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We study certain mod p differential operators that act on automorphic forms over Shimura varieties of type A or C. We show that, over the ordinary locus, these operators agree with the mod p reduction of the p -adic theta operators previously studied by some of the authors. In the characteristic 0, p -adic case, there is an obstruction that makes it impossible to extend the theta operators to the whole Shimura variety. On the other hand, our mod p operators extend (“analytically continue”, in the language of de Shalit and Goren) to the whole Shimura variety. As a consequence, motivated by their use by Edixhoven and Jochnowitz in the case of modular forms for proving the weight part of Serre’s conjecture, we discuss some effects of these operators on Galois representations.

Our focus and techniques differ from those in the literature. Our intrinsic, coordinate-free approach removes difficulties that arise from working with q -expansions and works in settings where earlier techniques, which rely on explicit calculations, are not applicable. In contrast with previous constructions and analytic continuation results, these techniques work for any totally real base field, any weight, and all signatures and ranks of groups at once, recovering prior results on analytic continuation as special cases.

1. Introduction

Fix an odd prime number p . This paper concerns certain mod p differential operators that act on automorphic forms over Shimura varieties of types A and C. Key motivation comes from the setting of modular forms and associated Galois representations. Even though some of the ingredients present in the case of modular forms have been established in broad generality, extending them to the higher-rank setting involves various challenges.

1.1. Context from the special case of GL_2 . Katz [1977] constructed a mod p differential operator θ that acts on mod p modular forms. Given a mod p modular form f , the operator θ acts on the q -expansion $f(q) = \sum_{n \geq 0} a_n q^n$ by

$$\theta f(q) = \sum_{n \geq 0} n a_n q^n. \quad (1)$$

Mod p modular forms of weight k and level N , with $p \nmid N$, arise as global sections of a certain line bundle $\omega^{\otimes k}$ over a curve $\mathcal{M}_N$ parametrizing elliptic curves with level N structure. That is, modular

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forms of weight k and level N are elements of $H^0(\mathcal{M}_N, \omega^{\otimes k})$. The operator θ is, a priori, constructed as a map $H^0(\mathcal{M}_N^{\text{ord}}, \omega^{\otimes k}) \rightarrow H^0(\mathcal{M}_N^{\text{ord}}, \omega^{\otimes k+2})$, i.e., only over the ordinary locus $\mathcal{M}_N^{\text{ord}}$ of $\mathcal{M}_N$. Given any modular form $f \in H^0(\mathcal{M}_N, \omega^{\otimes k})$, however, Katz explains how to extend (“analytically continue”, in the terminology of [de Shalit and Goren 2019]) the section $\theta(f|_{\mathcal{M}_N^{\text{ord}}})$ to a classical modular form of weight $k + p + 1$, i.e., an element of $H^0(\mathcal{M}_N, \omega^{\otimes k+p+1})$ whose q -expansions are the same as those of $\theta(f|_{\mathcal{M}_N^{\text{ord}}})$.

Associated to a cuspidal eigenform f with Fourier expansion $f(q) = \sum_{n \geq 1} a_n q^n$, normalized so that $a_1 = 1$, there is a continuous, semisimple representation

$$\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p), \quad (2)$$

unramified at all primes ℓ coprime to p and to the level of f , satisfying

$$\text{Trace}(\rho_f(\text{Frob}_\ell)) = a_\ell \quad \text{and} \quad \det(\rho_f(\text{Frob}_\ell)) = \ell^{k-1} \psi(\ell)$$

at such primes, where k is the weight of f and ψ is the nebentypus of f [Eichler 1954; Shimura 1958a; 1958b; Igusa 1959; Deligne 1971; 1973; Deligne and Serre 1974]. Equivalently, using the description of Hecke operators on q -expansions, this statement can be reformulated in terms of Hecke eigenvalues in place of Fourier coefficients. Serre’s conjectures (formulated in [Serre 1975; 1987] and proved in generality in [Khare and Wintenberger 2009a; 2009b; 2009c] and in certain cases in [Dieulefait 2007; 2009]) associate to a representation

$$\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F}),$$

with $\mathbb{F}$ a finite field of characteristic p , a cuspidal eigenform f such that $\bar{\rho}_f \cong \rho$, where $\bar{\rho}_f$ denotes the reduction of $\rho_f \bmod p$. Serre’s conjectures also specify the weight and level of f .

The effect of iterations of θ (“theta cycles”) plays a key role in the proof of the weight part of Serre’s conjecture [Edixhoven 1992; Jochnowitz 1982]. Via the action of Hecke operators T_ℓ on q -expansions, it is simple to observe that

$$T_\ell \theta = \ell \theta T_\ell. \quad (3)$$

As a consequence, the effect of θ on the mod p Galois representation attached to a mod p eigenform f is

$$\rho_{\theta f} = \chi \otimes \rho_f, \quad (4)$$

where $\chi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{F}_p^\times$ is the mod p cyclotomic character.

Continuing with the case of GL_2 , Andreatta and Goren studied mod p theta operators in the case of Hilbert modular forms over a totally real field L . They also obtained several results on theta cycles in [Andreatta and Goren 2005, Section 16], building on earlier work of Jochnowitz [1982], at least assuming p remains inert in L and the weight is parallel (although they also obtain preliminary results in the nonparallel case).

1.2. The case of higher rank (especially unitary and symplectic) groups. Given recent developments for automorphic forms on higher-rank groups and given the significance of mod p theta operators in the case of modular forms, it is natural to try to extend results on mod p theta operators to the setting of higher-rank groups. In particular, one desires

- (i) a mod p theta operator that is defined over the whole Shimura variety (and, ideally, the relationship with the mod p reduction of the p -adic operators constructed over the ordinary locus, as, for example, in [Eischen 2012; Eischen et al. 2018; Eischen and Mantovan 2021b]),
- (ii) knowledge of the amount by which this mod p theta operator increases the weight of an automorphic form,
- (iii) connections with Galois representations.

Since various ingredients exist in broad generality for higher-dimensional groups, it is tempting to think one can handle all settings at once. In fact, the arguments in the literature are delicate and have involved imposing restrictions specific to the cases at hand. To give the reader an idea of the state of the art prior to this paper, we briefly summarize the most recent developments (primarily from the past year):

- Siegel modular forms (symplectic groups): Yamauchi [2014] recently produced results for mod p Siegel modular forms, in the case of GSp_4 over $\mathbb{Q}$. His clever computations rely on the numerics of the case at hand, including conditions on the weights, and do not naturally generalize.
- Unitary groups: Via techniques relying on Fourier–Jacobi expansions, de Shalit and Goren [2016; 2019] achieved analytic continuation when the base real field is $\mathbb{Q}$ (so the CM field associated to the unitary group is imaginary quadratic) and the weight is scalar. It is not immediately clear how to use this approach when the degree of the field extension is greater or the weight is nonscalar. Furthermore, under these conditions, they achieved new, interesting results on theta cycles, which require that p be inert in the imaginary quadratic field and the signature of the unitary group be $(n, 1)$, and the numerics there are particular to these conditions being met.

Our focus and techniques differ from those recently introduced in the literature, and the principal achievements of the present paper include removal of some of these conditions for construction and analytic continuation of theta operators. Our intrinsic, coordinate-free approach avoids a need for q -expansions and works in settings where their techniques, which include explicit calculations, are not applicable. In particular, for analytic continuation, our techniques immediately

- work for any totally real base field E (while their approaches require $E = \mathbb{Q}$),
- allow all vector and scalar weights (in contrast to the restrictions above),
- handle all signatures and ranks of groups at once (whereas Yamauchi employs GSp_4 and the approach of de Shalit and Goren divides signatures into two cases).

When the ordinary locus is nonempty and their constraints are also met, our operators agree with theirs. As explained in [Eischen and Mantovan 2021a], the present approach can also be extended (with additional

work) to the case where the ordinary locus is empty. Now that we have defined this operator in much more generality, we show that results of Katz from the case of modular forms (which can be viewed as preliminary results toward theta cycles) extend to Hermitian (i.e., unitary of signature (n, n)) and Hilbert–Siegel modular forms when the weight is parallel scalar but the base field can be of arbitrary degree. We also achieve other nearly immediate results for Galois representations and set the stage for further developments in [Eischen and Mantovan 2021a].

Our proofs are entirely intrinsic to the geometry of the underlying Shimura varieties. This allows us to overcome what might appear — from the setup for modular forms in Section 1.1 — to be an obstruction to working with more general groups: We do not necessarily have q -expansions in our setting, thus depriving us of analogues of (1). By exploiting geometry, however, we obtain clean, intrinsic descriptions without reference to q -expansions (and without reference to Serre–Tate or Fourier–Jacobi expansions, which might at first appear to be reasonable substitutes but turn out not to be similarly suitable for studying the action of the Hecke algebras).

Our first result below concerns operators $\mathcal{D}^\lambda$, generalizing θ , that have been previously constructed over the ordinary locus of unitary and symplectic Shimura varieties (e.g., in [Eischen 2012; Eischen et al. 2018], when the ordinary locus is nonempty, building on ideas introduced by Katz [1978] and Harris [1981; 1986]). More precisely, in [Eischen 2012; Eischen et al. 2018], the operators $\mathcal{D}^\lambda$ are defined on p -adic forms over the ordinary locus. Here, we are concerned with their reduction modulo p , which provides an analogous weight-raising operator for p sufficiently large (as we explain in Remark 3.2.2).

Note that, in contrast to the conditions forced by the approaches of prior work in the literature, there are no restrictions on the degree of the base field and no restrictions on the weight of the automorphic forms here. As a consequence of Theorem 3.4.1, we obtain the following result.

Theorem A (analytic continuation of mod p differential operators). *In the symplectic and unitary cases, the differential operators $\mathcal{D}^\lambda$, defined a priori only over the ordinary locus, can be analytically continued to the whole mod p Shimura variety to give a differential operator Θ^λ on mod p automorphic forms. These operators specialize in the case of $\mathrm{GSp}_2 = \mathrm{GL}_2$ to Katz’s operator θ described in (1).*

Similarly to Katz’s case of raising the weight by $p + 1$, our operators also raise the weight as specified in Theorem B. With these operators, we can reformulate equations (3) and (4) to obtain Theorem B. (Below, $\widehat{G}$ denotes the dual group of G , $\widehat{\nu}$ is the cocharacter dual to the similitude character ν of G , and $\underline{k}$ is the parallel weight with entries all equal to $k \in \mathbb{Z}$.)

Theorem B (action of differential operators on mod p Galois representations). *Let G be a symplectic or unitary group over $\mathbb{Q}$, split over a number field F . Let f be a mod p Hecke eigenform on G of weight κ , and let λ be a weight that is admissible (as in Definition 2.2.3). Assume $\Theta^\lambda(f)$ is nonzero. Then $\Theta^\lambda(f)$ is a Hecke eigenform of weight $\kappa + \lambda + \underline{(p - 1)|\lambda/2|}$.*

Furthermore, for a continuous representation $\rho : \mathrm{Gal}(\overline{F}/F) \rightarrow \widehat{G}(\overline{\mathbb{F}}_p)$, the Frobenius eigenvalues of ρ agree with the Satake parameters of f (as defined in Conjecture 2.1.1) if and only if the Frobenius

eigenvalues of $(\hat{v}^{|\lambda|/2} \circ \chi) \otimes \rho$ agree with the Satake parameters of $\Theta^\lambda(f)$. In particular, if ρ is modular of weight κ , then $(\hat{v}^{|\lambda|/2} \circ \chi) \otimes \rho$ is modular of weight $\kappa + \lambda + \underline{(p-1)|\lambda|/2}$.

Remark 1.2.1. The reader who prefers to work with representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ instead of $\text{Gal}(\overline{F}/F)$ can do so by replacing the split version of the Satake isomorphism described in [Section 2.1](#) (where we follow [\[Gross 1998\]](#)) with the nonsplit version from [\[Treumann and Venkatesh 2016, Section 7\]](#). The Galois representations are then of the form $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow {}^L G(\overline{\mathbb{F}}_p)$, taking values in the Langlands dual group of G .

Remark 1.2.2. In the classical GL_2 case described in [Section 1.1](#), passing from the commutation relation between differential and Hecke operators [\(3\)](#) to the twisting effect on Galois representations [\(4\)](#) is immediate, given the well-known expression of the characteristic polynomial of Frobenius as an explicit quadratic polynomial. The same approach can be used in the GSp_4 case [\[Yamauchi 2014\]](#), where the characteristic polynomial of Frobenius can be written down as an explicit quartic polynomial. Beyond these two low-rank cases, the explicit approach is impractical given that the polynomial for GSp_{2g} has degree 2^g . We therefore adopt a more conceptual view and use the Satake isomorphism to establish the effect of differential operators on Galois representations in [Theorem B](#).

Remark 1.2.3. Katz [\[1977, Section I\]](#) discusses the *exact filtration* of a mod p modular form f : the smallest k such that f is not divisible by the Hasse invariant, or equivalently, the smallest k such that there is no modular form of weight $k' < k$ whose q -expansion at some cusp agrees with the q -expansion of f . The notion of exact filtration is related to questions about the kernel of the differential operator (times the Hasse invariant) and to the proof of the weight part of Serre's conjecture, as described in, e.g., [\[Edixhoven 1992, Section 3\]](#). As an indication of potential future applications of [Theorem B](#), [Section 5.2](#) contains some remarks on the effect of the differential operator on the exact filtration, in the special case of Siegel or Hermitian forms of parallel weight.

Remark 1.2.4. As noted above, when working with unitary groups, we assume the ordinary locus is nonempty. Fully extending our results to the case where the ordinary locus is empty, the μ -ordinary setting, requires substantial additional work and is the subject of [\[Eischen and Mantovan 2021a\]](#). The μ -ordinary case is also addressed, via different techniques, when the base field is $\mathbb{Q}$ (so the field associated to the unitary group is quadratic imaginary) and the weight is scalar in [\[de Shalit and Goren 2016; 2019\]](#).

Remark 1.2.5. The differential operator θ on classical modular forms can be constructed via several other methods, some of which have been generalized to certain groups of higher rank. The interested reader can consult the survey in [\[Ghitza 2019\]](#).

1.3. Obstructions to analytic continuation in characteristic 0. Since the representation [\(2\)](#) lives in characteristic 0, it is natural to ask about the possibility of lifting [Theorems A](#) and [B](#) from the mod p setting to the p -adic setting.

As an intermediate step, one might try to lift to characteristic p^m for $m > 1$ an integer. Chen and Kiming [\[2016, Theorem 1\]](#) have accomplished this for mod p^m modular forms f , and they have proved

that if f has weight filtration k , then θf has weight filtration $k + 2 + 2p^{m-1}(p - 1)$. This shows that, for modular forms over $\mathbb{Z}_p$, there is no way to analytically continue the operator θ from the ordinary locus to the entire Shimura variety over $\mathbb{Z}_p$. Indeed, if it were possible to analytically continue θ , then the weight filtration of $\theta f \bmod p^m$ would be bounded above by the weight of the characteristic 0 form θf ; but as Chen and Kiming have shown, the weight filtration of $\theta f \bmod p^m$ is unbounded as m goes to infinity.

Observing this obstruction to analytically continuing the entire form in characteristic 0, one might try just to analytically continue the *ordinary* part of $\mathcal{D}_\kappa^\lambda f$. More precisely, let e denote Hida's ordinary projector (defined in [Hida 2002, Sections 6–8]), which acts on the space of p -adic automorphic forms. A p -adic automorphic form is *ordinary* if it is in the image of e . Every ordinary p -adic automorphic form of sufficiently regular algebraic weight is, in fact, an ordinary classical automorphic form defined over the whole Shimura variety $\mathcal{X}$, by [Hida 2002, Theorems 6.8(4) and 7.1(4)]. So even though $\mathcal{D}_\kappa^\lambda f$ cannot be extended to all of $\mathcal{X}$, we at least have that $e\mathcal{D}_\kappa^\lambda f$ extends to all of $\mathcal{X}$.

As we shall see in Corollary 4.3.5, though, the proof of Theorem B (an analysis of the interaction between Hecke operators and differential operators) also shows that $e\mathcal{D}_\kappa^\lambda = 0$. So the ordinary part of $\mathcal{D}_\kappa^\lambda f$ is 0. In the case of modular forms in characteristic 0, Coleman, Gouvêa, and Jochnowitz [1995, Corollary 10] also showed that θ destroys overconvergence, due to a relationship with the weight 2 Eisenstein series E_2 (a relationship that actually enables the mod p^m liftings in [Chen and Kiming 2016]).

Remark 1.3.1. Readers who are primarily familiar with the characteristic 0, p -adic setting should be careful not to equate the objectives of this paper with recent work on overconvergence, which concerns extending modular forms past the ordinary locus (but not to the whole moduli space) as a step toward extending certain constructions of p -adic L -functions. As the above shows, in characteristic 0, there is no hope of extending our operator to the whole Shimura variety. Furthermore, recent work on overconvergence and theta operators has primarily focused on GL_2 , whereas we are interested in higher-rank groups.

1.4. Organization. The meat of the paper is in Sections 3 and 4. First, though, Section 2 introduces necessary conventions and background (i.e., previously established results and machinery on which our present paper relies), divided into two parts. The first focuses on Hecke algebras and Galois representations. The second provides background on Shimura varieties and automorphic forms (including Hasse invariants, a key ingredient in the analytic continuation of differential operators).

Section 3 establishes the analytic continuation of the modulo p reduction of p -adic differential operators. Those operators were previously constructed over the ordinary locus in [Eischen 2012; Eischen et al. 2018]. The key work in this section lies in producing another construction of differential operators in characteristic p that is defined over the whole Shimura variety and then showing that it agrees with the mod p reduction of the p -adic differential operators we originally constructed. Theorem A is a consequence of the main result of this section, Theorem 3.4.1, which demonstrates the role of Hasse invariants in analytically continuing differential operators, in analogy with Katz's approach [1977].

Section 4 describes the interaction between differential operators and Hecke operators. Our proofs rely entirely on the geometry of Shimura varieties and do not require q -expansions. Theorem 4.2.4 shows that

the differential operators commute with prime-to- p Hecke operators up to scalar multiples, in analogy with (3), a key step toward describing the effect on Galois representations.

Section 5 establishes consequences of the previous two sections, applied to Galois representations. The section begins, though, with an axiomatic result, Theorem 5.1.1, on Galois representations and associated Hecke eigenvalues for general reductive groups. Theorem B is a consequence of Theorem 5.1.1, as the relevant conditions hold by Theorem 4.2.4. We conclude by showing that some results on iterations of theta (preliminary results on theta cycles) from modular forms naturally extend to Hilbert–Siegel and Hermitian modular forms when the weight is parallel scalar, without restriction on the degree of the base field.

2. Background and conventions

This section recalls key background information about Galois representations (Section 2.1) and associated automorphic forms over Shimura varieties (Section 2.2).

2.1. Hecke algebras and Galois representations. We recall properties of algebraic groups, Hecke algebras, and Galois representations associated with systems of Hecke eigenvalues. For clarity, here and in Section 5.1 we work with a general connected reductive group G , although in the rest of the paper we deal exclusively with G either symplectic or unitary (as in Section 2.2).

Our main reference is [Gross 1998], with some aspects borrowed from [Treumann and Venkatesh 2016] and [Buzzard and Gee 2014].

2.1.1. Dual group. Let G be a connected reductive algebraic group over a field F . **We assume that G is split over F** , that is, there exists a maximal torus of G that is isomorphic over F to a product of copies of $\mathbb{G}_m$.

We fix a split maximal torus T contained in a Borel subgroup B of G , all defined over F . The Weyl group of T is $W = N_G(T)/T$. Let $X^\bullet = X^\bullet(T) = \text{Hom}(T_{\bar{F}}, \mathbb{G}_m)$ be the character lattice and $X_\bullet = X_\bullet(T) = \text{Hom}(\mathbb{G}_m, T_{\bar{F}})$ be the cocharacter lattice. There is a natural pairing

$$\langle \cdot, \cdot \rangle : X^\bullet \times X_\bullet \rightarrow \text{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$$

given by $\langle \chi, \mu \rangle = \chi \circ \mu$.

Given a representation of G , its restriction to T is a direct sum of characters (the *weights* of the representation). The *roots* of G are the nontrivial weights of the adjoint representation $\text{Ad} : G \rightarrow \text{Aut}(\text{Lie}(G))$. If one instead considers the adjoint action on $\text{Lie}(B)$, one obtains the *positive roots*. The set $\Delta^\bullet$ of simple roots consists of those positive roots that cannot be written as the sum of other positive roots. The coroot basis $\Delta_\bullet$ is the set of simple coroots. The *root datum* of G is the quadruple $\Psi(G) = (X^\bullet, \Delta^\bullet, X_\bullet, \Delta_\bullet)$.

We consider $\widehat{G}$, the dual group of G , viewed as a reductive group over k , a field to be specified later. (In the automorphic literature, k is often taken to be $\mathbb{C}$, but our main focus will be on $\overline{\mathbb{F}}_p$. For the existence of $\widehat{G}$ over a general base, see [Conrad 2014, Theorem 6.1.16].) After fixing $\widehat{T} \subset \widehat{B} \subset \widehat{G}$, there is an identification $X^\bullet(\widehat{T}) \cong X_\bullet(T)$ that maps the positive roots with respect to $\widehat{B}$ to the positive coroots with respect to B . The root datum of $\widehat{G}$ is $\Psi(G)^\vee = (X_\bullet, \Delta_\bullet, X^\bullet, \Delta^\bullet)$.

Under the duality between G and $\widehat{G}$, a character $\chi : T \rightarrow \mathbb{G}_m$ in $X^\bullet(T)$ corresponds to a cocharacter $\hat{\chi} : \mathbb{G}_m \rightarrow \widehat{T}$ in $X_\bullet(\widehat{T}) = X^\bullet(T)$. Similarly, the character of $\widehat{T}$ corresponding to $\mu \in X_\bullet(T)$ is denoted by $\hat{\mu}$. Note that, given $\chi \in X^\bullet(T)$ and $\mu \in X_\bullet(T)$, the two morphisms

$$\chi \circ \mu : \mathbb{G}_m \rightarrow T \rightarrow \mathbb{G}_m \quad \text{and} \quad \hat{\mu} \circ \hat{\chi} : \mathbb{G}_m \rightarrow \widehat{T} \rightarrow \mathbb{G}_m$$

agree; i.e., $\chi \circ \mu = \hat{\mu} \circ \hat{\chi}$.

Let

$$P^+ := \{\chi \in X_\bullet \mid \langle \alpha, \chi \rangle \geq 0 \text{ for all } \alpha \in \Delta^\bullet\}.$$

The elements of P^+ are called the *dominant weights* of (the maximal torus $\widehat{T}$ of) $\widehat{G}$. There is a partial ordering $\geq$ on P^+ , where $\lambda \geq \mu$ if

$$\lambda - \mu = \sum_{\alpha^\vee \in \Delta_\bullet} n_{\alpha^\vee} \alpha^\vee \quad \text{with } n_{\alpha^\vee} \in \mathbb{Z}_{\geq 0}.$$

By Chevalley's theorem [Jantzen 2003, Corollary 2.7 of Part II], the set P^+ is in bijection with the set of irreducible finite-dimensional $\widehat{G}$ -modules over k . For any $\lambda \in P^+$, we denote by $(\rho_\lambda, V_\lambda)$ the irreducible $\widehat{G}$ -module of highest weight λ .

Writing $R(\widehat{G})$ for the representation ring of $\widehat{G}$, we have an identification [Gross 1998, Section 1]

$$R(\widehat{G}) \cong \mathbb{Z}[X^\bullet(\widehat{T})]^W \cong \mathbb{Z}[X_\bullet(T)]^W,$$

where W acts on $X_\bullet(T)$ by postcomposition and on $X^\bullet(\widehat{T})$ by the identification $X^\bullet(\widehat{T}) \cong X_\bullet(T)$ above. Using this we can view the elements of the representation ring as sums of characters of $\widehat{T}$ or cocharacters of T . Given $\lambda \in P^+$, let $\chi_\lambda := \text{Tr}(\rho_\lambda|_{\widehat{T}}) \in \mathbb{Z}[X^\bullet(\widehat{T})]$ denote the character of the irreducible representation $\rho_\lambda : \widehat{G} \rightarrow \text{GL}(V_\lambda)$ of highest weight λ . Using the above, we can express χ_λ as

$$\chi_\lambda = \sum_{\mu \leq \lambda} (\dim V_\lambda(\mu)) \hat{\mu},$$

where $\hat{\mu} \in X^\bullet(\widehat{T})$ is the character corresponding to $\mu \in X_\bullet(T)$ by duality, and $V_\lambda(\mu)$ is the weight space of weight μ in V_λ . For further discussion see the remark after [Jantzen 2003, Corollary 2.7 of Part II].

2.1.2. Local Hecke algebra and Satake isomorphism. We now take G to be a connected split reductive group over a local field F_v with ring of integers $\mathcal{O}_v$. We choose a uniformizer π_v of $\mathcal{O}_v$, and we let q_v denote the cardinality of the residue field $\mathcal{O}_v/\pi_v\mathcal{O}_v$.

As we assume G to be split, there exists a group scheme $\mathcal{G}$ over $\mathcal{O}_v$ whose generic fiber is G and whose special fiber is reductive. We set $G_v = \mathcal{G}(F_v)$ and $K_v = \mathcal{G}(\mathcal{O}_v)$, and then K_v is a hyperspecial maximal compact subgroup of the locally compact group G_v .

Given a commutative ring R , the R -valued Hecke algebra of the pair (G_v, K_v) is

$$\mathcal{H}(G_v, K_v; R) = \{h : K_v \backslash G_v / K_v \rightarrow R \mid h \text{ locally constant, compactly supported}\}$$

with multiplication given by

$$(h_1 * h_2)(K_v g K_v) = \sum_{x K_v \in G_v / K_v} h_1(K_v x K_v) h_2(K_v x^{-1} g K_v).$$

We work with the basis of $\mathcal{H}(G_v, K_v; R)$ consisting of the characteristic functions

$$c_\lambda = \text{char}(K_v \lambda(\pi_v) K_v) \quad \text{for } \lambda \in P^+.$$

The Satake transform is a ring isomorphism

$$\mathcal{S}_v : \mathcal{H}(G_v, K_v; \mathbb{Z}[q_v^{\pm 1/2}]) \rightarrow \mathbb{Z}[q_v^{\pm 1/2}][X^\bullet(\widehat{T})]^W = R(\widehat{G}) \otimes \mathbb{Z}[q_v^{\pm 1/2}].$$

If $\lambda \in P^+$ then the image of the basis element c_λ can be written¹

$$\mathcal{S}_v(c_\lambda) = \sum_{\mu \leq \lambda} b_\lambda(\mu) q_v^{\langle \rho, \mu \rangle} \chi_\mu, \quad (5)$$

where μ runs over the elements in P^+ such that $\mu \leq \lambda$, $b_\lambda(\mu) \in \mathbb{Z}$ and $b_\lambda(\lambda) = 1$.

There is a similar identity expressing χ_λ in terms of images of characteristic functions,

$$\chi_\lambda = q_v^{-\langle \rho, \lambda \rangle} \sum_{\mu \leq \lambda} d_\lambda(\mu) \mathcal{S}_v(c_\mu),$$

where $d_\lambda(\mu) \in \mathbb{Z}$ and $d_\lambda(\lambda) = 1$. For future use, let us record a consequence of this identity:

$$\mathcal{S}_v^{-1}(\chi_\lambda) = q_v^{-\langle \rho, \lambda \rangle} \sum_{\mu \leq \lambda} d_\lambda(\mu) c_\mu. \quad (6)$$

In most of the paper we work with $\overline{\mathbb{F}}_p$ -coefficients, for a fixed prime p . If v does not divide p , then, after making a choice of square root² of q_v in $\overline{\mathbb{F}}_p$, we get a mod p version of the Satake transform by tensoring with $\overline{\mathbb{F}}_p$:

$$\mathcal{S}_v : \mathcal{H}(G_v, K_v; \overline{\mathbb{F}}_p) \rightarrow \overline{\mathbb{F}}_p[X^\bullet(\widehat{T})]^W = R(\widehat{G}) \otimes \overline{\mathbb{F}}_p.$$

Equation (6) continues to hold in this setting, with $d_\lambda(\mu) \in \mathbb{F}_p$ and $d_\lambda(\lambda) = 1$.

2.1.3. Galois representations associated to automorphic forms mod p . We summarize the (conjectural) correspondence between Galois representations and automorphic representations following [Buzzard and Gee 2014].

Let G be a connected reductive group over $\mathbb{Q}$ and let $F/\mathbb{Q}$ be the extension over which G splits. Henceforth we let G refer to the base change $G \times_{\text{Spec } \mathbb{Q}} \text{Spec } F$. Fix a prime p . Given a level, i.e., an open compact subgroup $K = \prod_v K_v \subset G(\mathbb{A}_f)$, we say that a finite place v of F is *bad* (with respect to p and K) if v lies above p or if K_v is *not* a hyperspecial maximal compact subgroup of $G_v = G(F_v)$.

¹The element ρ is the half-sum of the positive roots of G , but we will not need to know this, only that it is the same in all the identities related to the Satake isomorphism $\mathcal{S}_v$.

²For the subtleties of this choice of square root, look for local and global pseudoroots in [Treumann and Venkatesh 2016, Section 7].

Otherwise we say that v is *good*, in which case we are in the situation described in [Section 2.1.2](#). All but finitely many places are good with respect to p and K .

Let f be a Hecke eigenform mod p of level K on G , that is, a class in the cohomology with $\overline{\mathbb{F}}_p$ -coefficients of the locally symmetric space defined by G and K (see [\[Treumann and Venkatesh 2016, Section 5\]](#) for details). (NB: The examples of interest to this paper are the automorphic forms coming from Shimura varieties, as defined in [Section 2.2.6](#).)

For every good place v , f defines a character of the local Hecke algebra $\mathcal{H}(G_v, K_v; \overline{\mathbb{F}}_p)$, namely the *Hecke eigensystem*

$$\Psi_{f,v} : \mathcal{H}(G_v, K_v; \overline{\mathbb{F}}_p) \rightarrow \overline{\mathbb{F}}_p$$

which takes an operator T to its eigenvalue:

$$Tf = \Psi_{f,v}(T)f.$$

Using the Satake isomorphism $\mathcal{S}_v$ from [Section 2.1.2](#), we can define a character $\omega_f : R(\widehat{G}) \otimes \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p$ by

$$\omega_f(\chi_\lambda) = \Psi_{f,v}(\mathcal{S}_v^{-1}(\chi_\lambda)).$$

The characters of the representation ring $R(\widehat{G}) \otimes \overline{\mathbb{F}}_p$ are indexed by the semisimple conjugacy classes in $\widehat{G}(\overline{\mathbb{F}}_p)$. Given such a class s , the corresponding character ω_s is determined by

$$\omega_s(\chi_\lambda) = \chi_\lambda(s).$$

In particular, the character ω_f corresponding to f is indexed by the conjugacy class $s_{f,v}$ of some semisimple element of $\widehat{G}(\overline{\mathbb{F}}_p)$; we call $s_{f,v}$ the *v -Satake parameter* of f .

We are ready to state the expected relation between Hecke eigenforms and Galois representations:

Conjecture 2.1.1 (positive characteristic form of [\[Buzzard and Gee 2014, Conjecture 5.17\]](#)). *Let f be a mod p Hecke eigenform of level K on a connected reductive group G over $\mathbb{Q}$, split over a number field F . There exists a continuous representation*

$$\rho : \text{Gal}(\overline{F}/F) \rightarrow \widehat{G}(\overline{\mathbb{F}}_p)$$

that is unramified outside the finite set Σ of places that are bad with respect to p and the level K . If $v \notin \Sigma$ is a finite place of F , then $\rho(\text{Frob}_v) = s_{f,v}$, the v -Satake parameter of f .

As noted in [\[Buzzard and Gee 2014, Remark 5.19\]](#), the representation ρ need not be unique up to conjugation. Given a Hecke eigenform f , we will denote by $R(f)$ the set of Galois representations attached to f as in [Conjecture 2.1.1](#).

Remark 2.1.2. The reader looking to reconcile the statement of [Conjecture 2.1.1](#) with that of [\[Buzzard and Gee 2014, Conjecture 5.17\]](#) will note the following differences:

- We work with Galois representations mod p rather than p -adic.

- Our assumption that G is split over F means that the Galois representations land in the dual group $\widehat{G}$ rather than the L -group ${}^L G$.
- We relate the Hecke eigensystem of f with the images of Frobenii by using Satake parameters (as defined above) rather than by comparing representations of Weil groups.
- Since Buzzard and Gee work in the more general setting of automorphic representations, they need to impose the condition of L -algebraicity, which is automatically satisfied in the setting of our automorphic forms.

For the current state of the art regarding [Conjecture 2.1.1](#), we have great achievements by Scholze [\[2015\]](#) for the case $G = \mathrm{GL}_n$. (The conjectural correspondence in this setting was first formulated by Ash [\[1992\]](#).) Scholze's work gives the following.

Theorem 2.1.3 [\[Scholze 2015, Theorem I.3\]](#). *If F is a CM field that contains an imaginary quadratic field (or more generally, assuming the statements in [\[Arthur 2013\]](#), if F is totally real or CM), then for any system of Hecke eigenvalues occurring in the singular cohomology $H^i(X_K, \mathbb{F}_p)$ of the locally symmetric space X_K attached to GL_n over F , there is a continuous semisimple representation $\mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ whose Frobenius eigenvalues agree with that system of Hecke eigenvalues.*

The method involves recognizing the cohomology of X_K within the cohomology of a Shimura variety attached to a symplectic or unitary group, which are our primary cases of interest. For results directly in those settings, see [\[Kret and Shin 2016\]](#) and [\[Chenevier and Harris 2013\]](#), respectively.

2.2. Shimura varieties and automorphic forms. This section introduces the Shimura varieties and spaces of automorphic forms with which we will work.

2.2.1. Shimura varieties. In this paper, we consider PEL-type Shimura varieties of either unitary (A) or symplectic (C) type. Here, we briefly introduce the Shimura datum $(D, *, V, \langle \cdot, \cdot \rangle, h)$ of PEL type needed for our work later in the paper. We refer to [\[Kottwitz 1992\]](#) for a more detailed treatment.

Let D be a finite-dimensional simple $\mathbb{Q}$ -algebra, and let F be its center. Let $*$ be a positive involution on D over $\mathbb{Q}$, and let F_0 be the fixed field of $*$ on F . Since $*$ is positive on F , its fixed field F_0 is totally real. We say that $*$ is of the first kind if $F = F_0$, and of the second kind if F is a quadratic imaginary extension of F_0 (in which case F is a CM field).

In the following, we denote by $\mathcal{T}_{F_0}$ (resp. $\mathcal{T}_F$) the set of embeddings $\tau : F_0 \rightarrow \mathbb{R}$ (resp. $\tau : F \rightarrow \mathbb{C}$). If $*$ is of the second kind (case (A)), then for each $\tau \in \mathcal{T}_{F_0}$, we fix an extension $F \rightarrow \mathbb{C}$ in $\mathcal{T}_F$, which by abuse of notation, we still denote by τ , and we write τ^* for the other embedding $F \rightarrow \mathbb{C}$ that restricts to τ on F_0 . We shall also fix a choice Σ_F of CM type, i.e., a set consisting of exactly one of τ, τ^* for each $\tau \in \mathcal{T}_F$.

Let V be a nonzero finitely generated left D -module, and let $\langle \cdot, \cdot \rangle$ be a nondegenerate $\mathbb{Q}$ -valued alternating form on V such that $\langle bv, w \rangle = \langle v, b^*w \rangle$ for all $v, w \in V$ and all $b \in D$. Let C be the $\mathbb{Q}$ -algebra $\mathrm{End}_D(V)$; it is a simple algebra with center F , and has an involution $*$ coming from the form $\langle \cdot, \cdot \rangle$. Let $h : \mathbb{C} \rightarrow C_{\mathbb{R}}$ be a $*$ -homomorphism such that the symmetric real-valued bilinear form $\langle \cdot, h(i) \cdot \rangle$ on $V_{\mathbb{R}}$ is positive-definite.

Associated with the above data, we define the algebraic group G over $\mathbb{Q}$ whose points on a $\mathbb{Q}$ -algebra R are given by

$$G(R) := \{x \in C \otimes_{\mathbb{Q}} R \mid xx^* \in R^\times\}.$$

We denote by $\nu : G \rightarrow \mathbb{G}_m$ the morphism $x \rightarrow xx^*$, called the *similitude factor*, by G_1 its kernel, and by $\hat{\nu} : \mathbb{G}_m \rightarrow \widehat{G}$ the cocharacter of $\widehat{G}$ corresponding to ν under duality. (Note that for any integer $m \geq 1$, the cocharacter of $\widehat{G}$ dual to ν^m is $\hat{\nu}^m$.) Then G_1 is obtained from an algebraic group G_0 over F_0 by restriction of scalars from F_0 to $\mathbb{Q}$. If $*$ is of the second kind—case (A)—then G_0 is an inner form of a quasisplit unitary group over F_0 . If $*$ is of the first kind, then over an algebraic closure of F_0 , the group G_0 is either orthogonal—case (D)—or symplectic—case (C). **Going forward, we assume we are in case (A) or (C).**

The endomorphism $h_{\mathbb{C}} = h \times_{\mathbb{R}} \mathbb{C}$ of $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C} = V \otimes_{\mathbb{Q}} \mathbb{C}$ gives rise to a decomposition $V_{\mathbb{C}} = V_1 \oplus V_2$, where for all $z \in \mathbb{C}$, $(h(z), 1) = h(z) \times 1$ acts by z on V_1 and by $\bar{z}$ on V_2 . The *reflex field* E of the Shimura datum $(D, *, V, \langle, \rangle, h)$ is the field of definition of the $G(\mathbb{C})$ -conjugacy class of V_1 .

Let p be a rational prime. We write $\mathbb{Z}_{(p)}$ for the localization of $\mathbb{Z}$ at p . We choose our data so that the following conditions are met:

- (i) The prime p is unramified in F .
- (ii) The algebra D is split at p ; i.e., $D_{\mathbb{Q}_p}$ is a product of matrix algebras over (unramified) extensions of $\mathbb{Q}_p$.
- (iii) There exists a $\mathbb{Z}_{(p)}$ -order $\mathcal{O}_D$ in D that is preserved by $*$ and whose p -adic completion is a maximal order in $D_{\mathbb{Q}_p}$.
- (iv) There exists a $\mathbb{Z}_p$ -lattice L in $V_{\mathbb{Q}_p}$ that is self-dual for $\langle, \rangle$ and preserved by $\mathcal{O}_D$.

We also specify a *level* K , an open compact subgroup of $G(\mathbb{A}_f)$, where $\mathbb{A}_f$ denotes the ring of finite adeles of $\mathbb{Q}$. We further assume that K is neat (as defined in [Lan 2013, Definition 1.4.1.8]) and that it decomposes as $K = K^p K_p$, where $K^p \subset G(\mathbb{A}_f^{(p)})$ and $K_p \subset G(\mathbb{Q}_p)$ is hyperspecial, namely K_p is the stabilizer of L in $V_{\mathbb{Q}_p}$.

We define $\mathcal{X} = \mathcal{X}_K$ to be the PEL-type moduli space of abelian varieties of level K , associated with the datum $(D, *, V, \langle, \rangle, h)$. Under our assumptions, $\mathcal{X}_K$ extends canonically to a smooth quasiprojective scheme over $\mathcal{O}_E \otimes \mathbb{Z}_{(p)}$, which by abuse of notation we still denote by $\mathcal{X}$. The set $\mathcal{X}(\mathbb{C})$ is a (finite) disjoint union of Shimura varieties obtained from the data (G, h, K) . In the following, by abuse of language, we refer to $\mathcal{X}$ as the PEL-type Shimura variety of level K .

Given an abelian variety A , we denote by A^t its dual (and, likewise, in the context of sheaves, we also use a superscript t to denote the dual).

2.2.2. Simplifying conditions. We assume p is totally split in the reflex field E . Then, by [Wedhorn 1999], the ordinary locus of the modulo p reduction X of the Shimura variety $\mathcal{X}$ is nonempty. We fix a prime $\mathfrak{p}$ of E above p , and we regard $\mathcal{X} = \mathcal{X}_K$ as a scheme over $\mathbb{W} = \mathbb{Z}_p = \mathcal{O}_{E, \mathfrak{p}}$.

For convenience, we further *assume that $D = F$ and that p splits completely in F* . These two assumptions are not necessary, but they allow us to simplify notation. (By assumption (ii) above, Morita equivalence reduces the general case of $D \neq F$ to that of $D = F$; we refer to [Eischen and Mantovan 2021b, Section 6] for an explanation of how the case of p totally split in E , but not in F , relates to the case of p totally split in F .)

2.2.3. Signatures and Levi subgroups. The decomposition $F \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\tau \in \mathcal{T}_F} \mathbb{C}$ induces a decomposition $V_1 = \bigoplus_{\tau \in \mathcal{T}_F} V_{1,\tau}$. The *signature* of the Shimura datum is the collection of integers

$$a_{\tau} := \dim_{\mathbb{C}} V_{1,\tau} \quad \text{for all } \tau \in \mathcal{T}_F.$$

Let $2n = \dim_{F_0} V$. In the unitary case (A), we have $a_{\tau} \oplus a_{\tau^*} = n$ for all $\tau \in \mathcal{T}_F$, and the tuple of pairs (a_{τ}, a_{τ^*}) , for $\tau \in \Sigma_F$, is the signature of the unitary group G_0/F_0 . In the symplectic case (C), we have $a_{\tau} = n$ for all $\tau \in \mathcal{T}_{F_0}$. (For $F_0 = \mathbb{Q}$, we refer to case (C) as the *Siegel case*, and to case (A) as the *Hermitian case* if $a_{\tau} = a_{\tau^*}$.)

Letting $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$, we denote by $\mathbb{C}_p$ its p -adic completion. We fix an isomorphism $\iota : \mathbb{C} \rightarrow \mathbb{C}_p$, compatible with the choice of a prime $\mathfrak{p}$ of E above p . The map $\tau \mapsto \iota \circ \tau$ defines a bijection between the complex embeddings $\tau : F \rightarrow \mathbb{C}$ and the p -adic embeddings $F \rightarrow \overline{\mathbb{Q}}_p$. By assumption, p splits completely in F , and hence we deduce an identification of $\mathcal{T}_F$ with the set of primes of F above p (resp. $\mathcal{T}_{F_0}$ with the set of primes of F_0 above p), and write $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p = \bigoplus_{\tau \in \mathcal{T}_F} \mathbb{Z}_p$.

Starting from the signature of the Shimura datum, we define an algebraic group H over $\mathbb{Z}$ by

$$H := \prod_{\tau \in \mathcal{T}_F} \mathrm{GL}_{a_{\tau}} = \begin{cases} \prod_{\tau \in \mathcal{T}_{F_0}} (\mathrm{GL}_{a_{\tau}} \times \mathrm{GL}_{a_{\tau^*}}) \subseteq \prod_{\tau \in \mathcal{T}_{F_0}} \mathrm{GL}_n & \text{in the unitary case (A),} \\ \prod_{\tau \in \mathcal{T}_{F_0}} \mathrm{GL}_n & \text{in the symplectic case (C).} \end{cases}$$

Note that $H_{\mathbb{C}}$ can be identified with the Levi subgroup of $G_{1,\mathbb{C}}$ that preserves the decomposition $V_{\mathbb{C}} = V_1 \oplus V_2$. Via ι , we also identify $H_{\mathbb{Z}_p}$ with a Levi subgroup of $G_{1,\mathbb{Z}_p}$. (Here, with abuse of notation, we still denote by G the reductive model over $\mathbb{Z}_p = \mathcal{O}_{E,\mathfrak{p}}$ associated with the $\mathbb{Z}_p$ -lattice L in $V_{\mathbb{Q}_p}$. Note that all groups are split over $\mathbb{Z}_p$.)

We denote by $T \subseteq B$ the diagonal maximal torus inside the Borel subgroup of upper-triangular matrices of H , and by N the unipotent subgroup of B .

2.2.4. Weights and Schur functors. We recall relevant aspects of the theory of algebraic representations of general linear groups, adapted to the group $H = \prod_{\tau \in \mathcal{T}_F} \mathrm{GL}_{a_{\tau}}$. We refer to [Eischen et al. 2018, Section 2.4] for a more detailed discussion.

Over an algebraically closed field of characteristic 0, in particular over $\overline{\mathbb{Q}}_p$, the irreducible algebraic representations of H (up to isomorphism) are in one-to-one correspondence with the dominant weights of T . For each dominant weight κ , let $(\rho_{\kappa}, V_{\kappa})$ denote the irreducible algebraic representation of H over $\overline{\mathbb{Q}}_p$ of highest weight κ . Via Schur functors, we construct $\mathbb{Z}_p$ -models of the representations $(\rho_{\kappa}, V_{\kappa})$.

We identify the set of dominant weights of (the maximal torus T of) H with the set

$$X(T)_+ := \left\{ (\kappa_{1,\tau}, \dots, \kappa_{a_\tau,\tau}) \in \prod_{\tau \in \mathcal{T}_F} \mathbb{Z}^{a_\tau} \mid \kappa_{i,\tau} \geq \kappa_{i+1,\tau} \text{ for all } i \right\},$$

via the morphism $\prod_{\tau \in \mathcal{T}_F} \text{diag}(t_{1,\tau}, \dots, t_{a_\tau,\tau}) \mapsto \prod_{\tau \in \mathcal{T}_F} \prod_{1 \leq i \leq a_\tau} t_{i,\tau}^{\kappa_{i,\tau}}$. For $\kappa = (\kappa_\tau) \in X(T)_+$, we write

$$|\kappa| := \sum_{\tau \in \mathcal{T}_F} |\kappa_\tau| \quad \text{and} \quad |\kappa_\tau| = \sum_{1 \leq i \leq a_\tau} \kappa_{i,\tau}.$$

We call a dominant weight $\kappa \in X(T)_+$ *positive* if $\kappa \neq 0$ and $\kappa_{a_\tau,\tau} \geq 0$ for all $\tau \in \mathcal{T}_F$.

Let κ be positive dominant weight (using twists by appropriate powers of the determinant, all constructions and definitions below extend to nonpositive dominant weights). Let y_κ be the associated *Young symmetrizer*. More precisely, we define $y_\kappa = \bigotimes_{\tau \in \mathcal{T}_F} y_{\kappa_\tau}$, where for each $\tau \in \mathcal{T}_F$, $y_{\kappa_\tau} \in \mathbb{Z}[\mathfrak{S}_{|\kappa_\tau|}]$ is the Young symmetrizer associated with the dominant weight κ_τ of $H_\tau = \text{GL}_{a_\tau}$, in the group algebra of the symmetric group $\mathfrak{S}_{|\kappa_\tau|}$ on $|\kappa_\tau|$ letters.

The κ_τ -Schur functor on the category of R -modules, for R any ring,³ is defined by

$$\mathbb{S}_{\kappa_\tau}(M) = M^{\otimes |\kappa_\tau|} \cdot y_{\kappa_\tau} \subset M^{\otimes |\kappa_\tau|}.$$

When R is a $\mathbb{Z}_p$ -algebra, or an algebraically closed field of characteristic 0, the module $\mathbb{S}_{\kappa_\tau}(R^{a_\tau})$ is an irreducible representation of H_τ over R , with highest weight κ_τ .

For each positive weight $\kappa \in X(T)_+$, we define the κ -Schur functor on the category of R -modules M endowed with a decomposition $M = \bigoplus_{\tau \in \mathcal{T}_F} M_\tau$ by

$$\mathbb{S}_\kappa(M) = \bigotimes_{\tau \in \mathcal{T}_F} \mathbb{S}_{\kappa_\tau}(M_\tau) = \left(\bigotimes_{\tau \in \mathcal{T}_F} M^{\otimes |\kappa_\tau|} \right) \cdot y_\kappa \subset M^{\otimes |\kappa|}.$$

Let $W = \bigoplus_{\tau \in \mathcal{T}_F} \mathbb{Z}_p^{a_\tau}$ be the standard representation of H over $\mathbb{Z}_p$. Then the module $\mathbb{S}_\kappa(W)$ is a $\mathbb{Z}_p$ -model of the irreducible representation of H with highest weight κ . In the following, with abuse of notation and language, for each dominant weight κ , we still denote by (ρ_κ, V_κ) the representation $\mathbb{S}_\kappa(W)$, and refer to it as the irreducible representation of highest weight κ .

Remark 2.2.1. By construction, the Young symmetrizer defines an epimorphism

$$y_\kappa : W^{\otimes |\kappa|} \rightarrow \mathbb{S}_\kappa(W).$$

By [Eischen et al. 2018, Lemma 2.4.6], if κ and λ are two positive dominant weights, then $|\kappa + \lambda| = |\kappa| + |\lambda|$ and there exists a projection map $\pi_{\kappa,\lambda} : \mathbb{S}_\kappa(W) \otimes \mathbb{S}_\lambda(W) \rightarrow \mathbb{S}_{\kappa+\lambda}(W)$ such that the morphism $y_{\kappa+\lambda} : W^{\otimes |\kappa+\lambda|} \rightarrow \mathbb{S}_{\kappa+\lambda}(W)$ factors via the morphism

$$y_\kappa \otimes y_\lambda : W^{|\kappa+\lambda|} = W^{\otimes |\kappa|} \otimes W^{\otimes |\lambda|} \rightarrow \mathbb{S}_\kappa(W) \otimes \mathbb{S}_\lambda(W),$$

or in other words, $y_{\kappa+\lambda} = \pi_{\kappa,\lambda} \circ (y_\kappa \otimes y_\lambda)$.

³Schur functors are also defined on the category of locally free sheaves on a scheme S .

It is important to note that, as detailed in [Eischen et al. 2018, Section 2.4], all these maps are defined over $\mathbb{Z}_p$.

Remark 2.2.2. The Schur functors and the various projection maps are an important ingredient of our construction of the differential operators on p -adic automorphic forms in Section 3.2. For the construction of the differential operators on mod p automorphic forms in Section 3.3.2, we use⁴ the reductions modulo p of the Schur functors and projection maps defined above over $\mathbb{Z}_p$. As indicated in Remark 3.2.2, while this provides us with a definition of the necessary maps in characteristic p , some of these maps may be zero when p is small.

2.2.5. Admissible weights. We now introduce the set of dominant weights by which Maass–Shimura differential operators (as well as the analogous differential operators studied in Section 3) can raise the weight of an automorphic form.

We call a dominant weight $\kappa \in X(T)_+$ *even* if $\kappa_{i,\tau} \equiv 0 \pmod{2}$ for all $\tau \in \mathcal{T}_F$ and all $1 \leq i \leq a_\tau$, and we call κ *sum-symmetric* if $\sum_{1 \leq i \leq a_\tau} \kappa_{i,\tau} = \sum_{1 \leq i \leq a_{\tau^*}} \kappa_{i,\tau^*}$ for all $\tau \in \mathcal{T}_F$.

Let $W = \bigoplus_{\tau \in \mathcal{T}_F} W_\tau$, where $W_\tau = \mathbb{Z}_p^{a_\tau}$, be the standard representation of H over $\mathbb{Z}_p$.

Definition 2.2.3. A dominant weight λ is called *admissible of depth* $e_\lambda = e$, for $e \in \mathbb{Z}_{>0}$, if the corresponding irreducible algebraic representation of H occurs as a constituent of the representation $(W^2)^{\otimes e}$ for

$$W^2 := \begin{cases} \bigoplus_{\tau \in \mathcal{T}_{F_0}} \text{Sym}^2 W_\tau & \text{in the symplectic case,} \\ \bigoplus_{\tau \in \mathcal{T}_{F_0}} (W_\tau \otimes W_{\tau^*}) & \text{in the unitary case.} \end{cases}$$

By definition, admissible weights are positive. Furthermore, a dominant weight λ is *admissible* in the symplectic case if and only if it is positive and even; and in the unitary case, a dominant weight λ is *admissible* if and only if it is positive and sum-symmetric.

Note that if λ is admissible, then $|\lambda|$ is positive and even, and the Young symmetrizer

$$y_\lambda : W^{\otimes |\lambda|} \rightarrow W_\lambda$$

induces an epimorphism $(W^2)^{\otimes |\lambda|/2} \rightarrow W_\lambda$, which, by abuse of notation, we still denote by y_λ . Hence, in particular, λ is admissible of depth $e_\lambda = |\lambda|/2$.

Remark 2.2.4. We denote scalar weights by $(\underline{k}_\tau)_{\tau \in \mathcal{T}_F} := (k_\tau, \dots, k_\tau)_{\tau \in \mathcal{T}_F}$ for integers k_τ . Furthermore, if there exists an integer k such that $k_\tau = k$ for all τ , then we write $\underline{k}$ for $(\underline{k}_\tau)_{\tau \in \mathcal{T}_F}$. In the symplectic case, a scalar weight $(\underline{d}_\tau)_{\tau \in \mathcal{T}_F}$ is admissible if and only if d_τ is nonnegative and even for all $\tau \in \mathcal{T}_F$. Hence, in the Siegel case, the admissible scalar weights are exactly the positive multiples of $\underline{2}$. In the unitary case, a scalar weight $(\underline{d}_\tau)_{\tau \in \mathcal{T}_F}$ is admissible if and only if $a_\tau d_\tau = a_{\tau^*} d_{\tau^*}$ for all $\tau \in \mathcal{T}_F$. Hence, in the Hermitian case, a scalar weight is admissible if and only if $d_\tau = d_{\tau^*}$ for all $\tau \in \mathcal{T}_F$.

⁴There are other approaches, for instance via the characteristic-free theory of Schur functors [Akin et al. 1982] and the corresponding Littlewood–Richardson rule [Boffi 1988], but using reduction modulo p from the $\mathbb{Z}_p$ -models we defined is sufficient for our purposes.

2.2.6. Automorphic forms. We recall the construction of automorphic sheaves over $\mathcal{X}$. We refer to [Caraiani et al. 2016, Section 3.2] for details.

Given a characteristic p field $\mathbb{F}$, we denote its Witt vectors by $W(\mathbb{F})$. We also set $\mathbb{W} := W(\mathbb{Z}/p\mathbb{Z})$ and identify $\mathbb{W}$ with $\mathbb{Z}_p$. Following the conventions of Section 2.2.3, we identify $\mathcal{T}_{F_0}$ (resp. $\mathcal{T}_F$) with $\mathrm{Hom}(\mathcal{O}_{F_0}, W(\overline{\mathbb{F}}_p))$ (resp. $\mathrm{Hom}(\mathcal{O}_F, W(\overline{\mathbb{F}}_p))$).

Let $\pi : \mathcal{A} \rightarrow \mathcal{X}$ denote the universal abelian scheme, of relative dimension $g = n[F_0 : \mathbb{Q}]$ over $\mathcal{X}$, and set $\omega := \pi_* \Omega_{\mathcal{A}/\mathcal{X}}^1$, which is a locally free sheaf of rank g .

Set $\mathcal{T} := \mathcal{T}_F$. The action of $\mathcal{O}_F$ on ω induces a decomposition of ω as

$$\omega = \bigoplus_{\tau \in \mathcal{T}} \omega_\tau,$$

where for each $\tau \in \mathcal{T}$ the sheaf ω_τ is locally free of rank a_τ .

For any dominant weight κ of T , the *automorphic sheaf* ω^κ of weight κ over $\mathcal{X}$ is defined as follows. Let $\mathcal{E}/\mathcal{X}$ be the sheaf

$$\mathcal{E} := \bigoplus_{\tau \in \mathcal{T}} \underline{\mathrm{Isom}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}^{a_\tau}, \omega_\tau);$$

it admits a natural left action of H . Given an irreducible representation (ρ_κ, V_κ) of H of highest weight κ , we define $\omega^\kappa := \mathcal{E} \times^{\rho_\kappa} V_\kappa$ so that $\omega^\kappa(R) := (\mathcal{E} \times V_\kappa \otimes R) / ((\ell, m) \equiv (g\ell, \rho_\kappa({}^t g^{-1})m))$ for each $\mathcal{O}_{E,p}$ -algebra R .

An *automorphic form* of weight κ and level K , defined over an $\mathcal{O}_{E,p}$ -algebra R , is a global section of the sheaf ω^κ on $\mathcal{X}_K \times_{\mathcal{O}_{E,p}} R$.

2.2.7. Hasse invariants and p -adic automorphic forms. We recall the construction and properties of Hasse invariants relevant to our settings. Details most pertinent to the present paper are available, for various contexts, in [Goren 2001; Conrad 2005; Goldring and Nicole 2017] and [Andreatta and Goren 2005, Section 7]. (More general constructions have been given more recently, e.g., in [Koskivirta and Wedhorn 2018], but such generality is unnecessary for the present paper.)

Let $\pi : \mathcal{A} \rightarrow X$ denote the universal abelian scheme over the mod p reduction X of the Shimura variety $\mathcal{X}$. Setting $\omega_{\mathcal{A}/X} := \pi_* \Omega_{\mathcal{A}/X}^1$, we have the Hodge filtration of $H_{\mathrm{dR}}^1(\mathcal{A}/X)$ over X :

$$0 \rightarrow \omega_{\mathcal{A}/X} \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/X) \rightarrow R^1 \pi_* \mathcal{O}_{\mathcal{A}} \rightarrow 0.$$

Let $F : X \rightarrow X$ denote the absolute Frobenius on X ; we denote by $\mathcal{A}^{(p)} := \mathcal{A} \times_{X,F} X$ the pullback of $\mathcal{A}$ under F , and by $\mathrm{Fr} : \mathcal{A} \rightarrow \mathcal{A}^{(p)}$ the relative Frobenius of $\mathcal{A}$. The morphism Fr induces an $\mathcal{O}_X$ -linear map

$$\mathrm{Fr}^* : R^1 \pi_* \mathcal{O}_{\mathcal{A}^{(p)}} = R^1 \pi_* \mathcal{O}_{\mathcal{A}}^{(p)} \rightarrow R^1 \pi_* \mathcal{O}_{\mathcal{A}};$$

we denote the dual map by

$$h : \omega_{\mathcal{A}/X} \rightarrow \omega_{\mathcal{A}^{(p)}/X} = \omega_{\mathcal{A}/X}^{(p)}.$$

Definition 2.2.5. The *Hasse invariant* is the automorphic form

$$E := \det h \in H^0(X, |\omega_{\mathcal{A}/X}|^{p-1})$$

(where $|\cdot|$ denotes the top exterior power).

By construction, the ordinary locus S of X agrees with the complement of the vanishing locus of the Hasse invariant E .

Recall that a sufficiently large power of E is known to lift to characteristic zero. For any $m \geq 1$, set $\mathcal{X}_m := \mathcal{X} \times_{\mathbb{W}} \mathbb{W}/p^m$, and denote by $\mathcal{S}_m$ the locus of $\mathcal{X}_m$ where (a sufficiently large power of) the Hasse invariant E is invertible. Define $\mathcal{S} := \varinjlim_m \mathcal{S}_m$ as a formal scheme over $\mathbb{W}$. The formal scheme $\mathcal{S}$ is the *formal ordinary locus* over $\mathbb{W}$.

Definition 2.2.6. For any dominant weight κ , a *p -adic automorphic form of weight κ* is a section of

$$H^0(\mathcal{S}, \omega^\kappa) := \varprojlim_m H^0(\mathcal{S}_m, \omega^\kappa).$$

3. Analytic continuation of differential operators

This section establishes analytic continuation of the mod p reduction of certain p -adic differential operators $\mathcal{D}_\kappa^\lambda$ (analogues of C^∞ Maass–Shimura differential operators), a priori defined over the ordinary locus, to the whole Shimura variety. (Recall from the previous section that we assume we are in case (A) or (C), i.e., the unitary or symplectic case.) This section's main result is [Theorem 3.4.1](#), of which [Theorem A](#) is a consequence. Via different methods, de Shalit and Goren [\[2019\]](#) obtained related results on analytic continuation, when one restricts to quadratic imaginary fields F and scalar weights. As discussed in [Section 1.2](#), unlike that approach, the techniques below handle all signatures and all weights simultaneously. The techniques below also do not rely on explicit computations via q -expansions (and consequently handle fields F of any degree and automorphic forms of any weight).

3.1. Key ingredients for the construction of our differential operators. Let T be a scheme, and let Y be a smooth scheme over T . Suppose $\pi : \mathcal{A} \rightarrow Y$ is a polarized abelian scheme. In particular, π is a smooth and proper morphism. Let $\omega := \omega_{\mathcal{A}/Y} := \pi_* \Omega_{\mathcal{A}/Y}^1 \subseteq H_{\mathrm{dR}}^1(\mathcal{A}/Y)$, and consider the Hodge filtration

$$0 \rightarrow \omega \hookrightarrow H_{\mathrm{dR}}^1(\mathcal{A}/Y) \rightarrow R^1 \pi_* \mathcal{O}_{\mathcal{A}} \rightarrow 0.$$

Our differential operators are built from the Gauss–Manin connection

$$\nabla = \nabla_{\mathcal{A}/Y} : H_{\mathrm{dR}}^1(\mathcal{A}/Y) \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/Y) \otimes \Omega_{Y/T}^1$$

and the Kodaira–Spencer morphism

$$\mathrm{KS} = \mathrm{KS}_{\mathcal{A}/Y} : \omega \otimes \omega \rightarrow \Omega_{Y/T}^1.$$

By definition, $\mathrm{KS}_{\mathcal{A}/Y} := \langle \cdot, \nabla(\cdot) \rangle_{\mathcal{A}}$, where $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is the pairing induced by the polarization on $\mathcal{A}$ and extended linearly in the second variable to a pairing between ω and $\nabla(\omega)$ (and using the fact that ω is

an isotropic subspace of $H_{\mathrm{dR}}^1(\mathcal{A}/Y)$ under this pairing). The Kodaira–Spencer morphism induces an isomorphism

$$\mathrm{ks} : \omega^2 \xrightarrow{\sim} \Omega_{Y/T}^1,$$

where

$$\omega^2 := \begin{cases} \bigoplus_{\tau \in \mathcal{T}_{F_0}} \mathrm{Sym}_{\mathcal{O}_Y}^2(\omega_\tau) & \text{in the symplectic case,} \\ \bigoplus_{\tau \in \mathcal{T}_{F_0}} (\omega_\tau \otimes_{\mathcal{O}_Y} \omega_{\tau^*}) & \text{in the unitary case.} \end{cases}$$

For details on the Gauss–Manin connection and the Kodaira–Spencer (iso)morphism, we refer the reader to [Lan 2013, Sections 2.1.7 and 2.3.5] and [Faltings and Chai 1990, Section 9].

3.2. Differential operators on p -adic automorphic forms. We briefly recall a construction of p -adic differential operators $\mathcal{D}_\kappa^\lambda$, analogues of C^∞ Maass–Shimura operators, introduced in [Katz 1978, Chapter II] for Hilbert modular forms and extended to the Siegel and unitary cases in [Courtieu and Panchishkin 1991; Panchishkin 2005] and [Eischen 2012; Eischen et al. 2018], respectively.

Let S denote the formal ordinary locus over $W(\overline{\mathbb{F}}_p)$. Write $\omega = \omega_{\mathcal{A}/S}$. Define

$$U \subseteq H_{\mathrm{dR}}^1(\mathcal{A}/S)$$

to be Dwork’s *unit root* subcrystal⁵ (introduced in [Katz 1973]). Then U is a complement of ω , and the inclusion $\omega \subseteq H_{\mathrm{dR}}^1(\mathcal{A}/S)$ composed with the projection $\pi_U : H_{\mathrm{dR}}^1(\mathcal{A}/S) \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/S)/U$ is an isomorphism. We denote the induced morphism by

$$\Pi_U : H_{\mathrm{dR}}^1(\mathcal{A}/S) \rightarrow \omega_{\mathcal{A}/S}.$$

Define

$$\mathcal{D} := (\Pi_U \otimes \mathrm{ks}^{-1}) \circ \nabla|_\omega : \omega \subseteq H_{\mathrm{dR}}^1(\mathcal{A}/S) \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/S) \otimes \Omega_{S/W(\overline{\mathbb{F}}_p)}^1 \rightarrow \omega \otimes \omega^2.$$

For any $d \geq 1$, by the Leibniz rule (the product rule), we obtain an operator

$$\mathcal{D}_d : \omega^{\otimes d} \rightarrow \omega^{\otimes d} \otimes \omega^2 \subseteq \omega^{\otimes(d+2)}.$$

By the construction of Schur functors (see also [Eischen et al. 2018, Proposition 3.3.1]), for any dominant weight κ , the operator $\mathcal{D}_d$ for $d = |\kappa|$ induces an operator

$$\mathcal{D}_\kappa : \omega^\kappa \rightarrow \omega^\kappa \otimes \omega^2.$$

For any admissible weight λ , the corresponding weight-raising differential operators

$$\mathcal{D}_\kappa^\lambda : \omega^\kappa \rightarrow \omega^{\kappa+\lambda}$$

are induced via Schur functors by the e -th iterations $\mathcal{D}_d^{(e)} := \mathcal{D}_{d+2(e-1)} \circ \cdots \circ \mathcal{D}_{d+2} \circ \mathcal{D}_d$,

$$\mathcal{D}_d^{(e)} : \omega^{\otimes d} \rightarrow \omega^{\otimes d} \otimes (\omega^2)^{\otimes e} \subseteq \omega^{\otimes d+2e},$$

composed with the Young symmetrizer $y_\lambda : (\omega^2)^{\otimes e} \rightarrow \omega^\lambda$, for $e = |\lambda|/2$ (see Remark 2.2.1).

⁵Under the identification of $H_{\mathrm{dR}}^1(\mathcal{A}/S)$ with the Dieudonné crystal of $\mathcal{A}[p^\infty]/S$, the subcrystal U is the maximal étale subcrystal of $H_{\mathrm{dR}}^1(\mathcal{A}/S)$.

Remark 3.2.1. For any admissible weight λ and any p -adic form f of weight κ , $\mathcal{D}_\kappa^\lambda(f)$ is a p -adic form of weight $\kappa + \lambda$. In particular, for any integer $m \geq 1$ and any mod p^m automorphic form $f \in H^0(\mathcal{X}_m, \omega^\kappa)$, we obtain $\mathcal{D}_\kappa^\lambda(f) \in H^0(\mathcal{S}_m, \omega^{\kappa+\lambda})$. By construction of the Hasse invariant E , for a sufficiently large integer $N \gg 0$, the section $E^N \cdot \mathcal{D}_\kappa^\lambda(f) \in H^0(\mathcal{S}_m, \omega^{\kappa+\lambda+(p-1)N})$ extends (uniquely) to all of $\mathcal{X}_m$, i.e., $E^N \cdot \mathcal{D}_\kappa^\lambda(f)$ is, in fact, the restriction of an element of $H^0(\mathcal{X}_m, \omega^{\kappa+\lambda+(p-1)N})$.

We shall prove in [Theorem 3.4.1](#) that when $m = 1$, for each admissible weight λ , there exists an explicit integer, namely $N = |\lambda|/2$, such that for all dominant weights κ and all characteristic p automorphic forms f of weight κ , the sections $E^N \cdot \mathcal{D}_\kappa^\lambda(f)$, defined a priori over the ordinary locus, extend to all of $X := \mathcal{X}_1$.

Remark 3.2.2. Recall that $2n := \dim_{F_0} V$ as defined in [Section 2.2.3](#). By [\[Eischen et al. 2018, Section 5.2, especially the proof of Proposition 5.2.4\]](#), when $p > n$, the weight-raising differential operators $\mathcal{D}^\lambda$ modulo p , for λ symmetric, exist and are nonzero. In fact, by loc. cit., in the unitary case, it suffices to work with $p > \max_{\tau \in \mathcal{T}}(\min(a_\tau, a_{\tau^*}))$.

3.3. Differential operators on automorphic forms modulo p . Let X denote the reduction modulo p of the Shimura variety $\mathcal{X}$, i.e., $X := \mathcal{X}_1$, and set $\omega = \omega_{\mathcal{A}/X}$. We now construct a new class of weight-raising differential operators Θ_κ^λ on the space $H^0(\mathcal{X}_1, \omega_{\mathcal{A}/X}^\kappa)$ of automorphic forms in characteristic p . For the special case $G = \mathrm{GSp}_4$, Yamauchi [\[2014\]](#) constructed and studied similar operators.

3.3.1. Adjugates and the Hasse invariant. For a morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ of locally free $\mathcal{O}_X$ -modules of rank g , the *adjugate* of f is the morphism

$$*f : \mathcal{G} \otimes |\mathcal{F}| \rightarrow \mathcal{F} \otimes |\mathcal{G}|$$

(recall $|\cdot|$ denotes the top exterior power) obtained from the dual map

$$\left(\wedge^{g-1} f\right)^t : (\mathcal{G} \otimes |\mathcal{G}|^{-1}) \simeq \left(\wedge^{g-1} \mathcal{G}\right)^t \rightarrow \left(\wedge^{g-1} \mathcal{F}\right)^t \simeq \mathcal{F} \otimes |\mathcal{F}|^{-1}$$

after tensoring with the identity map on $|\mathcal{F}| \otimes |\mathcal{G}|$. It satisfies the property that

$$*f \circ (f \otimes \mathrm{id}_{|\mathcal{F}|}) = \mathrm{id}_{\mathcal{F}} \otimes \det(f) : \mathcal{F} \otimes |\mathcal{F}| \rightarrow \mathcal{F} \otimes |\mathcal{G}|. \quad (7)$$

Let $\mathrm{Fr} : \mathcal{A} \rightarrow \mathcal{A}^{(p)}$ denote the relative Frobenius morphism on the universal abelian scheme over X . Denote by $h : \omega_{\mathcal{A}/X} \rightarrow \omega_{\mathcal{A}/X}^{(p)}$ the dual of the morphism $\mathrm{Fr}^* : R^1\pi_*\mathcal{O}_{\mathcal{A}}^{(p)} \rightarrow R^1\pi_*\mathcal{O}_{\mathcal{A}}$ introduced in [Section 2.2.7](#). In the following, by abuse of notation, we still denote by

$$*h : \omega_{\mathcal{A}/X}^{(p)} \rightarrow \omega_{\mathcal{A}/X} \otimes |\omega_{\mathcal{A}/X}|^{p-1}$$

the map obtained by tensoring the adjugate of h with the identity on $|\omega_{\mathcal{A}/X}|^{-1}$ and then composing with the identification $|\omega_{\mathcal{A}/X}^{(p)}| \simeq |\omega_{\mathcal{A}/X}|^p$. From equality (7), we deduce that

$$*h \circ h : \omega_{\mathcal{A}/X} \rightarrow \omega_{\mathcal{A}/X} \otimes |\omega_{\mathcal{A}/X}|^{p-1}$$

is equal to multiplication by the Hasse invariant $E = \det(h)$, as introduced in [Definition 2.2.5](#).

Define⁶

$$\mathcal{U} := \text{Image}(\text{Fr}^* : H_{\text{dR}}^1(\mathcal{A}/X)^{(p)} \rightarrow H_{\text{dR}}^1(\mathcal{A}/X)).$$

Lemma 3.3.1. *The relative Frobenius $\text{Fr} : \mathcal{A} \rightarrow \mathcal{A}^{(p)}$ induces an isomorphism*

$$H_{\text{dR}}^1(\mathcal{A}/X)/\mathcal{U} \simeq \omega^{(p)}. \quad (8)$$

Under this identification, the inclusion $\omega \subseteq H_{\text{dR}}^1(\mathcal{A}/X)$ composed with the projection

$$\pi_{\mathcal{U}} : H_{\text{dR}}^1(\mathcal{A}/X) \rightarrow H_{\text{dR}}^1(\mathcal{A}/X)/\mathcal{U}$$

agrees with the morphism $h : \omega \rightarrow \omega^{(p)}$.

Proof. By Katz’s work on the conjugate Hodge–de Rham spectral sequence [1972, Section 2.3] (see also [Wedhorn 2008, Corollary 1.6., §5.1, Proposition 5.1] and [Berthelot et al. 1982, §5.1]), the sheaf $\mathcal{U}$ and the quotient $H_{\text{dR}}^1(\mathcal{A}/X)/\mathcal{U}$ are locally free of rank g and dual to each other under the pairing $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ on $H_{\text{dR}}^1(\mathcal{A}/X)$ induced by the polarization $\mu_{\mathcal{A}}$ of $\mathcal{A}$. Furthermore, it follows from this work of Katz that the relative Frobenius

$$\text{Fr}^* : H_{\text{dR}}^1(\mathcal{A}/X)^{(p)} \rightarrow H_{\text{dR}}^1(\mathcal{A}/X)$$

induces an isomorphism

$$R^1\pi_*\mathcal{O}_{\mathcal{A}}^{(p)} \simeq \mathcal{U}. \quad (9)$$

Dualizing isomorphism (9), we obtain isomorphism (8), as well as that the inclusion $\omega \subseteq H_{\text{dR}}^1(\mathcal{A}/X)$ composed with the projection $\pi_{\mathcal{U}} : H_{\text{dR}}^1(\mathcal{A}/X) \rightarrow H_{\text{dR}}^1(\mathcal{A}/X)/\mathcal{U}$ agrees with the morphism $h : \omega_{\mathcal{A}/X} \rightarrow \omega_{\mathcal{A}/X}^{(p)}$ dual to $\text{Fr}^* : R^1\pi_*\mathcal{O}_{\mathcal{A}}^{(p)} \rightarrow R^1\pi_*\mathcal{O}_{\mathcal{A}}$. $\square$

3.3.2. A construction of differential operators in characteristic p . We continue to set $\omega = \omega_{\mathcal{A}/X}$. Consider the morphism

$$\Pi_{\mathcal{U}} := {}^*\!h \circ \pi_{\mathcal{U}} : H_{\text{dR}}^1(\mathcal{A}/X) \rightarrow H_{\text{dR}}^1(\mathcal{A}/X)/\mathcal{U} \simeq \omega^{(p)} \rightarrow \omega \otimes |\omega|^{p-1}.$$

Define

$$\Theta := (\Pi_{\mathcal{U}} \otimes \text{ks}^{-1}) \circ \nabla|_{\omega} : \omega \rightarrow H_{\text{dR}}^1(\mathcal{A}/X) \otimes \Omega_{X/W(\overline{\mathbb{F}}_p)}^1 \rightarrow (\omega \otimes |\omega|^{p-1}) \otimes \omega^2 = \omega \otimes (|\omega|^{p-1} \otimes \omega^2).$$

More generally, we construct an operator

$$\Theta_d : \omega^{\otimes d} \rightarrow \omega^{\otimes d} \otimes (|\omega|^{p-1} \otimes \omega^2) \subseteq \omega^{\otimes d+p+1}$$

as follows. As in [Eischen and Mantovan 2021a, Section 3.1] (or similarly [Eischen 2012, Section 3.1.2] and [Harris 1981, Section 4.3]), we use the product rule to extend the Gauss–Manin connection to a connection

$$\nabla_{\otimes d} : (H_{\text{dR}}^1(\mathcal{A}/X))^{\otimes d} \rightarrow (H_{\text{dR}}^1(\mathcal{A}/X))^{\otimes d} \otimes \Omega_{X/W(\overline{\mathbb{F}}_p)}^1,$$

⁶More precisely, $\mathcal{U}$ is the sheafification of the image of Fr^* .

via

$$\nabla_{\otimes d}(f_1 \otimes \cdots \otimes f_d) = \sum_{j=1}^d \iota_j(f_1 \otimes \cdots \otimes \nabla(f_j) \otimes \cdots \otimes f_d),$$

where ι_j is the isomorphism

$$\begin{aligned} \iota_j : (H_{\mathrm{dR}}^1(\mathcal{A}/X))^{\otimes j} \otimes \Omega_{Y/W(\overline{\mathbb{F}}_p)}^1 \otimes (H_{\mathrm{dR}}^1(\mathcal{A}/X))^{\otimes (d-j)} &\xrightarrow{\sim} (H_{\mathrm{dR}}^1(\mathcal{A}/X))^{\otimes d} \otimes \Omega_{X/W(\overline{\mathbb{F}}_p)}^1, \\ e_1 \otimes \cdots \otimes e_j \otimes u \otimes e_{j+1} \otimes \cdots \otimes e_d &\mapsto e_1 \otimes \cdots \otimes e_d \otimes u. \end{aligned}$$

So composing with the inclusion $\omega^{\otimes d} \hookrightarrow (H_{\mathrm{dR}}^1(\mathcal{A}/X))^{\otimes d}$, we get a morphism

$$\nabla_{\otimes d} : \omega^{\otimes d} \rightarrow \sum_{j=1}^d (\omega^{\otimes (j-1)} \otimes H_{\mathrm{dR}}^1(\mathcal{A}/X) \otimes \omega^{\otimes (d-j)} \otimes \Omega_{X/W(\overline{\mathbb{F}}_p)}^1). \quad (10)$$

For any $d \geq 1$, we then obtain an operator

$$\Theta_d := (\Pi_{\mathcal{U}} \otimes \mathrm{ks}^{-1}) \circ \nabla_{\otimes d} : \omega^{\otimes d} \rightarrow \omega^{\otimes d} \otimes (|\omega|^{p-1} \otimes \omega^2) \subseteq \omega^{\otimes d+p+1}.$$

This is similar to Katz's construction of the theta operator [1977, Remark, p. Ka-5 to Ka-6] and relies on the fact that the image of $\nabla_{\otimes d}$ restricted to $\omega^{\otimes d}$ is as in (10).

By the construction of Schur functors, for any dominant weight κ , the operator Θ_d for $d = |\kappa|$ induces an operator

$$\Theta_{\kappa} : \omega^{\kappa} \rightarrow \omega^{\kappa} \otimes (|\omega|^{p-1} \otimes \omega^2).$$

For any admissible weight λ of depth $e = |\lambda|/2$, the corresponding differential operators

$$\Theta_{\kappa}^{\lambda} : \omega^{\kappa} \rightarrow \omega^{\kappa+\lambda+(p-1)e}$$

are induced via Schur functors applied to the e -th iteration $\Theta_d^{(e)} := \Theta_{d+(p+1)(e-1)} \circ \cdots \circ \Theta_{d+p+1} \circ \Theta_d$,

$$\Theta_d^{(e)} : \omega^{\otimes d} \rightarrow \omega^{\otimes d} \otimes (|\omega|^{p-1} \otimes \omega^2)^{\otimes e} \subseteq \omega^{\otimes d+(p+1)e},$$

composed with the projection

$$\mathrm{id}_{|\omega|^{(p-1)e}} \otimes y_{\lambda} : (|\omega|^{p-1} \otimes \omega^2)^{\otimes e} = |\omega|^{(p-1)e} \otimes (\omega^2)^{\otimes e} \rightarrow |\omega|^{(p-1)e} \otimes \omega^{\lambda} = \omega^{\lambda+(p-1)e}.$$

Remark 3.3.2. In Section 3.4, we shall be particularly interested in the action of $\Theta_{\kappa}^{\lambda}$ on the restriction $\omega_{\mathcal{A}/X}|_S = \omega_{\mathcal{A}/S}$ to the ordinary locus $S := S_1 \subset X$. For convenience of notation, we write $\Theta_{\kappa}^{\lambda}|_S$ to denote the resulting morphism of sheaves $\omega_{\mathcal{A}/S}^{\kappa} \rightarrow \omega_{\mathcal{A}/S}^{\kappa+\lambda+(p-1)|\lambda|/2}$ over S . Similarly, we write $\Pi_{\mathcal{U}}|_S$ for the morphism $H_{\mathrm{dR}}^1(\mathcal{A}/S) \rightarrow \omega_{\mathcal{A}/S} \otimes |\omega_{\mathcal{A}/S}|^{p-1}$ obtained by restriction to sheaves over $S \subset X$.

3.4. Analytic continuation modulo p . We now prove the operators $\Theta_{\kappa}^{\lambda}$ analytically continue the mod p reduction of the p -adic differential operators $\mathcal{D}_{\kappa}^{\lambda}$, a priori defined only over the ordinary locus $S := S_1$, to all of $X := \mathcal{X}_1$. More precisely, we establish the following result, of which Theorem A from Section 1.2 is a consequence.

Theorem 3.4.1 (analytic continuation). *For any admissible weight λ and dominant weight κ ,*

$$\Theta_\kappa^\lambda|_S \equiv E^{|\lambda|/2} \cdot \mathcal{D}_\kappa^\lambda \pmod{p}$$

as morphisms $\omega_{\mathcal{A}/S}^\kappa \rightarrow \omega_{\mathcal{A}/S}^{\kappa+\lambda+(p-1)|\lambda|/2}$ of sheaves over S .

In particular, for any classical automorphic form f of weight κ , the mod p reduction of the p -adic automorphic form $E^{|\lambda|/2} \cdot \mathcal{D}_\kappa^\lambda f$ is classical.

Proof. Comparing the constructions of the operators Θ_κ^λ and $\mathcal{D}_\kappa^\lambda$, the statement reduces to the following lemma. □

Lemma 3.4.2. *Maintaining the above notation,*

$$\Pi_{\mathcal{U}}|_S \equiv E \cdot \Pi_U \pmod{p}$$

as maps $H_{\mathrm{dR}}^1(\mathcal{A}/S) \rightarrow \omega_{\mathcal{A}/S} \otimes |\omega_{\mathcal{A}/S}|^{p-1}$.

Proof. We work over S and write $\omega = \omega_{\mathcal{A}/S}$. Consider the pullback of the slope filtration to $H_{\mathrm{dR}}^1(\mathcal{A}/S)^{(p)}$, namely $0 \subseteq U^{(p)} \subset H_{\mathrm{dR}}^1(\mathcal{A}/S)^{(p)}$. By construction of the unit root subcrystal U , the restriction of the relative Frobenius map Fr^* to $U^{(p)}$ induces an isomorphism onto U . Since Fr^* is equal to 0 on $\omega^{(p)}$, it follows from the definition of $\mathcal{U}$ that $\mathcal{U} = U$ over S .

By [Lemma 3.3.1](#), the induced morphism

$$\omega \simeq H_{\mathrm{dR}}^1(\mathcal{A}/S)/U \rightarrow H_{\mathrm{dR}}^1(\mathcal{A}/S)/\mathcal{U} \simeq \omega^{(p)}$$

agrees with the restriction to S of the morphism $h : \omega \rightarrow \omega^{(p)}$. We deduce that $\pi_{\mathcal{U}} \equiv h \circ \Pi_U$ over S , so

$$\Pi_{\mathcal{U}} = {}^*h \circ \pi_{\mathcal{U}} \equiv {}^*h \circ h \circ \Pi_U \equiv E \cdot \Pi_U \pmod{p}. \quad \square$$

Remark 3.4.3. Since the weights of $\Theta_\kappa^\lambda|_S(f)$ and $\mathcal{D}_\kappa^\lambda(f)$ are different, we can only compare the values of $\Theta_\kappa^\lambda|_S(f)$ and $\mathcal{D}_\kappa^\lambda(f)$ at a point of S after identifying modular forms of different weights in a larger, common space of p -adic modular forms V (as in [\[Hida 2004, Section 8.1.3\]](#) or [\[Caraiani et al. 2016, Section 4.2.1\]](#)). For the purpose of the present paper, we need not concern ourselves with such details of comparisons between forms of different weights. We note, though, that since the q -expansions of the Hasse invariant E are identically 1 mod p , the q -expansions of $\Theta_\kappa^\lambda|_S(f)$ and $\mathcal{D}_\kappa^\lambda(f)$ agree mod p , which implies $\Theta_\kappa^\lambda|_S(f) \equiv \mathcal{D}_\kappa^\lambda(f) \pmod{p}$ inside V .

Remark 3.4.4. In [Theorem 3.4.1](#), we show that in order to analytically continue $\mathcal{D}^\lambda$ to the whole Shimura variety, it suffices to multiply $\mathcal{D}^\lambda$ by a power of E which depends only on the depth of λ (in the sense of [Definition 2.2.3](#)). This guarantees that the resulting operator is defined on automorphic forms on the whole mod p Shimura variety. If one specializes to a specific weight λ of a given depth, though, it is possible that a smaller power of E suffices to cancel the poles introduced by $\mathcal{D}^\lambda$.

For example, this is seen for scalar weights in [\[Yamauchi 2014, Proposition 3.9\]](#). Working with $\mathrm{GSp}_4(\mathbb{Q})$ and toward a different goal from us, Yamauchi shows that $E\mathcal{D}^2(f)$ is entire for any classical form f (by comparison, [Theorem 3.4.1](#) gives the operator $E^2\mathcal{D}^2$). Although only carried out in that work for $\mathrm{GSp}_4(\mathbb{Q})$,

Yamauchi's ideas might be considered more generally, and offer a possible route to determining a lower power of E by which one needs to multiply the operator $\mathcal{D}^\lambda$. In particular, for a scalar weight $\lambda = 2k$, with $k \in \mathbb{Z}_{>0}$, one would expect the operator $E^k \mathcal{D}^{2k}$ to be entire. In the spirit of Katz's approach [1977, proof of Lemma 3] (which covers the case of $\mathrm{GSp}_2(\mathbb{Q})$), the key input is a careful analysis of ∇ , KS, and Frobenius, working locally on the base, with explicit coordinates. Indeed, Proposition 3.4.5 illustrates that in the special case of scalar-weight Siegel modular forms over $\mathbb{Q}$ of sufficiently large weight, we need not have multiplied by such a high power of E , as one copy suffices.

3.4.1. Relation to Böcherer and Nagaoka's work with Rankin–Cohen brackets. In the case of scalar-valued classical Siegel modular forms (when the totally real field is $\mathbb{Q}$) of level 1 that arise as the reduction mod p of classical characteristic 0 Siegel modular forms, work of Böcherer and Nagaoka [2007, Theorem 4] yields a different approach to the analytic continuation of the operator $\mathcal{D}^2$ modulo p for $p \geq n + 3$. We denote by $[\cdot, \cdot]$ the generalized Rankin–Cohen bracket constructed for pairs of Siegel modular forms of weight $\geq n/2$ in [Ibukiyama 1999] and [Eholzer and Ibukiyama 1998]. Böcherer and Nagaoka [2007, Section 4] define a theta operator (denoted here by Θ_{BN}) on classical Siegel modular forms f of level 1 and scalar weight k that arise as the mod p reduction of classical characteristic 0 Siegel modular forms. (As noted in Remark 3.4.6, not all intrinsic mod p Siegel modular forms are reductions from characteristic zero.)

When $2k \geq n$, Böcherer and Nagaoka [2007, proof of Theorem 4] express $\Theta_{\mathrm{BN}}(f)$ in terms of the normalized pairing $\{\cdot, \cdot\} := (2\pi\sqrt{-1})^{-n} [\cdot, \cdot]$. As noted there, $\Theta_{\mathrm{BN}}(f) = 0$ when $k < n/2$. This vanishing follows immediately from the formula for the action of Θ_{BN} on q -expansions $\sum a(T)q^T$ of mod p reductions of characteristic 0 (archimedean) Siegel modular forms, combined with the fact that classical characteristic 0 (archimedean) Siegel modular forms are singular, i.e., their Fourier coefficients $a(T)$ vanish at all half-integral positive T (as explained in [Klingen 1990, Section 1] and first proved in [Resnikoff 1975; Freitag 1975]).

Proposition 3.4.5. *Assume $p \geq n + 3$. Then, for any scalar-valued Siegel modular form f (where the totally real field is $\mathbb{Q}$) of level 1 and weight k ,*

$$\Theta^2(f) = \begin{cases} ((-1)^n/(n+1))E^{n-1}\{f, E\} \bmod p & \text{if } k \geq n/2, \\ 0 & \text{if } k < n/2. \end{cases}$$

Proof. First we deal with the case $k \geq n/2$, i.e., the range where $\{\cdot, \cdot\}$ is defined. Under the assumptions that $p \geq n + 3$ and $2k \geq n$, both $E^{n-1}\{f, E\}$ and $\Theta^2 \equiv E^n \cdot \mathcal{D}^2(f)$ are Siegel modular forms, of scalar weight $k + 2 + n(p - 1)$. Hence, by the q -expansion principle, it suffices to compare their q -expansions. By comparing the action of $\{\cdot, E\}$ in [Böcherer and Nagaoka 2007, proof of Theorem 4] with the action of $\mathcal{D}^2$ on q -expansions in [Eischen 2012, Theorem 9.2], combined with the formula for the effect of the Schur functors on coefficients (as provided, e.g., in [Eischen et al. 2018, Corollary 5.2.10]), we deduce

$$\frac{(-1)^n}{(n+1)!} \{\cdot, E\}|_S \equiv \frac{1}{n!} E \mathcal{D}^2 \bmod p.$$

(Note that $\Theta^2 f(q) = n! \Theta_{\text{BN}} f(q)$.) Under the assumption $p > n + 3$, both constants $(-1)^n/(n+1)!$ and $1/n!$ are p -adic units, so their quotient $(-1)^n/(n+1)$ is a p -adic unit as well. So, applying [Theorem 3.4.1](#), we have

$$\Theta^2 \equiv E^n \mathcal{D}^2 \equiv \frac{(-1)^n}{(n+1)} E^{n-1} \{ \cdot, E \} \pmod{p}.$$

In the case $k < n/2$, as noted above, f is singular. Similarly to the argument of [\[Böcherer and Nagaoka 2007\]](#) mentioned above, we immediately see from the action of Θ^2 on q -expansions that $\Theta^2(f) = 0$ for all singular forms. $\square$

Remark 3.4.6. As noted in the remark at the end of [\[Böcherer and Nagaoka 2007, Section 1\]](#), the Böcherer–Nagaoka operator applies to reductions modulo p of scalar-weight classical Siegel modular forms (when the totally real field is $\mathbb{Q}$) in characteristic zero, whereas our operators apply to intrinsic characteristic p Siegel modular forms. (As explained in [Remark 3.4.4](#), specializing to a specific weight in our setting can enable lowering our exponent on E to match theirs.) It is expected that, in low weights, the space of intrinsic forms is strictly bigger than the space of reductions. Stroh [\[2013, Théorème 1.3; 2010, Théorème 1.1\]](#) and Lan and Suh [\[2011, Corollary 4.3\]](#) show that in the scalar-valued case, the two spaces coincide when $k \geq n + 2$ and $p \geq n(n+1)/2$.

Our focus on the geometry provides results for a broader set of cases, including vector weights (as well as lower scalar weights, since we do not restrict ourselves to reductions mod p of characteristic 0 forms), Siegel–Hilbert modular forms associated with totally real fields of arbitrary degree (as opposed to degree 1 in [\[Böcherer and Nagaoka 2007\]](#)), and unitary groups of arbitrary signature.

4. Commutation relations with Hecke operators

The goal of this section is to explain how the differential operators Θ_κ^λ (resp. $\mathcal{D}_\kappa^\lambda$) interact with Hecke operators. Following the convention of [\[Edixhoven 1992, Section 3.1\]](#), we use the terminology “commutation relations” to describe this interaction. We shall see that the two commute up to (explicit) scalar multiples. Hence, in particular, we shall deduce that the differential operators Θ_κ^λ (resp. $\mathcal{D}_\kappa^\lambda$) map Hecke eigenforms to Hecke eigenforms. Furthermore, in the case of Hecke operators at p , we shall prove that the scalar multiple is a power of p , and as an application, deduce the vanishing of operators $e\mathcal{D}_\kappa^\lambda = 0$, where e denotes Hida’s ordinary projector. For modular forms, such vanishing was also noted in [\[Coleman et al. 1995, p. 29\]](#) and [\[Coleman 1995, p. 345\]](#).

4.1. Interaction with isogenies. Throughout this section, let Y be a scheme that is smooth over a scheme T .

Definition 4.1.1. Given two polarized abelian varieties (A, μ_A) , (B, μ_B) , we say an isogeny $\phi : A \rightarrow B$ preserves the polarizations up to a scalar multiple if there exists $\nu(\phi) \in \mathbb{Q}^\times$, called the *similitude factor*, such that

$$\phi^t \circ \mu_B \circ \phi = [\nu(\phi)]_{A^t} \circ \mu_A.$$

Let (A, μ_A) and (B, μ_B) be polarized abelian schemes of the same relative dimension g over the scheme Y/T . An isogeny $\phi : A \rightarrow B$ defined over Y induces a morphism $\phi^* : H_{\text{dR}}^1(B/Y) \rightarrow H_{\text{dR}}^1(A/Y)$ which preserves the Hodge filtration, that is, $\phi^*(\omega_{B/Y}) \subseteq \omega_{A/Y}$. By abuse of notation, for each dominant weight κ , we still denote by

$$\phi^* : \omega_{B/Y}^\kappa \rightarrow \omega_{A/Y}^\kappa$$

the morphism induced by the map ϕ^* via Schur functors. Note that if the polarizations μ_A and μ_B have the same degree, then $v \in \mathbb{Z}$ and $\deg(\phi) = v(\phi)^g$. In particular, for $(A, \mu_A) = (B, \mu_B)$, if $\phi = [n]_A$, then $v(\phi) = n^2$; in this case, the morphism $(\phi^*)^{\otimes d}$ is multiplication by n^d .

Lemma 4.1.2. *Let (A, μ_A) and (B, μ_B) be polarized abelian schemes of the same relative dimension over Y/T , and let $\phi : A \rightarrow B$ be an isogeny preserving their polarization up to multiplication by a scalar $v(\phi) \in \mathbb{Q}^\times$. Then*

$$\text{KS}_{A/Y} \circ (\phi^* \otimes \phi^*) = v(\phi) \text{KS}_{B/Y}.$$

Proof. By the definition of the Kodaira–Spencer morphism

$$\text{KS}_{A/Y} := \langle \cdot, \nabla_A(\cdot) \rangle_A,$$

the statement is equivalent to the equality

$$\langle \phi^*(\cdot), \nabla_A \phi^*(\cdot) \rangle_A = v(\phi) \langle \cdot, \nabla_B(\cdot) \rangle_B.$$

By the functoriality of the Gauss–Manin connection, we deduce $\nabla_A \circ \phi^* = (\phi^* \otimes \text{id}) \circ \nabla_B$, and reduce the statement to the equality

$$\langle \phi^*(\cdot), \phi^*(\cdot) \rangle_A = v(\phi) \langle \cdot, \cdot \rangle_B.$$

The latter follows from $\phi^t \circ \mu_B \circ \phi = v(\phi) \mu_A$. □

4.2. Commutation relations with prime-to- p Hecke operators. Following [Faltings and Chai 1990, Chapter VII, §3], we define the action of prime-to- p algebraic correspondences, and of prime-to- p Hecke operators, on automorphic forms. For the construction of the moduli spaces of isogenies in the PEL-type case, see [Bütel and Wedhorn 2006, §1.6–1.7].

Fix a rational prime $\ell \neq p$. **We assume ℓ is good** (see Section 2.1.3 for definition). We denote by $\mathcal{H}_0(G_\ell, \mathbb{Q})$ the $\mathbb{Q}$ -subalgebra of the local Hecke algebra $\mathcal{H}(G_\ell, K_\ell; \mathbb{Q})$ generated by locally constant functions supported on cosets $K_\ell \gamma K_\ell$, for $\gamma \in G_\ell$ an integral matrix (see Section 2.1.2 for notation).

Definition 4.2.1. We denote by ℓ -Isog the moduli scheme⁷ of ℓ -isogenies over $\mathcal{X}$, and by $\phi : \text{pr}_1^* \mathcal{A} \rightarrow \text{pr}_2^* \mathcal{A}$ the universal ℓ -isogeny for

$$\text{pr} = (\text{pr}_1, \text{pr}_2) : \ell\text{-Isog} \rightarrow \mathcal{X} \times \mathcal{X}$$

the natural structure morphism.

⁷In general, ℓ -Isog is a stack, but it is relatively representable over $\mathcal{X}$; as noted in [Bütel and Wedhorn 2006, §1.7], the latter is a scheme in our setting since we assume that the level is neat.

Note that the degree of the universal isogeny is locally constant on ℓ -Isog. For any connected component Z of ℓ -Isog, the two projections $\mathrm{pr}_i : Z \rightarrow \mathcal{X}$ are proper, and they are finite étale over $\mathcal{X}[1/\deg(Z, \phi)]$, where $\deg(Z, \phi)$ denotes the degree of ϕ on Z . In particular, they are finite and étale over $\mathcal{X}/\mathcal{O}_{E, \mathfrak{p}}$.

Definition 4.2.2. For any dominant weight κ , there is a natural action of (Z, ϕ) on the space $H^0(\mathcal{X}, \omega^\kappa)$ of automorphic forms of weight κ , defined as

$$T_\phi = T_{(Z, \phi), \kappa} := \mathrm{tr} \circ \phi^* \circ \mathrm{pr}_2^* : H^0(\mathcal{X}, \omega^\kappa) \rightarrow H^0(Z, \mathrm{pr}_2^* \omega^\kappa) \rightarrow H^0(Z, \mathrm{pr}_1^* \omega^\kappa) \rightarrow H^0(\mathcal{X}, \omega^\kappa).$$

It follows from the definition that, for any dominant weight κ , the operator $T_{(Z, \phi), \kappa}$ induces an action of (Z, ϕ) on the space $H^0(X, \omega^\kappa)$ of mod p automorphic forms of weight κ (by reduction modulo p), and also on the space $H^0(\mathcal{S}, \omega^\kappa)$ of p -adic automorphic forms of weight κ (by restriction).

Definition 4.2.3. Let Y denote a base scheme such that ℓ is invertible in $\mathcal{O}_Y$, equipped with a map $f : Y \rightarrow \mathcal{X}$ (e.g., $Y = \mathcal{X}$ over $\mathcal{O}_{E, \mathfrak{p}}$, $Y = X$ over $\mathbb{F}_p$, or $Y = \mathcal{S}$ over $\mathbb{Z}_p$). There is a natural $\mathbb{Q}$ -linear map

$$h = h_\ell : \mathcal{H}_0(G_\ell, \mathbb{Q}) \rightarrow \mathbb{Q}[\ell\text{-Isog}/Y]$$

which to any double coset $K_\ell \gamma K_\ell$, with γ an integral matrix in G_ℓ , associates the union $Z(K_\ell \gamma K_\ell)$ of those connected components of ℓ -Isog where the universal isogeny is an ℓ -isogeny of type $K_\ell \gamma K_\ell$.

By definition, the action on automorphic forms of the prime-to- p Hecke operators agrees with that of the prime-to- p algebraic correspondences (via pullback under h).

Theorem 4.2.4. *Let (Z, ϕ) be a connected component of ℓ -Isog. For any dominant weight κ and any admissible weight λ , we have*

$$(i) \quad T_\phi \circ \Theta_\kappa^\lambda = v(\phi)^{|\lambda|/2} \Theta_\kappa^\lambda \circ T_\phi,$$

$$(ii) \quad T_\phi \circ \mathcal{D}_\kappa^\lambda = v(\phi)^{|\lambda|/2} \mathcal{D}_\kappa^\lambda \circ T_\phi,$$

where $v(\phi)$ denotes the similitude factor of ϕ .

Proof. For (i), by the functoriality of the construction of the operators Θ_κ^λ , it suffices to establish the equality

$$T_\phi \circ \Theta = v(\phi) \Theta \circ T_\phi.$$

By definition, $\Theta := (\Pi_{\mathcal{U}} \otimes \mathrm{ks}^{-1}) \circ \nabla|_{\omega_{\mathcal{A}/X}}$. By the functoriality of the Gauss–Manin connection ∇ , and of the definition of $\Pi_{\mathcal{U}}$, it suffices to prove the equality

$$v(\phi) \mathrm{KS} = \mathrm{KS} \circ (\phi^* \otimes \phi^*) : \mathrm{pr}_2^* H_{\mathrm{dR}}^1(\mathcal{A}/X) \otimes \mathrm{pr}_2^* H_{\mathrm{dR}}^1(\mathcal{A}/X) \rightarrow \mathrm{pr}_1^* \Omega_{X/T}^1.$$

This follows from [Lemma 4.1.2](#).

For (ii), the statement reduces to the equality $T_\phi \circ \mathcal{D} = v(\phi) \mathcal{D} \circ T_\phi$, and the same argument applies to the operator $\mathcal{D}$, with the morphism Π_U in place of $\Pi_{\mathcal{U}}$. $\square$

Remark 4.2.5. In the Siegel case, for an admissible scalar weight $\lambda = \underline{d}$, we have that d is even, $|\lambda| = gd$, and $v(\phi)^{|\lambda|/2} = \deg(\phi)^{d/2}$. In particular, for $g = 2$, [Theorem 4.2.4](#) specializes to Yamauchi's result [2014, Proposition 3.9].

4.3. Commutation relations with Hecke operators at p . Following [Faltings and Chai 1990, Chapter VII, §4], we define the action of p -power algebraic correspondences, and of Hecke operators at p , on automorphic forms over the ordinary locus $\mathcal{S}$. Following the same work, we identify $H \times \mathbb{G}_m$ with the appropriate maximal Levi subgroup M of $\mathcal{G}$ over $\mathbb{Z}_p$ (see [Section 2.2.3](#)).

We write $M_p := M(\mathbb{Q}_p)$ and express $\gamma \in M_p$ as $\gamma = (\alpha, p^d)$ with $\alpha \in H(\mathbb{Q}_p)$ and $d \in \mathbb{Z}$. Note that $\gamma = (\alpha, p^d) \in M_p$ is an integral matrix if and only if $d \geq 0$ and both α and $p^d \alpha^{-1}$ are integral matrices in $H(\mathbb{Q}_p)$. Also, by definition, $M(\mathbb{Z}_p) = K_p \cap M_p$, and hence the local Hecke algebra $\mathcal{H}(M_p, M(\mathbb{Z}_p); \mathbb{Q})$ is a subalgebra of $\mathcal{H}(G_p, K_p; \mathbb{Q})$ (see [Section 2.1.2](#) for notation).

We denote by $\mathcal{H}_0(M_p, \mathbb{Q})$ the $\mathbb{Q}$ -subalgebra of $\mathcal{H}(M_p, M(\mathbb{Z}_p); \mathbb{Q})$ generated by locally constant functions supported on cosets $M(\mathbb{Z}_p)\gamma M(\mathbb{Z}_p)$, for $\gamma \in M_p$ an integral matrix.

Definition 4.3.1. We denote by $p\text{-Isog}^o$ the moduli scheme of p -isogenies over the ordinary locus $\mathcal{S}$, and by $\phi : \text{pr}_1^* \mathcal{A} \rightarrow \text{pr}_2^* \mathcal{A}$ the universal p -isogeny for

$$\text{pr} = (\text{pr}_1, \text{pr}_2) : p\text{-Isog}^o \rightarrow \mathcal{S} \times \mathcal{S}$$

the natural structure morphism.

By [Faltings and Chai 1990, §VII.4, Proposition 4.1], for any connected component Z of $p\text{-Isog}^o$, the two projections $\text{pr}_i : Z \rightarrow \mathcal{S}$ are finite and flat over $\mathcal{S}/W(\overline{\mathbb{F}}_p)$.

Definition 4.3.2. For any dominant weight κ , there is a natural action of (Z, ϕ) on the space $H^0(\mathcal{S}, \omega^\kappa)$ of p -adic automorphic forms of weight κ , defined as

$$T_\phi = T_{(Z, \phi)} := \text{tr} \circ \phi^* \circ \text{pr}_2^* : H^0(\mathcal{S}, \omega^\kappa) \rightarrow H^0(Z, \text{pr}_2^* \omega^\kappa) \rightarrow H^0(Z, \text{pr}_1^* \omega^\kappa) \rightarrow H^0(\mathcal{S}, \omega^\kappa).$$

Theorem 4.3.3. Let (Z, ϕ) be a connected component of $p\text{-Isog}^o$. For any dominant weight κ and any admissible weight λ , we have

$$T_\phi \circ \mathcal{D}_\kappa^\lambda = v(\phi)^{|\lambda|/2} \mathcal{D}_\kappa^\lambda \circ T_\phi,$$

where $v(\phi)$ denotes the similitude factor of ϕ . In particular, if $v(\phi) > 1$, then $T_\phi \circ \mathcal{D}_\kappa^\lambda \equiv 0 \pmod{p}$.

Proof. The commutation relations follow by the same argument as in the proof of part (ii) of [Theorem 4.2.4](#). The vanishing in positive characteristic follows from the equality $v(\phi) = p^d$ for $d \geq 1$. $\square$

Consider the mod p -reduction

$$\text{pr}_{\overline{\mathbb{F}}_p} : p\text{-Isog}^o \otimes_{W(\overline{\mathbb{F}}_p)} \overline{\mathbb{F}}_p \rightarrow \mathcal{S} \times \mathcal{S}.$$

Note that the degree, similitude factor, and p -type of the universal isogeny are locally constant on $p\text{-Isog}^o$. (Recall that, by definition, the p -type of a p -isogeny with similitude factor p^d is a coset $H(\mathbb{Z}_p)\alpha H(\mathbb{Z}_p)$ of an integral matrix $\alpha \in H(\mathbb{Q}_p)$ such that $p^d \alpha^{-1}$ is also integral.) Faltings and Chai [1990, §VII.4,

Proposition 4.1] compute, for any connected component Z of $p\text{-Isog}^o \otimes_{W(\overline{\mathbb{F}}_p)} \overline{\mathbb{F}}_p$, the purely inseparable multiplicity $\mu(Z)$ of the geometric fibers of $\text{pr}_i : Z \rightarrow S$ in terms on the degree and the p -type of the universal isogeny on Z . Furthermore, they prove there is a well-defined integral action of the operator $\mu(Z)^{-1}T_{(Z,\phi)}$ on the space of mod p automorphic forms of weight κ , for all dominant weights κ .

We define the *normalized* action of (Z, ϕ) on mod p automorphic forms as $t_\phi = t_{(Z,\phi)} := \mu(Z)^{-1}T_{(Z,\phi)}$.

Definition 4.3.4. Let Y denote a base scheme of characteristic p , equipped with a map $f : Y \rightarrow S$. There is a natural $\mathbb{Q}$ -linear map

$$h = h_p : \mathcal{H}_0(M_p, \mathbb{Q}) \rightarrow \mathbb{Q}[p - \text{Isog}^o/Y]$$

which to any double coset $M(\mathbb{Z}_p)\gamma M(\mathbb{Z}_p)$ with $\gamma = (\alpha, p^d)$ an integral matrix in M_p , associates $\mu(Z(\alpha, d))^{-1} \cdot Z(\alpha, d)$, where $Z(\alpha, d)$ denotes the union of those connected components of $p - \text{Isog}^o$ where the universal isogeny is a p -isogeny of p -type $H(\mathbb{Z}_p)\alpha H(\mathbb{Z}_p)$ and similitude factor p^d .

By definition, the action of the Hecke operators at p on mod p automorphic forms agrees with the *normalized* action of p -power algebraic correspondences.

4.3.1. Hida’s ordinary projector. Hida [2004, §8; 2002, §3.6] establishes the divisibility of the Hecke operators at p on p -adic automorphic forms by a given power of p , keeping their integrality. More precisely, he proves that the normalized action on mod p automorphic forms lifts to p -adic automorphic forms.

For $1 \leq j \leq n$, let $\alpha_j \in H(\mathbb{Q}_p)$ be defined by $\alpha_{j,\tau} := \text{diag}[\mathbb{I}_{n-j}, p\mathbb{I}_j]$ for $\tau \in \mathcal{T}_0$. (For any positive integer k , $\mathbb{I}_k$ denotes the $k \times k$ identity matrix.) Let Z_{α_j} denote the union of those connected components of $p\text{-Isog}^o$ where the universal isogeny has p -type $H(\mathbb{Z}_p)\alpha_j H(\mathbb{Z}_p)$ and similitude factor p . For $j = 1, \dots, n$, Hida proves the integrality of the operators $\mu(\alpha_j)^{-1}\mathbb{U}(\alpha_j) := \mu(Z_{\alpha_j})^{-1}T_{(Z_{\alpha_j},\phi)}$ on the space of p -adic automorphic forms over the ordinary locus.

By definition [Hida 2004, §8.3.1, Lemma 8.12], the ordinary projector e on the space of p -adic automorphic forms is

$$e := \varinjlim_m \mathbb{U}(p)^{m!},$$

where $\mathbb{U}(p) := \prod_1^n \mu(\alpha_j)^{-1}\mathbb{U}(\alpha_j)$.

Hence, by Theorem 4.3.3, we deduce the following result.

Corollary 4.3.5. For any dominant weight κ , and admissible weight λ , we have

$$e\mathcal{D}_\kappa^\lambda = 0.$$

5. Applications to Galois representations

Theorem 4.2.4 implies that the mod p differential operators from Section 3 map Hecke eigenforms to Hecke eigenforms. When there are Galois representations arising from the corresponding systems of Hecke eigenvalues, it is natural to ask how our differential operators affect the Galois representations.

5.1. An axiomatic theorem. We start with a result on Galois representations attached to systems of Hecke eigenvalues for a general algebraic reductive group G defined over $\mathbb{Q}$ (and split over a number field F), as introduced in [Section 2.1.3](#).

We fix a prime p and write χ for the mod p cyclotomic character. Given a mod p Hecke eigenform f , we denote by Σ_f the set of places that are bad with respect to p and the level of f , and by $R(f)$ the set of Galois representations $\rho : \text{Gal}(\bar{F}/F) \rightarrow \widehat{G}(\bar{\mathbb{F}}_p)$ that are (conjecturally) attached to f as in [Conjecture 2.1.1](#).

Theorem 5.1.1 (axiomatic theorem). *Let G be a connected reductive group over $\mathbb{Q}$, split over the number field F . For $i = 1, 2$, let f_i be a mod p Hecke eigenform on G . For $v \notin \Sigma_i := \Sigma_{f_i}$, let $\Psi_{i,v} : \mathcal{H}(G_v, K_v; \bar{\mathbb{F}}_p) \rightarrow \bar{\mathbb{F}}_p$ denote the Hecke eigensystem of f_i at v . Then*

$$\Psi_{2,v}(c_\lambda) = \eta(\lambda(\pi_v))\Psi_{1,v}(c_\lambda) \quad \text{for all } \lambda \in P^+ \text{ and } v \notin \Sigma_1 \cup \Sigma_2 \quad (11)$$

for a character η of G if and only if

$$(\hat{\eta} \circ \chi) \otimes \rho_1 \in R(f_2) \quad \text{for all } \rho_1 \in R(f_1) \quad (12)$$

for the cocharacter $\hat{\eta}$ of $\widehat{G}$ corresponding to η by duality.

Remark 5.1.2. For Galois representations arising from Hecke eigenforms on symplectic and unitary groups, [Theorem 5.1.1](#) specializes to [Theorem B](#) from [Section 1.2](#), as we shall observe in [Section 5.2](#).

Remark 5.1.3. The use of the tensor product sign in $(\hat{\eta} \circ \chi) \otimes \rho_1$ is generally an abuse of notation and should be understood as in [Lemma 5.1.5](#). We retain the $\otimes$ notation because it is evocative of the case when $\widehat{G}$ is a group of matrices, where we are indeed dealing with a tensor product of representations.

The proof of [Theorem 5.1.1](#) uses the following lemmas.

Lemma 5.1.4. *Let G be a reductive group and let $\eta : G \rightarrow \mathbb{G}_m$ be a character of G . Then the cocharacter $\hat{\eta} : \mathbb{G}_m \rightarrow \widehat{G}$ has image in the center $Z(\widehat{G})$ and*

$$\eta \circ \mu = \eta \circ \lambda$$

for any dominant weights $\mu, \lambda \in P^+$ such that $\mu \leq \lambda$.

Proof. The image of η is abelian. So $\eta(G') = 1$, where G' denotes the derived subgroup of G . So η induces a character of the abelianization of G , hence the dual cocharacter $\hat{\eta}$ lands in the center of $\widehat{G}$. Moreover, since $\eta(G') = 1$, and the coroots of G are the same as the coroots of G' , we see that $\eta \circ \alpha^\vee = 1$ for any coroot $\alpha^\vee$. If μ and λ are comparable, they differ by a linear combination of coroots, so $\eta \circ \mu = \eta \circ \lambda$. $\square$

Lemma 5.1.5. *Let G be a reductive group over a field k and let $\eta : G \rightarrow \mathbb{G}_m$ be a character of G . Let $\rho : \Gamma \rightarrow \widehat{G}(k)$ be a representation of a group Γ into the k -valued points of the dual group $\widehat{G}$. Let $\chi : \Gamma \rightarrow \mathbb{G}_m(k)$ be a k -valued character of Γ . Then the map $(\hat{\eta} \circ \chi) \otimes \rho : \Gamma \rightarrow \widehat{G}(k)$ defined by*

$$((\hat{\eta} \circ \chi) \otimes \rho)(\gamma) := \hat{\eta}(\chi(\gamma))\rho(\gamma) \quad (13)$$

is a representation of Γ .

Proof. In order to show that (13) defines a group homomorphism, it suffices to prove that $\hat{\eta}(\chi(\gamma))$ is in the center of $\widehat{G}(k)$, which was done in Lemma 5.1.4. $\square$

We are ready to prove the main result of this section.

Proof of Theorem 5.1.1. Let $\rho_1 \in R(f_1)$ and define $\rho_2 := (\hat{\eta} \circ \chi) \otimes \rho_1$. Then ρ_1 and ρ_2 are both unramified at primes $v \notin \Sigma := \Sigma_1 \cup \Sigma_2$, and $\rho_2(\text{Frob}_v) = \hat{\eta}(\pi_v) \rho_1(\text{Frob}_v)$ for all $v \notin \Sigma$.

For $v \notin \Sigma$, let $s_{i,v}$ denote the v -Satake parameter of f_i , for $i = 1, 2$. Then $\rho_i \in R(f_i)$ if and only if $s_{i,v} = \rho_i(\text{Frob}_v)$ for all $v \notin \Sigma$. Hence (12) is equivalent to the equalities

$$s_{2,v} = \hat{\eta}(\pi_v) s_{1,v} \quad \text{for all } v \notin \Sigma.$$

Fix $v \notin \Sigma$. Recalling the Satake parameters from Section 2.1.2, note that the equality $s_{2,v} = \hat{\eta}(\pi_v) s_{1,v}$ is equivalent to

$$\chi_\lambda(s_{2,v}) = \chi_\lambda(\hat{\eta}(\pi_v) s_{1,v}) \quad \text{for all } \lambda \in P^+.$$

Let $\lambda \in P^+$, and let $\rho_\lambda : \widehat{G}(\overline{\mathbb{F}}_p) \rightarrow \text{GL}(V_\lambda)$ be the irreducible representation of $\widehat{G}(\overline{\mathbb{F}}_p)$ of highest weight λ . As a function on the maximal torus $\widehat{T}$, the character of ρ_λ can be written as

$$\chi_\lambda = \sum_{\mu \leq \lambda} (\dim V_\lambda(\mu)) \hat{\mu}.$$

In particular,

$$\begin{aligned} \chi_\lambda(\hat{\eta}(\pi_v) s_{f_1,v}) &= \sum_{\mu \leq \lambda} (\dim V_\lambda(\mu)) \hat{\mu}(\hat{\eta}(\pi_v) s_{f_1,v}) = \sum_{\mu \leq \lambda} (\dim V_\lambda(\mu)) \hat{\mu}(\hat{\eta}(\pi_v)) \hat{\mu}(s_{f_1,v}) \\ &= \sum_{\mu \leq \lambda} (\dim V_\lambda(\mu)) \eta(\mu(\pi_v)) \hat{\mu}(s_{f_1,v}) = \eta(\lambda(\pi_v)) \sum_{\mu \leq \lambda} (\dim V_\lambda(\mu)) \hat{\mu}(s_{f_1,v}) \\ &= \eta(\lambda(\pi_v)) \chi_\lambda(s_{f_1,v}), \end{aligned}$$

where we used that $\eta(\mu(\pi_v)) = \eta(\lambda(\pi_v))$ whenever $\mu \leq \lambda$ (see Lemma 5.1.4).

Hence, (12) holds if and only if

$$\chi_\lambda(s_{2,v}) = \eta(\lambda(\pi_v)) \chi_\lambda(s_{1,v}) \quad \text{for all } \lambda \in P^+ \text{ and } v \notin \Sigma. \quad (14)$$

We use (6) to show that (11) implies (14) (and hence (12)). So

$$\begin{aligned} \chi_\lambda(s_{2,v}) &= \omega_2(\chi_\lambda) = \Psi_{2,v}(S_v^{-1}(\chi_\lambda)) = \Psi_{2,v}\left(q_v^{(-\rho, \lambda)} \sum_{\mu \leq \lambda} d_\lambda(\mu) c_\mu\right) \\ &= q_v^{(-\rho, \lambda)} \sum_{\mu \leq \lambda} d_\lambda(\mu) \Psi_{2,v}(c_\mu) = q_v^{(-\rho, \lambda)} \sum_{\mu \leq \lambda} d_\lambda(\mu) \eta(\mu(\pi_v)) \Psi_{1,v}(c_\mu) \\ &= \eta(\lambda(\pi_v)) q_v^{(-\rho, \lambda)} \sum_{\mu \leq \lambda} d_\lambda(\mu) \Psi_{1,v}(c_\mu) = \eta(\lambda(\pi_v)) \chi_\lambda(s_{1,v}), \end{aligned}$$

after another appeal to Lemma 5.1.4. Similarly, (5) can be used to show that (14) implies (11). $\square$

5.2. Effects of differential operators on Galois representations. We assume the group G is symplectic or unitary.

Proof of Theorem B. (For consistency with the notation in Section 5.1, here λ denotes a dominant weight in P^+ , and the admissible weight in the statement of Theorem B — previously denoted by λ — is κ_0 .) Theorem 4.2.4 implies that the assumptions in Theorem 5.1.1 are satisfied by the systems of Hecke eigenvalues associated with a mod p automorphic form f of weight κ and its image under $\Theta^{\kappa_0} = \Theta_{\kappa}^{\kappa_0}$. More precisely, Theorem 4.2.4 implies

$$\Psi_{\Theta^{\kappa_0}(f),v}(c_{\lambda}) = v(\lambda(\pi_v))^{| \kappa_0 | / 2} \Psi_{f,v}(c_{\lambda})$$

for all $\lambda \in P^+$ and all but finitely many places v (e.g., all places v not above p and of good reduction for the Shimura variety $\mathcal{X}$). Hence, the theorem is an immediate consequence of Theorem 5.1.1. $\square$

Theorem B could play a role in the study of Galois representations in positive characteristic, and also in generalizations of Serre's weight conjecture. For modular forms on GL_2 over $\mathbb{Q}$, the theory of Θ -operators, including an analogue of Theorem B, was used in the study of the modularity of 2-dimensional Galois representations in positive characteristic. Serre's weight conjecture predicted that every odd irreducible continuous 2-dimensional mod p Galois representation ρ that is unramified outside p has a twist by a power of the cyclotomic character that is modular of weight at most $p + 1$. Using Θ -operators, it was shown that the statement also implies that ρ itself is modular, of weight at most $p^2 - 1$.

On the other hand, Theorem B is only a first step in this direction, as more can be inferred from the existence of mod p weight-raising differential operators. More precisely, the study of the image (and kernel) of the theta operators would yield a range for weights of modularity for mod p Galois representations, while the study of theta cycles describes all weights occurring by twisting the Galois representation by powers of the cyclotomic characters. For modular forms, this is carried out in [Katz 1977], [Johnowitz 1982] and [Edixhoven 1992]; for Siegel modular forms of degree 2, we refer to [Yamauchi 2014]; for Hilbert modular forms, we refer to [Andreatta and Goren 2005, Section 16].

To exemplify potential future applications, we consider the case of Siegel and Hermitian modular forms of scalar, parallel weight, and of Θ -operators that raise the weight by a parallel weight. That is, we consider iterations of the operator Θ , where $\Theta = \Theta^2$ in the Siegel case and $\Theta = \Theta^1$ in the Hermitian case. For $d \geq 1$, we write $\Theta^d = \Theta \circ \cdots \circ \Theta$ (d times). We assume $p > n$.

Remark 5.2.1. By Theorem B, Θ acts on mod p Galois representations by twisting by the n -th power of the cyclotomic character, that is,

$$\rho_{\Theta(f)} = (\hat{v} \circ \chi)^n \otimes \rho_f.$$

In particular, when $(n, p - 1) = 1$, if the twist of ρ by some power of the cyclotomic character is modular, then ρ itself is modular. Also, if ρ is a Galois representation associated with a mod p automorphic form f , then any twist of ρ by powers of the cyclotomic character is associated with the image of f under some iteration of Θ . That is, if $(\hat{v} \circ \chi)^m \otimes \rho = \rho_f$ for some modular form f and integer $m \geq 1$, then $\rho = \rho_{\Theta^d(f)}$ for d such that $dn \equiv m \pmod{p - 1}$.

Let $f \neq 0$ be a mod p Siegel or Hermitian automorphic form of parallel weight. We write $w(f)$ for the *minimal weight* of f (in the terminology of [Katz 1977, Section I] recalled in Remark 1.2.3, the minimal

weight is called the exact filtration). By definition, $w(f)$ is the smallest integer $k > 0$ for which there exists a mod p automorphic form g of weight $\underline{k}$ whose (mod p) q -expansion agrees with that of f , that is, $g(q) \equiv f(q) \pmod{p}$. In particular, $w(f) = w(Ef)$.

Suppose f is in the image of Θ . We define the Θ -cycle of f as

$$(w(f), w(\Theta(f)), \dots, w(\Theta^{p-2}(f))).$$

By [Eischen et al. 2018, Theorem 5.2.6], for any mod p automorphic form g , we have $\Theta(g)(q) \equiv \Theta^p(g)(q) \pmod{p}$. We deduce that if f is a mod p automorphic form in the image of the operator Θ , then $\Theta(f) \neq 0$ and $w(f) = w(\Theta^{p-1}f)$. In particular, the minimal weights of f and of its images under iterations of the operator Θ all occur in the Θ -cycle of f .

Remark 5.2.2. By the p -adic and algebraic q -expansion principles ([Hida 2004, Section 8.4] and [Lan 2013, Proposition 7.1.2.15]; also [Ichikawa 2008, Theorem 1] for Siegel modular forms of level 1 and $p > n+2$), if f and g are two mod p automorphic forms of scalar parallel weights k and h respectively such that $f(q) \equiv g(q) \pmod{p}$ and $h \leq k$, then $h \equiv k \pmod{p-1}$ and $f = E^n g$ for $n = (k-h)/(p-1)$. We deduce $w(f) = k - b(p-1)$ for $b \geq 0$ the exponent of the power of E that exactly divides f . In particular, if f is a modular form of weight k , where $1 \leq k \leq p-2$, then $w(f) = k$. Also, if f is a cuspidal modular form of weight $k = p-1$, then $w(f) = p-1$.

Assume $\Theta(f) \neq 0$. By Theorem 3.4.1, $\Theta(f)$ is an automorphic form of weight $k+2+n(p-1)$; hence, $w(\Theta(f)) = w(f) + 2 + (n-a)(p-1)$, where $a \geq 0$ is the exponent of the power of E that exactly divides $\Theta(f)$. By Remark 3.4.4, we expect $a \geq n-1$; by Section 3.4.1, the statement is true if f is the mod p reduction of a scalar-weight Siegel modular form of level 1 and $p > n+2$. Indeed, by Proposition 3.4.5, $\Theta(f)$ is divisible by E^{n-1} ; hence, $a \geq n-1$ and

$$w(\Theta(f)) = w(f) + p + 1 - b(p-1)$$

for some integer $b \geq 0$. (Note that if f is the mod p reduction of a scalar-weight Siegel modular form of level 1, the assumption $\Theta(f) \neq 0$ also implies $w(f) \geq n/2$.)

For modular forms, the integer $b \geq 0$ is explicitly computed by Jochnowitz [1982] (level 1) and by Edixhoven [1992, §3.2] (general level). In particular, they prove that $b = 0$ unless $w(f) \equiv 0 \pmod{p}$.

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