided by empty wells. The droplets communicate through their common wave field, and the intervening empty wells serve as a resonant transmission line. The droplets exhibit correlated dynamical features, even when separated by a considerable distance. Specifically, the position-velocity histograms of the two droplets display statistical coherence, and the droplets' complex distributions in phase space are statistically indistinguishable. Removing one drop drastically alters the phase space signature, which the author argues is reminiscent of quantum entanglement. The author forges provocative new links between the pilot-wave hydrodynamic system considered and both the Kuramoto model of coupled oscillators[63] and entanglement in stochastic electrodynamics.[11]

III. OPEN SYSTEMS

The paper "The interaction of a walking droplet and a submerged pillar: From scattering to the logarithmic spiral" by Harris *et al.*[64] presents a striking puzzle. Their experiments demonstrate that droplets aimed at a submerged pillar scatter at low memory but execute an expanding logarithmic spiral around the pillar at higher memory. The form of the spiral is independent of the droplet's impact parameter: there is a universal spiral emerging for a given drop size, pillar size, and memory. The system behavior is captured numerically using the theoretical model of Faria,[49] in which the pillar is treated as a region of decreased wave speed. Because the droplet speed remains equal to its free walking speed along the spiral, the system was ideally suited to application of the boost model,[65] which was used to infer an effective force due to the presence of the pillar. Remarkably, the pillar-induced force is found to be a lift force proportional to the product of the drop's walking speed and its instantaneous angular speed around the post. This system thus presents a macroscopic example of pilot-wave-mediated forces giving rise to apparent "action at a distance."

In 2008, Eddi *et al.*[23] demonstrated that neighboring droplets of unequal size interact through their common wave field, forming a bound pair that self-propels through a ratcheting mechanism. In their contribution to this focus issue, Galeano-Rios *et al.*[46] present an integrated experimental and theoretical investigation of such ratcheting pairs, detailing the dependence of the ratcheting behavior on the droplet sizes and vibrational acceleration. Their experiments demonstrate that the ratcheters exhibit quantized inter-drop distances, with this distance depending on the vibrational acceleration. For a given pair, as the vibrational acceleration increases progressively, the direction of the ratcheting motion may reverse up to four times. Numerical simulations based on recent theoretical developments[41,43] reproduce and rationalize the system behavior. Through demonstrating the critical importance of the vertical bouncing dynamics and specifically the transient wave generated during impact on the ratcheting behavior, this study underscores the shortcomings of the stroboscopic models in capturing close-range droplet-droplet interactions.

The paper "Bouncing droplet dynamics above the Faraday threshold" by Tambasco *et al.*[66] is the first to systematically characterize experimentally the behavior of walkers on a bath vibrated above the Faraday threshold. Several new dynamical regimes are reported, including meandering, zigzagging, erratic bouncing, coalescence, and trapping. The erratic bouncing state is similar to a random walk in which the elementary steps are of the order of the Faraday wavelength. Similar random walks have been reported below the Faraday threshold for walkers in confined geometries[34,35] and open systems.[67,68] The study suggests the possibility of tuning the relative magnitudes of the background walker wave fields, and that new HQAs may be discovered just above the Faraday threshold, where the walker navigates a background wave field prescribed by the system geometry. Such a scenario might be particularly useful in the context of diffraction, as it would bear a strong resemblance to the physical pictures proposed by both Bohmian mechanics[69] and stochastic electrodynamics.[70]

In "Hong-Ou-Mandel-like two-droplet correlations," Valani *et al.*[54] present the results of a numerical study of two-droplet correlations generated by a pair of walkers launched toward a common origin. By virtue of their pilot-wave coupling, the walkers may either scatter or form one of three two-droplet bound states, specifically orbiting, promenading, or drafting pairs. The probability of such pairing is quantified as a function of the parameters of their pilot-wave model. For certain parameters, the droplets may become correlated for certain initial paths and remain uncorrelated for others, while in other cases, the droplets may never produce bound states. The study suggests new directions in generating and rationalizing strongly correlated states in pilot-wave hydrodynamics.

Tadrist *et al.*[47] present the results of a combined experimental and theoretical study of walker-walker interactions. In their experiments, walkers are launched toward each other and either scatter or lock into bound states, specifically orbiting or promenading pairs. They introduce a new experimental technique that allows them to deduce the walkers' impact phase from visualization of the wave field, a technique that will prove valuable in resolving the vertical dynamics in many hydrodynamic pilot-wave systems. They demonstrate that the behavior during collisions cannot be predicted simply on the basis of the impact parameter; rather, the system is chaotic, the origins of the unpredictability being uncertainty in the vertical bouncing dynamics. They introduce a theoretical model in which they model the droplet as an inelastic ball riding a field of Faraday waves and demonstrate that the system behavior depends critically on the vertical dynamics of the walkers, specifically, the relative phase of the walker pair. Their study demonstrates that uncertainty in the vertical bouncing dynamics is responsible for the chaotic behavior in their system. In so doing, it introduces a new means of promoting chaos through vertical phase variations in a number of open hydrodynamic pilot-wave systems, including single- and double-slit diffraction.[17–19]

IV. GENERALIZED PILOT-WAVE DYNAMICS

In their contribution to this focus issue, Oza *et al.*[31] present theoretical results concerning the stability of hydrodynamic spin states. Under the action of a Coriolis force, the walking droplet executes a circular orbit whose radius becomes quantized at high memory owing to the droplet's interaction with its circularly polarized wave field.[26,27] Oza *et al.*[71] rationalized the observed stability of these orbits and the onset of chaos arising at high memory. They further demonstrated theoretically that, if the walker's wave force is sufficiently strong, it may result in a circular orbit even in the absence of rotation. While such hydrodynamic spin states have not been reported in the laboratory,[30] their stability is considered here in the context of a generalized pilot-wave framework[2] based on the stroboscopic model of Oza *et al.*[72] The authors characterize the parameter regime in which spin states would be stable and demonstrate that the application of weak rotation would then result in an analog of Zeeman splitting, as has been achieved experimentally with orbiting pairs.[28]

The comprehensive theoretical and numerical study of Valani and Slim,[57] "Pilot-wave dynamics of two identical, in-phase bouncing droplets," explores the range of behavior of this particular droplet pair configuration. Rather than constraining themselves to experimentally realizable parameters, they explore the generalized pilot-wave framework proposed by Bush,[2] wherein two control parameters prescribe the relative magnitudes of the inertial force, the drag force, and the wave force. In so doing, they discover a remarkably rich zoology of dynamical states, including experimentally achievable bound states such as in-line oscillations, drafting pairs, orbiting pairs, and promenading pairs. More exotic bound states hitherto unobserved in the laboratory were also discovered, including polygonal orbits of promenading pairs, as well as wandering, lopsided, and reversing promenading pairs. The study serves to demonstrate that pilot-wave systems have a fascinatingly rich range of behaviors not necessarily accessible to the hydrodynamic system.

In their paper "Exploring orbital dynamics and trapping with a generalized pilot wave framework," Tambasco and Bush[56] examine pilot wave dynamics in the presence of a central "well" of deeper fluid. Since the fluid in the well is locally above the Faraday threshold, the well creates a Faraday wave field that acts as an external potential with a Bessel form. Experiments show that this wave field may trap walking droplets on stable circular orbits. The authors then explore theoretically a generalized pilot-wave framework[2] in order to obtain richer dynamical behavior. Specifically, they alter the relative strengths of the drop inertia, the propulsive pilot-wave force, and the imposed Bessel potential. Orbits may then become unstable at high memory and transition to chaos through a path reminiscent of the Ruelle-Takens-Newhouse scenario. A parameter regime is found in which the drop switches chaotically between unstable circular orbits and ultimately converges to a statistically steady state. The form of the emergent statistically stationary probability distribution of the drops position reflects the relative instability of the circular orbits. The timescale of statistical convergence is characterized, and the mean wavefield in the chaotic regime is shown to be related to the droplet histogram through the convolution theorem of Durey *et al.*[59] reported in this issue.

In their contribution, "Bouncing ball on a vibrating periodic surface," Halev and Harris[73] explore a dynamical system representing an elastic ball bouncing on a vertically vibrating rigid surface with periodic topography. In addition to classical period-doubling bifurcations, they report horizontal symmetry breaking that yields a stable "walking" regime. In this regime, the ball motion consists of a horizontal translation of one wavelength of the topography at each bounce. Particular attention is given to characterizing the influence of the bottom topography, specifically the wave amplitude, wavelength, and wave shape, on the bouncing behavior.

A hydrodynamic analog to the optical Talbot effect was recently realized using a periodic array of pillars protruding from the surface of a vibrating fluid bath.[38] When the pillar spacing is simply related to the Faraday wavelength, repeated images of the pillars are projected in front of the array. These self-images have their origins in the sloshing inter-pillar menisci that act as sources of Faraday waves. The resulting self-images represent surface wave features that may serve to trap walkers. In this issue, Sungar *et al.*[74] build upon this previous study in order to explore the emergence of Faraday-Talbot patterns when the sloshing ridges between successive pillar pairs are out of phase. A simple theoretical model rationalizes the observed self-image locations for both linear and circular arrays of pillars. In the latter, array curvature allows for magnification and demagnification of the self-imaging pattern.

V. OUTLOOK

This focus issue reflects the current state of experimental, theoretical, and numerical investigations of hydrodynamic pilot-wave dynamics. Experimental studies have introduced new phenomenology[46,48,64] that will motivate theoretical developments, as well as new experimental techniques that will find wide application in the HQA community. Noteworthy is the technique of Tadrist *et al.*[47] for resolving the drop's impact phase through direct observation of its wave field and the use of bottom topography to exceed the Faraday threshold in a localized region and so induce an effective background potential.[56] The hierarchy of theoretical models developed by the HQA community[55] has been called upon to explore both established[46,48] and new[60,62,64] walker systems. The limitations of the stroboscopic models in a number of settings underscore the importance of resolving the fast dynamical scale,[43] specifically that of droplet bouncing, in order to rationalize the dynamics and stability of bound states,[46] wave-induced trapping,[38,66] and routes to chaos in both open and closed systems.[47,48]

The contribution of Durey *et al.*[59] represents a significant advance in our attempts to connect the dynamical and statistical behavior of classical pilot-wave systems and to better understand their relation to realist models of quantum dynamics.[8–11] While contributions to this special issue demonstrate how pilot-wave-mediated forces can generate both apparent "action at a distance"[64] and the long-range statistical indistinguishability of droplet pairs,[62] the definition of entanglement measures in the walker system is a subject of current interest and activity.[6] This special issue has also broadened the range of the walker system beyond traditional quantum[31,48,54] and optical analog systems[74] to touch upon quantum chaos,[58,60] statistical mechanics,[59] and the theory of coupled oscillators.[62] Finally, investigations of a generalized pilot-wave framework[31,56,57] provide early indication that this vast theoretical landscape will be fertile soil for novel dynamical systems and new HQAs and so will serve to further expand the phenomenological range of classical systems. It is hoped that this focus issue will attract the attention and interest of workers from the dynamical systems community and inspire some to join our exploration of this exciting new class of problems.

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