

## STATIONARY REFLECTION

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**Abstract.** We improve the upper bound for the consistency strength of stationary reflection at successors of singular cardinals.

**§1. Introduction.** Stationary reflection is an important compactness principle in set theory. Its failure, the existence of a nonreflecting stationary set, is sufficient for the construction of objects which witness the noncompactness of various properties. Examples include freeness of abelian groups and metrizability of topological spaces [16] and chromatic number of graphs [20, 21].

We recall the basic definitions:

**DEFINITION 1.** Let  $\kappa$  be a regular cardinal. A set  $S \subseteq \kappa$  reflects at  $\alpha$  if  $S \cap \alpha$  is stationary at  $\alpha$ , where  $\text{cf } \alpha > \omega$ . We say that a stationary set  $S \subseteq \kappa$  reflects if it reflects at  $\alpha$  for some  $\alpha < \kappa$ .

**DEFINITION 2.** For a stationary set  $S \subseteq \kappa$ , we denote by  $\text{Refl}(S)$  the assertion:  $\forall T \subseteq S$  stationary,  $T$  reflects.

The main theorem of this paper deals with the consistency strength of stationary reflection at  $\aleph_{\omega+1}$ . Until our work the best known upper bound was due to Magidor [14].

**THEOREM 3 (Magidor).**  $\text{Refl}(\aleph_{\omega+1})$  is consistent relative to the existence of  $\omega$ -many supercompact cardinals.

We prove:

**THEOREM 4.**  $\text{Refl}(\aleph_{\omega+1})$  is consistent relative to the existence of a cardinal  $\kappa$  which is  $\kappa^+$ - $\Pi_1^1$ -subcompact.

Subcompact cardinals were defined by Jensen, and  $\kappa^+$ - $\Pi_1^1$ -subcompact cardinals were defined by Neeman and Steel (denoted  $\Pi_1^2$ -subcompact in [17]). Under GCH, the large cardinal assumption in our theorem is weaker than the assumption that  $\kappa$  is  $\kappa^+$ -supercompact. Subcompactness and its variations were defined during the investigation of square principles in core models. See Section 3 for the exact definitions, and more details.

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Our construction is motivated by an analogy with the consistency of stationary reflection at  $\aleph_2$ . Reflection of stationary sets is an instance of reflection of a  $\Pi_1^1$ -statement. Namely, if  $S$  is a subset of  $\kappa$ , then the statement “ $S$  is stationary” is a  $\Pi_1^1$ -statement in the model  $\langle H(\kappa), \in, S \rangle$ . If  $\kappa$  is weakly compact, then this  $\Pi_1^1$ -statement will reflect to a smaller ordinal  $\alpha$ . So  $S \cap \alpha$  is stationary and  $S$  reflects.

Thus, it was natural to suspect that the consistency strength of  $\text{Refl}(S_\omega^{\omega_2})$  is a weakly compact cardinal. Baumgartner [1] showed that after collapsing a weakly compact to be  $\aleph_2$ ,  $\text{Refl}(S_\omega^{\omega_2})$  holds and even any collection of  $\aleph_1$  stationary subsets of  $S_\omega^{\omega_2}$  will reflect at a common point. This thesis was supported by a result of Jensen that stationary reflection is possible in  $L$  only at weakly compact cardinals. Moreover, Magidor [14] showed that if any two stationary subsets of  $S_\omega^{\omega_2}$  have a common reflection point then  $\omega_2$  is weakly compact in  $L$ .

Surprisingly, in [10], Harrington and Shelah proved that the consistency strength of  $\text{Refl}(S_\omega^{\omega_2})$  is only a Mahlo cardinal. An important part of their result is the idea that one must iterate to destroy the stationarity of certain “bad” sets to achieve stationary reflection. These results show that there is a gap in the consistency strength between stationary reflection and simultaneous reflection for collections of stationary sets. This gap is explained by the difference between Jensen’s square  $\square_\kappa$  and Todorčević’s square  $\square(\kappa^+)$ . See [11] for more details.

In our work, we exploit the strong analogy between weak compactness and  $\Pi_1^1$ -subcompactness in order to get the consistency of stationary reflection at  $\aleph_{\omega+1}$ . Our argument is similar to Baumgartner’s in the sense that we do not need to iterate to destroy bad stationary sets. This analogy suggests that our assumption is not quite optimal.

There is a vast gap between the strength in the large cardinal axioms which are needed for stationary reflection at  $\aleph_2$  and at  $\aleph_{\omega+1}$ . This gap is related to the problem of controlling the successor of a singular cardinal. The Weak Covering Lemma [13] states that if there is no transitive model with a Woodin cardinal then there is a definable class  $K$  which is generically absolute and for every  $\kappa$  which is a strong limit singular cardinal in  $V$ ,  $(\kappa^+)^V = (\kappa^+)^K$ . In  $K$ ,  $\square_\kappa$  holds for all infinite  $\kappa$  by a result of Schimmerling and Zeman [19]. Since  $\square_\kappa$  is upwards absolute between models that agree on  $\kappa^+$ , we conclude that if there is no inner model with a Woodin cardinal, then  $\square_\kappa$  holds at every successor of a singular cardinal and therefore stationary reflection fails by a standard argument.

Thus, in order to obtain stationary reflection principles at the successor of a singular cardinal, one needs either to violate weak covering or to start with a model in which square principles fail. This requires large cardinal axioms which are much stronger than the ones which are required for the parallel treatment of the successors of regular cardinals.

The paper is organized as follows. In Section 2 we prove some standard facts about Prikry forcing with collapses. In Section 3, we give the definitions of subcompact and  $\Pi_1^1$ -subcompact and calibrate the extent to which they imply stationary reflection. In Section 4, we prove our main theorem.

**§2. Prikry forcing.** In this section we will review some facts about Prikry forcing which are useful in this paper. We refer the reader to [9] for the proofs of the facts

cited in this section. For this section we assume that  $\kappa$  is a measurable cardinal and  $2^\kappa = \kappa^+$ . Let  $\mathcal{U}$  be a normal ultrafilter on  $\kappa$  and  $j : V \rightarrow M$  be the ultrapower embedding.

FACT 5. *We have the following:*

- (1) *If  $a = \langle \alpha_\xi \mid \xi < \kappa \rangle \in V \cap {}^\kappa M$  then  $a \in M$ .*
- (2)  $|\mathcal{P}^M(j(\kappa))|^V = |j(\kappa^+)|^V = \kappa^+$ .
- (3) *If  $\mathbb{P} \in M$  such that*

$M \models \mathbb{P} \text{ is } j(\kappa^+)-\text{cc, } \kappa^+ \text{-closed forcing notion, } |\mathbb{P}| \leq j(\kappa^+)$ ,

*then there is  $K \in V$  which is an  $M$ -generic filter for  $\mathbb{P}$ .*

Using part 5, let  $K \subseteq \text{Col}^M(\kappa^+, < j(\kappa))$  be an  $M$ -generic filter. We define a forcing  $\mathbb{P}$  called Prikry forcing over the measure  $\mathcal{U}$  with interleaved collapses using the guiding generic  $K$ .

DEFINITION 6. Let  $\mathbb{P}$  be the following forcing notion with

$$p = \langle c_{-1}, \rho_0, c_0, \rho_1, c_1, \dots, \rho_{n-1}, c_{n-1}, A, C \rangle \in \mathbb{P},$$

if and only if

- (1)  $0 \leq n < \omega$ .  $n$  is called the *length* of  $p$ , and we write  $\text{len } p = n$ .
- (2)  $\rho_0 < \rho_1 < \dots < \rho_{n-1} < \kappa$  are called the *Prikry points* of the condition  $p$ .
- (3) For  $0 \leq i \leq n$ ,  $c_{i-1} \in \text{Col}(\rho_{i-1}^+, < \rho_i)$  where for notational convenience we set  $\rho_{-1} = \omega$  and (temporarily)  $\rho_n = \kappa$ .
- (4)  $A \in \mathcal{U}$ ,  $\min A > \rho_{n-1}$ ,  $\sup \text{dom } c_{n-1}$ .
- (5)  $C$  is a function with domain  $A$ , for all  $\alpha \in A$ ,  $C(\alpha) \in \text{Col}(\alpha^+, < \kappa)$ , and  $[C]_{\mathcal{U}} \in K$ .

For a condition  $p$  as above we write  $p_m^p$ ,  $A^p$ ,  $c_i^p$ , and  $C^p$  with the obvious meaning.

We define two orderings. The *direct extension*,  $\leq^*$ , is defined as follows.  $p \leq^* q$  if  $\text{len } p = \text{len } q$ ,  $c_i^p \supseteq c_q^i$  for  $i \in \{-1, 0, \dots, n-1\}$ ,  $C^p$  is stronger than  $C^q$  pointwise, and  $A^p \subseteq A^q$ . For a condition  $p$  of length  $n$  and  $\rho \in A^p$ , we denote by  $p \frown \rho$  the condition of length  $n+1$  with  $\rho_i = \rho_i^p$  for  $i < n$ ,  $\rho_n = \rho$ ,  $c_i = c_i^p$  for  $i < n$ ,  $c_n = C(\rho)$ , measure one set  $A^p \setminus \sup \text{dom } c_n$  and the natural restriction of  $C$ . The forcing ordering  $\leq$  is given by a combination of direct extensions and adjoining points as above. Namely,  $\leq$  is the transitive closure of the relation

$$\{(p, q) \in \mathbb{P}^2 \mid p \leq^* q \text{ or } \exists \rho \in A^p, q = p \frown \rho\}.$$

For a condition  $p \in \mathbb{P}$  as above, the *stem* of  $p$  is  $\langle c_{-1}, \rho_0, c_0, \dots, \rho_{n-1}, c_{n-1} \rangle$ . Clearly, if  $p, p' \in \mathbb{P}$  have the same stem then they are compatible. In particular,  $\mathbb{P}$  is  $\kappa$ -centered. We also note that  $\leq^*$  is only  $\sigma$ -closed.

LEMMA 7.  *$\mathbb{P}$  satisfies the Prikry Property. Namely, for every statement in the forcing language  $\Phi$  and condition  $p \in \mathbb{P}$  there is  $q \leq^* p$  such that  $q \Vdash \Phi$  or  $q \Vdash \neg \Phi$ .*

Using the Prikry Property and a standard factorization argument, one can show that the set of cardinals below  $\kappa$  in the generic extension is exactly  $\{\omega, \omega_1\} \cup \{\rho_n, \rho_n^+ \mid n < \omega\}$ . In particular,  $\kappa$  is forced to be  $\aleph_\omega$  of the generic extension.

Let  $p$  be a condition with stem  $s$ . The set of stems of conditions  $q \leq p$ ,  $\text{len } q = \text{len } p$  is naturally isomorphic to a finite product of Levy collapses. We will say that a set of stems  $D$  is dense if it is dense with respect to this order.

Lemma 7 has several stronger versions which are called the Strong Prikry Property. The version which we need is the following:

**LEMMA 8.** *Let  $D \subseteq \mathbb{P}$  be a dense open set. There are a large set  $A \in \mathcal{U}$  and a condition  $[C] \in K$  such that the following holds. For every condition of the form  $p = \text{stem}(p) \cap \langle A, C \rangle$ , there is a dense set of extensions for the stem (with the same length),  $E$ , such that for every  $q \leq^* p$  with  $\text{stem } q \in E$ , there is a natural number  $m$  such that for every  $q' \leq q$ , with  $\text{len } q' = \text{len } q + m$ ,  $q' \in D$ .*

Let  $p$  be a condition of length  $n$ ,

$$p = \langle c_{-1}, \rho_0, c_0, \rho_1, c_1, \dots, \rho_{n-1}, c_{n-1}, A, C \rangle.$$

Let  $\mathbb{P} \upharpoonright p$  be the set of conditions in  $\mathbb{P}$  which are stronger than  $p$ . Let  $\mathbb{P}_n \upharpoonright p$  be the forcing

$$\text{Col}(\omega_1, < \rho_0) \times \text{Col}(\rho_0^+, < \rho_1) \times \dots \times \text{Col}(\rho_{n-1}^+, < \kappa)$$

below the condition  $(c_{-1}, \dots, c_{n-1})$ . Note that we include in the definition of  $\mathbb{P}_n$  the last collapse of all cardinals between  $\rho_{n-1}^+$  and  $\kappa$ . This will be useful later.

Let  $W$  be a model of set theory,  $V \subseteq W$ . In  $W$ , let  $\langle \rho_n \mid n < \omega \rangle \in W$  be a sequence of  $V$ -regular cardinals below  $\kappa$  and let  $C_n \subseteq \text{Col}^V(\rho_n^+, < \rho_{n+1})$ ,  $C_{-1} \subseteq \text{Col}^V(\omega_1, < \rho_0)$  be filters.

Let  $\mathcal{C} = \langle C_n \mid -1 \leq n < \omega \rangle$  and  $P = \langle \rho_n \mid n < \omega \rangle$ . Let us denote by  $G(\mathcal{C}, P)$  the filter which is defined from  $\mathcal{C}$  and  $P$ . Namely,  $G(\mathcal{C}, P) \subseteq \mathbb{P}$  is defined by:

$$p = \langle c_{-1}, \eta_0, \dots, c_{n-2}, \eta_{n-1}, c_{n-1}, A, F \rangle \in G(\mathcal{C}, P),$$

if and only if

- (1)  $p \in \mathbb{P}$ .
- (2) For all  $m \in \omega$  with  $m < n$ ,  $\rho_m = \eta_m$  and  $c_m \in C_m$  (in particular, the domain of  $c_{n-1}$  is a subset of  $\rho_{n-1}^+ \times \rho_n$ ).
- (3)  $c_{-1} \in C_{-1}$ .
- (4) For all  $m \geq n$ ,  $\rho_m \in A$  and  $F(\rho_m) \in C_m$ .

**THEOREM 9.** *Let  $\mathcal{C}, P \in W$  be as above.  $G(\mathcal{C}, P)$  is  $V$ -generic if and only if*

- (1) *For every  $m \in \{-1\} \cup \omega$ ,  $C_m$  is  $V$ -generic.*
- (2) *For every  $A \in \mathcal{U}$ , there is  $n < \omega$  such that for all  $m \geq n$ ,  $\rho_m \in A$ .*
- (3) *For every  $C: \kappa \rightarrow V$  such that  $[C] \in K$  there is  $n < \omega$  such that for all  $m \geq n$ ,  $C(\rho_m) \in C_m$ .*

**PROOF.** The forward direction is clear.

For the backwards direction, let  $G$  be the filter generated by  $\mathcal{C}, P$ . Let  $D$  be a dense open subset of  $\mathbb{P}$ . We will find a condition  $r \in G \cap D$ .

By the Strong Prikry Property (Lemma 8), there are a large set  $A$  and a member  $[C]$  of  $K$  such that for every condition  $q$  of the form  $\text{stem}(q) \cap \langle B, F \rangle$ , with  $B \subseteq A$  and  $\forall \alpha \in \text{dom}(F), F(\alpha) \leq C(\alpha)$ , there is dense subset  $E$  of the stems of  $\mathbb{P}$  below  $\text{stem}(q)$  as in the conclusion of Lemma 8.

Let  $q \in G$  of some length  $n$  such that for all  $m \geq n$ ,  $\rho_m \in A$  and  $C(\rho_m) \in C_m$ . Let  $E$  be the witnessing dense open set of stems as above. By a standard argument using Easton's lemma,  $C_{-1} \times C_0 \times \dots \times C_{n-1}$  is  $V$ -generic. For notational convenience we call this generic  $C_n^*$ . Since  $(c_{-1}^q, c_0^q, \dots, c_{n-1}^q) \in C_n^*$ , there is some extension  $(c_{-1}, c_0, \dots, c_{n-1})$  of it in  $C_n^* \cap E$ . Let  $q'$  be the strengthening of  $q$  by  $(c_{-1}, c_0, \dots, c_{n-1})$ .

By the conclusion of Lemma 8, there is a natural number  $m$  such that any  $m$ -step extension of  $q'$  is in  $D$ . So if we take a condition  $r \in G$  with  $r \leq q'$  of length  $n+m$  then it follows that  $r \in D$ .  $\dashv$

Let  $M_0 = V$  and let  $j_{n,n} = id$  for all  $n < \omega$ . Let us define, by induction on  $n$ , transitive classes  $M_n$  and elementary embeddings  $j_{m,n} : M_m \rightarrow M_n$ . Let us denote  $j_n = j_{0,n}$ . Let  $j_{n,n+1} : M_n \rightarrow M_{n+1}$  be the ultrapower by  $j_{0,n}(\mathcal{U})$  and let  $j_{m,n+1} = j_{n,n+1} \circ j_{m,n}$  for every  $m < n$ .

Let  $j_\omega : V \rightarrow M_\omega$  be the direct limit of the directed system  $\langle M_m, j_{m,n} \mid m \leq n < \omega \rangle$ . Let  $j_{n,\omega} : M_n \rightarrow M_\omega$  be the corresponding elementary embeddings.

**THEOREM 10** (Gaifman).  *$M_\omega$  is well founded.*

The following fact is well-known:

**LEMMA 11.**  *$M_\omega[\langle j_n(\kappa) \mid n < \omega \rangle]$  is closed under  $\kappa$ -sequences.*

**PROOF.** Since  $W = M_\omega[\langle j_n(\kappa) \mid n < \omega \rangle]$  is a model of ZFC, it is enough to show that for every  $\kappa$ -sequence of ordinals from  $V$ ,  $s = \langle \alpha_\xi \mid \xi < \kappa \rangle$  belongs to  $W$ .

Let us fix for every  $\xi < \kappa$ , a natural number  $n_\xi$  and a function  $f : \kappa^{n_\xi} \rightarrow \text{Ord}$  such that  $j_\omega(f)(\kappa, j_1(\kappa), \dots, j_{n_\xi-1}(\kappa)) = \alpha_\xi$ .

Let  $F = \langle f_\xi \mid \xi < \kappa \rangle$ .  $j_\omega(F) \upharpoonright \kappa = \langle j_\omega(f_\xi) \mid \xi < \kappa \rangle$ . By applying each function from the sequence  $j_\omega(F) \upharpoonright \kappa$  to the corresponding initial segment of  $\langle j_n(\kappa) \mid n < \omega \rangle$ , we get  $s$ . Since  $j_\omega(F) \upharpoonright \kappa \in W$ , we conclude that  $s \in W$ .  $\dashv$

**DEFINITION 12.** For a subset  $X$  of a partial order  $\mathbb{X}$  we will denote by  $\langle X \rangle$  the upwards closure of  $X$ :

$$\langle X \rangle = \{x \in \mathbb{X} \mid \exists y \in X, x \geq y\}.$$

The following well-known fact will play a major role in Section 4.

**LEMMA 13.** *Let  $p \in \mathbb{P}$  and  $\text{len } p = n$ . Let  $G' \subseteq \mathbb{P}_n \upharpoonright p$  be a  $V$ -generic filter. In  $V[G']$  there is an  $M_\omega$ -generic filter for  $j_\omega(\mathbb{P})$  that contains  $j_\omega(p)$ .*

**PROOF.** Let  $n > 0$ . Let  $K_n \subseteq \text{Col}(j_{n-1}(\kappa)^+, \langle j_n(\kappa) \rangle^{M_n})$  be  $\langle j_{n-1} \rangle K$ , i.e., the filter generated from  $j_{n-1} \rangle K$ . Note that  $K_n = j_{n-1}(K)$ . The following argument is standard, see [4].  $\dashv$

**CLAIM 14.**  *$K_n$  is  $M_n$ -generic.*

**PROOF.** Let  $D \in M_n$  be a dense open set. Then there is a function  $f : \kappa^n \rightarrow V$  such that for all  $a \in \kappa^n$ ,  $f(a)$  is a dense open subset of the forcing  $\text{Col}(a_{n-1}^+, \kappa)$  where we write  $a = \{a_0, a_1, \dots, a_{n-1}\}$  listed in an increasing order. By the distributivity of the forcing  $\text{Col}(a_{n-1}^+, \kappa)$ , for every  $\alpha < \kappa$ , the set  $D_\alpha = \bigcap_{b \in \alpha^{n-1}} f(b \cap \langle \alpha \rangle)$  is dense open in  $\text{Col}(\alpha^+, \kappa)$ . Let  $q$  be a function with domain  $\kappa$  such that  $[q]_{\mathcal{U}} = j_1(q)(\kappa) \in K$ ,

and  $\{\alpha < \kappa \mid q(\alpha) \in D_\alpha\} \in \mathcal{U}$ . Such a condition exists, since  $K$  is  $M_1$ -generic. Let us consider the function  $\tilde{q}: \kappa^n \rightarrow V$  which is defined as  $\tilde{q}(a) = q(a_{n-1})$ .

Let  $r = j_n(\tilde{q})(\kappa, j_1(\kappa), \dots, j_{n-1}(\kappa))$ .  $r = j_{n-1}(j_1(q)(\kappa))$  since:

$$\begin{aligned} j_{n-1}(j_1(q)(\kappa)) &= j_{n-1}([q]_{\mathcal{U}}) \\ &= [j_{n-1}(q)]_{j_{n-1}(\mathcal{U})} \\ &= j_{n-1,n}(j_{n-1}(q))(j_{n-1}(\kappa)) \\ &= j_n(q)(j_{n-1}(\kappa)) \end{aligned}$$

and by the definition of  $\tilde{q}$ ,  $j_n(\tilde{q})(\kappa, \dots, j_{n-1}(\kappa)) = j_n(q)(j_{n-1}(\kappa))$ . We conclude that  $r \in D$ .

Since  $r = j_{n-1}([q]_{\mathcal{U}})$ ,  $r \in K_n$ .  $\dashv$

Note that  $j_{n,\omega}'' K_n = K_n$  and that  $K_n$  is also an  $M_\omega$ -generic filter.

Let  $\mathcal{C}$  be the sequence of generic collapses from  $G'$ , augmented by the sequence  $\langle K_1, K_2, \dots \rangle$ . Let  $P = \langle \rho_0, \dots, \rho_{n-1}, \kappa, j_1(\kappa), j_2(\kappa), \dots \rangle$ , where  $\rho_0, \dots, \rho_{n-1}$  are the Prikry points in the condition  $p$ . Let  $G = G(\mathcal{C}, P) \subseteq j_\omega(\mathbb{P})$ , as in Theorem 9.

For every  $A \in j_\omega(\mathcal{U})$  there is  $m < \omega$  such that  $A = j_{m,\omega}(A')$ . Note that the tail of the sequence  $P$ , starting at point  $n+m$ , is contained in  $A$ . Similarly, if  $q' \in j_\omega(K)$  then  $q' = j_{m,\omega}(q)$  for some  $q$  and therefore for every  $k \geq m$ ,  $q(j_k(\kappa)) \in K_k$ . Finally, each  $K_n$  is  $M_\omega$ -generic. Thus, the conditions of Theorem 9 hold and  $G$  is  $M_\omega$ -generic for  $j_\omega(\mathbb{P})$ .

**2.1. Splitting generic filters.** During the proof of the main theorem, Theorem 38, we will need to analyze models of the form  $M_\omega[P][\mathcal{H}]$  such that  $P$  is the critical sequence and  $\mathcal{H}$  has the form  $\langle < j_{n,\omega}'' H > \mid n < \omega \rangle$  where  $H$  is a  $V$ -generic filter for some  $\kappa^+$ -closed forcing notion in  $M_\omega[P]$ .

Let  $\mathbb{A}$  be a forcing notion that has unique greatest lower bounds and a  $\kappa^+$ -closed dense subset, which is closed under those greatest lower bounds. Let us assume that  $j_n(\mathbb{A}) = \mathbb{A}$  for all  $n < \omega$  (in particular,  $\mathbb{A} \in M_\omega[P]$ ). In this subsection we will define and analyze a forcing notion,  $\mathbb{H}$ , which will have the property that  $P$  and  $\mathcal{H}$  generate an  $M_\omega$ -generic filter for  $j_\omega(\mathbb{H})$ .

A lot of information on the model  $M_\omega[P][\mathcal{H}]$  can be extracted without understanding the forcing  $\mathbb{H}$ . In particular, one can prove Lemmas 20 and 23 without mentioning  $\mathbb{H}$ . Moreover, using Bukovský's Theorem [2], one can deduce the *existence* of some forcing notion that introduces  $\mathcal{H}$  over  $M_\omega[P]$ , without knowing what precisely this forcing is. In particular, one can prove Claim 40, which is central in the Theorem 38, without explicitly constructing the forcing notion  $\mathbb{H}$ . Despite this, we prefer to construct the forcing  $\mathbb{H}$  in detail, since we believe that its structure helps to unravel some of the mysterious properties of  $\mathcal{H}$ .

In the following definition we will use the convention that a finite sequence  $s$  is an end extension of a sequence  $t$  if  $t = s \upharpoonright \text{len } t$ . In this case we will write  $t \trianglelefteq s$ . Note that  $s \trianglelefteq s$  always holds. We will denote by  $s \perp t$  if  $s \not\trianglelefteq t$  and  $t \not\trianglelefteq s$ .

The conditions of  $\mathbb{H}$  are pairs of the form  $p = \langle T, F \rangle$  where:

- (1)  $T \subseteq \kappa^{<\omega}$ . For every  $\eta \in T$  and  $n < \text{len } \eta$ ,  $\eta \upharpoonright n \in T$ . Let us order  $T$  by  $\trianglelefteq$ .
- (2) Any element of  $T$  is a strictly increasing finite sequence of regular cardinals.
- (3) There is a single element  $s \in T$  such that every  $t \in T$  is comparable with  $s$  and  $\text{len } s$  is maximal. Let us denote  $\text{stem}(T) = s$ .

(4) For every  $t \in T$ , if  $\text{stem}(T) \leq t$  then

$$\{\alpha < \kappa \mid t^\frown \langle \alpha \rangle \in T\} \in \mathcal{U}.$$

(5)  $F$  is a function,  $F: T \rightarrow \mathbb{A}$ .

(6) (Stabilization) Let  $t \in T$  such that  $\text{stem}(T) \leq t$ . Let  $g_t$  be the function  $g_t(\alpha) = F(t^\frown \langle \alpha \rangle)$  for all  $\alpha < \kappa$  such that  $t^\frown \langle \alpha \rangle \in T$ . Then  $j(g_t)(\kappa) = F(t)$ .

For a condition  $p = \langle T, F \rangle \in \mathbb{H}$  we write  $T^p = T$ ,  $F^p = F$ , and  $\text{len } p = \text{len stem}(T)$ . We denote  $\text{stem}(p) = \langle F(t) \mid t \leq \text{stem}(T) \rangle$ .

For  $p, q \in \mathbb{H}$ , we define  $p \leq q$  ( $p$  extends  $q$ ) if  $T^p \subseteq T^q$  and  $F^p(\eta) \leq_{\mathbb{A}} F^q(\eta)$  for all  $\eta \in T^p$ . We define  $p \leq^* q$  ( $p$  is a direct extension of  $q$ ) if  $p \leq q$  and  $\text{len } p = \text{len } q$ .

**LEMMA 15 (Strong Prikry Property).** *Let  $D \subseteq \mathbb{H}$  be a dense open set and let  $p \in \mathbb{H}$ . There is a direct extension  $p^* \leq p$  and a natural number  $n < \omega$  such that any  $q \leq p^*$  with  $\text{len } q \geq n$  is in  $D$ .*

**PROOF.** Let  $D \subseteq \mathbb{H}$  be dense open and  $p \in \mathbb{H}$  be a condition.

Let  $\langle \eta_\alpha \mid \alpha < \kappa \rangle$  be an enumeration of  $\kappa^{<\omega}$  such that if  $\eta_\alpha \leq \eta_\beta$ , then  $\alpha \leq \beta$ . Let us define, by recursion, a decreasing sequence of conditions  $p_\alpha = \langle T_\alpha, F_\alpha \rangle$ ,  $\alpha < \kappa$ , such that  $\text{stem}(T_\alpha) = \text{stem}(T_\beta)$  for all  $\alpha < \beta$ . For all such conditions, the range of  $F_\alpha$  is always chosen to be included in the  $\kappa^+$ -closed subset of  $\mathbb{A}$ .

Let  $p = p_0$ . Let  $s = \text{stem}(T_0)$ . For each  $\alpha$ , if  $\eta_\alpha \in T_\alpha$  and extends  $s$ , we look at the tree  $T_{\eta_\alpha} = \{\eta \in T_\alpha \mid \eta \not\leq \eta_\alpha\}$  and check if there is a condition  $q_\alpha = \langle T', F' \rangle \in D$ , which is a direct extension of  $p_\alpha \upharpoonright T_{\eta_\alpha}$ . If there is no such condition, we let  $p_{\alpha+1} = p_\alpha$ .

Otherwise, let us define:

$$\begin{aligned} T'' &= \{\eta \in T_\alpha \mid \eta \perp \eta_\alpha \text{ or } \eta \in T'\}, \\ F'' &= F_\alpha \upharpoonright \{\eta \in T_\alpha \mid \eta \perp \eta_\alpha\} \cup F'. \end{aligned}$$

For  $\eta \in T'' \setminus T'$ , requirement 2.1 might fail. Since  $F''(\eta)$  is stronger than  $F_\alpha(\eta)$ , we may find a condition  $r_{\eta^\frown \langle \alpha \rangle}$  such that  $r_{\eta^\frown \langle \alpha \rangle} \leq F''(\eta^\frown \langle \alpha \rangle)$  for all  $\alpha$  and  $j(r)_{\eta^\frown \langle \kappa \rangle} = F''(\eta)$ . Continue this way for  $\omega$  many steps we construct a function  $F'''$  with domain  $T''$  such that  $p_{\alpha+1} = \langle T'', F''' \rangle$  is a condition.

For a limit ordinal  $\alpha < \kappa$ , let  $p_\alpha = \langle T_\alpha, F_\alpha \rangle$  be the pair,  $T_\alpha = \bigcap_{\beta < \alpha} T_\beta$ , and  $F_\alpha(\eta)$  is the greatest lower bound of  $F_\beta(\eta)$  for all  $\eta \in T_\alpha$  (this lower bound exists by the closure of the forcing  $\mathbb{A}$ ).

Let us verify that for all  $\alpha < \kappa$ ,  $p_\alpha$  is a condition. For limit ordinal  $\alpha$ ,  $T_\alpha$  is  $\mathcal{U}$ -splitting, using the closure of the measure  $\mathcal{U}$ .  $F_\alpha$  satisfies condition 2.1, since for all  $\eta \in T_\alpha$ ,  $F_\alpha(\eta)$  is the greatest lower bound of a decreasing sequence of length  $\alpha < \kappa$ . Since  $\alpha < \text{crit } j$ , applying  $j$  does not change this fact. For successor ordinals  $\alpha$ , the requirement follows from the construction.

We would like to continue and construct a condition  $p_\kappa$ , which is a lower bound for the sequence  $p_\alpha$ .

Let  $T_\kappa = \bigcap_{\alpha < \kappa} T_\alpha$ . Since the set of successors of each element in the tree is modified only finitely many times,  $T_\kappa$  is a  $\mathcal{U}$ -splitting tree with stem  $s$ .

Let us consider for each  $\eta \in T_\kappa$  the following sequence of functions. Let  $g_\eta^n: \kappa^n \rightarrow \mathbb{A}$  be the function defined by  $g_\eta^n(v) = \lim_\alpha F_\alpha(\eta^\frown v)$ . Let

$$p_\eta^n = j_n(g_\eta^n)(\kappa, j_1(\kappa), \dots, j_{n-1}(\kappa)).$$

Let us claim that for each  $\eta$ , the sequence  $\langle p_\eta^n \mid n < \omega \rangle$  is decreasing. Indeed,

$$\begin{aligned} p_\eta^1 &= j_1(g_\eta^1)(\kappa) &= \lim_{\alpha < j_1(\kappa)} j_1(F)_\alpha(\eta^\frown \langle \kappa \rangle) \\ &\leq \lim_{\alpha < \kappa} j_1(F)_\alpha(\eta^\frown \langle \kappa \rangle) &= \lim_{\alpha < \kappa} j_1(F)_\alpha(\eta^\frown \langle \kappa \rangle) \\ &= \lim_{\alpha < \kappa} F_\alpha(\eta) &= p_\eta^0. \end{aligned}$$

By the elementarity of  $j_n$ ,

$$j_n(\langle F_\alpha \mid \alpha < \kappa \rangle) = \langle F_\alpha^{j_n} \mid \alpha < j_n(\kappa) \rangle,$$

and for all  $\alpha < j_n(\kappa)$ , for all  $\eta \in \text{dom } F_\alpha^{j_n}$ ,

$$j_{n,n+1}(\langle F^{j_n}(\eta^\frown \langle \zeta \rangle) \mid \zeta < j_n(\kappa) \rangle)(j_n(\kappa)) = F^{j_n}(\eta),$$

and therefore, we conclude that in general:

$$\begin{aligned} p_\eta^{n+1} &= j_{n+1}(g_\eta^{n+1})(\kappa, \dots, j_n(\kappa)) \\ &= \lim_{\alpha < j_{n+1}(\kappa)} F_\alpha^{j_{n+1}}(\eta^\frown \langle \kappa, \dots, j_n(\kappa) \rangle) \\ &\leq \lim_{\alpha < j_n(\kappa)} j_{n+1}(F)_\alpha(\eta^\frown \langle \kappa, \dots, j_n(\kappa) \rangle) \\ &= \lim_{\alpha < j_n(\kappa)} j_{n,n+1}(F_\alpha^{j_n})(\eta^\frown \langle \kappa, \dots, j_{n-1}(\kappa) \rangle^\frown \langle j_n(\kappa) \rangle) \\ &= \lim_{\alpha < j_n(\kappa)} (F_\alpha^{j_n})(\eta^\frown \langle \kappa, \dots, j_{n-1}(\kappa) \rangle) \\ &= j_n(g_\eta^n)(\eta^\frown \langle \kappa, \dots, j_{n-1}(\kappa) \rangle) &= p_\eta^n. \end{aligned}$$

Let  $F(\eta)$  be the greatest lower bound of  $\langle p_\eta^n \mid n < \omega \rangle$ . Let us claim that requirement 2.1 holds for  $F$ . Namely, that for any  $\eta$ , if we let  $h_\eta(\alpha) = F(\eta^\frown \langle \alpha \rangle)$ , then  $j(h_\eta)(\kappa) = F(\eta)$ . By elementarity,  $j(h_\eta)(\kappa)$  is the greatest lower bound of the sequence  $q_\eta^n$  where  $q_\eta^n = j(\langle p_{\eta^\frown \langle \alpha \rangle}^n \mid \alpha < \kappa \rangle)(\kappa)$ . Let us compute:

$$\begin{aligned} q_\eta^n &= j(\langle p_{\eta^\frown \langle \alpha \rangle}^n \mid \alpha < \kappa \rangle)(\kappa) \\ &= j_1(\langle j_n(g_{\eta^\frown \langle \alpha \rangle}^n)(\kappa, \dots, j_{n-1}(\kappa)) \mid \alpha < \kappa \rangle)(\kappa) \\ &= j_{n+1}(g_{\eta^\frown \langle \kappa \rangle}^n)(j_1(\kappa), \dots, j_n(\kappa)) \\ &= j_{n+1}(g_\eta^{n+1})(\kappa, j_1(\kappa), \dots, j_n(\kappa)) &= p_\eta^{n+1}. \end{aligned}$$

Thus,  $F(\eta)$  satisfies requirement 2.1 in the definition of  $\mathbb{H}$ . Let  $p_\kappa = \langle T_\kappa, F \rangle$ .

Let us consider the condition  $p_\kappa$ . By narrowing down the tree  $T = T_\kappa$ , we may assume that for any  $\eta \in T$  one of the following two holds: Either for every  $\alpha < \kappa$ , if  $\eta^\frown \langle \alpha \rangle \in T$  then the extension of  $p$  by picking  $\eta^\frown \langle \alpha \rangle$  is in  $D$ , or that all of them are not in  $D$ . By narrowing the tree  $T$  again, we may assume that for any element of the tree  $\eta$  the minimal level of the tree that enters  $D$  above  $\eta$  is fixed. This induces a coloring of  $T$  which is (by induction) fixed on levels. Clearly, if an element  $\eta \in T$  was colored by the number  $n$  then its successors are colored by  $n-1$ . Let  $n$  be the color of  $\text{stem}(T)$ . Then, every direct extension of  $p_\kappa$  with at least  $n$  new points is in  $D$ , as required.  $\dashv$

The generic filter can be described compactly using a branch in the tree,  $P$ , and a sequence of filters  $\mathcal{H} = \langle H_n \mid n < \omega \rangle$  of  $\mathbb{A}$ . Let  $G(P, \mathcal{H})$  be the set of all conditions  $p \in \mathbb{H}$  such that for all  $n < \omega$ ,  $P \upharpoonright n \in T^p$  and  $F^p(P \upharpoonright n) \in H_n$ .

**LEMMA 16.** *For any increasing  $\omega$ -sequence in  $\kappa$  and collection of filters  $\mathcal{H} = \langle H_n \mid n < \omega \rangle$ ,  $G(P, \mathcal{H})$  is a filter.*

PROOF. Let  $p, q \in G(P, \mathcal{H})$ . We want to show that they are compatible and have a common lower bound in  $G(P, \mathcal{H})$ . Let us assume, without loss of generality, that  $\text{len } p \leq \text{len } q$ . Then  $\text{stem}(T^p) \trianglelefteq \text{stem}(T^q) \trianglelefteq P$ . In particular, the intersection of  $T^p$  and  $T^q$  is a  $\mathcal{U}$ -branching tree. Thus we may assume without loss of generality that  $T^p = T^q = T$ .

Let us consider  $F^p$  and  $F^q$ . For every element below the stem of the tree, the values of those two functions are compatible since each  $H_m$  is a filter. For elements above the stem, one can show, by induction on the height of  $\eta \in T$ , that the compatibility of  $F^p(\eta)$  and  $F^q(\eta)$  implies (using requirement 2.1 in the definition of the forcing) that for a large set of extension of  $\eta$ ,  $F^q(\eta \cap \langle \alpha \rangle)$  is compatible with  $F^p(\eta \cap \langle \alpha \rangle)$ . Moreover, by the definition of the filter  $G(P, \mathcal{H})$ , if  $\eta$  is an initial segment of  $P$  then  $F^p(\eta)$  is compatible with  $F^q(\eta)$ . Narrowing down  $T$ , we may assume that for any  $\eta \in T$ ,  $F^p(\eta)$  is compatible with  $F^q(\eta)$ , while the initial segments of  $P$  are all in  $T$ . Let  $F(\eta)$  be the greatest lower bound of  $F^p(\eta)$  and  $F^q(\eta)$ . For every  $\eta$ , let  $g_\eta(\alpha) = F(\eta \cap \langle \alpha \rangle)$  and let  $g^p, g^q$  be the analogous functions with respect to  $p$  and  $q$ . Then,  $j(g_\eta)(\kappa)$  is the greatest lower bound of  $j(g_\eta^p)(\kappa)$  and  $j(g_\eta^q)(\kappa)$  and therefore is the same as  $F(\eta)$ .

We conclude that this condition satisfies requirement 2.1. Thus, it is in  $G(P, \mathcal{H})$ , as wanted.  $\dashv$

We wish to generate an  $M_\omega$ -generic filter for  $j_\omega(\mathbb{H})$ . Let  $H \subseteq \mathbb{A}$  be a  $V$ -generic filter. Let  $\mathcal{H}_n = \langle \langle j_{m,n} \rangle H \mid m \leq n \rangle$  and let  $\mathcal{H} = \langle \langle j_{n,\omega} \rangle H \mid n < \omega \rangle$ . Let  $P$  be the critical sequence  $\langle j_n(\kappa) \mid n < \omega \rangle$ . It is immediate that  $P, \mathcal{H} \in M_n[\mathcal{H}_n]$ .

We will show that  $G(P, \mathcal{H})$  is  $M_\omega$ -generic for  $j_\omega(\mathbb{H})$ . We start by showing that  $\mathcal{H}_n$  is generic over  $M_n$ . To do so we need the following lemma which is attributed to Woodin in a paper of Cummings, see [4, Fact 2].

LEMMA 17 (Cummings–Woodin). *Let  $W$  be a model of ZFC. In  $W$ , let  $\mu$  be a measurable cardinal, and let  $\mathcal{U}$  be a normal measure on  $\mu$ . Let  $j: W \rightarrow N$  be the ultrapower embedding.*

*Let  $\mathbb{A}$  be a forcing notion such that  $\mathbb{A} = j(\mathbb{B})$  and  $\mathbb{B}$  is a  $\mu$ -closed forcing notion. Let  $G_A \subseteq \mathbb{A}$  be a  $W$ -generic filter. Then  $j$  extends to an embedding :*

$$j^*: W[G_A] \rightarrow N[\langle j \rangle^* G_A],$$

*which is definable in  $W[G_A]$ , and  $G_A$  is  $N[\langle j \rangle^* G_A]$ -generic.*

CLAIM 18.  $\mathcal{H}_n$  generates a generic filter for  $\mathbb{A}^{n+1}$  over  $M_n$ .

PROOF. We go by induction on  $n < \omega$ . For  $n = 0$ , this is true as  $H$  is a  $V$ -generic for  $\mathbb{A}$ . Assume that the claim holds for  $n < \omega$ .

Consider the elementary embedding:

$$j_{n,n+1}: M_n \rightarrow M_{n+1}.$$

This is an ultrapower embedding, using the measure  $j_n(\mathcal{U})$  over  $M_n$ . We apply Lemma 17 with  $W = M_n$ ,  $\mu = j_n(\kappa)$ ,  $j = j_{n,n+1}$ ,  $\mathbb{B} = \mathbb{A}^{n+1}$ , and  $G_A = \mathcal{H}_n$ . We conclude that there is an elementary embedding:

$$j_{n,n+1}^*: M_n[\mathcal{H}_n] \rightarrow M_{n+1}[\langle j_{n,n+1} \rangle^* \mathcal{H}_n].$$

By the definition of  $\mathcal{H}_n$ ,  $\langle j_{n,n+1} \rangle^* \mathcal{H}_n = \mathcal{H}_{n+1} \upharpoonright [1, n+1]$ : the last  $n+1$  coordinates of  $\mathcal{H}_{n+1}$ .

By the second part of the lemma,  $\mathcal{H}_n$  is  $M_{n+1}[\mathcal{H}_{n+1} \upharpoonright [1, n+1]]$ -generic. In particular,  $H$  is generic over this model. Since  $H$  is the first component of  $\mathcal{H}_{n+1}$  we conclude that  $\mathcal{H}_{n+1}$  is  $M_{n+1}$ -generic for the forcing  $\mathbb{A}^{n+2}$ .  $\dashv$

**THEOREM 19.**  *$G(P, \mathcal{H})$  is  $M_\omega$ -generic for  $j_\omega(\mathbb{H})$ .*

**PROOF.** Let  $D \in M_\omega$  be dense open and let  $p \in G(P, \mathcal{H})$ . Let  $n < \omega$  be large enough so that there is  $D' \subseteq j_n(\mathbb{H})$  and  $p' \in j_n(\mathbb{H})$  such that  $D = j_{n,\omega}(D')$  and  $p = j_{n,\omega}(p')$ . Without loss of generality,  $\text{len } p = n$ .

Let

$$E = \{p^* \in j_n(\mathbb{H}) \mid p^* \leq^* p', \exists m < \omega, \forall r \leq p^*, \text{len } r \geq m \implies r \in D'\}.$$

By the Strong Prikry Property in  $M_n$ ,  $E$  is dense and open in  $\leq^*$ .

Let us claim that there is a condition  $r \in E$  such that  $j_{n,\omega}(r) \in G(P, \mathcal{H})$ . Indeed, since  $E$  is dense open in the direct extension relation, the collection of stems of elements of  $E$  is dense open in the forcing  $\mathbb{A}^{n+1}$ . By the genericity of  $\mathcal{H}_n$ , there is  $s \in \mathcal{H}_n$  which is a stem of an element in  $E$ . Let  $r$  be any element in  $E$  with  $\text{stem}(r) = s$ . Let us claim that  $j_{n,\omega}(r) \in G(P, \mathcal{H})$ . Indeed, working in  $V$  fixes a condition  $h \in \mathbb{H}$  such that  $s' = \langle j_{m,n}(h) \mid m \leq n \rangle$  is stronger than  $s$ . Let  $r' = \langle T', F' \rangle$  be any condition of  $\mathbb{H}$  such that  $\text{stem}(r') = s'$ . Then, by induction on  $k < \omega$ ,

$$j_{n,n+k}(F')(\text{stem}(T') \cap \langle j_n(\kappa), \dots, j_{n+k-1}(\kappa) \rangle) = h.$$

We conclude that  $j_{n,\omega}(F')(\langle \kappa, \dots, j_{k-1}(\kappa) \rangle) \in j_{k,\omega}''H$  for all  $k$ . Thus,  $j_{n,\omega}(r') \in G(P, \mathcal{H})$ . Following the arguments of Lemma 16, by narrowing down the tree of  $r'$  in  $M_n$ , we obtain a condition  $r'' \leq^* r'$ ,  $F'' = F'^* \upharpoonright T''$ , and  $r'' \leq r$ . Since the critical sequence starting from  $n$  enters any  $\mathcal{U}$ -splitting tree from  $M_n$ , we get that  $j_{n,\omega}(r'') \in G(P, \mathcal{H})$  as well.

We conclude that  $j_{n,\omega}(r) \in G(P, \mathcal{H})$ . Let  $m < \omega$  be the natural number that witnesses  $r \in E$ . That is, for all  $q \leq r$  with  $\text{len } q \geq m$ ,  $q \in D'$ . By elementarity, any extension of  $j_{n,\omega}(r)$  of length  $m$  is in  $D$ . In particular,  $G(P, \mathcal{H}) \cap D \neq \emptyset$ .  $\dashv$

**LEMMA 20.**  *$M_\omega[P][\mathcal{H}]$  has the same  $j_\omega(\kappa)$ -sequences of ordinals as  $M_\omega[P]$ .*

**PROOF.** Since  $j_\omega(\kappa)$  is singular in  $M_\omega[P]$ , it is enough to show that there is no new sequence of ordinals in  $M_\omega[P][\mathcal{H}]$  of length  $\rho < j_\omega(\kappa)$ .

Let us fix  $n < \omega$  large enough so that  $\rho < j_n(\kappa)$ .  $M_\omega[P]$  and  $M_n$  have the same  $j_n(\kappa)$ -sequences of ordinals by Lemma 11. Let us assume that there is a  $\rho$ -sequence of ordinals,

$$x \in M_\omega[P][\mathcal{H}] \setminus M_\omega[P].$$

Recall that  $\mathcal{H}_n = \langle j_{m,n} \mid m \leq n \rangle$ . By Claim 18,  $\mathcal{H}_n$  generates a generic filter for  $\mathbb{A}^{n+1}$  which is a  $j_n(\kappa^+)$ -closed forcing in  $M_n$ . In particular, since  $\mathcal{H}_n$  is an  $M_n$ -generic filter for a  $j_n(\kappa^+)$ -distributive forcing,  $M_n$  has the same  $\rho$ -sequences as  $M_n[\mathcal{H}_n]$  and in particular, any  $\rho$ -sequence in  $M_\omega[P][\mathcal{H}]$  belongs to  $M_n$ . But since  $M_\omega[P]$  contains all  $j_n(\kappa)$  sequence of ordinals from  $M_n$  (by applying Lemma 11 in  $M_n$ ), we conclude that  $x \in M_\omega[P]$ .  $\dashv$

Let us give another argument for the distributivity of the extension by  $\mathcal{H}$ .

DEFINITION 21. Let  $p, q \in \mathbb{H}$ .  $p \leq^{**} q$  if there is a  $\mathcal{U}$ -splitting tree  $T$  with  $\text{stem}(T) = \text{stem}(T^p) = \text{stem}(T^q)$  and  $p' = \langle T, F^p \upharpoonright T \rangle \leq^* q$ .

Using diagonal intersections, the partial order  $\leq^{**}$  is  $\kappa^+$ -closed. Let  $\langle D_i \mid i < \kappa \rangle$  be a sequence of dense open subsets of  $\mathbb{H}$ . Using the Strong Prikry Property, we can construct a sequence of conditions  $\langle p_i \mid i < \kappa \rangle$  which is decreasing in  $\leq^*$ . There is no  $\leq^*$ -lower bound for those conditions, but there is a  $\leq^{**}$ -lower bound,  $p_\kappa$ . For every  $\alpha < \kappa$ , there is a natural number  $n_\alpha$  such that for every increasing sequence  $s \in T^{p_\kappa}$ , with  $\text{len } s \geq n_\alpha$ ,  $\max s \geq \alpha$ , the condition  $p' = \langle T_s^{p_\kappa}, F^{p_\kappa} \upharpoonright T_s^{p_\kappa} \rangle$  is in  $D_\alpha$ , where  $T_s$  is the restriction of the tree  $T$  to its elements above  $s$ . This implies that for any name for a  $\kappa$ -sequence of ordinals there is a condition that forces it to be equivalent to a name relative to the Prikry forcing.

Recall that  $\mathbb{P}$  is the Prikry forcing with interleaved collapses and that  $G$  is an  $M_\omega$ -generic filter for  $j_\omega(\mathbb{P})$ .

CLAIM 22. *In  $M_\omega$ ,  $P$  and  $\mathcal{H}$  generate a generic filter for  $j_\omega(\mathbb{H})$  which is mutually generic to the quotient forcing for adding collapses over the standard Prikry forcing. Moreover, the filter  $\mathcal{H}$  does not add any  $j_\omega(\kappa)$ -sequences of ordinals over  $M_\omega[G]$ .*

PROOF. Work over  $V$ . Let  $\tilde{P}$  be a  $V$ -generic Prikry sequence. By elementarity, there is a condition in  $\mathbb{H}/\tilde{P}$  that forces that any  $\kappa$ -sequence of ordinals in the generic extension is already in  $V[\tilde{P}]$ . The quotient forcing for adding the interleaved collapses over the Prikry sequence,  $\mathbb{P}/\tilde{P}$ , is  $\kappa$ -centered and in particular  $\kappa^+$ -cc, also in the extension by  $\mathbb{H}$ .

Thus, by the distributivity of  $\mathbb{H}/\tilde{P}$  over  $V[\tilde{P}]$ , every maximal antichain of  $\mathbb{P}/\tilde{P}$  belongs to  $V[\tilde{P}]$ . Therefore if  $G \subseteq \mathbb{P}$  is a  $V$ -generic and  $\tilde{\mathcal{H}} \subseteq \mathbb{H}/\tilde{P}$  is  $V[\tilde{P}]$ -generic, then it is also  $V[G]$ -generic.

The arguments for the distributivity of  $j_\omega(\mathbb{H})/P$  over  $M_\omega[G]$  are the same as in Lemma 20. Using the fact that  $M_\omega[G][\mathcal{H}] \subseteq \bigcap_{n < \omega} M_n[G \upharpoonright n+1][\mathcal{H}_n]$  and that the forcing that adds  $G \upharpoonright n+1$  is  $j_n(\kappa)$ -cc, we can trace back any name for a new sequence of ordinals which is shorter than  $j_\omega(\kappa)$  to one of the  $M_n$  and use the distributivity of  $\mathbb{A}^{n+1}$  in  $M_n$  in order to conclude that this name appears already in  $M_\omega[P]$ .  $\dashv$

The following lemma is a generalization of the classical theorem of Bukovský [3] and independently Dehornoy [7]. We will follow Bukovský's proof.

LEMMA 23.  $M_\omega[P][\mathcal{H}] = \bigcap_{n < \omega} M_n[\mathcal{H}_n]$ .

PROOF. We already know that  $M_\omega[P][\mathcal{H}] \subseteq \bigcap_{n < \omega} M_n[\mathcal{H}_n]$ . Let us show the other direction.

Let  $x$  be a set of ordinals in the intersection of all  $M_n[\mathcal{H}_n]$ . Since  $j_{n,\omega}$  is definable in  $M_n$ , we may define:

$$x_n = \{\zeta \mid j_{n,\omega}(\zeta) \in x\} \in M_n[\mathcal{H}_n].$$

The embedding  $j_{n,\omega}$  extends to an embedding:

$$j_{n,\omega}^* : M_n[\mathcal{H}_n] \rightarrow M_\omega[\langle j_{n,\omega} \rangle \mathcal{H}_n] \subseteq M_\omega[P][\mathcal{H}].$$

Therefore,  $j_{n,\omega}(x_n)$  is well-defined and belongs to  $M_\omega[P][\mathcal{H}]$ . The model  $M_\omega[P]$  is closed under  $\omega$ -sequences. Since any  $H$  does not add any  $\omega$ -sequence of ordinals to  $V$ , the same holds in  $M_\omega[P][\mathcal{H}]$ . Thus,  $\langle j_{n,\omega}(x_n) \mid n < \omega \rangle \in M_\omega[P][\mathcal{H}]$ .

Now we can reconstruct  $x$  as follows:  $\zeta \in x$  if and only if for all but finitely many  $n < \omega$ ,  $\zeta \in j_{n,\omega}(x_n)$ .  $\dashv$

We conclude that  $H \in M_\omega[P][\mathcal{H}]$  (as it belongs to any of the models  $M_n[\mathcal{H}_n]$ ).

**§3. Subcompact cardinals.** In this section we will discuss the relationship between subcompactness of cardinals and stationary reflection. Both of these concepts are related to Jensen's square principle.

**DEFINITION 24** (Jensen). Let  $\kappa$  be a cardinal. A sequence  $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$  is a  $\square_\kappa$ -sequence if:

- (1)  $C_\alpha$  is a closed unbounded subset of  $\alpha$ .
- (2)  $\text{otp } C_\alpha \leq \kappa$ .
- (3) For all  $\beta \in \text{acc } C_\alpha$ ,  $C_\beta = C_\alpha \cap \beta$ .

$\square_\kappa$  is a strong non-compactness principle, see [15] and [6]. For example:

**LEMMA 25.** *Let  $\kappa$  be a cardinal such that  $\square_\kappa$  holds. For every stationary subset  $S \subseteq \kappa^+$  there is a stationary subset  $T \subseteq S$  that does not reflect.*

Let  $V$  be a model of ZFC, such that  $\kappa \in V$  is an infinite cardinal and  $\square_\kappa$  holds. If  $W$  is a larger model,  $V \subseteq W$ , and  $(\kappa^+)^V = (\kappa^+)^W$ , then  $W \models \square_{|\kappa|W}$ . Thus, in order to obtain a model in which some type of stationary reflection holds at  $\kappa^+$ , without collapsing  $\kappa^+$ , we must start from a model in which either  $\kappa$  is inaccessible or  $\square_\kappa$  fails. Since we are aiming towards stationary reflection at the successor of a singular cardinal, the second possibility seems to be more natural.

The principle  $\square_\kappa$  was originated from the study of the fine structure of  $L$ . Jensen proved that  $\square_\kappa$  holds in  $L$  for all infinite  $\kappa$  and more sophisticated arguments provide square sequences in larger inner models. While studying the properties that imply the failure of square, Jensen isolated the notion of subcompactness.

**DEFINITION 26** (Jensen). Let  $\kappa$  be a cardinal.  $\kappa$  is *subcompact* if for every  $A \subseteq H(\kappa^+)$  there is  $\rho < \kappa$  and  $B \subseteq H(\rho^+)$  such that there is an elementary embedding:

$$j: \langle H(\rho^+), \in, B \rangle \rightarrow \langle H(\kappa^+), \in, A \rangle,$$

with  $\text{crit } j = \rho$ .

Note that  $j(\rho) = \kappa$ , thus  $\rho$  is 1-extendible with target  $\kappa$ . Moreover, the set of ordinals  $\delta < \kappa^+$  which are the  $\sup j'' \rho^+$  for some subcompact embedding with critical point  $\rho$  is stationary.

Assuming GCH, the first subcompact cardinal is smaller than the first cardinal  $\kappa$  which is  $\kappa^+$ -supercompact. Moreover, the first subcompact is weakly compact but not measurable. Nevertheless, subcompact cardinals are still strong enough in order to imply the failure of  $\square_\kappa$ .

In [19], Schimmerling and Zeman proved that  $\square_\kappa$  holds for every  $\kappa$  which is not subcompact in models of the form  $L[\bar{E}]$  which satisfy some modest iterability and solidity requirements.

**THEOREM 27** (Jensen). *If  $\kappa$  is subcompact then  $\square_\kappa$  fails.*

PROOF. Assume otherwise, and let  $\mathcal{C}$  be  $\square_\kappa$ -sequence. Let  $\rho < \kappa$ , and  $\tilde{\mathcal{C}}$  be such that there is an elementary embedding:

$$j: \langle H(\rho^+), \in, \tilde{\mathcal{C}} \rangle \rightarrow \langle H(\kappa^+), \in, \mathcal{C} \rangle.$$

Let  $\mathcal{A}$  be  $j''\rho^+$ . For every  $\alpha \in \mathcal{A}$ , if  $\text{cf } \alpha \neq \kappa$  then  $\text{cf } \alpha < \rho$  and therefore  $\text{otp } C_\alpha < \rho$ .

Let  $\delta = \sup \mathcal{A}$ , and let us look at  $C_\delta$ . Since  $\text{cf } \delta = \rho^+$ ,  $\text{otp } C_\delta \geq \rho^+$ . On the other hand for every  $\alpha \in \text{acc } C_\delta$  with  $\text{cf } \alpha < \rho$ ,  $\text{otp}(C_\delta \cap \alpha) = \text{otp } C_\alpha < \rho$ , which is impossible.  $\dashv$

The same proof as above shows that the following stronger claim holds:

REMARK 28 (Zeman [22]). Let  $\kappa$  be a subcompact cardinal. Then, there is no sequence  $\langle C_\alpha \mid \alpha \in S_{<\kappa}^{\kappa^+} \rangle$  such that :

- (1) For all  $C \in \mathcal{C}_\alpha$ ,  $C$  is a club at  $\alpha$ ,  $\text{otp } C < \kappa$ .
- (2) For all  $\alpha \in S_{<\kappa}^{\kappa^+}$ ,  $0 < |C_\alpha| < \kappa$ .
- (3) For all  $C \in \mathcal{C}_\alpha$ ,  $\beta \in \text{acc } C$ ,  $C \cap \beta \in \mathcal{C}_\beta$ .

We note that  $\square(\kappa^+)$  can still hold where  $\kappa$  is subcompact. Indeed, subcompactness behaves much like Mahloness of  $\kappa^+$ , and cannot be destroyed by a forcing which is  $\kappa^+$ -strategically closed, such as the standard forcing to add  $\square(\kappa^+)$  by initial segments. Yet, the failure of  $\square_\kappa$  for a subcompact  $\kappa$  indicates that subcompactness has a deep connection to stationary reflection.

The following argument (essentially due to Zeman) is similar to the Harrington–Shelah [10] argument for obtaining  $\text{Refl}(S_\omega^{\omega_2})$  from a Mahlo cardinal. In [22], a similar theorem is proven when the subcompact cardinal is collapsed to be  $\omega_n$  for some  $n$ . For completeness we include a proof here for the case in which the subcompactness of  $\kappa$  is preserved.

THEOREM 29 (Zeman). *Let  $\kappa$  be subcompact and assume that  $2^\kappa = \kappa^+$  and let  $\eta < \kappa$ . Then, there is a forcing notion  $\mathbb{P}$  that does not collapse cardinals and forces that every stationary subset of  $S_{<\eta}^{\kappa^+}$  reflects at a point in  $S_{<\kappa}^{\kappa^+}$  of arbitrary high cofinality. Moreover,  $\kappa$  remains subcompact in the generic extension.*

PROOF. Let  $\mathbb{Q}_0$  be an Easton support iteration of length  $\kappa$ . In the  $\rho$  step, if  $\rho$  is not inaccessible, force with the trivial forcing. Otherwise, force with  $\text{Add}(\rho^+, \rho^{++})$ .

Let us define a forcing notion  $\mathbb{P}$ , which is essentially a  $\kappa$ -support iteration of length  $\kappa^{++}$ . Let us define, by induction on  $\alpha < \kappa^{++}$ ,  $\mathbb{P}_\alpha$  and  $\mathbb{Q}_\alpha$ . The forcing  $\mathbb{Q}_0$  was already defined. Let  $\mathbb{P}_0$  be the trivial forcing and  $\mathbb{P}_1 = \mathbb{Q}_0$ .

Let  $\alpha > 0$ . Let us pick a name  $\dot{S}_\alpha$  for a subset of  $S_\theta^{\kappa^+}$  for some  $\theta < \eta$ . If there is  $\bar{\kappa} < \kappa$  such that  $\mathbb{P}_\alpha$  forces that  $\dot{S}_\alpha$  does not reflect at any ordinal of cofinality between  $\bar{\kappa}$  and  $\kappa$ , then we let  $\mathbb{Q}_\alpha$  be the forcing that adds a club  $C_\alpha$  disjoint from  $S_\alpha$ , using bounded conditions from  $V^{\mathbb{Q}_0}$ .

For all  $\gamma < \kappa^{++}$ , let  $\mathbb{P}_\gamma$  be the collection of all sequences of length  $\gamma, p$ , such that  $\text{supp } p = \{\beta < \gamma \mid p(\beta) \neq \emptyset\}$  has size at most  $\kappa$  and for all  $\beta < \gamma$ ,  $p \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} p(\beta) \in \mathbb{Q}_\beta$ . We order  $\mathbb{P}_\gamma$  in the natural way. Let  $\mathbb{P} = \mathbb{P}_{\kappa^{++}}$ .

Since our forcing notions are going to be only distributive and not closed (or strategically closed), we wish to avoid the delicate point of whether the conditions

from  $V^{\mathbb{Q}_0}$  are a dense subset of the iteration, and thus  $\mathbb{P}$  is not defined as the standard iteration of the  $\mathbb{Q}_\alpha$ .  $\dashv$

LEMMA 30. *Every  $\mathbb{P}_\alpha$ -name for a  $\kappa$ -sequence of ordinals is forced to be a  $\mathbb{Q}_0$ -name.*

PROOF. Let us prove the lemma by induction. For  $\alpha = 1$ ,  $\mathbb{P}_\alpha = \mathbb{Q}_0$  and the statement is trivial.

Let us assume now that the claim is true for all  $\beta < \alpha$ . Since  $|\mathbb{P}_\alpha| \leq \kappa^+$ , we can code  $\mathbb{P}_\alpha$  as a subset of  $H(\kappa^+)$ . Let  $\eta < \rho < \kappa$  such that there is an elementary embedding:

$$j: \langle H(\rho^+), \in, \tilde{\mathbb{P}}_{\tilde{\alpha}} \rangle \rightarrow \langle H(\kappa^+), \in, \mathbb{P}_\alpha \rangle.$$

By elementarity,  $\tilde{\mathbb{P}}_{\tilde{\alpha}}$  codes an iteration for killing nonreflecting subsets of  $\rho^+$ , of length  $\tilde{\alpha}$  in the same way as  $\mathbb{P}_\alpha$ . Let us denote the components of the iteration by  $\tilde{\mathbb{Q}}_\beta$ . We note that if  $H_0$  is  $\mathbb{Q}_0$ -generic, then  $j$  lifts to the extension of  $H(\rho^+)$  by  $H_0 \upharpoonright \rho$ . In particular we can apply the elementarity of  $j$  to the coordinates  $\tilde{\mathbb{Q}}_\beta$ .

We build a generic filter  $\tilde{G}$  for  $\tilde{\mathbb{P}}_{\tilde{\alpha}}$  using the Cohen generic subsets of  $\rho^+$  added by the iteration  $\mathbb{Q}_0$ . In fact we will show that  $\tilde{\mathbb{P}}_\alpha$  is equivalent to the Cohen forcing of subsets of  $\rho^+$  over  $H(\rho^+)[H_0 \upharpoonright \rho]$ .

To do this we define clubs  $E_\gamma$  and  $C_\gamma$  for  $\gamma < \tilde{\alpha}$ . Intuitively, we will use the club  $E_\gamma$  in order to show that  $\tilde{\mathbb{Q}}_\gamma$  is equivalent (externally to  $H(\rho^+)$ ) to the forcing that adds a Cohen subset of  $\rho^+$ . We do this by taking the dense set of conditions  $p$  such that for all  $\gamma$ , either  $p(\gamma) = \emptyset$  or  $\max(p(\gamma)) \in E_\gamma$ .

We will also construct clubs  $C_\gamma$  for all  $\gamma < \tilde{\alpha}$ . Those clubs are going to be  $H(\rho^+)$ -generic in the following sense: The natural filter  $\tilde{G}_\beta \subseteq \tilde{\mathbb{P}}_\beta$  given by the set of all  $p$  in  $\tilde{\mathbb{P}}_\beta$  such that  $p(0) \in H_0 \upharpoonright \rho$  and

$$\forall 0 < \gamma < \beta, q(\gamma) = \emptyset \vee q(\gamma) = C_\gamma \cap \max(q(\gamma) + 1)$$

is  $H(\rho^+)$ -generic.

We go by induction on  $\beta$ . Suppose that we have constructed  $E_\gamma$  and  $C_\gamma$  for all  $\gamma < \beta$ .

If  $\beta = \zeta + 1$ , then by elementarity and the definition of the iteration we have that  $j(\dot{S}_\zeta) = \dot{S}_{j(\zeta)}$  is forced by  $\mathbb{P}_\zeta$  to be a set consisting of ordinals of cofinality less than  $\rho$  which does not reflect at  $\delta = \sup j'' \rho^+$ . Again by elementarity, we can interpret  $\dot{S}_{j(\zeta)} \cap j'' \rho^+$  using only  $\tilde{G}_\zeta$ .

By induction  $\tilde{\mathbb{P}}_\zeta$  is equivalent to adding Cohen subsets of  $\rho^+$  and using a straightforward density argument it follows that there is a dense subset of  $p$  in  $\tilde{\mathbb{P}}_\zeta$  such that each nontrivial coordinate of  $p$  has the same maximum element. It follows that the condition  $m$  in  $\mathbb{P}_{j(\zeta)}$  given by  $\text{supp}(m) = \bigcup_{q \in \tilde{G}_\zeta} \text{supp} j(q)$  and for all  $\gamma \in \text{supp}(m)$ ,  $m(\gamma) = \bigcup_{q \in \tilde{G}_\zeta} j(q)(\gamma) \cup \{\delta\}$  is a master condition for  $j'' \tilde{G}_\zeta$ . Indeed, cf  $\delta > \eta$ , so it is not a member of any of the nonreflecting stationary sets that we kill.

It follows that  $m$  decides  $\dot{S}_{j(\zeta)} \cap j'' \rho^+$ . This set is nonstationary in  $V[H_0 \upharpoonright \rho][\tilde{G}_\zeta]$ , since otherwise it would remain stationary in the full generic extension. It follows that we can find a club  $D_\zeta$  in  $\delta$  which is disjoint from it.

Let  $E_\zeta = \text{acc}\{\xi < \rho^+ \mid j(\xi) \in D_\zeta\}$ . Note that any  $\alpha \in E_\zeta$  such that  $j(\alpha) \notin D_\zeta$  would have cofinality  $\rho$ .

If we consider the dense subset of  $\tilde{\mathbb{Q}}_\zeta$  whose maximum element is in  $E_\gamma$ , then this forcing is isomorphic to adding a Cohen subset of  $\rho^+$  over  $V[H_0 \upharpoonright \rho][\tilde{G}_\zeta]$ . We stress that this isomorphism can be computed in the model  $V[H_0 \upharpoonright \rho][\tilde{G}_\zeta]$ . Let  $X_\zeta$  be a Cohen subset of  $\rho^+$  which is generic over  $V[H_0 \upharpoonright \rho][\tilde{G}_\zeta]$  (there are such sets since  $\tilde{G}_\zeta$  is equivalent to a subset of  $\zeta^+$  and  $\text{Add}(\zeta^+, \zeta^{++})$  is  $\zeta^{++}$ -cc). Applying the isomorphism between  $\text{Add}^{V[H_0 \upharpoonright \rho]}(\rho^+, 1)$  and  $\tilde{\mathbb{Q}}_\zeta$  on  $X_\zeta$  we obtain a club  $C_\zeta$  which is  $\tilde{\mathbb{Q}}_\zeta$ -generic over  $V[H_0 \upharpoonright \rho][\tilde{G}_\zeta]$ . In fact  $V[H_0 \upharpoonright \rho][\tilde{G}_\zeta][C_\zeta] = V[H_0 \upharpoonright \rho][\tilde{G}_\zeta][X_\zeta]$ . This completes the successor step.

If  $\beta$  is limit, then using  $E_\gamma$  for  $\gamma < \beta$  and induction there is an isomorphism between  $\tilde{\mathbb{P}}_\beta$  and  $\text{Add}(\rho^+, \beta)$  as computed in  $V[H_0 \upharpoonright \rho]$ . The fact that the sequence  $\langle X_\gamma \mid \gamma < \beta \rangle$  is generic for  $\text{Add}(\rho^+, \beta)$  implies that  $\tilde{G}_\beta$  is generic for  $\tilde{P}_\beta$ .

We conclude that there is a generic filter  $\tilde{G}$  for  $\tilde{\mathbb{P}}_\alpha$ . This generic filter is obtained in a  $\rho^+$ -distributive extension of  $V^{\mathbb{Q}_0 \upharpoonright \rho}$ . Thus, it does not introduce any new  $\rho$ -sequences of ordinals (recall that  $H(\rho^+)$  is closed under  $\rho$ -sequences and thus computes correctly  $\rho$ -distributivity). The lemma follows by elementarity.  $\dashv$

By the chain condition of  $\mathbb{P}$ , we can make sure that in the generic extension, if  $S$  is a subset of  $S_{<\kappa}^{\kappa^+}$  which does not reflect then  $\Vdash S = \dot{S}_\alpha$  for some  $\alpha < \kappa^{++}$  and therefore it is nonstationary.

LEMMA 31.  $\kappa$  is subcompact in the generic extension.

PROOF. Let  $p$  be a condition and let  $A$  be a name for a subset of  $H(\kappa^+)$ . By the chain condition of the iteration, there is an  $\alpha < \kappa^{++}$  such that  $\dot{A}$  is a  $\mathbb{P}_\alpha$ -name.

By the subcompactness of  $\kappa$  there is a cardinal  $\rho < \kappa$  and an elementary embedding

$$j: \langle H(\rho^+), \in, \bar{p}, \bar{\mathbb{P}}_\alpha, \bar{A} \rangle \rightarrow \langle H(\kappa^+), \in, p, \mathbb{P}_\alpha, A \rangle.$$

By the arguments of Lemma 30, we can find a master condition  $m$ , namely a condition  $m \leq p$  such that for any dense open set  $D \subseteq \bar{\mathbb{P}}$  which is definable from parameters in  $H(\rho^+)$  and  $A$ , there is  $q \in D$  such that  $m \leq j(q)$ . It is clear that in this case, if  $m$  belongs to the generic filter then  $j$  lifts to the generic extension.

Our argument shows that the set of such master conditions is dense in  $\mathbb{P}_\alpha$ , so the lemma follows.  $\dashv$

This finishes the proof of Theorem 29.

The proof of the above theorem only provides stationary reflection for sets of bounded cofinality. In [5], Cummings showed that one can start with a cardinal  $\kappa$  which is  $\kappa^+$ -supercompact, and force that every stationary subset of  $\kappa^+$  contains a non-reflection stationary subset, while preserving the subcompactness of  $\kappa$  and much more. Nevertheless, in Cummings model, it is possible that there is a generic extension that restores stationary reflection.

Similarly, in [18], Schimmerling analyzed the canonical square sequence in extender models,  $L[E]$ , and showed that the least cardinal  $\kappa$  in which stationary reflection holds for  $S_{<\kappa}^{\kappa^+}$  is much larger than the least subcompact.

Those results indicate that the consistency strength of  $\text{Refl}(S_{<\kappa}^{\kappa^+})$  might be actually larger than a single subcompact. Strengthening Cummings' result, the next theorem shows that it is consistent that  $\kappa$  is subcompact yet there is no forcing extension

that preserves  $\kappa$  and  $\kappa^+$  and forces full stationary reflection at  $S_{<\kappa}^{\kappa^+}$ . Schimmerling's results suggest that this would be the case also in an extender model,  $L[E]$ .

**THEOREM 32.** *Let  $\kappa$  be subcompact. There is a generic extension in which  $\kappa$  is subcompact and there is a nonreflecting stationary set  $S \subseteq S_{<\kappa}^{\kappa^+}$  and a partial square  $\langle C_\alpha \mid \alpha \notin S \rangle$ .*

**PROOF.** By preparing the ground model, if necessary, we may assume that for every  $\delta < \kappa$ , which is not subcompact,  $\square_\delta$  holds.

Let  $\mathbb{P}$  be a forcing notion which consists of pairs  $\langle s, c \rangle$  where:

- (1)  $s$  is a bounded subset of  $S_{<\kappa}^{\kappa^+}$  and for all limit  $\alpha \leq \sup s$ , there is a club  $d_\alpha$  in  $\alpha$ , which is disjoint from  $s$ .
- (2) If  $\beta \in s$  then  $c\beta$  is non-measurable.
- (3)  $c$  is a function and  $\text{dom } c$  is a successor ordinal between  $\sup s$  and  $\kappa^+$ .
- (4) For every  $\alpha \in \text{dom } c$ ,  $c(\alpha)$  is a closed subset of  $\alpha$  (possibly the empty set).
- (5) If  $\alpha \in \text{dom } c \setminus s$  then  $\sup c(\alpha) = \alpha$ .
- (6) If  $\beta \in \text{acc } c(\alpha)$  then  $c(\alpha) \cap \beta = c(\beta)$ .

We order  $\mathbb{P}$  by  $\langle s', c' \rangle \leq \langle s, c \rangle$  if and only if  $s \subseteq s'$ ,  $c = c' \upharpoonright \text{dom } c$  and  $(s' \setminus s) \cap (\text{dom } c) = \emptyset$  (note that  $s'$  is an end extension of  $s$  above the maximum of the domain of  $c$ , which is at least  $\sup s$ ).  $\dashv$

**CLAIM 33.** *The forcing  $\mathbb{P}$  is  $\kappa + 1$ -strategically closed.*

**PROOF.** We define a winning strategy for the good player. At successor stages, the good player does nothing. At limit stages, if the current stage of the game is  $\langle \langle s_\alpha, c_\alpha \rangle \mid \alpha < \beta \rangle$ , then setting  $\rho_\alpha = \max \text{dom } c_\alpha$  the good player plays

$$s_\beta = \bigcup_{\alpha < \beta} s_\alpha \text{ and } c_\beta = \bigcup_{\alpha < \beta} c_\alpha \cup \{ \langle \rho_\beta, \{ \rho_\alpha \mid \alpha < \beta \} \rangle \}.$$

It is clear that this choice is a condition in  $\mathbb{P}$  which is stronger than all previous conditions in the play provided that  $\beta \leq \kappa$ .  $\dashv$

By the proof of the claim, it is clear that the strategy is definable in  $H(\kappa^+)$ . Moreover, throughout the game the ordinals  $\rho_\alpha$  will be a club which witness the nonreflection of  $s$  at each limit point.

Let us show that  $\kappa$  is subcompact in the generic extension. Let  $\dot{A}$  be a name for a subset of  $H(\kappa^+)$  in the generic extension. Since  $\mathbb{P} \subseteq H(\kappa^+)$ , we have  $\dot{A} \subseteq H(\kappa^+)$ .

Let  $\rho < \kappa$  and  $\dot{B}, \tilde{\mathbb{P}} \subseteq H(\rho^+)$  be such that there is an elementary embedding:

$$j: \langle H(\rho^+), \in, \tilde{\mathbb{P}}, \dot{B} \rangle \rightarrow \langle H(\kappa^+), \in, \mathbb{P}, \dot{A} \rangle.$$

Moreover, let us assume that  $\rho$  is the minimal cardinal for which such  $\dot{B}$  and  $j$  exist.

**CLAIM 34.**  *$\rho$  is not subcompact.*

**PROOF.** Assume that  $\rho$  is subcompact. Then there is some  $\eta < \rho$  and  $\dot{C}, \tilde{\mathbb{P}}$  and an elementary embedding  $k$  such that:

$$k: \langle H(\eta^+), \in, \tilde{\mathbb{P}}, \dot{C} \rangle \rightarrow \langle H(\rho^+), \in, \tilde{\mathbb{P}}, \dot{B} \rangle.$$

Then  $j \circ k$  is elementary, which contradicts the assumption that  $\rho$  is minimal.

The forcing  $\tilde{\mathbb{P}}$  is  $\rho + 1$ -strategically closed by elementarity. By a theorem of Ishiu and Yoshinobu [12], since  $\square_\rho$  holds,  $\tilde{\mathbb{P}}$  is  $\rho^+$ -strategically closed. This strategy is combined from the strategies for the shorter games and thus we can verify that the sequence of  $\rho_\alpha$  which is constructed in the game is closed and disjoint from the constructed non-reflecting set. Let

$$\mathcal{D} = \{D_{\varphi,a} \mid \varphi(x, y) \text{ is a first order formula, } a \in H(\rho^+)\}$$

be the set of all dense open subsets of  $\tilde{\mathbb{P}}$  of the form

$$D_{\varphi,a} = \{p \in \tilde{\mathbb{P}} \mid p \Vdash \neg\varphi(a, \dot{B}) \text{ or } p \Vdash \varphi(a, \dot{B})\}.$$

Let  $\langle D_\alpha \mid \alpha < \rho^+ \rangle$  be an enumeration of  $\mathcal{D}$  with length  $\rho^+$ . Using the strategic closure of  $\tilde{\mathbb{P}}$  we can generate a decreasing sequence of conditions  $\langle p_\alpha \mid \alpha < \rho^+ \rangle$  such that  $p_\alpha \in D_\alpha$ .

Let  $\tilde{G} \subseteq \tilde{\mathbb{P}}$  be the filter generated from the sequence  $\langle p_\alpha \mid \alpha < \rho^+ \rangle$ . Let us show that there is a condition  $m \in \mathbb{P}$  such that  $\forall q \in \tilde{G}, m \leq j(q)$ . This implies that  $m$  forces that the embedding  $j$  lifts to the generic extension.

Indeed, let  $p_\alpha = \langle s_\alpha, c_\alpha \rangle$ . Then clearly, for  $\alpha < \beta$ ,  $s_\alpha$  is an initial segment of  $s_\beta$  and  $c_\alpha$  is an initial segment of  $c_\beta$ . Let  $\delta = \sup j'' \rho^+$  and let us consider

$$\begin{aligned} s &= \{\delta\} \cup \bigcup_\alpha j(s_\alpha), \\ c &= \bigcup_\alpha j(c_\alpha) \cup \{\langle \delta, \emptyset \rangle\}. \end{aligned}$$

The strategy enables us to obtain a club  $E \subseteq \rho^+$  which is disjoint from  $\bigcup_\alpha s_\alpha$ .  $j'' E$  is disjoint from  $s$ . Moreover, the closure of  $j'' E$  differs from  $j'' E$  only by points of cofinality  $\rho$ . Since  $\rho$  is measurable those points cannot appear at  $s$  and therefore also  $\text{acc } j'' E$  is disjoint from  $s$ .  $\dashv$

The theorem suggests that the consistency of full stationary reflection at a subcompact cardinal might exceed the consistency of subcompact cardinal. Moreover, since the forcing is  $\kappa^+$ -distributive, we can conclude that if  $\kappa$  is measurable subcompact or even more it will remain measurable subcompact after the forcing and there is no generic extension in which stationary reflection holds at  $S_{<\kappa}^{\kappa^+}$  and  $\kappa, \kappa^+$  are preserved.

The exact large cardinal assumption which is required in order to get stationary reflection at the set  $S_{<\kappa}^{\kappa^+}$  where  $\kappa$  is subcompact is unclear. In the previous theorem, the set of all  $\beta < \kappa^+$  such that there is an elementary embedding  $j: H(\rho^+) \rightarrow H(\kappa^+)$  with  $\sup j'' \rho^+ = \beta$  is stationary and non-reflecting. This is analogous to the case of Mahlo cardinal in a generic extension of  $L$  in which stationary sets of bounded cofinality might reflect at inaccessible cardinals but the set of inaccessible cardinals does not reflect.

The following definition, due to Neeman and Steel, will play a major role in our investigation of unbounded stationary reflection.

**DEFINITION 35.** A cardinal  $\kappa$  is  $\kappa^+$ - $\Pi_1^1$ -subcompact if for every set  $A \subseteq H(\kappa^+)$ , and every  $\Pi_1^1$ -statement  $\Phi$  such that  $\langle H(\kappa^+), \in, A \rangle \models \Phi$ , there is  $\rho < \kappa$ ,  $B \subseteq H(\rho^+)$ ,

and an elementary embedding:

$$j: \langle H(\rho^+), \in, B \rangle \rightarrow \langle H(\kappa^+), \in, A \rangle$$

such that  $\langle H(\rho^+), \in, B \rangle \models \Phi$ .

In their paper [17], this large cardinal notion is denoted by  $\Pi_1^2$ -subcompact. We feel that the notion  $\kappa^+$ -  $\Pi_1^1$ -subcompact is more appropriate as it emphasizes the resemblance between  $\kappa^+$  and a weakly compact cardinal.

**LEMMA 36.** *Let  $\kappa$  be  $\kappa^+$ -  $\Pi_1^1$ -subcompact. Then  $\kappa$  is measurable.*

**PROOF.** Let  $\Phi$  be the  $\Pi_1^1$ -statement “for every  $\mathcal{U} \subseteq \mathcal{P}(\kappa)$  which is an ultrafilter,  $\mathcal{U}$  is not  $\kappa$ -complete”. If  $\kappa$  is not measurable,  $\Phi$  holds. But for every  $\rho < \kappa$  such that there is an elementary embedding  $j: H(\rho^+) \rightarrow H(\kappa^+)$  with critical point  $\rho$ , one can obtain a measure of  $\rho$  by  $\mathcal{U}_\rho = \{A \subseteq \rho \mid \rho \in j(A)\}$ . So  $\Phi$  fails at  $H(\rho^+)$ .  $\dashv$

**LEMMA 37.** *Let  $\kappa$  be  $\kappa^+$ -  $\Pi_1^1$ -subcompact. Then every sequence of  $< \kappa$  many stationary subsets of  $S_{< \kappa}^{\kappa^+}$  has a common reflection point.*

**PROOF.** Let  $\mathcal{S}$  be a collection of stationary sets,  $|\mathcal{S}| < \kappa$ . Let us reflect the  $\Pi_1^1$ -statement: “ $\forall C \subseteq \kappa^+$ , which is closed and unbounded, for all  $S \in \mathcal{S}$ ,  $C \cap S \neq \emptyset$ ”.

Fix  $\rho > |\mathcal{S}|$  such that there is an elementary embedding:

$$j: \langle H(\rho^+), \in, \tilde{\mathcal{S}} \rangle \rightarrow \langle H(\kappa^+), \in, \mathcal{S} \rangle.$$

Note that if  $S \in \mathcal{S}$ , and  $\alpha \in S$  with  $\text{cf}(\alpha) = \eta$ , then  $\eta < \rho$ . The ordinal  $\delta = \sup j'' \rho^+$  will be a reflection point of every member of  $\mathcal{S}$ . Indeed, for every  $S \in \mathcal{S}$  there is a unique  $\tilde{S} \in \tilde{\mathcal{S}}$  such that  $j(\tilde{S}) = S$ . Every  $\tilde{S} \in \tilde{\mathcal{S}}$  is stationary at  $\rho^+$  of cofinality  $< \rho$ . Therefore,  $j'' \tilde{S} = S \cap j'' \rho^+$  is stationary at  $\delta$ .  $\dashv$

Neeman and Steel showed that the consistency strength of simultaneous stationary reflection at the successor of a threadable Woodin cardinal (indeed, threadable successor of a threadable Woodin cardinal) is  $\kappa^+$ -  $\Pi_1^1$ -subcompact under some iterability assumptions.

**§4. Stationary Reflection at  $\aleph_{\omega+1}$ .** In this section, we will prove the main theorem of the paper which improves the upper bound of the consistency strength of stationary reflection at the successor of a singular cardinal. The proof splits into two components: the first component is a general statement about preservation of some mildly indestructible reflection principles at the successor of a measurable cardinal  $\kappa$  under a forcing that changes the cofinality of  $\kappa$  to  $\omega$  and shoots a club through the stationary set  $(S_\kappa^{\kappa^+})^V$ . The second is to show how to obtain the hypothesis of the previous result from  $\kappa^+$ -  $\Pi_1^1$ -subcompactness. We formulate the result this way, since it may be possible to use a weaker large cardinal notion to obtain the hypothesis of the first step. Further, we are able to use the first step to give an application to stationary reflection for subsets of some bounded cofinality.

The idea to use Prikry forcing in order to force a measurable to be  $\aleph_\omega$  while preserving its successor and maintaining stationary reflection at almost all stationary subsets of  $\aleph_{\omega+1}$  appears in several places. In the paper of Cummings, Foreman, and Magidor [6, Section 11], they show that if  $\kappa$  is  $\kappa^+$ -supercompact then after forcing

with Prikry forcing stationary reflection holds outside the set of ordinals of ground model cofinality  $\kappa$ . This was later improved by Faubion in [8], who obtained the same result starting from a weaker assumption of a quasi-compact cardinal. Clearly to obtain full stationary reflection in such a model, we must destroy the stationarity of  $S_0$  and any other nonreflecting stationary sets which may appear in the extension.

We will state and prove the main theorem for simultaneous reflection of finitely many stationary sets. The proof adjusts easily to the case of stationary reflection of single sets.

**THEOREM 38.** *Assume GCH. Let  $\kappa$  be a measurable cardinal and let  $S \subseteq S_{<\kappa}^{\kappa^+}$  be stationary. Let us assume that for every  $\omega \leq \theta < \mu < \kappa$  regular cardinals,  $S_\theta^\mu \in I[\mu]$ .*

*Let us assume further that  $\text{Add}(\kappa^+, 1)$  forces that every finite sequence of stationary subsets of  $S$  reflects simultaneously at ordinals of unbounded cofinalities below  $\kappa$ . Then, there is a generic extension in which  $\kappa = \aleph_\omega$ ,  $\kappa^+ = \aleph_{\omega+1}$ , the ground model  $S_\kappa^{\kappa^+}$  is nonstationary, and simultaneous reflection holds for finite sequences of stationary subsets of  $S$ .*

By the assumption of GCH, the approachability requirement is not satisfied trivially only at successors of singular cardinals.

**PROOF.** In  $V$ , let us fix a normal ultrafilter on  $\kappa, \mathcal{U}$ . Let  $j: V \rightarrow M$  be the ultrapower embedding given by  $\mathcal{U}$ . Let  $\mathbb{P}$  be the Prikry forcing with interleaved collapses using a guiding generic  $K$  as defined in Section 2.

Let  $\dot{P}$  be the canonical name of the generic Prikry sequence added by  $\mathbb{P}$  (so  $\dot{P}$  does not include the generic filters for the collapses). Let  $\mathbb{Q}$  be the forcing notion for adding a club to  $(S_{<\kappa}^{\kappa^+})^V$  in  $V[\dot{P}]$ . Note that  $\mathbb{Q}$  is defined in a submodel of the generic extension of  $V$  by  $\mathbb{P}$ .

Let us start by analyzing  $j_\omega(\mathbb{Q})$ . ⊣

**CLAIM 39.**  $\text{cf}^V j_\omega(\kappa^+) = \kappa^+$ . Moreover, in  $V$  there is a closed unbounded set  $D \subseteq j_\omega(\kappa^+)$  such that  $D \subseteq (S_{\leq \kappa}^{j_\omega(\kappa^+)})^{M_\omega}$ .

**PROOF.** The sequence  $j_\omega \upharpoonright \kappa^+$  is cofinal at  $j_\omega(\kappa^+)$ . Indeed, let  $\beta < j_\omega(\kappa^+)$ . Then there is a function  $f: \kappa^n \rightarrow \kappa^+$  such that  $\beta = j_\omega(f)(\kappa, j_1(\kappa), \dots, j_{n-1}(\kappa))$ . Let  $\gamma = \sup_{a \in \kappa^n} f(a) < \kappa^+$ . Then  $j_\omega(\gamma) > \beta$ .

Let  $D = \text{acc } j_\omega \upharpoonright \kappa^+$ . For every  $\delta \in D$ , let us show that  $\text{cf}^V \delta = \text{cf}^{M_\omega} \delta$ . Clearly,  $\text{cf}^V \delta \leq \text{cf}^{M_\omega} \delta$ . Let us assume that  $\text{cf}^V \delta = \eta \leq \kappa$  and let  $\langle \gamma_i \mid i < \eta \rangle$  be a sequence of ordinals such that  $\sup_{i < \eta} j_\omega(\gamma_i) = \delta$ . The sequence  $\langle j_\omega(\gamma_i) \mid i < \eta \rangle$  belongs to  $M_\omega$  since it is  $j_\omega(\langle \gamma_i \mid i < \eta \rangle) \upharpoonright \eta$ . Therefore,  $M_\omega$  computes the cofinality of  $\delta$  correctly. ⊣

Let  $p_\star \in \mathbb{P}$  and let  $s_\star = \langle p_0, \dots, p_{n_\star-1} \rangle$  be the Prikry part of the stem of  $p_\star$ .

Let  $D$  be as in the conclusion of the claim. Let  $P = s_\star^\frown \langle j_n(\kappa) \mid n < \omega \rangle$ . By a theorem of Mathias,  $P$  is an  $M_\omega$ -generic Prikry sequence. So one can think of  $P$  as the realization of  $j_\omega(\dot{P})$  using the generic filter over  $M_\omega$  which is obtained from Lemma 13.

Since  $|j_\omega(\mathbb{Q})|^V = \kappa^+$ , in  $V$ , one can construct a tree of conditions in  $j_\omega(\mathbb{Q})^P$  which is isomorphic to  $(\kappa^+)^{<\kappa^+}$ . This is done by induction. Assume that for  $\eta \in (\kappa^+)^{<\kappa^+}$ ,  $q_\eta$  is defined. Let  $\langle r_\alpha^\eta \mid \alpha < \kappa^+ \rangle$  enumerate all conditions in  $j_\omega(\mathbb{Q})^P$  which are stronger

than  $q_\eta$ . For each  $\alpha < \kappa^+$ , let  $q_{\eta \cap \langle \alpha \rangle}$  be an extension of  $r_\alpha^\eta$ , such that  $\max q_{\eta \cap \langle \alpha \rangle} \in D$ . If  $\eta \in (\kappa^+)^{< \kappa^+}$ , and  $\text{len } \eta$  is a limit ordinal, we let  $q_\eta = \{\delta_\eta\} \cup \bigcup_{\gamma < \text{len } \eta} q_{\eta \upharpoonright \gamma}$  where  $\delta_\eta = \sup\{\max q_{\eta \upharpoonright \gamma} \mid \gamma < \text{len } \eta\}$ . Note that  $\delta_\eta \in D$  since  $D$  is club. Moreover,  $q_\eta \in M_\omega[P]$ , since this model is closed under  $\kappa$ -sequences from  $V$ .

Therefore, in  $V$ , there is a tree which is dense in  $j_\omega(\mathbb{Q})^P$  and isomorphic to the forcing  $\text{Add}(\kappa^+, 1)^V$ .

Let us remark that if  $P'$  is any other Prikry sequence such that  $P'$  differs from  $P$  by only finitely many ordinals, then the interpretation of  $\mathbb{Q}$  is the same. In particular, for every  $n$ ,  $j_n(j_\omega(\mathbb{Q})^P) = j_\omega(\mathbb{Q})^P$ .

Fix a  $V$ -generic for  $\mathbb{P}_n \upharpoonright p_*$ ,  $G'$ , and let  $\tilde{G}$  be the  $M_\omega$ -generic filter for  $j_\omega(\mathbb{P})$ , which is derived from it. Let  $H$  be a  $V$ -generic filter for  $j_\omega(\mathbb{Q})^P$  where  $P$  is derived from  $\tilde{G}$ .

We will apply the machinery of Section 2.1 for  $\mathbb{A} = j_\omega(\mathbb{Q})^P$ . Let  $\mathcal{H}$  be  $\langle <_{j_{n,\omega}} H > \mid n < \omega \rangle$  and let  $\mathcal{H}_n = \langle <_{j_{m,n}} H > \mid m \leq n \rangle$ . Recall that  $P * \mathcal{H}$  is generic for the forcing  $j_\omega(\mathbb{H})$  over  $M_\omega$ , and that  $H \in M_\omega[P][\mathcal{H}]$  by Lemma 23.

**CLAIM 40.** *In  $M_\omega[\tilde{G}][\mathcal{H}]$  every finite collection of stationary subsets of  $j_\omega(\kappa^+)$  reflects at a common point.*

**PROOF.** For  $i \leq k$ , let  $S_i \subseteq j_\omega(\kappa^+)$  be stationary. Since  $H \in M_\omega[P][\mathcal{H}]$ , we may assume that each  $S_i$  is disjoint from the set of ordinals that has cofinality  $j_\omega(\kappa)$  in  $M_\omega$ . Without loss of generality, the cofinality of the members of each  $S_i$  is fixed to be some  $\theta_i < \kappa_m$ .

Work in  $M_n[\tilde{G} \upharpoonright (n_* + n + 1)][\mathcal{H}_k]$ ,  $n \geq m$ . In this model, one can construct  $\tilde{G}$  and  $\mathcal{H}$  as well as  $M_\omega$ .

Let

$$T_n^i = \{\alpha < j_n(\kappa^+) \mid j_{n,\omega}(\alpha) \in \dot{S}_i^{\tilde{G} * \mathcal{H}}\}.$$

If the sequence  $T_n^i$  for  $i \leq k$  are all stationary in  $j_n(\kappa^+)$  then, since the forcing that introduces  $\tilde{G} \upharpoonright (n_* + n + 1)$  has cardinality  $j_n(\kappa)$  in  $M_n[\mathcal{H}_n]$ , one can find stationary subsets of  $T_n^i$ ,  $T^i$ , in  $M_n[\mathcal{H}_n]$ . Since  $\mathcal{H}_k$  is equivalent to a generic filter for  $\text{Add}(j_n(\kappa^+), n+1)$  over  $M_n$ , simultaneous stationary reflection holds in this model. In particular, the sequence  $T^i$  for  $i \leq k$  reflects at common ordinals of arbitrary large cofinalities. Let  $\delta$  be a common reflection point of  $T^i$ ,  $i \leq k$ , such that  $\kappa_n > \text{cf } \delta \geq \kappa_{n-1}$ . Recall that  $T^i$  consists of ordinals of cofinality  $\theta_i$  which is less than  $\kappa_m$  (in particular less than  $\kappa_{n-1}$ ). Let  $\{\beta_i \mid i < \text{cf } \delta\}$  be a continuous and increasing sequence of ordinals, cofinal at  $\delta$ . Let  $A^i = \{\gamma < \text{cf } \delta \mid \beta_\gamma \in T^i\}$ . By the assumption,  $A^i$  is stationary in  $S_{\theta_i}^{\text{cf } \delta}$ .

The forcing that introduces  $\tilde{G} \upharpoonright (n_* + n + 1)$  splits into a product of  $\kappa_{n-1}$ -cc forcing and  $\kappa_{n-1}$ -closed forcing. Using the approachability assumption  $\text{cf } \delta \in I[\text{cf } \delta]$  and the assumption that  $\theta_i < \kappa_{n-1} < \text{cf } \delta$ , we conclude that each  $A^i$  is stationary in  $M_n[\tilde{G} \upharpoonright (n_* + n + 1)]$  and in particular in  $M_n[\tilde{G} \upharpoonright (n_* + n + 1)][\mathcal{H}_n]$ . The set  $j_{n,\omega}(A^i) = j_{n,\omega}'' A^i$  belongs to  $M_\omega$  and by downwards absoluteness from  $M_n[\tilde{G} \upharpoonright (n_* + n + 1)][\mathcal{H}_n]$ , it is stationary in  $M_\omega[\tilde{G}][\mathcal{H}]$ .

Thus, if each  $T_n^i$  is stationary then there is a condition that forces that the sequence of  $S_i$  reflects at a common point. Therefore, we conclude that at least one of the  $T_n^i$  is non-stationary. Let  $C_n$  be a club in  $M_n[\tilde{G} \upharpoonright (n_* + n + 1)][\mathcal{H}_n]$  disjoint from  $T_n^{i_n}$  for the relevant  $i_n \leq k$ . By the chain condition of the forcing that introduces  $\tilde{G} \upharpoonright (n_* + n + 1)$ , we may assume that  $C_n \in M_n[\mathcal{H}_n]$ . Let  $\dot{C}_n$  be a name for the club  $C_n$ .

Let us consider  $C = \bigcap_{n > m} j_{n,\omega}(\dot{C}_n)^{< j_{n,\omega} \text{ " } \mathcal{H}_n >}$ . We claim that  $C \in M_\omega[P][\mathcal{H}]$ . Indeed, for each  $n$  the filter  $\langle j_{n,\omega} \text{ " } \mathcal{H}_n \rangle$  is simply an initial segment of  $\mathcal{H}, j_{n,\omega}(\dot{C}_n)$  is a member of  $M_\omega$ , and  $M_\omega[P][\mathcal{H}]$  is closed under  $\omega$ -sequences.

Let  $i \leq k$  be such that  $i_n = i$  for infinitely many  $n \geq m$ . Let us show that  $C$  is disjoint from  $S_i$ . Indeed, if  $\alpha \in C \cap S_i$  then  $\alpha = j_{n,\omega}(\alpha')$  for some  $n < \omega$ . Without loss of generality, we can take  $n$  to be such that  $i_n = i$ . Then  $\alpha' \in C_n$  and in  $T_n^i$ , a contradiction to the choice of  $C_n$ .

By elementarity, we conclude that when forcing over  $V$  with  $\mathbb{P} * \mathbb{H} / \dot{P}$  simultaneous stationary reflection holds at  $\kappa^+$  for finite collections.  $\dashv$

The proof shows that the forcing  $\mathbb{P} * \mathbb{H} / \dot{P}$  preserves the stationarity of subsets of  $S_{<\kappa}^{\kappa^+}$ . In particular, the conclusion is never vacuous, as the set  $S$  for which  $\text{Refl}(S)$  holds is stationary in the generic extension.

In order to use Theorem 38, we need to show that some indestructibility can be achieved at the level of subcompact cardinals.

**LEMMA 41.** *Let  $\kappa$  be  $\kappa^+$ - $\Pi_1^1$ -subcompact. There is a generic extension in which GCH holds,  $\kappa$  is  $\kappa^+$ - $\Pi_1^1$ -subcompact, and this property is indestructible under the forcing  $\text{Add}(\kappa^+, 1)$ .*

**PROOF.** Let us assume, by forcing if needed, that GCH holds in the ground model. Let  $\mathbb{L}$  be the Easton support iteration of  $\text{Add}(\kappa^+, 1)$  for all inaccessible  $\alpha \leq \kappa$ . Let  $V^\mathbb{L}$  be the generic extension.  $\dashv$

**LEMMA 42.** *In  $V^\mathbb{L}$ ,  $\kappa$  is  $\kappa^+$ - $\Pi_1^1$ -subcompact. Moreover, this remains true after further forcing with  $\text{Add}(\kappa^+, 1)$ .*

**PROOF.** Work in  $V$ . Let  $\dot{A}$  be a name for a subset of  $\kappa^+$ . Let  $\Phi$  be a  $\Pi_1^1$ -statement with parameter  $\dot{A}$  which is true in the generic extension. Thus, the following  $\Pi_1^1$ -statement holds in the structure  $\langle H(\kappa^+), \in, \mathbb{L}, \dot{A}, \Vdash_{\mathbb{L}} \rangle$ :

$$\forall \dot{X} \subseteq \mathbb{L} \times H(\kappa^+), \Vdash_{\mathbb{L}} \varphi(\dot{X}, \dot{A}),$$

where  $\varphi$  is a first order statement in the language of forcing.

Since  $\kappa$  is  $\kappa^+$ - $\Pi_1^1$ -subcompact in  $V$ , we can find some cardinal  $\rho < \kappa$ , and  $\tilde{\mathbb{L}}, \tilde{\dot{A}}$  such that there is an elementary embedding:

$$j: \langle H(\rho^+), \in, \tilde{\mathbb{L}}, \tilde{\dot{A}}, \Vdash_{\tilde{\mathbb{L}}} \rangle \rightarrow \langle H(\kappa^+), \in, \mathbb{L}, \dot{A}, \Vdash_{\mathbb{L}} \rangle.$$

It is clear that  $\tilde{\mathbb{L}} = \mathbb{L} \upharpoonright \rho + 1$ .

Let  $p$  be a condition in  $\mathbb{L}$ . Without loss of generality,  $p \in \text{im } j$ , and let  $q \in \tilde{\mathbb{L}}$  such that  $p = j(q)$ . Let  $\tilde{G}$  be a generic filter for  $\tilde{\mathbb{L}}$  that contains  $q$ . Let  $m = \bigcup_{r \in \tilde{G}} j(r(\rho))$ , namely the union over the last coordinate of the  $j$ -image of all conditions in  $\tilde{G}$ . By the directed closure of the forcing  $\text{Add}(\kappa^+, 1)$ ,  $m$  is a condition. Let  $G$  be a generic that contains  $\tilde{G} \upharpoonright \rho + 1$  and  $m$ . Note that  $p \in G$ . By Silver's criterion,  $j$  extends to an elementary embedding between  $H(\rho^+)$  and  $H(\kappa^+)$  of the generic extension. Since we assumed that  $q$  forces that  $\Phi$  holds at  $H(\rho^+)$  and that  $p$  forces that  $\Phi$  holds at  $H(\kappa^+)$ , the conclusion of the lemma follows.  $\dashv$

**THEOREM 43.** *Simultaneous reflection for finite collections of stationary subsets of  $\aleph_{\omega+1}$  is consistent relative to a cardinal  $\kappa$  which is  $\kappa^+$ - $\Pi_1^1$ -subcompact.*

**PROOF.** By Lemmas 36 and 37,  $\kappa$  is a measurable cardinal and every collection of fewer than  $\kappa$  many stationary subsets of  $S_{<\kappa}^{\kappa^+}$  reflects simultaneously at arbitrarily high cofinalities below  $\kappa$ . By Lemma 41, this property of  $\kappa$  can be forced to be indestructible under the forcing  $\text{Add}(\kappa^+, 1)$ . By standard arguments, we may assume that  $S_\theta^\mu \in I[\mu]$  for every  $\theta < \mu < \kappa$  regular. Finally, by applying Theorem 38 (with  $S = S_{<\kappa}^{\kappa^+}$ ), the conclusion holds.  $\dashv$

**REMARK 44.** In the model for the main theorem, there is a very good scale of length  $\kappa^+$  by [6, Theorem 20]. By [6, Theorem 5], it follows that simultaneous reflection for countable collections of stationary sets fails in the final model in a strong way.

By using Theorem 29, if there is a measurable subcompact  $\kappa$  and  $\eta < \kappa$  then there is a generic extension in which every stationary subset of  $\kappa^+$  of cofinality  $< \eta$  reflects. In this model, we obtain that any stationary subset reflects at ordinals of arbitrary high cofinality. We may also assume that the approachability holds everywhere below  $\kappa$ . By the proof of Theorem 29, in this model, we obtain that stationary reflection for stationary subsets of  $S_{<\eta}^{\kappa^+}$  is indestructible under the forcing  $\text{Add}(\kappa^+, 1)$ . Thus, we conclude:

**THEOREM 45.** *Let  $\kappa$  be a measurable subcompact cardinal and let  $n < \omega$ . There is a generic extension in which every stationary subset of  $S_{\leq\aleph_n}^{\aleph_{\omega+1}}$  reflects.*

It is interesting to compare Theorem 45 to Zeman's Theorem on the upper bound for the consistency strength of the failure of  $\square_{\aleph_\omega}$  [22].

By Theorem 38, the consistency of stationary reflection at the successor of a singular cardinal is bounded from above by the consistency of mildly indestructible stationary reflection at the successor of a measurable cardinal.

**QUESTION.** Assume that  $\text{Refl}(S)$  holds for some stationary subset of  $\aleph_{\omega+1}$ . Is there an inner model with a measurable subcompact cardinal?

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