Short communication

Structured autocorrelation matrix estimation for coprime arrays

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A B S T R A C T

A coprime array receiver processes a collection of received-signal snapshots to estimate the autocorrelation matrix of a larger (virtual) uniform linear array, known as coarray. By the received-signal model, this matrix has to be (i) Positive Definite, (ii) Hermitian, (iii) Toeplitz, and (iv) its noise-subspace eigenvalues have to be equal. Existing coarray autocorrelation matrix estimates satisfy a subset of the above conditions. In this work, we propose an optimization framework which offers a novel estimate satisfying all four conditions: we propose to iteratively solve a sequence of distinct structure-optimization problems and show that, upon convergence, we provably obtain a single estimate satisfying (i)-(iv). Numerical studies illustrate that the proposed estimate outperforms standard counterparts, both in autocorrelation matrix estimation error and Direction-of-Arrival estimation.

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1. Introduction

I Direction-of-Arrival (DoA) estimation, coprime arrays offer increased Degrees-of-Freedom (DoF) and enable the identification of more sources than sensors compared to equal-length uniform linear arrays [1–13]. Coprime arrays have been successfully employed in applications such as beamforming design [14–16] and space-time processing [17], to name a few. Other non-uniform arrays with increased DoF and closed-form expressions are the nested and MISC arrays [18,19]. Intelligent processing of the autocorrelations of the physical array's elements enables the estimation of a signal subspace corresponding to a larger (virtual) array known as 'coarray' which, in general, is non-uniform. Commonly, only a segment of the coarray is retained wherein the elements are uniformly spaced. Alternatively, some works employ interpolation methods to fill the 'gaps' of the full coarray. In this work, we consider only the uniform segment of the coarray for simplicity. Our result is straightforwardly extended to the full coarray after its gaps are filled by existing interpolation methods (e.g., [20–22]).

A coprime array receiver processes the autocorrelations of the physical-array's elements and estimates the autocorrelation matrix of the coarray. By the received-signal model, the nominal autocorrelation matrix of the coarray has a specific structure: it is (i) Positive Definite (PD), (ii) Hermitian, (iii) Toeplitz, and (iv) its noise-subspace eigenvalues are equal. In practice, the autocorrelations of the physical-array's elements are estimated by processing a col-

lection of received-signal snapshots and diverge from the nominal ones. Accordingly, existing approaches offer autocorrelation-matrix estimates which diverge from the nominal one, while at the same time, violate at least one of the above structure-properties.

In this work, we propose an optimization framework for computing an improved coarray autocorrelation-matrix estimate that satisfies properties (i)-(iv). In practice, we iteratively solve a sequence of distinct structure-optimization problems, obtaining upon convergence, an improved estimate that satisfies properties (i)-(iv). The proposed framework is accompanied by formal convergence analysis. Our studies illustrate that the proposed method outperforms standard counterparts, both in estimation error and DoA estimation.

2. Signal model and problem statement

We consider coprime integers $M < N$ and design coprime array with $L \stackrel{\text{def}}{=} 2M + N - 1$ elements at locations $\mathcal{L} \stackrel{\text{def}}{=} \{(i-1)M\delta\}_{i=1}^{N} \bigcup \{iN\delta\}_{i=1}^{2M-1}$, where $\delta = \frac{\lambda}{2}$ is the reference inter-element spacing and λ is the wavelength [23]. Narrow-band signals impinge on the array from $K < MN + M$ sources with DoAs $\Theta \stackrel{\text{def}}{=} \{\theta_1, \theta_2, \ldots, \theta_K\}$ under carrier frequency f_c and propagation speed c. Under far-field conditions, the kth source-signal impinges on the array from direction $\theta_k \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ with respect to the broadside. Sensor element-locations are described by $\mathbf{p} \stackrel{\text{def}}{=} \text{sort}(\mathcal{L})$, where $\text{sort}(\cdot)$ sorts the elements of its input argument in ascending order. The array-response vector of source k is $\mathbf{s}(\theta_k) \stackrel{\text{def}}{=} [\nu(\theta_k)^{[\mathbf{p}]_1}, \nu(\theta_k)^{[\mathbf{p}]_2}, \ldots, \nu(\theta_k)^{[\mathbf{p}]_L}]^{\top}$, where $\nu(\theta_k) \stackrel{\text{def}}{=} \exp(-\frac{i 2\pi f_c}{c} \sin(\theta_k))$ and $(\cdot)^{\top}$ denotes the transpose operation. Accordingly, the receiver collects received-signal snap-

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shots of the form

$$\mathbf{y}_q \stackrel{\text{def}}{=} \sum_{k=1}^{K} \mathbf{s}(\theta_k) x_{q,k} + \mathbf{n}_q \in \mathbb{C}^L, \tag{1}$$

where $x_{q,k} \sim \mathcal{CN}(0, d_k)$ and $\mathbf{n}_q \sim \mathcal{CN}(\mathbf{0}_L, \sigma^2 \mathbf{I}_L)$, model the qth symbol transmitted by the kth signal-source (power-scaled and flat-fading-channel processed) and Additive White Gaussian Noise, respectively. $\mathcal{CN}(\boldsymbol{\mu}, \mathbf{C})$ denotes the complex normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$. For any integer $x \geq 1$, $\mathbf{0}_x = [0, 0, \ldots, 0]^\top \in \mathbb{R}^x$. Received-symbols are uncorrelated across snapshots and sources. Noise-variables are uncorrelated from received symbols. The receiver's objective is to identify Θ from the collected snapshots. Next, we briefly review standard coprime array processing.

Physical-Array Autocorrelation Matrix: The nominal received-signal autocorrelation matrix of the physical array is given by $\mathbf{R}_y \stackrel{\text{def}}{=} \mathbb{E}\{\mathbf{y}_q \mathbf{y}_q\} = \mathbf{S} \text{diag}(\mathbf{d}) \mathbf{S}^H + \sigma^2 \mathbf{I}_L$, where $(\cdot)^H$ denotes the conjugate-transpose (or, Hermitian) operation. $\mathbf{S} \stackrel{\text{def}}{=} [\mathbf{s}(\theta_1), \ldots, \mathbf{s}(\theta_K)]$ and $\mathbf{d} \stackrel{\text{def}}{=} [d_1, \ldots, d_K]^\top$ denote the array-response matrix and source-power vector, respectively. Since Θ, $\mathbf{d}$, and σ^2 are in practice unknown to the receiver, $\mathbf{R}_y$ can not be directly computed and is estimated based on Q received-signal snapshots by $\widehat{\mathbf{R}}_y \stackrel{\text{def}}{=} \frac{1}{Q} \sum_{q=1}^{Q} \mathbf{y}_q \mathbf{y}_q^H$.

Autocorrelation Sampling: Nominally, the receiver processes $\mathbf{R}_y$ and computes the autocorrelation-vector $\mathbf{r} \stackrel{\text{def}}{=} \text{vec}(\mathbf{R}_y) = \sum_{k=1}^{K} \mathbf{a}(\theta_k) d_k + \sigma^2 \mathbf{i}_L$, where $\mathbf{a}(\theta_k) \stackrel{\text{def}}{=} \mathbf{s}(\theta_k)^* \otimes \mathbf{s}(\theta_k)$ and $\mathbf{i}_L \stackrel{\text{def}}{=} \text{vec}(\mathbf{I}_L)$. Operators'$\otimes$' and $\text{vec}(\cdot)$ denote the Kronecker product [24] of matrices and vectorization operation of a matrix, respectively. For $j \in [L^2] \stackrel{\text{def}}{=} \{1, \ldots, L^2\}$, it holds $[\mathbf{a}(\theta_k)]_j = \nu(\theta_k)^n$, $n \in \mathcal{A} \stackrel{\text{def}}{=} \text{sort}(\{n_1 - n_2 \mid n_1, n_2 \in \mathcal{L}\})$. The element-locations of the uniform segment of the coarray are described by

$$\mathcal{B} \stackrel{\text{def}}{=} \{n \in \mathcal{A} \mid 1 - L' \leq n \leq L' - 1\}, \tag{2}$$

where $L' \stackrel{\text{def}}{=} MN + M$. For every $n \in \mathcal{B}$, the receiver discards duplicates by selecting any single index $j_n \in [L^2]$ such that $[\mathbf{a}(\theta_k)]_{j_n} = \nu(\theta_k)^n$. That is, the receiver forms selection-sampling matrix

$$\mathbf{E}_{\text{sel}} \stackrel{\text{def}}{=} [\mathbf{e}_{j_{1-L'}, L^2}, \ldots, \mathbf{e}_{j_{L'-1}, L^2}], \tag{3}$$

where for any $p \leq P \in \mathbb{N}_+$, $\mathbf{e}_{p,P}$ is the pth column of $\mathbf{I}_P$, and computes $\mathbf{r}_{\text{sel}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{sel}}^\top \mathbf{r} = \sum_{k=1}^{K} \mathbf{a}_{\text{sel}}(\theta_k) d_k + \sigma^2 \mathbf{e}_{L', 2L'-1}$, where $\mathbf{a}_{\text{sel}}(\theta_k) \stackrel{\text{def}}{=} \mathbf{E}_{\text{sel}}^\top \mathbf{a}(\theta_k) = [\nu(\theta_k)^{1-L'}, \ldots, \nu(\theta_k)^{L'-1}]^\top$. In practice, the autocorrelation-vector $\mathbf{r}$ is estimated by $\widehat{\mathbf{r}} \stackrel{\text{def}}{=} \text{vec}(\widehat{\mathbf{R}}_y)$ and $\mathbf{r}_{\text{sel}}$ is estimated by $\widehat{\mathbf{r}}_{\text{sel}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{sel}}^\top \widehat{\mathbf{r}}$.

Coarray Autocorrelation Matrix: The receiver applies a rank-enhancement approach on $\mathbf{r}_{\text{sel}}$ (or, $\widehat{\mathbf{r}}_{\text{sel}}$ in practice) to form the autocorrelation matrix of the coarray. Commonly, the *Augmented Matrix* [25] and *Spatial Smoothing* [23] approaches are employed. According to the augmented matrix approach, the receiver computes

$$\mathbf{R}_{\text{am}} \stackrel{\text{def}}{=} \mathbf{F}(\mathbf{I}_{L'} \otimes \mathbf{r}_{\text{sel}}) \in \mathbb{C}^{L' \times L'}, \tag{4}$$

where $\mathbf{F} \stackrel{\text{def}}{=} [\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_{L'}]$ and, for every $m \in [L']$, $\mathbf{F}_m \stackrel{\text{def}}{=} [\mathbf{0}_{L' \times (L'-m)}, \mathbf{I}_{L'}, \mathbf{0}_{L' \times (m-1)}]$. $\mathbf{R}_{\text{am}}$ has full-rank, is PD, Hermitian, Toeplitz, and coincides with the autocorrelation matrix of the coarray

$$\mathbf{R}_{\text{co}} = \mathbf{S}_{\text{co}} \text{diag}(\mathbf{d}) \mathbf{S}_{\text{co}}^H + \sigma^2 \mathbf{I}_{L'}, \tag{5}$$

where $[\mathbf{S}_{\text{co}}]_{m,k} \stackrel{\text{def}}{=} \nu(\theta_k)^{m-1}$, for every $m \in [L']$ and $k \in [K]$. According to the spatial-smoothing approach [23], in the case of known statistics, the receiver computes the spatially-smoothed matrix $\mathbf{R}_{\text{ss}} \stackrel{\text{def}}{=} \frac{1}{L'} \mathbf{R}_{\text{am}} \mathbf{R}_{\text{am}}^H$ which is not an autocorrelation matrix but an autocorrelation matrix is extracted from it as a scaled version of its

principal square root

$$\mathbf{R}_{\text{psr}} \stackrel{\text{def}}{=} \sqrt{L'} \mathbf{R}_{\text{ss}}^{\frac{1}{2}}. \tag{6}$$

We notice that $\mathbf{R}_{\text{am}} \mathbf{R}_{\text{am}}^H = \mathbf{R}_{\text{am}}^2 = L' \mathbf{R}_{\text{ss}}$. Moreover, $\mathbf{R}_{\text{ss}}$ admits Singular Value Decomposition (SVD) $\mathbf{R}_{\text{ss}} \stackrel{\text{svd}}{=} \mathbf{U} \boldsymbol{\Lambda} \mathbf{V}^H$ which implies that $\mathbf{R}_{\text{psr}} = \mathbf{U}(\sqrt{L'} \boldsymbol{\Lambda}^{\frac{1}{2}}) \mathbf{V}^H = \mathbf{R}_{\text{am}}$. That is, $\mathbf{R}_{\text{psr}}$ and $\mathbf{R}_{\text{am}}$ both coincide with $\mathbf{R}_{\text{co}}$. Here, we note that in the (ideal) case of known statistics to the receiver, all estimates above coincide with the nominal autocorrelation matrix of the coarray and satisfy (i)-(iv). However, in the practical case of unknown statistics (case of interest) to the receiver, the estimates above diverge from $\mathbf{R}_{\text{co}}$ and satisfy only a subset of (i)-(iv): The augmented matrix approach of [25] proposed to substitute the sampling matrix $\mathbf{E}_{\text{sel}}$ by the averaging sampling matrix

$$\mathbf{E}_{\text{avg}} \stackrel{\text{def}}{=} \left[\frac{1}{|\mathcal{J}_{1-L'}|} \sum_{j \in \mathcal{J}_{1-L'}} \mathbf{e}_{j,L^2}, \ldots, \frac{1}{|\mathcal{J}_{L'-1}|} \sum_{j \in \mathcal{J}_{L'-1}} \mathbf{e}_{j,L^2} \right], \tag{7}$$

where for every $n \in \mathcal{B}$, $\mathcal{J}_n = \{j \in [L^2] \mid [\mathbf{a}(\theta_k)]_j = \nu(\theta_k)^n\}$, substituting $\widehat{\mathbf{r}}_{\text{sel}}$ by $\widehat{\mathbf{r}}_{\text{avg}} \stackrel{\text{def}}{=} \mathbf{E}_{\text{avg}}^\top \widehat{\mathbf{r}}$.[1] Accordingly, $\mathbf{R}_{\text{co}}$ is estimated by $\widehat{\mathbf{R}}_{\text{am}} \stackrel{\text{def}}{=} \mathbf{F}(\mathbf{I}_{L'} \otimes \widehat{\mathbf{r}}_{\text{avg}})$. Importantly, it holds that $\widehat{\mathbf{R}}_{\text{am}}$ is Hermitian and Toeplitz, however, it's not guaranteed to be PD. That is, $\widehat{\mathbf{R}}_{\text{am}}$ can be an indefinite estimate of $\mathbf{R}_{\text{co}}$ [25]. Similarly, $\mathbf{R}_{\text{ss}}$ and $\mathbf{R}_{\text{psr}}$ are estimated by $\widehat{\mathbf{R}}_{\text{ss}} \stackrel{\text{def}}{=} \frac{1}{L'} \widetilde{\mathbf{R}}_{\text{am}} \widetilde{\mathbf{R}}_{\text{am}}^H$ and $\widehat{\mathbf{R}}_{\text{psr}} \stackrel{\text{def}}{=} \sqrt{L'} \widehat{\mathbf{R}}_{\text{ss}}^{\frac{1}{2}}$, respectively, where $\widetilde{\mathbf{R}}_{\text{am}} = \mathbf{F}(\mathbf{I}_{L'} \otimes \widehat{\mathbf{r}}_{\text{sel}})$.[2] $\widetilde{\mathbf{R}}_{\text{am}}$ can be an indefinite matrix. In view of the above, $\widehat{\mathbf{R}}_{\text{psr}}$ is by construction a PD and Hermitian matrix estimate of the coarray autocorrelation matrix, however, it violates the Toeplitz structure-property of $\mathbf{R}_{\text{co}}$. It follows that $\widehat{\mathbf{R}}_{\text{psr}}$ and $\widetilde{\mathbf{R}}_{\text{am}}$ no longer coincide, however, their left-hand singular-valued singular vectors span the same signal subspace. For the unknown statistics case, we summarize the above estimates in Table 1, where for each estimate we mention the employed autocorrelation sampling approach. Moreover, for each structure property guaranteed to be satisfied, we place a '√', otherwise, we place a '✗'. Given a coarray autocorrelation matrix estimate $\widehat{\mathbf{R}} \in \{\widehat{\mathbf{R}}_{\text{am}}, \widetilde{\mathbf{R}}_{\text{am}}, \widehat{\mathbf{R}}_{\text{psr}}\}$, a standard DoA estimation approach–e.g., MUltiple SIgnal Classification (MUSIC)–is applied for identifying the DoAs in Θ.

3. Proposed autocorrelation-matrix estimate

We propose an algorithm which iteratively solves a sequence of optimization problems returning, upon convergence, an improved coarray autocorrelation matrix estimate. Motivated by [26], where it was formally proven that averaging autocorrelation sampling attains superior autocorrelation estimates compared to selection sampling with respect to the Mean-Squared-Error (MSE) metric, we propose to initialize the proposed algorithm to $\mathbf{P}_0 = \sqrt{\frac{1}{L'} \widehat{\mathbf{R}}_{\text{am}} \widehat{\mathbf{R}}_{\text{am}}^H}$. At iteration $i \geq 0$, the proposed algorithm computes

$$\mathbf{Q}_i = \Phi(\mathbf{P}_i), \tag{8}$$

$$\mathbf{R}_i = \Psi(\mathbf{Q}_i), \tag{9}$$

$$\mathbf{P}_{i+1} = \Omega(\mathbf{R}_i), \tag{10}$$

where for any $\mathbf{X} = \mathbf{X}^H \in \mathbb{C}^{D \times D}$ with Eigenvalue Decomposition (EVD) $\mathbf{X} \stackrel{\text{evd}}{=} \mathbf{U} \text{diag}(\boldsymbol{\lambda}_x) \mathbf{U}^H$ the following hold.

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[1] If the nominal statistics are known, $\mathbf{r}_{\text{sel}}$ and $\mathbf{r}_{\text{avg}} = \mathbf{E}_{\text{avg}}^\top \mathbf{r}$ coincide. The latter does not hold if $\mathbf{r}_{\text{sel}}$ and $\mathbf{r}_{\text{avg}}$ are estimated by $\widehat{\mathbf{r}}_{\text{sel}}$ and $\widehat{\mathbf{r}}_{\text{avg}}$, respectively.

[2] $\widehat{\mathbf{R}}_{\text{am}}$ and $\widetilde{\mathbf{R}}_{\text{am}}$ denote the augmented matrix approach estimates combined with averaging and selection sampling, respectively.

Table 1
Comparison of coarray autocorrelation matrix estimates: autocorrelation sampling approach and structure properties.

Matrix estimate	Autocorrelation sampling approach	Positive Definite	Toeplitz	Hermitian	Equal noise-subspace eigenvalues
$\widetilde{\mathbf{R}}_{am}$	Selection	✗	✓	✓	✗
$\widehat{\mathbf{R}}_{am}$	Averaging	✗	✓	✓	✗
$\widehat{\mathbf{R}}_{psr}$	Averaging	✓	✗	✓	✗
Structured (proposed)	Averaging	✓	✓	✓	✓

Fig. 1. Illustration of the ith diagonal of $\mathbf{X} \in \mathbb{C}^{4 \times 4}$, $\mathbf{d}_i(\mathbf{X})$, $i \in \{0, \pm 2\}$.

Definition 1. $\Phi(\mathbf{X})$ returns the nearest Toeplitz matrix, in the Euclidean norm sense,[3] to $\mathbf{X}$: $\Phi(\mathbf{X}) \overset{\text{def}}{=} \underset{\mathbf{T}_0 \in \mathcal{T}^D}{\operatorname{argmin}} \|\mathbf{X} - \mathbf{T}_0\|_F^2$, where $\mathcal{T}^D \overset{\text{def}}{=} \{\mathbf{T} \in \mathbb{C}^{D \times D} \mid \mathbf{T} \text{ is Toeplitz}\}$.

Definition 2. $\Psi(\mathbf{X})$ returns the nearest Positive Semidefinite (PSD) matrix to $\mathbf{X}$: $\Psi(\mathbf{X}) \overset{\text{def}}{=} \underset{\mathbf{X}_0 \in \mathbb{S}_+^D}{\operatorname{argmin}} \|\mathbf{X} - \mathbf{X}_0\|_F^2$, where $\mathbb{S}_+^D \overset{\text{def}}{=} \{\mathbf{A} \in \mathbb{C}^{D \times D} \mid \mathbf{A} = \mathbf{A}^H \succeq \mathbf{0}\}$.

Definition 3. $\Omega(\mathbf{X})$ performs an eigenvalue-correction operation to the $D - \rho$ smallest eigenvalues of $\mathbf{X}$. For some general $\rho \in \{1, \ldots, D - 1\}$, $\Omega(\mathbf{X}) \overset{\text{def}}{=} \mathbf{U} \operatorname{diag}(\overline{\boldsymbol{\lambda}}_x) \mathbf{U}^H$, where

$$[\overline{\boldsymbol{\lambda}}_x]_i = \begin{cases} [\boldsymbol{\lambda}_x]_i, & i \leq D - \rho + 1, \\ \frac{1}{D - \rho} \sum_{j=\rho+1}^{D} [\boldsymbol{\lambda}_x]_j, & i > D - \rho + 1. \end{cases} \tag{11}$$

In view of the above, the proposed algorithm seeks to optimize the $D - \rho$ smallest eigenvalues of the autocorrelation matrix estimate at which it is initialized while preserving the PSD, Hermitian, and Toeplitz structure. Next, we conduct formal convergence analysis of the proposed algorithm.

Consider arbitrary $\mathbf{X} = \mathbf{X}^H \in \mathbb{C}^{D \times D}$ and let $\mathbf{d}_i(\mathbf{X})$ denote a diagonal of $\mathbf{X}$ (see Fig. 1) such that

$$[\mathbf{d}_i(\mathbf{X})]_j = \begin{cases} [\mathbf{X}]_{j-i,j}, & i \leq 0, \\ [\mathbf{X}]_{j,j+i}, & i > 0, \end{cases} \tag{12}$$

for any $j \in \{1, 2, \ldots, D - |i|\}$. The following remarks hold.

Remark 1. It holds that $\mathbf{d}_i(\mathbf{X}) = \mathbf{d}_{i'}^*(\mathbf{X})$, if $|i| = |i'|$.

Remark 2. Let $\mathbf{T} \overset{\text{def}}{=} \Phi(\mathbf{X}) = \min_{\mathbf{T}_0 \in \mathcal{T}^D} \|\mathbf{X} - \mathbf{T}_0\|_F^2$. For any $i \in \{1 - D, \ldots, D - 1\}$, it holds that $\mathbf{d}_i(\mathbf{T}) = \frac{1}{D-|i|} \mathbf{1}_{D-|i|}^\top \mathbf{d}_i(\mathbf{X}) \mathbf{1}_{D-|i|}$ where, for any integer $x \geq 1$, $\mathbf{1}_x = [1, 1, \ldots, 1]^\top \in \mathbb{R}^x$.

Remark 2 illustrates that the nearest Toeplitz matrix to $\mathbf{X}$ can be found in closed-form.

Remark 3. Let $\mathbf{T} \overset{\text{def}}{=} \Phi(\mathbf{X})$. It holds that $\|\mathbf{T}\|_F^2 \leq \|\mathbf{X}\|_F^2$.

Proof.

$$\|\mathbf{T}\|_F^2 = \sum_{i=1-D}^{D-1} \|\mathbf{d}_i(\mathbf{T})\|_2^2 = \sum_{i=1-D}^{D-1} \| \frac{1}{D-|i|} (\mathbf{1}_{D-|i|}^\top \mathbf{d}_i(\mathbf{X})) \mathbf{1}_{D-|i|} \|_2^2$$

$$= \sum_{i=1-D}^{D-1} \frac{(\mathbf{1}_{D-|i|}^\top \mathbf{d}_i(\mathbf{X}))^2}{(D-|i|)^2} \|\mathbf{1}_{D-|i|}\|_2^2 \tag{13}$$

$$= \sum_{i=1-D}^{D-1} \frac{(\mathbf{1}_{D-|i|}^\top \mathbf{d}_i(\mathbf{X}))^2}{D-|i|} \leq \sum_{i=1-D}^{D-1} \frac{\|\mathbf{1}_{D-|i|}\|_2^2 \|\mathbf{d}_i(\mathbf{X})\|_2^2}{D-|i|} = \|\mathbf{X}\|_F^2. \tag{14}$$

$\square$

By Remark 3, $\mathbf{T}$ is the nearest Toeplitz matrix to $\mathbf{X}$ and exhibits squared Frobenius norm lower or equal than that of $\mathbf{X}$.

Remark 4. Let $\mathbf{T} \overset{\text{def}}{=} \Phi(\mathbf{X})$ admit EVD[4] $\mathbf{T} = \mathbf{U} \operatorname{diag}(\boldsymbol{\lambda}_T) \mathbf{U}^H$.[5] It holds that $\|\mathbf{T}\|_F^2 = \operatorname{Tr}(\mathbf{U} \operatorname{diag}(\boldsymbol{\lambda}_T) \operatorname{diag}(\boldsymbol{\lambda}_T) \mathbf{U}^H) = \|\boldsymbol{\lambda}_T\|_2^2$.

Remark 5. Let $\mathbf{T} \overset{\text{def}}{=} \Phi(\mathbf{X})$ admit EVD $\mathbf{T} = \mathbf{U} \operatorname{diag}(\boldsymbol{\lambda}_T) \mathbf{U}^H$ and define $\mathbf{P}$ such that $\mathbf{P} = \mathbf{U} \operatorname{diag}(\boldsymbol{\lambda}_P) \mathbf{U}^H$, where[6] $\boldsymbol{\lambda}_P = \boldsymbol{\lambda}_T^+$ –i.e., $\forall i \in [D]$, $[\boldsymbol{\lambda}_P]_i = \max\{[\boldsymbol{\lambda}_T]_i, 0\}$. It holds that $\mathbf{P}$ is the solution to $\min_{\mathbf{P}_0 \in \mathbb{S}_+^D} \|\mathbf{T} - \mathbf{P}_0\|_F^2$. A proof for Remark 5 was first offered for real matrices in [27]. For completeness purposes, we offer an analogous proof for complex-valued matrices.

Proof. Consider Hermitian $\mathbf{T} \in \mathbb{C}^{D \times D}$ with EVD $\mathbf{T} = \mathbf{U} \operatorname{diag}(\boldsymbol{\lambda}_T) \mathbf{U}^H$, $\mathbf{U}$ is unitary (i.e., $\mathbf{U} \mathbf{U}^H = \mathbf{U}^H \mathbf{U} = \mathbf{I}_D$). Let $\mathbf{H} = \mathbf{U}^H \mathbf{P}_0 \mathbf{U}$ which implies that $\mathbf{P}_0 = \mathbf{U} \mathbf{H} \mathbf{U}^H$. It holds

$$\min_{\mathbf{P}_0 \in \mathbb{S}_+^D} \|\mathbf{T} - \mathbf{P}_0\|_F^2 = \min_{\mathbf{H} \in \mathbb{C}^{D \times D}} \|\operatorname{diag}(\boldsymbol{\lambda}_T) - \mathbf{H}\|_F^2$$

$$= \min_{\mathbf{H} \in \mathbb{C}^{D \times D}} \sum_{\substack{i,j \\ i \neq j}} [\mathbf{H}]_{i,j}^2 + \sum_{i=1}^{D} ([\boldsymbol{\lambda}_T]_i - [\mathbf{H}]_{i,i})^2 \tag{15}$$

$$\geq \sum_{\substack{i,j \\ i \neq j}} [\mathbf{H}]_{i,j}^2 + \sum_{i=1}^{D} ([\boldsymbol{\lambda}_T]_i - [\mathbf{H}]_{i,i}) \geq \sum_{i \in \{1,2,\ldots,D \mid [\boldsymbol{\lambda}_T]_i < 0\}} [\boldsymbol{\lambda}_T]_i^2. \tag{16}$$

Similar to [27], the lower bound in (16) is attained by matrix $\mathbf{H} = \operatorname{diag}(\boldsymbol{\lambda}_P)$ for $\boldsymbol{\lambda}_P$ such that $[\boldsymbol{\lambda}_P]_i = \max\{[\boldsymbol{\lambda}_T]_i, 0\}$. $\square$

By Remark 5, the nearest PSD matrix to $\mathbf{T}$ can be found in closed-form.

Remark 6. For $\mathbf{P} = \Psi(\mathbf{T})$, it holds that $\|\mathbf{P}\|_F^2 \leq \|\mathbf{T}\|_F^2$. Formally, $\|\mathbf{P}\|_F^2 = \|\boldsymbol{\lambda}_P\|_2^2 = \sum_{i=1}^{D} [\boldsymbol{\lambda}_P]_i^2 \leq \sum_{i=1}^{D} [\boldsymbol{\lambda}_T]_i^2 = \|\mathbf{T}\|_F^2$.
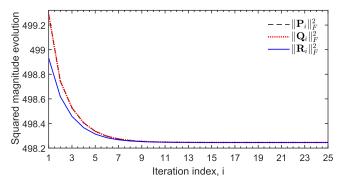
By Remark 6, $\mathbf{P}$ is the nearest PSD matrix to $\mathbf{T}$ and attains squared Frobenius norm lower or equal than that of $\mathbf{T}$.

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[3] Otherwise known as Frobenius norm: $\|\cdot\|_F^2$ returns the sum of the squared entries of its argument.

[4] A Hermitian matrix $\mathbf{A}$ can be expressed as $\mathbf{A} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^H$, where $\boldsymbol{\Lambda}$ is an upper diagonal with the eigenvalues of $\mathbf{A}$ in its main diagonal. If $\mathbf{A}$ is normal (i.e., $\mathbf{A} \mathbf{A}^H = \mathbf{A}^H \mathbf{A}$), then $\boldsymbol{\Lambda}$ is diagonal. Every Hermitian matrix is normal.

[5] For any normal matrix $\mathbf{W}$, we denote its eigenvalues by vector $\boldsymbol{\lambda}_W$. For instance, $\boldsymbol{\lambda}_T$ denotes the eigenvalues of $\mathbf{T}$ while $\boldsymbol{\lambda}_P$ denotes the eigenvalues of $\mathbf{P}$.

[6] $\boldsymbol{\lambda}_T^+$ returns a vector with length equal to that of $\boldsymbol{\lambda}_T$ with the positive entries of $\boldsymbol{\lambda}_T$ intact and its negative entries set to zero.

Fig. 2. Visual illustration of Proposition 1. $(M, N) = (2, 3)$, $\Theta = \{-43°, -21°, -10°, 17°, 29°, 54°\}$, $d_k = 0$ dB $\forall k$, $\sigma^2 = 1$, $Q = 50$.

Remark 7. Let $\mathbf{A} = \Omega(\mathbf{P}) = \mathbf{U}\text{diag}(\lambda_A)\mathbf{U}^H$, where

$$[\lambda_A]_i = \begin{cases} [\lambda_P]_i, & i \leq D - \rho + 1, \\ \frac{1}{D-\rho} \sum_{j=\rho+1}^{D} [\lambda_P]_j, & i > D - \rho + 1, \end{cases} \quad (17)$$

for $\rho \in \{1, \ldots, D-1\}$. It holds that $\|\mathbf{A}\|_F^2 \leq \|\mathbf{P}\|_F^2$.

Proof.

$$\|\mathbf{A}\|_F^2 = \sum_{i=1}^{D} [\lambda_A]_i^2 = \sum_{i=1}^{D-\rho} [\lambda_P]_i^2 + \rho \left(\frac{1}{\rho} \sum_{j=D-\rho+1}^{D} [\lambda_P]_j \right)^2$$
$$\leq \sum_{i=1}^{D-\rho} [\lambda_P]_i^2 + \sum_{j=D-\rho+1}^{D} [\lambda_P]_j^2 = \|\mathbf{P}\|_F^2. \quad (18)$$

$\square$

By Remark 7, the eigenvalue correction operation on a normal matrix returns a matrix with squared-Frobenius norm lower or equal than that of its input argument. In view of Remarks 1–7, the following Proposition derives.

Proposition 1. *For $\mathbf{Q}_i$, $\mathbf{R}_i$, and $\mathbf{P}_{i+1}$ in (8)-(10), it holds that $\|\mathbf{Q}_i\|_F^2 \geq \|\mathbf{R}_i\|_F^2 \geq \|\mathbf{P}_{i+1}\|_F^2 \geq \|\mathbf{Q}_{i+1}\|_F^2 \geq \ldots \geq 0 \forall i \geq 0$.*

Proposition 1 states that the proposed algorithm is guaranteed to converge. In practice, one can terminate the iterations when $\|\mathbf{P}_{i+1} - \mathbf{P}_i\| \leq \epsilon$, for some $\epsilon \geq 0$. For sufficiently small ϵ, Proposition 1 implies that, at convergence, $\mathbf{P}_{i+1} = \mathbf{R}_i = \mathbf{Q}_i$ which, in turn, implies that the algorithm converged to a PD, Hermitian, and Toeplitz matrix the noise-subspace eigenvalues of which are equal. A visual illustration of Proposition 1 and a pseudocode of the proposed algorithm are offered in Fig. 2 and Fig. 3, respectively. Importantly, $\forall i \geq 0$, the Algorithm of Fig. 3 computes $\mathbf{Q}_i, \mathbf{R}_i, \mathbf{P}_{i+1}$ by closed-form expressions with cost at most the cost of EVD– i.e., $\mathcal{O}(D^3)$. Overall, the cost of the proposed algorithm is $\mathcal{O}(TD^3)$ where T is the number of iterations required for convergence.

4. Numerical studies

We consider coprime naturals $(M, N) = (3, 5)$ and form coprime array with $L = 10$ elements. Source-signals impinge on the array from $K = 13$ DoAs $\{\theta_k\}_{k=1}^{13}$, $\theta_k = (-75 + (k-1)12)°$. The noise variance is set to $\sigma^2 = 0$dB. All sources emit signals with equal power $d_k = \alpha^2$dB. Accordingly, the Signal-to-Noise Ratio SNR $= \alpha^2$. The receiver collects $Q \in \{150, 300, 450, 600\}$ received-signal snapshots. For every Q, we consider $\Delta = 3000$ statistically independent realizations of noise; i.e., $\{\mathbf{y}_{r,1}, \ldots, \mathbf{y}_{r,Q}\}_{r=1}^{\Delta}$. At every realization r, we compute coarray autocorrelation matrix estimates corresponding to the augmented matrix approach (AM), principal square root of the spatial smoothed matrix (PSR), nearest Hermitian, PSD,

and Toeplitz (H-PSD-T) approach of [22][7], and the proposed structured estimate. We take a moment and discuss similarities and differences between H-PSD-T and the proposed framework: (1) H-PSD-T is guaranteed to satisfy the Toeplitz, Hermitian, and Positive Semidefinite structure properties while the proposed estimate is guaranteed to satisfy all four, (2) H-PSD-T requires tuning of an ad-hoc parameter that moderates overfitting while the proposed framework is ad-hoc parameter free, and (3) H-PSD-T computes an estimate by iterative solvers for convex optimization problems while the proposed approach enjoys closed-form solutions for each individual optimization problem in its sequence. For every method and estimate $\widehat{\mathbf{R}}_r$ at realization r, we compute the Normalized Squared Error

$$\text{NSE}(\widehat{\mathbf{R}}_r) = \|\widehat{\mathbf{R}}_r - \mathbf{R}_{co}\|_F^2 \|\mathbf{R}_{co}\|_F^{-2}. \quad (19)$$

Then, we compute the Root Mean Normalized Squared Error

$$\text{RMNSE} = \sqrt{\frac{1}{\Delta} \sum_{r=1}^{\Delta} \text{NSE}(\widehat{\mathbf{R}}_r)}. \quad (20)$$

In Fig. 4(a) and Fig. 4(d), we plot the RMNSE versus sample support, Q, for SNR $= -4$dB and SNR $= 2$dB, respectively. Expectedly, we observe that all methods employing averaging-sampling perform similarly well. The proposed estimate attains superior estimation performance across the board. Moreover, we notice the sensitivity of H-PSD-T with respect to the ad-hoc parameter μ; e.g., for SNR $= -4$dB, H-PSD-T with $\mu = 1.5$ exhibits low performance while for SNR $= 2$dB it exhibits high estimation performance.

Thereafter, we consider that the nominal coarray autocorrelation matrix admits SVD $\mathbf{R}_{co} = \mathbf{Q}_{co}\mathbf{\Sigma}_{co}\mathbf{V}_{co}^H + \sigma^2\bar{\mathbf{Q}}_{co}\bar{\mathbf{Q}}_{co}^H$, where $\mathbf{Q}_{co}$ and $\bar{\mathbf{Q}}_{co}$ correspond to the signal and noise subspace bases, respectively. Similarly, every coarray autocorrelation matrix estimate $\widehat{\mathbf{R}}_r$ admits SVD $\widehat{\mathbf{R}}_r = \mathbf{Q}_r\mathbf{\Sigma}_r\mathbf{V}_r^H + \bar{\mathbf{Q}}_r\bar{\mathbf{\Sigma}}_r\bar{\mathbf{V}}_r^H$, where $\mathbf{Q}_r$ denotes the signal-subspace-basis of the K dominant left-hand singular valued singular vectors of $\widehat{\mathbf{R}}_r$. At each realization and for every value of Q, we compute the Normalized Squared Subspace Error

$$\text{NSSE}(\widehat{\mathbf{Q}}_r) = \|\widehat{\mathbf{Q}}_r\widehat{\mathbf{Q}}_r^H - \mathbf{Q}_{co}\mathbf{Q}_{co}^H\|_F^2 (2K)^{-1}. \quad (21)$$

Then, we compute the Root Mean Normalized Squared Subspace Error

$$\text{RMN-SSE} = \sqrt{\frac{1}{\Delta} \sum_{r=1}^{\Delta} \text{NSSE}(\widehat{\mathbf{Q}}_r)}. \quad (22)$$

In Fig. 4(b) and Fig. 4(e), we plot the RMN-SSE versus sample support for SNR $= -4$dB and 2dB, respectively. We notice the influence of the ad-hoc parameter μ with respect to H-PSD-T and observe that the proposed structured estimate clearly outperforms all counterparts across the board in subspace estimation performance.

Next, for every value of sample support and realization r, we conduct DoA estimation by applying MUSIC on the estimate $\widehat{\mathbf{R}}_r$ which returns estimates $\{\widehat{\theta}_{k,r}\}_{k=1}^K$ [26]. Then, we measure the Root Mean Squared Error

$$\text{RMSE} = \sqrt{\frac{1}{\Delta} \frac{1}{K} \sum_{r=1}^{\Delta} \sum_{k=1}^{K} (\theta_k - \widehat{\theta}_{k,r})^2} \quad (23)$$

and illustrate the corresponding RMSE curves versus sample support Q, in Fig. 4(c) and Fig. 4(f), for SNR $= -4$dB and 2dB, respectively. We include the Cramér Rao Lower Bound (CRLB) curves as benchmarks [28]. We notice that the performances of standard

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[7] H-PSD-T seeks a Hermitian-PSD-Toeplitz matrix which fills the gaps of the coarray. When the uniform segment of the coarray is considered, H-PSD-T returns $\underset{\mathbf{R} \in \mathbb{S}_+^{L'}}{\arg\min} \|\mathbf{R} - \widehat{\mathbf{R}}_{am}\|_F^2 + \mu \|\mathbf{R}\|_*$, where $\mu \|\mathbf{R}\|_*$ is a regularization term that moderates overfitting.

Algorithm 1. Structured coarray autocorrelation matrix estimation

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Input: Coarray autocorrelation matrix estimate $\widehat{\mathbf{R}}$

0:     $\mathbf{P}_0 \leftarrow \widehat{\mathbf{R}}$ % Initialization

1:     Until convergence/termination

2:        $\mathbf{Q}_i \leftarrow \Phi(\mathbf{P}_i)$ % Nearest Toeplitz to $\mathbf{P}_i$

3:        $\mathbf{R}_i \leftarrow \Psi(\mathbf{Q}_i)$ % Nearest PSD to $\mathbf{Q}_i$

4:        $\mathbf{P}_{i+1} \leftarrow \Omega(\mathbf{R}_i)$ % Eigenvalue-correction

Return: $\mathbf{R} \leftarrow \mathbf{P}_i$

Fig. 3. Proposed structured coarray autocorrelation matrix estimation.
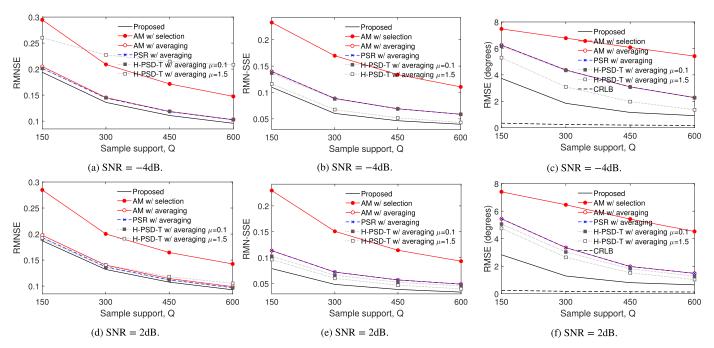

Fig. 4. Root Mean Normalized Squared Error (RMNSE) with respect to (w.r.t.) $\mathbf{R}_{co}$, Root Mean Normalized - Subspace Squared Error (RMN-SSE) w.r.t. $\mathbf{Q}_{co}$, and Root Mean Squared Error (RMSE) w.r.t. Θ versus sample support for varying SNR $\in \{-4, 2\}$dB.

counterparts (AM, PSR) deviate significantly from the CRLB. In contrast, the proposed coarray autocorrelation matrix estimate outperforms all counterparts by at least 0.3° and at most 2°. In addition, as Q increases, its performance curves approach the CRLB curves.

5. Conclusions

We proposed an optimization framework which computes a structured coarray autocorrelation matrix estimate. The proposed algorithm is accompanied by convergence analysis and is guaranteed to return a coarray autocorrelation matrix estimate satisfying all structure properties of the true autocorrelation matrix. Numerical studies illustrate the enhanced performance of the proposed estimate compared to standard counterparts, both in autocorrelation matrix estimation error and DoA estimation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Dimitris G. Chachlakis: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Writing - original

draft, Writing - review & editing. **Panos P. Markopoulos:** Conceptualization, Methodology, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Project administration, Funding acquisition, Supervision, Resources.

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