Chapter 1

Topological Defects in General Quantum LDPC Codes

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We consider the structure of defects carrying quantum information in general quantum low-density parity-check (LDPC) codes. These generalize the corresponding constructions for topological quantum codes, without the need for locality. Relation of such defects to (generalized) topological entanglement entropy is also discussed.

1.1 Introduction

Quantum computation offers exponential algorithmic speed-up for some classically hard problems. It relies on multi-particle quantum-correlated states which are very fragile and are destroyed rapidly in the presence of errors — noise, environment, or just random control errors. Quantum error correction gives a unique way of dealing with such a fragility. Selective measurements are performed to separate just one of exponentially many possible outcomes. As a result, instead of the net error probability which grows linearly or faster with the number of qubits n and saturates rapidly, one only has to worry about the error probability per qubit, p. This leads to the threshold theorem [1, 2], basically stating that an arbitrarily long quantum computation is possible when p is below certain threshold, $p_c > 0$.

Most important class of quantum error-correcting codes are the stabilizer codes [3,4]. Of these, most studied, and closest to a practical implementation, are the surface codes [5, 6]. A surface code can be viewed as the degenerate ground-state manifold of certain quantum spin model with the Hamiltonian formed by a sum of commuting terms local on a two-dimensional lattice. One of the many advantages of surface codes is the flexibility they offer in the code parameters and the structure of logical operators. To add an

extra qubit, one may simply create a hole in the surface. A larger hole, well separated from other defects, offers better protection (larger minimal code distance). Pairs of such holes can be moved around to perform encoded Clifford gates, etc. [6–8].

On the other hand, a substantial disadvantage of surface codes, or any stabilizer code with generators local on a D-dimensional Euclidean lattice, is that such codes necessarily have small rates $R = k/n$ whenever the code distance d gets large [9, 10]. Here k is the number of encoded qubits and n is the block length of the code. To get a finite asymptotic rate, one needs more general quantum codes. In particular, any family of *w-bounded quantum LDPC codes* with stabilizer generators of weight not exceeding $w > 0$ and distances divergent as a logarithm or a power of n has a nonzero asymptotic error correction threshold even in the presence of measurement errors [11,12]. Several families of bounded-weight quantum LDPC codes with finite rates have been constructed. Best-known constructions are quantum hypergraph-product (qHP) and related codes [13–15], and various hyperbolic codes [16–19].

The biggest obstacle to practical use of finite-rate quantum LDPC codes is that their stabilizer generators must include far separated qubits, regardless of the qubit layout in a D-dimensional space [9, 10]. Error correction requires frequent measurement of all stabilizer generators, and measuring such non-local generators is not practical if the hardware only allows local measurements. Nevertheless, there is a question of whether other advantages of surface codes, e.g., the ability to perform protected Clifford gates by code deformations, can be extended to more general quantum LDPC codes.

Such a construction generalizing the surface-code defects and gates by code deformations to the family of qHP codes [13] has been recently proposed by Krishna and Poulin [20]. However, their defect construction is very specific to qHP codes. Second, Krishna and Poulin do not discuss the distance of the defect codes they construct, even though it is important for the accuracy of the resulting gates. Indeed, since gates by code deformation are relatively slow, the distance has to be large enough to suppress logical errors.

The purpose of this work is to give a general defect construction applicable to any stabilizer code. In the simplest form, one may just remove a stabilizer generator which produces an additional logical qubit, $k \to k+1$. However, the distance d' of such a code will not exceed the maximum stabilizer generator weight, $d' \le w$. Given a degenerate quantum LDPC code with the stabilizer generator weights bounded by w and a distance $d > w$, we would actually like to construct a related code encoding more

qubits but retaining degeneracy, i.e., with a distance $d' > w$. We propose a three-step defect construction: remove qubits in an erasable region to obtain a subsystem code, do gauge-fixing to obtain a stabilizer code with some generators of weight exceeding w, and promote one or more such generators of the resulting code to logical operators. The choice of the gauge-fixing prescription is easier in the case of Calderbank, Shor, and Steane (CSS) codes [21, 22], which makes the construction more explicit. For such codes, with some additional assumptions, we give a lower bound on the distance of the defect code. This shows that defect codes with unbounded distances can be constructed, as is also the case with surface codes.

An interesting and a rather unexpected application of this analysis is the relation of qubit-carrying capacity of a defect to its (generalized) topological entanglement entropy [23–25] (TEE), denoted γ. Namely, a degenerate defect code with distance $d' > w$ can only be created when $\gamma > 0$. Further, when distance d' is large, the TEE γ acquires stability: it remains nonzero whenever the defect is deformed within certain bounds.

1.2 Defect construction

Generally, an n-qubit quantum code is a subspace of the n-qubit Hilbert space $\mathbb{H}_2^{\otimes n}$. A quantum $[[n, k, d]]$ stabilizer code is a 2^k-dimensional subspace $\mathcal{Q} \subseteq \mathbb{H}_2^{\otimes n}$ specified as a common $+1$ eigenspace of all operators in an Abelian *stabilizer* group $\mathcal{S} \in \mathcal{P}_n$, $-\mathbf{1} \notin \mathcal{S}$, where $\mathcal{P}_n$ denotes the n-qubit Pauli group generated by tensor products of single-qubit Pauli operators. The stabilizer is typically specified in terms of its generators, $\mathcal{S} = \langle S_1, \ldots, S_r \rangle$. If the number of independent generators is $r \equiv \operatorname{rank} \mathcal{S}$, the code encodes $k = n - r$ qubits. The weight of a Pauli operator is the number of qubits that it affects. The distance d of a quantum code is the minimum weight of a Pauli operator $L \in \mathcal{P}_n$ which commutes with all operators from the stabilizer $\mathcal{S}$, but is not a part of the stabilizer, $L \notin \mathcal{S}$. Such operators act nontrivially in the code and are called logical operators.

An n-qubit CSS stabilizer code $\mathcal{Q} \equiv \texttt{CSS}(P, Q)$ is specified in terms of two n-column binary stabilizer generator matrices $H_X \equiv P$ and $H_Z \equiv Q$. Rows of the matrices correspond to stabilizer generators of X- and Z-type, respectively, and the orthogonality condition $PQ^T = 0$ is required to ensure commutativity. The code encodes $k = n - \operatorname{rank} P - \operatorname{rank} Q$ qubits, and has the distance $d = \min(d_X, d_Z)$,

$$d_X = \min_{b \in \mathcal{C}_Q^\perp \setminus \mathcal{C}_P} \operatorname{wgt}(b), \quad d_Z = \min_{c \in \mathcal{C}_P^\perp \setminus \mathcal{C}_Q} \operatorname{wgt}(c). \tag{1.1}$$

Here $\mathcal{C}_Q \in \mathbb{F}_2^{\otimes n}$ is the binary linear code (linear space) generated by the rows of Q, and $\mathcal{C}_Q^\perp$ is the corresponding dual code formed by all vectors in $\mathbb{F}_2^{\otimes n}$ orthogonal to the rows of Q. Matrix Q is the parity-check matrix of the code $\mathcal{C}_Q^\perp$. A generating matrix of $\mathcal{C}_Q^\perp$, Q^*, has $\operatorname{rank} Q^* = n - \operatorname{rank} Q$ and is called *dual* to Q. Also, if $V = \{1, \ldots, n\}$ is the set of indices and $B \subset V$ its subset, for any vector $b \in \mathbb{F}_2^{\otimes n}$, we denote $b[B]$ the corresponding *punctured* vector with positions outside B dropped. Similarly, $Q[B]$ (with columns outside of B dropped) generates the code $\mathcal{C}_Q$ *punctured* to B. We will also use the notion of a binary code $\mathcal{C}$ *shortened* to B, which is formed by puncturing only vectors in $\mathcal{C}$ supported inside B,

$$\text{Code } \mathcal{C} \text{ shortened to } B = \{c[B] : c \in \mathcal{C} \wedge \operatorname{supp}(c) \in B\}.$$

We will denote Q_B a generating matrix of the code $\mathcal{C}_Q$ shortened to B. If G and $H = G^*$ is a pair of mutually dual binary matrices, i.e., $GH^T = 0$ and $\operatorname{rank} G + \operatorname{rank} H = n$, then H_B is a parity-check matrix of the punctured code $\mathcal{C}_{G[B]}$, and [26]

$$\operatorname{rank} G[B] + \operatorname{rank} H_B = |B|. \tag{1.2}$$

The distance d of a linear code $\mathcal{C}$ is the minimal Hamming weight of a nonzero vector in $\mathcal{C}$. In general puncturing reduces the code distance. More precisely, if d and d' are the distances of the original and the punctured code, respectively, they satisfy $d - |A| \le d' \le d$, where $A = V \setminus B$ is the complement of B. On the other hand, the minimum distance d'' of a shortened code is not smaller than that of the original code, $d'' \ge d$.

For a quantum code, if A is a set of qubits and $B = V \setminus A$ its complement, the stabilizer group $\mathcal{S}$ can also be punctured to B, by dropping all positions outside B. With the exception of certain special cases [27, 28], the resulting group $\mathcal{G} \equiv \mathcal{S}[B]$ will not be Abelian, and can be viewed as a gauge group of a subsystem code [29, 30] called the *erasure* code. A stabilizer code can be obtained by removing some of the generators from $\mathcal{G}$ to make it Abelian; such a procedure is called gauge-fixing. In the case of a CSS code with stabilizer generator matrices $H_X = P$ and $H_Z = Q$, the punctured group has generators $P[B]$ and $Q[B]$, while a gauge-fixed stabilizer code can be obtained, e.g., by replacing punctured matrix $Q[B]$ with the corresponding shortened matrix, Q_B. This latter construction can be viewed as a result of measuring qubits outside B in the X-basis. Qubits in an erasable set A can be removed without destroying quantum information. In this case, according to the cleaning lemma [10], the logical operators of the original code can all

be chosen with the support outside A. From here, with the help of Eqs. (1.1) and (1.2), we obtain the following statement (see Appendix A for all proofs).

Statement 1. *Consider a CSS code $\mathcal{Q} \equiv \mathtt{CSS}(P, Q)$ on qubit set V of cardinality $|V| = n$, encoding k qubits and with the CSS distances d_X, d_Z. Let $A \subset V$ be an erasable in $\mathcal{Q}$ set of qubits, and $B \equiv V \setminus A$ its complement. Then, the length-$|B|$ code $\mathcal{Q}' \equiv \mathtt{CSS}(P[B], Q_B)$ encodes the same number of qubits, $k' = k$, and has the CSS distances d'_X, d'_Z such that*:

$$d_X - |A| \leq d'_X \leq d_X, \quad d'_Z \geq d_Z. \tag{1.3}$$

The statement about the number of encoded qubits is true in general: an erasure code and any of the corresponding gauge-fixed codes encode the same number of qubits as the original code as long as the set A of removed qubits is erasable. (And, of course, we want to stick to erasable sets since we do not want to lose quantum information). To construct a code that encodes $k'' > k$ qubits, it is not sufficient to just remove some qubits, one has to also remove some group generators. If we do not care about the weight of stabilizer generators and start with a generic stabilizer code, a code with a decent distance may be obtained simply by dropping one of the existing stabilizer generators. Our general construction below is focused on quantum LDPC codes with weight-limited stabilizer generators:

Construction 1. *Given an original $[[n, k, d]]$ degenerate code with stabilizer generator weights bounded by some $w < d$, in order to create a degenerate "defect" code with $k' > k$ and $d' > w$,* (**i**) *remove some qubits in an erasable set,* (**ii**) *gauge fix the resulting subsystem code, and then* (**iii**) *drop one or more stabilizer generators with weights bigger than w.*

The gauge group $\mathcal{G} = \mathcal{S}[B]$ of the erasure code in step (**i**) has generators of weights w or smaller; generators of weight greater than w are obtained after gauge fixing in step (**ii**). This construction does not guarantee whether we get a degenerate code or not. Below, with the help of some additional assumptions, we prove several inequalities that guarantee the existence of not only degenerate defect codes with $d' > w$, but also highly-degenerate defect codes with unbounded distances.

1.3 Distance bounds for a defect in a CSS code

First, let us get general expressions for the distances d'_X, d'_Z of a CSS code with a removed Z-type generator. Given the original code $\mathtt{CSS}(P, Q)$, we choose a linearly-independent row of Q, u_0, as the additional type-Z logical

operator, and denote Q' the corresponding matrix with the row dropped (and of the rank reduced by one). Denote

$$d_Z^{(0)} = \min_{\alpha} \mathrm{wgt}(u_0 + \alpha Q'), \tag{1.4}$$

the minimum weight of a linear combination of u_0 with the rows of Q'. Then, Eq. (1.1) gives

$$d'_Z = \min(d_Z, d_Z^{(0)}). \tag{1.5}$$

The additional type-X logical operator has to be taken from the set of detectable errors of the original code. Specifically, it has to anticommute with the element of the stabilizer being removed, but commute with the remaining operators in the stabilizer and all logical operators of the original code. In addition to the X-type logical operators of the original code, the logical operators of the new code include all errors with the same syndrome as the chosen canonical operator. Respectively, the expression for the distance reads:

$$d'_X = \min(d_X, d_X^{(0)}), \quad d_X^{(0)} = \min_{b\,:\,u_0 b^T=1 \wedge Q' b^T=0} \mathrm{wgt}(b). \tag{1.6}$$

The lower bounds constructed in the following two subsections both rely on geometry in a bipartite (Tanner) graph associated with the Z-type generator matrix $H_Z = Q$. Namely, given its row-set U (check-nodes) and column-set V (value-nodes), the Tanner graph has the union $U \cup V$ as its vertex set, and an undirected edge $(u, v) \in U \times V$ for each nonzero matrix element Q_{uv}. On a graph there is a natural notion of the distance between a pair of nodes, the number of edges in the shortest path between them; a ball $\Omega_R(u_0)$ of radius R centered around u_0 is the set of all vertices at distance R or smaller from u_0. Then, an erasable region $A = \Omega_R(u_0) \cap V$ is chosen as a set of value nodes within the radius R from a check node $u_0 \in U$, subject to the condition that a row of the shortened matrix Q_B contains u_0 in its expansion over the rows of Q.

The condition is not a trivial one, as it is actually equivalent to region A being erasable in the code $\mathtt{CSS}(P, Q')$ formed by the original matrix $H_X \equiv P$ and the matrix Q', the original matrix Q with the row u_0 (considered linearly independent) dropped, same code as in Eqs. (1.4)–(1.6). As an equivalent but easier to check condition, one may request that row $u_0[A]$ be a linear combination of the rows of the punctured matrix $Q'[A]$ (remember that the support of u_0 is a subset of A, while u_0 is linearly independent from the rows of Q'). In addition, we use a corresponding sufficient condition as a part of lower X-distance bound in Statement 2, and formulate a related necessary

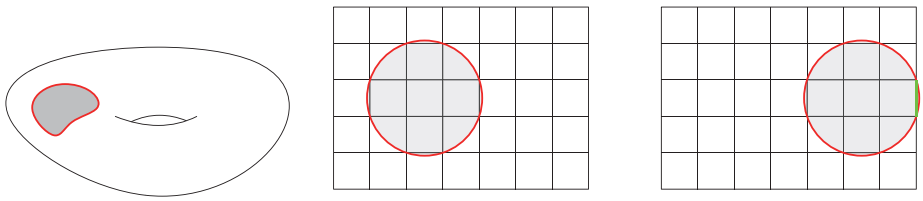
condition in terms of the topological entanglement entropy associated with the defect A in Sec. 1.4.

1.3.1 *Code with locally linearly-independent generators*

We need a condition to guarantee a lower bound on the weight of the operator conjugate to the row u_0 removed from the matrix Q_B, see Eq. (1.6). Here, we will assume that the set of Z-type stabilizer generators forming the rows of the matrix $H_Z = Q$ be overcomplete. That is, there be one or more linear relations between the rows of Q, and that we start with a row u_0 which takes part in such a relation.

In the case of the toric code (or any surface code on a locally planar graph without boundaries), see Fig. 1.1(a), the linear relation is simply the statement that the sum of all rows of H_Z be zero (necessarily so since each column has weight two). Such a relation exists for any matrix with even column weights, e.g., qHPs from (ℓ, m)-regular binary codes with both ℓ and m even. Further, many such linear relations exist for CSS codes forming chain complexes of length 3 or more, e.g., the D-dimensional hyperbolic [18, 31] and higher-dimensional qHP codes [15] with $D > 2$.

Statement 2. *Given a CSS code* $\texttt{CSS}(P, Q)$ *and a natural* R_1*, consider the bipartite Tanner graph associated with the matrix* Q*, and a ball* $W = \Omega_{2R_1}(u_0)$ *of radius* $2R_1$ *centered around the row* $u_0 \in U$*. Assume* (**a**) *that the row* u_0 *is involved in at least one linear relation with other rows of* Q*, and* (**b**) *there exists* $R_2 > R_1$ *such that all rows within radius* $2R_2$ *from the center be linearly independent of each other. Let* Q_1 *denote a full-row-rank matrix obtained from* Q *by removing some (linearly-dependent) rows outside* W*. Then weight of any* $b \in \mathbb{F}_2^{\otimes n}$ *such that the syndrome* $Q_1 b^T$ *has*



(a) (b) (c)

Fig. 1.1. (a) Homologically trivial hole on a torus. (b) Surface code with a smooth boundary. Removing qubits (edges) inside of the circle we get a nontrivial defect. (c) This circle contains a boundary edge with no neighboring plaquette; removing the corresponding edges we again get a trivial defect.

the only nonzero bit at the check node u_0 satisfies $\mathrm{wgt}(b) \geq R_2$, *and for the complement* $B = V \setminus A$ *of any region* $A \subseteq W$, $\mathrm{wgt}(b[B]) \geq R_2 - R_1$.

The punctured vector $b[B]$ is a representative of the new X-type codeword in the defect code $\mathtt{CSS}(P[B], Q'_B)$, where Q' is obtained from Q_1 by removing u_0; the constructed bound gives $d_X^{(0)} \geq R_2 - R_1$ for the distance in Eq. (1.6).

We also note that additional, linearly-dependent with u_0, rows in Q need not have bounded weight, as long as on the Tanner graph they are located outside the ball W_2. The corresponding requirement is of course equivalent to any of the two conditions discussion above this subsection title, with $A = W_2 \cap V$. In the case of a surface code with smooth boundary, see Figs. 1.1(b) and 1.1(c), the extra row may be chosen as the product of all plaquette generators, with the support along the actual boundary. In such a case, the lower distance bound in Statement 2 is saturated.

1.3.2 *Stabilizer group with an expansion*

Here we construct a simple lower bound on the Z-distance of the defect, in essence, relying on the monotonicity of the distance d_Z with respect to X-basis measurement of qubits in an erasable set, see Statement 1. To make it nontrivial, we assume that Z-type stabilizer generators of the original code satisfy an expansion condition, namely, there exists an increasing real-valued function f such that a product Π_m of any m distinct generators has weight bounded by $f(m)$,

$$\mathrm{wgt}(\Pi_m) \geq f(m), \quad f(m+1) > f(m). \tag{1.7}$$

Such a *global* condition on code generators guarantees that the boundary condition is good for the defect we are trying to construct. For example, in case of the toric code on an $L \times L$ square lattice with periodic boundary conditions, there are L^2 plaquette generators but only $L^2 - 1$ of them are independent. Namely, the product of all plaquette generators is an identity, so $m \to L^2 - m$ is a symmetry of the weight distribution. Necessarily, the function f in Eq. (1.7) has a trivial maximum, $f(L^2) \leq 0$. Respectively, a single hole in Fig. 1.1(a) has a homologically trivial boundary — meaning that it can be pushed out and eventually contracted to nothing by a sequence of single-plaquette steps. On the other hand, for a planar smooth-boundary surface code configuration as in Fig. 1.1(b), one gets $f(m)$ scaling as a perimeter of m plaquettes with a non-trivial maximum.

Generally, as one increases the set A of removed qubits, there will be rows in the shortened matrix Q_B formed as linear combinations of increasing numbers of rows of the original matrix Q. The expansion condition (1.7)

with $\max_m f(m) > 0$ guarantees that the corresponding rows cannot be contracted to nothing. For example, when we remove a single qubit corresponding to a weight-ℓ column of Q, if the corresponding adjacent rows all have weights w and do not overlap (in the case of a surface code $\ell \leq 2$ and w is the number of sides in the corresponding plaquette), the shortened matrix Q_B necessarily has $\kappa = \ell - 1$ rows of weight $2w - 2$. Assuming $f(2) = 2w - 2$, at any $w > 2$ this is already sufficient to guarantee the existence of a degenerate defect code with $d'_Z > w$.

With larger defects, combinations of larger numbers of rows may become necessary. If so, the expansion condition (1.7) will also guarantee that codes with Z-type distances (1.5) much greater than w can be constructed (assuming big enough original code distance d_Z).

Statement 3. *Given a code* $\mathtt{CSS}(P, Q)$ *and a natural* R_1*, consider the bipartite Tanner graph associated with the matrix* Q*, and a ball* $W = \Omega_{2R_1}(u_0)$ *of radius* $2R_1$ *centered around the row* $u_0 \in U$*. Denote* Q' *the matrix obtained by removing* u_0 *from* Q*. Assume* (**a**) *that* $A \equiv W \cap V$ *is erasable in the code* $\mathtt{CSS}(P, Q')$*, and* (**b**) *that the set of* Z*-generators defined by the rows of matrix* Q *satisfies the expansion condition* (1.7) *with* $f(2) \geq 1$*. Then, weight of any linear combination of* u_0 *with rows of the matrix* Q' *supported on the complement* $B \equiv V \setminus A$ *satisfies* $d_Z^{(0)} \geq f(R_1)$*.*

Notice that the condition (**a**) here is the same as discussed above Sec. 1.3.1 title; the corresponding sufficient condition is a part of Statement 2, where any $R_2 > R_1$ will do.

1.3.3 *Defect codes with arbitrary large distances*

Notice that Statement 2 requires a linear *dependence* between generators of Q, while Statement 3 requires the expansion condition (1.7) with nontrivial f which is stronger than just linear *independence*. Nevertheless, these conditions are not necessarily incompatible. The condition in Statement 2 only needs to be satisfied for some parent code. For example, in the case of a toric code, *two* distinct holes are needed in order to create a defect code with distance $d' = \min(d'_Z, d'_X) > 4$. Here a linear dependence between plaquette operators can be found in the parent toric code, one hole is needed to satisfy the conditions of Statement 3, while the other one is the actual erasable set in Construction 1.

Generally, suppose we have a parent CSS code $\mathtt{CSS}(H_X, H_Z)$ with bounded-weight generators, sufficiently large distance, and matrix H_Z with even-weight columns so that the sum of all rows be zero. Such a pair of

matrices satisfies conditions of Statement 2 but not of Statement 3. Similar to the toric code, where one needs two holes to create a single-qubit defect, here we also may need to take an erasable set A formed by two or more disjoint erasable defects, e.g., balls as in Statement 3; that the set A be erasable can be guaranteed by the union lemma (Lemma 2 in Ref. [9]). Then, one (or more if needed) balls can be used to ensure the existence of the function f in Eq. (1.7) with sufficiently large $\max_m f(m)$, while the qubits in the last remaining ball would be used as the erasable set.

Explicitly, as a parent code family, one can use, e.g., qHP codes [13] created from random matrices with even-valued row and column weights. For example, $(4, 6)$-regular random matrices would do well, leading to qHP codes with asymptotically finite rates, whose CSS generator matrices have column weights 4 and 6, regular row weights $w = 10$, and $\mathcal{O}(n^{1/2})$ linear relations between the rows of generator matrices with number of nonzero coefficients in each linear relation scaling linearly with block length n of the resulting code. The distance of such parent codes grows as $\mathcal{O}(n^{1/2})$; this is sufficient to ensure that for any $d_0 > 0$ one can choose n large enough so that sufficiently large erasable balls exist to guarantee the existence of defect codes with $d' \equiv \max(d'_X, d'_Z) \geq d_0$.

1.4 Relation with topological entanglement entropy

There exists a suggestive parallel between the structure of a large-distance qubit-carrying defect we discussed, and (generalized) topological entanglement entropy (TEE) which can be associated with such a defect [23,24]. The latter may be defined in terms of the usual entanglement entropy (EE), which characterizes what happens when some of the qubits carrying a normalized quantum state $|\psi\rangle \in \mathcal{H}_2^{\otimes n}$ are erased (traced over). Namely, if the set of qubits is decomposed into A and its complement $B = V \setminus A$, one considers the binary von Neumann entropy $\Upsilon(A; B) \equiv -\operatorname{tr}_B \rho_B \log_2 \rho_B$, where the density matrix $\rho_B = \operatorname{tr}_A |\psi\rangle\langle\psi|$ is obtained by tracing over the qubits in A. The definition is actually symmetric with respect to interchanging A and B, $\Upsilon(A; B) = \Upsilon(B; A)$.

The entanglement entropy has a particularly simple form when $|\psi\rangle$ is a stabilizer state [32]. Such a state is just a stabilizer code encoding no qubits, so that its dimension is $2^0 = 1$. With $n = |V|$ total qubits, this requires a stabilizer group with n independent generators. According to Fattal *et al.* [32], the EE of any stabilizer state $|\psi\rangle \in \mathcal{Q}$ is uniquely determined by the decomposition of the stabilizer group $\mathcal{S} = \mathcal{S}_A \times \mathcal{S}_B \times \mathcal{S}_{AB}$, where nontrivial elements of subgroups $\mathcal{S}_A$ and $\mathcal{S}_B$ are supported only

on A and only on B, respectively, and those of $\mathcal{S}_{AB}$ are necessarily split between A and B. Namely, $\operatorname{rank}\mathcal{S}_{AB} = 2p$ is always even, and it is this p (or, equivalently, the number of EPR pairs split between A and B) that determines the entanglement entropy,

$$\Upsilon(A;B) = p \equiv \tfrac{1}{2}\operatorname{rank}\mathcal{S}_{AB}. \tag{1.8}$$

Given a stabilizer code $\mathcal{Q}$ with parameters $[[n,k,d]]$ and a stabilizer group $\mathcal{S}$ of rank $n-k$, a stabilizer state $|\psi\rangle \in \mathcal{Q}$ can be formed by adding any k mutually commuting logical Pauli operators to the stabilizer group. Then, if set $A \subset V$ is erasable, according to the cleaning lemma [10], we can select all logical operators with the support in $B = V \setminus A$. With the logical operators in $\mathcal{S}_B$, both $\mathcal{S}_A$ and $\mathcal{S}_{AB}$ are subgroups of the stabilizer group $\mathcal{S}$ of our original code, and the entanglement entropy is given by Eq. (1.8). The same quantity p can also be expressed in terms of the punctured stabilizer group $\mathcal{S}[B] = \mathcal{S}_B \times \mathcal{S}_{AB}[B]$ (gauge group of the subsystem erasure code), written as a product of its center, the (shortened) stabilizer group $\mathcal{S}_B$, and p pairs of canonically conjugated "gauge" qubits which generate the (punctured) subgroup $\mathcal{S}_{AB}[B]$.

In the case of a CSS code, such a decomposition exists for both X-type and Z-type subgroups of the stabilizer, e.g., $\mathcal{S}^{(Z)} = \mathcal{S}_A^{(Z)} \times \mathcal{S}_B^{(Z)} \times \mathcal{S}_{AB}^{(Z)}$, with $\operatorname{rank}\mathcal{S}_{AB}^{(Z)} = \operatorname{rank}\mathcal{S}_{AB}^{(X)} = p$. These p independent generators are obtained from the rows of the original generator matrices that are split between A and B. In a weight-limited LDPC code, the total number of such rows, e.g., in H_Z, can be called the perimeter $L(A;B)$ of the cut. However, the number of generators of $\mathcal{S}_{AB}^{(Z)}$ can actually be smaller than $L(A;B)$ since some linear combination(s) of the generators split between A and B combined with other generators may form an element of $\mathcal{S}_A$ or $\mathcal{S}_B$. Thus, we can write EE as

$$\Upsilon(A;B) = L - \gamma, \tag{1.9}$$

with $L = L(A;B)$ the perimeter of the cut and some integer $\gamma \equiv \gamma(A;B) \geq 0$. While this expression strongly resembles the Kitaev–Preskill definition of TEE [23, 24], for now γ is just a parameter associated with the particular cut.

Let us now consider weights of generators of $\mathcal{S}_B^{(Z)}$. These correspond to the rows of Q_B. Clearly, each row of the original matrix Q may be supported in A, or in B, or be split between the two sets. Rows already supported in B can be moved directly to Q_B and preserve their original weights. Thus no more than $\gamma \geq 0$ generators of $\mathcal{S}_B^{(Z)}$ may need to have larger weights. Necessarily, if we want to construct a defect forming a degenerate code with

the distance $d' > w$, the additional number of qubits is bounded by

$$\kappa \equiv k' - k \leq \gamma. \tag{1.10}$$

Thus, with $\gamma = 0$, the defect cannot support a degenerate code with $\kappa > 0$. However, whether or not a particular defect does, in fact, support $\kappa > 0$, also depends on the global structure of the code, e.g., the boundary conditions.

Now, let us imagine that we have a defect code with a sufficiently large distance d. Then, such a defect is also *stable* to small deformations, e.g., when B is changed to some B' as a result of up to $M < d$ steps, where at each step a single position is added or removed from the set. That is, our defect code retains the same number κ of additional qubits when we change the set B to a set B', $|B \triangle B'| \leq M$, where $B \triangle B' = (B \setminus B') \cup (B' \setminus B)$ is the symmetric set difference. For deformations such that $M + w < d$, the inequality $\gamma \geq \kappa$ must be satisfied in the course of deformations.

Now, TEE is normally considered a property of ground-state wavefunction of some many-body Hamiltonian, while our focus was on quantum LDPC codes with bounded-weight but not necessarily local generators. Different terms in a Hamiltonian can be viewed as generators of the code. However, in the absence of locality, why would we care about weights of terms in a quantum spin Hamiltonian?

In a physical system, multi-qubit Pauli operators may appear as terms in an n-spin quantum Hamiltonian, e.g.,

$$H_0 = -A \sum_a P_a - B \sum_b Q_b, \tag{1.11}$$

where and $A > 0$ and $B > 0$ are the coupling constants, and, to connect with our discussion of CSS codes, P_a and Q_b could be Pauli operators of X- and Z-type, respectively, specified by rows of the binary matrices P and Q. Then, if all terms in the Hamiltonian commute, i.e., $PQ^T = 0$, the ground-state space of H_0 is exactly the code with the stabilizer group generated by these operators.

Any simple spin Hamiltonian (1.11) is usually just the leading-order approximation to a real problem. Even at zero temperature, additional interaction terms are virtually always present. Such terms may break the degeneracy of the ground-state of the Hamiltonian H_0. The effect is weak if the code has a large distance, while perturbations be small and local. The standard example is the effect of an external magnetic field $\mathbf{h} = (h_x, h_y, h_z)$, which can be introduced as an additional perturbation Hamiltonian

$$H_1 = -\frac{1}{2} \sum_i (h_x X_i + h_y Y_i + h_z Z_i). \tag{1.12}$$

For a code with distance d, only a Pauli operator of weight d or larger may act within the code. Respectively, assuming the magnetic field small, degenerate perturbation theory gives the ground state subspace energy splitting scaling as $\mathcal{O}(h^d)$, where $h = |\mathbf{h}|$ is the field magnitude.

However, the code distance d gives only a part of the story. Large-weight operators appearing in H_0 make the ground-state order particularly susceptible to local perturbations such as the magnetic field. In this case, the relevant scale for the magnetic field is $Wh \sim \max(A, B)$, that is, the effect of the magnetic field may be magnified by the operator weight W. Indeed, if we start with the spin-polarized ground state of H_1, a weight-W Pauli operator will generically flip W spins, producing a state with the energy increased by $\mathcal{O}(Wh)$. The effect of such a perturbation will be small as long as the corresponding coefficient, A or B in Eq. (1.11), remains small compared to Wh. Thus, with W large, the ground state of the spin Hamiltonian H_0 gets destroyed already with very small $h \sim \max(A, B)/W$. The same estimate can be also obtained with the help of an exact operator map similar to that used by Trebst *et al.* [33].

1.5 Conclusions

To summarize, we discussed a general approach to adding logical qubits to an existing quantum stabilizer code, with the focus on quantum LDPC codes with weight-limited stabilizer generators. In short, a stabilizer generator needs to be promoted to a logical operator, which puts a bound on the distance of the obtained code in terms of the generator weight w. As in a surface code, a degenerate code can be obtained by removing some qubits in an erasable set, and gauge-fixing the resulting subsystem code in such a way as to ensure that stabilizer generators of sufficiently large weight be created. We also constructed some lower bounds on the distance of thus obtained defect codes which show that construction can in principle be used to obtain highly degenerate codes with distances much larger than w.

An interesting observation is a relation between the ability of a particular defect (erasable set of qubits) to support an additional logical qubit in a degenerate code, and a quantity analogous to TEE, γ. A degenerate defect code can be only created with $\gamma > 0$. Further, when a defect code has a large distance d', a lower bound on $\gamma > 0$ is maintained in the course of deformations, not unlike for the conventionally defined TEE.

Many open problems remain. First, our lower distance bounds are constructed by analogy with surface codes. In particular, the lower bound

in Statement 2 applies only for a single qubit. In addition, we do not have good lower distance bounds for defects in non-CSS codes.

Second, the notion of generalized TEE γ in Eq. (1.9) needs to be cleaned up. Here we are working with lattice systems, not necessarily local, and the usual expansions in term of $1/L$ do not necessarily help. Further, as defined, γ certainly depends of the chosen set of generators. Redundant sets of small-weight generators imply the existence of higher homologies, as in higher-dimensional toric codes; it would be nice to be able to interpret values of γ, as, e.g., was done by Grover *et al.* in a field theory setting [25].

Third, if we start with a finite-rate family of codes, are there defects of size $|A|$ with $\gamma = \mathcal{O}(|A|)$? Coming back to defect codes, it appears that a typical defect with large γ would generically lead to an entire spectrum of operator weights in the generators of $\mathcal{S}_B$. Is there a situation when there is a large gap in this weight distribution, as in the surface codes with $\gamma = 1$, where only one high-weight operator may exist?

Appendix: All the proofs

Proof of Statement 1. The number of encoded qubits follows from the identity (1.2). Namely, the exact dual Q^* of the matrix Q can be obtained from P by adding k rows corresponding to inequivalent codewords $b \in \mathcal{C}_Q^{\perp} \setminus \mathcal{C}_P$. According to the cleaning lemma [10], these can be chosen with the support outside of an erasable set A. Dropping these k rows from $Q^*[B]$ recovers the punctured matrix $P[B]$ with the correct rank to ensure $k' = k$. The distance inequalities are obtained from Eq. (1.1) by considering removal of a single qubit at a time.

Proof of Statement 2. Indeed, since Q' differs from Q only by some rows outside the ball $W_2 \equiv \Omega_{2R_2}(u_0)$ which are linearly dependent with u_0, the corresponding full-matrix syndrome Qb^T must have nonzero bits outside $W_2 \cap U$. With the exception of u_0, any row in $W_2 \cap U$ must be incident on an even number of set bits in b, and there must be a continuous path on the graph from u_0 to outside W_2 formed by pairs of set bits in b (otherwise b could be separated into a pair of vectors with nonoverlapping supports, $b = b_1 + b_2$, such that $Qb_1^T = 0$ outside W_2 and $Qb_2^T = 0$ inside W_2, which would contradict the assumptions). This guarantees that at any odd distance from u_0 (up to $2R_2 - 1$), b has at least one set bit, which recovers the two lower bounds.

Proof of Statement 3. The inequality follows from the locality of row operations on the Tanner graph: a nonzero bit v in some vector $c \in \mathbb{F}_2^{\otimes n}$

can only be removed by adding a row $u \in U$ neighboring with v. The lower bound on f equivalent to linear independence of rows of Q guarantees that rows of Q' be linearly independent from u_0, thus weight must remain nonzero at every step. The condition (**a**) guarantees the existence of a linear combination in question.

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References

[1] P. W. Shor, Fault-tolerant quantum computation, in *Proc. 37th Ann. Symp. Fundamentals of Comp. Sci.*, Los Alamitos, IEEE (IEEE Computer Society Press, 1996), pp. 56–65. URL http://arxiv.org/abs/quant-ph/9605011v2.

[2] J. Preskill, Fault-tolerant quantum computation, in *Introduction to Quantum Computation*, eds. H.-K. Lo, S. Popescu and T. P. Spiller (World Scientific, 1998).

[3] D. Gottesman, *Stabilizer codes and quantum error correction*, Ph.D. thesis, Caltech (1997); URL http://arxiv.org/abs/quant-ph/9705052.

[4] A. R. Calderbank, E. M. Rains, P. M. Shor and N. J. A. Sloane, Quantum error correction via codes over GF(4), *IEEE Trans. Inform. Theory* **44**, 1369 (1998); URL http://dx.doi.org/10.1109/18.681315.

[5] A. Y. Kitaev, Fault-tolerant quantum computation by anyons, *Ann. Phys.* **303**, 2 (2003); URL http://arxiv.org/abs/quant-ph/9707021.

[6] E. Dennis, A. Kitaev, A. Landahl and J. Preskill, Topological quantum memory, *J. Math. Phys.* **43**, 4452 (2002); URL http://dx.doi.org/10.1063/1.1499754.

[7] H. Bombin and M. A. Martin-Delgado, Quantum measurements and gates by code deformation, *J. Phys. A* **42**(9), 095302 (2009); doi:10.1088/1751-8113/42/9/095302.

[8] H. Bombin, Topological subsystem codes, *Phys. Rev. A* **81**, 032301 (2010); doi:10.1103/PhysRevA.81.032301; URL http://link.aps.org/doi/10.1103/PhysRevA.81.032301.

[9] S. Bravyi and B. Terhal, A No-Go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes, *New J. Phys.* **11**(4), 043029 (2009); URL http://stacks.iop.org/1367-2630/11/i=4/a=043029.

[10] S. Bravyi, D. Poulin and B. Terhal, Tradeoffs for reliable quantum information storage in 2D systems, *Phys. Rev. Lett.* **104**, 050503 (2010); doi:10.1103/PhysRevLett.104.050503; URL http://link.aps.org/doi/10.1103/PhysRevLett.104.050503.

[11] A. A. Kovalev and L. P. Pryadko, Fault tolerance of quantum low-density parity check codes with sublinear distance scaling, *Phys. Rev. A* **87**, 020304(R) (2013); doi:10.1103/PhysRevA.87.020304; URL http://link.aps.org/doi/10.1103/PhysRevA.87.020304.

[12] I. Dumer, A. A. Kovalev and L. P. Pryadko, Thresholds for correcting errors, erasures, and faulty syndrome measurements in degenerate quantum codes, *Phys. Rev. Lett.* **115**, 050502 (2015); doi:10.1103/PhysRevLett.115.050502; URL http://link.aps.org/doi/10.1103/PhysRevLett.115.050502.

[13] J.-P. Tillich and G. Zemor, Quantum LDPC codes with positive rate and minimum distance proportional to $\sqrt{n}$, in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)* (June, 2009); pp. 799–803; doi:10.1109/ISIT.2009.5205648.

[14] A. A. Kovalev and L. P. Pryadko, Quantum Kronecker sum–product low-density parity-check codes with finite rate, *Phys. Rev. A* **88** 012311 (2013); doi:10.1103/PhysRevA.88.012311; URL http://link.aps.org/doi/10.1103/PhysRevA.88.012311.

[15] W. Zeng and L. P. Pryadko, Higher-dimensional quantum hypergraph-product codes with finite rates, *Phys. Rev. Lett.* **122**, 230501 (2019); doi:10.1103/PhysRevLett.122.230501; URL https://link.aps.org/doi/10.1103/PhysRevLett.122.230501.

[16] G. Zémor, On Cayley graphs, surface codes, and the limits of homological coding for quantum error correction, in *Proc. Coding and Cryptology: Second International Workshop, IWCC 2009*, eds. Y. M. Chee, C. Li, S. Ling, H. Wang and C. Xing, pp. 259–273 (Springer, 2009); doi:10.1007/978-3-642-01877-0_21; URL http://dx.doi.org/10.1007/978-3-642-01877-0_21.

[17] N. Delfosse, Tradeoffs for reliable quantum information storage in surface codes and color codes, in *2013 IEEE Int. Symp. Information Theory Proceedings (ISIT)*, pp. 917–921; (IEEE, 2013) pp. 917–921; doi:10.1109/ISIT.2013.6620360.

[18] L. Guth and A. Lubotzky, Quantum error correcting codes and 4-dimensional arithmetic hyperbolic manifolds, *J. Math. Phy.* **55**(8), 082202, (2014); doi:http://dx.doi.org/10.1063/1.4891487; URL http://scitation.aip.org/content/aip/journal/jmp/55/8/10.1063/1.4891487.

[19] N. P. Breuckmann and B. M. Terhal, Constructions and noise threshold of hyperbolic surface codes, *IEEE Trans. Inform. Theory* **62**(6), 3731–3744 (2016); doi:10.1109/TIT.2016.2555700.

[20] A. Krishna and D. Poulin, Fault-tolerant gates on hypergraph product codes, To be published in Phys. Rev. X (2021).

[21] A. R. Calderbank and P. W. Shor, Good quantum error-correcting codes exist, *Phys. Rev. A* **54**(2), 1098 (1996); doi:10.1103/PhysRevA.54.1098.

[22] A. M. Steane, Simple quantum error-correcting codes, *Phys. Rev. A* **54**, 4741 (1996); URL http://dx.doi.org/10.1103/PhysRevA.54.4741.

[23] M. Levin and X.-G. Wen, Detecting topological order in a ground state wave function, *Phys. Rev. Lett.* **96**, 110405 (2006); doi:10.1103/PhysRevLett.96.110405; URL https://link.aps.org/doi/10.1103/PhysRevLett.96.110405.

[24] A. Kitaev and J. Preskill, Topological entanglement entropy, *Phys. Rev. Lett.* **96**, 110404 (2006); doi:10.1103/PhysRevLett.96.110404; URL https://link.aps.org/doi/10.1103/PhysRevLett.96.110404.

[25] T. Grover, A. M. Turner and A. Vishwanath, Entanglement entropy of gapped phases and topological order in three dimensions, *Phys. Rev. B* **84**, 195120 (2011); doi:10.1103/PhysRevB.84.195120; URL https://link.aps.org/doi/10.1103/PhysRevB.84.195120.

[26] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes.* (North-Holland, 1981).

[27] E. M. Rains, Nonbinary quantum codes, *IEEE Trans. Inform. Theory* **45**(6), 1827 (1999); doi:10.1109/18.782103.

[28] P. K. Sarvepalli, *Quantum stabilizer codes and beyond*, PhD thesis, Texas A&M University (2008); URL http://hdl.handle.net/1969.1/86011.

[29] D. Poulin, Stabilizer formalism for operator quantum error correction, *Phys. Rev. Lett.* **95**, 230504 (2005); doi:10.1103/PhysRevLett.95.230504.

[30] D. Bacon, Operator quantum error-correcting subsystems for self-correcting quantum memories, *Phys. Rev. A* **73**, 012340 (2006); doi:10.1103/PhysRevA.73.012340.

[31] N. P. Breuckmann, *Homological quantum codes beyond the toric code*, PhD thesis, RWTH Aachen University (2017).

[32] D. Fattal, T. S. Cubitt, Y. Yamamoto, S. Bravyi and I. L. Chuang. Entanglement in the stabilizer formalism, preprint (2004); URL http://arXiv.org/abs/quant-ph/0406168; arXiv:quant-ph/0406168.

[33] S. Trebst, P. Werner, M. Troyer, K. Shtengel and C. Nayak, Breakdown of a topological phase: Quantum phase transition in a loop gas model with tension, *Phys. Rev. Lett.* **98**, 070602 (2007); doi:10.1103/PhysRevLett.98.070602; URL http://link.aps.org/doi/10.1103/PhysRevLett.98.070602.