Erratum: Minimal distances for certain quantum product codes and tensor products of chain complexes [Phys. Rev. A 102, 062402 (2020)]

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Lemma 1 in our paper is not valid as originally stated. Here we give a formulation with a condition added. The condition is satisfied by all code families considered in the paper; hence it does not affect other results. We restate the lemma and its proof as follows.

Lemma 1 (Z-puncturing bound). Consider a stabilizer code $\mathcal{Q} = \text{CSS}(H_X, H_Z)$ with the parameters $[[n, k, (d_X, d_Z)]]_q$ and a qudit index set $V = \{1, 2, \ldots, n\}$. Given a partition into complementary sets $I \subset V$ and $J = V \setminus I$, suppose a logical generator matrix L_X can be chosen so that none of its k rows is supported both in I and in J. Let $\mathcal{Q}' = \text{CSS}\,((H_X)_I, H_Z[I])$ and $\mathcal{Q}'' = \text{CSS}\,((H_X)_J, H_Z[J])$ be the codes whose X generator matrices are shortened and Z generator matrices punctured to I and J, respectively. Then the Z distances of the three codes satisfy the inequality $d_Z \geqslant \min(d'_Z, d''_Z)$.

Proof. The case $k = 0$ is trivial since it gives infinite d_Z. Assume $k > 0$. The distance d_Z of the code is the minimum weight in the set $\mathcal{Q}_Z = \mathcal{C}_{H_X}^{\perp} \setminus \mathcal{C}_{H_Z}$ of all nontrivial Z-like codewords and their equivalent vectors. For any $c \in \mathcal{Q}_Z$, the punctured vectors $c[I]$ and $c[J]$ are orthogonal to the rows of $(H_X)_I$ and $(H_X)_J$, respectively; the corresponding Pauli errors are undetectable. Further, since $L_X c^T \neq 0$, it is impossible that $c[I]$ is orthogonal to the rows of $(L_X)_I = L_X[I]$ and at the same time $c[J]$ is orthogonal to the rows of $(L_X)_J = L_X[J]$. Therefore, at most one of the vectors $c[I]$ and $c[J]$ can be trivial in the corresponding code.

Now consider the identity $\text{wgt}\, c[I] + \text{wgt}\, c[J] = \text{wgt}\, c > 0$. The punctured pieces $c[I]$ and $c[J]$ contribute to the distances d'_Z and d''_Z, respectively, only if the corresponding vectors are nontrivial. Let $d(c)$ equal infinity if c is trivial in $\mathcal{Q}$ and $\text{wgt}\, c \geqslant 1$ otherwise; define similar functions $d'(c)$ and $d''(c)$ for vectors corresponding to undetectable errors in $\mathcal{Q}'$ and $\mathcal{Q}''$, respectively. Then $d'_Z \leqslant \min_{c \in \mathcal{Q}_Z} d'(c[I])$ and $d''_Z \leqslant \min_{c \in \mathcal{Q}_Z} d''(c[J])$. The stated result is obtained by minimizing the inequality $\min\,(d'(c[I]), d''(c[J])) \leqslant d(c)$ over all $c \in \mathcal{Q}_Z$. ■

The additional condition is needed to exclude the case where a codeword in $\mathcal{Q}$ becomes trivial both in $\mathcal{Q}'$ and in $\mathcal{Q}''$ after puncturing. For example, consider a qubit code $[[4, 1, 2]]_2$ with

$$H_X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad H_Z = (1 \quad 1 \quad 1 \quad 1).$$

Then, with index sets $I = \{1, 2\}$ and $J = \{3, 4\}$, we get two trivial codes with no logical qubits and infinite distances, $d'_Z = d''_Z = \infty$, larger than d_Z. Indeed, it is easy to check by explicit enumeration that an X codeword of the original code cannot be entirely supported on either I or J. In comparison, with index sets $I = \{1, 2, 3\}$ and $J = \{4\}$, we may take $L_X = (0\ 1\ 1\ 0)$ and $L_Z = (1\ 1\ 0\ 0)$ to satisfy the conditions of the lemma; in this case we get $d'_Z = 2$ and a trivial code with $d''_Z = \infty$.

In our paper Lemma 1 is only used in Sec. IV C to get lower bounds on the homological distance of a product of chain complexes. Here decomposition is only done along the block boundaries; the condition in Lemma 1 can be verified by explicitly constructing bases of the product chain and product cochain complexes and using the Künneth formula to make sure that no vectors are lost.

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