A Framework for Predictive Control of Sampled-Data Systems Using Sporadic Model Approximation

Lixing Yang, Samuel Dauchert, and Xiaofeng Wang

***Abstract*— This paper studies stability of sampled-data systems controlled by discrete-time model predictive control (MPC) algorithms. We consider a general framework for the co-design of discrete-time MPC law and the associated scheduling schemes, where the discrete-time model used in MPC approximates the behavior of the plant with state-dependent sampling periods. Sufficient conditions are derived to guarantee uniform ultimate boundedness of the resulting closed-loop system. The results can be applied to most existing model approximation methods with either fixed or time-varying sampling rates. It is shown that to stabilize the system, the model used in the FHOCP does not have to be very accurate as long as the associated scheduling scheme matches the model and the approximation error is below a threshold related to the running cost function.**

I. Introduction

Model predictive control (MPC) has been widely applied in many applications such as process control, power grids, transportation, robotics, and manufacturing, to name a few. It solves a finite horizon optimal control problem (FHOCP) at each sampling instant and applies a part of the optimal solution to the plant as the control input. When implemented in computers, the MPC algorithms must be discrete-time, even though the plant is continuous-time, which motivates the research on sampled-data MPC.

Conventional approaches sample and solve FHOCP in a periodic manner [1]–[5]. To make sure that the discrete-time models used in the FHOCP is close to the behavior of the plant, the sampling period (or the upper bound on the sampling period) is fixed and usually very small. Such an selection could be very conservative with respect to computation efficiency. In general, solving an FHOCP is computationally expensive. Small sampling periods imply frequently solving FHOCPs, which will generate a significant number of control tasks, which could place a heavy computational burden on the processor and introduce significant computation delays. Consequently, the system performance may be degraded and sometimes the system can even be unstable.

To reduce the computation frequency, researchers began to investigate sampled-data MPC with aperiodic sampling, including event-triggered and self-triggered MPC approaches. Event-triggered MPC solves the FHOCP when some predefined events take place [6]–[10], while self-triggered MPC predicts the next the sampling instant based on the state and input information [11]–[14]. In both approaches, the sampling time instants are determined by both the system dynamics and the current system status. It allows the controller to dynamically adjust the sampling/computation frequency, according to what really happens in the system, and therefore avoid unnecessary computation.

It is worth mentioning that in all aforementioned work the FHOCP is still continuous-time. Recently, a Lebesgue approximation based MPC approach was proposed in [15], [16] for nonlinear systems, where the sampling time and the state transitions in the approximation model are both aperiodic and state-dependent. It can enlarge the inter-sampling intervals, while reducing the number of steps in the discrete-time FHOCP to prediction the same length of horizon in continuous-time domain. Therefore, both frequency and complexity of solving the FHOCP can be reduced. This result was extended later to stabilize nonlinear systems with measurement noises [17]. Both works are limited to first-order discrete-time approximation model in the FHOCP.

This paper relaxes the assumption of first-order approximation and considers a more general framework of discrete-time MPC for continuous-time systems that is suitable to various approximation models with either time-triggered or event/self-triggered scheduling schemes. The discrete-time model used in MPC can approximate the behavior of the plant with state-dependent sampling periods. The allowable model approximation error is also state-dependent and therefore admits a tighter error bound in MPC framework, compared with the error bound related to period only. We provide rigorous analysis on system stability and show that, as long as the model approximation errors at the scheduled time instants are bounded by a threshold related the cost function, the resulting closed-loop system will be uniformly ultimately bounded. The developed sufficient conditions can be used as the guidelines to design the discrete-time FHOCP in sampled-data MPC.

II. Problem Formulation

Notations: We denote by $\mathbb{R}^n$ the n-dimensional real vector space, by $\mathbb{R}^+$ the set of real positive numbers, and by $\mathbb{Z}_0^+$ the set of nonnegative integers. We use $\|\cdot\|$ to denote the Euclidean norm of a vector and the induced 2-norm of a matrix. Given a positive constant d, let $\mathcal{B}(d) = \{x \in \mathbb{R}^n \mid \|x\| \leq d\}$. Given two sets $\mathcal{X}, \mathcal{S} \subseteq \mathbb{R}^n$, $\mathcal{X} + \mathcal{S}$ is the Minkowski sum of these two sets.

Definition 2.1: A continuous function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Definition 2.2: The state $x(t)$ of a system $\dot{x} = f(x)$ is called uniformly ultimately bounded (UUB) with ultimate

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bound b if there exist positive constants b and c, independent of $t_0 \geq 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0 , such that $\|x(t_0)\| \leq a$ implies $\|x(t)\| \leq b$ for any $t \geq t_0 + T$.

Consider a nonlinear continuous-time dynamical system:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$ are the state and input of the system, respectively, $\mathcal{U} \subseteq \mathbb{R}^m$ be the input constraint set including $\{0\}$, $x_0 \in \mathbb{R}^n$ is the initial state, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function that describes the system dynamics satisfying $f(0, 0) = 0$.

When implementing state-feedback MPC algorithms in digital environments, the controller receives measurements in discrete-time. The basic idea is described as follows: At the time instant t_k, the system samples the state and the controller obtains the sampled state $x(t_k)$. Then the controller solves an N-step discrete-time FHOCP at time t_k with $x(t_k)$ as the initial condition. The FHOCP at time t_k can be stated as follows:

$$V(x(t_k)) = \min_{\hat{u}_k^i \in \mathcal{U},\ i=0,\ldots,N-1} J\left[\{\hat{u}_k^i\}_{i=0}^{N-1} | x(t_k)\right] \tag{2a}$$

$$\text{s.t.} \quad \hat{x}_k^{i+1} = \hat{f}(\hat{x}_k^i, \hat{u}_k^i) \tag{2b}$$

$$\hat{x}_k^0 = x(t_k), \quad \hat{x}_k^N \in \mathcal{X_T} \tag{2c}$$

where $J\left[\{\hat{u}_k^i\}_{i=0}^{N-1} | x(t_k)\right] = \sum_{i=0}^{N-1} \kappa(\hat{x}_k^i, \hat{u}_k^i) + V_f(\hat{x}_k^N)$, $\kappa : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_0^+$ is the running cost function, $V_f : \mathbb{R}^n \to \mathbb{R}_0^+$ is the terminal cost function, $\mathcal{X_T} \subset \mathbb{R}^n$ is the terminal set, $\hat{x}_k^i$ and $\hat{u}_k^i$ are the predicted state and input, respectively, and $\hat{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function to describe the approximation model. All functions will be determined later to ensure stability of the closed-loop system.

Let $\{\hat{u}_k^{i,*}\}_{i=0}^{N-1}$ be the optimal control inputs of the FHOCP at t_k and $\hat{x}_k^{i,*}$ be the corresponding optimal states. Then $\hat{u}_k^{0,*}$ will be actuated in the actual plant over the time interval $[t_k, t_{k+1})$, i.e., $u(t) = \hat{u}_k^{0,*}$ for $\forall t \in [t_k, t_{k+1})$. We can define

$$t_{k+1} = t_k + \hat{g}(x(t_k), \hat{u}_k^{0,*}) \tag{3}$$

where $\hat{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ is a positive function to be determined, and $u(t) = \hat{u}_k^{0,*}$ over $[t_k, t_{k+1})$. The next computation cycle starts at t_{k+1}. The overall sampled-data MPC algorithm is summarized as follows: At time $t = t_k$,

- (i) sample the state and obtain $x(t_k)$;
- (ii) solve the FHOCP in equation (2) for $\hat{x}_k^{i,*}$ and $\hat{u}_k^{i,*}$;
- (iii) set t_{k+1} using equation (3);
- (iv) send the optimal solution $\hat{u}_k^{0,*}$ to the plant, i.e, set $u(t) = \hat{u}_k^{0,*}$ over $[t_k, t_{k+1})$; and
- (v) start the next sampling/computation cycle at t_{k+1}.

Focusing on such a framework, we are interested in the conditions on the functions $\hat{f}$, $\hat{g}$, κ, and V_f, under which the closed-loop system defined by (1) – (2) can be stabilized.

III. Stability Analysis

This section discusses stability of the closed-loop system, which relies on the appropriate selection of the functions $\hat{f}$, $\hat{g}$, κ, and V_f. Several assumptions are made as follows.

Assumption 3.1: Given a pair $(x, u) \in \mathbb{R}^n \times \mathcal{U}$, let $z(t)$ be the solution satisfying $\dot{z}(t) = f(z(t), u)$ with $z(0) = x$. There exists a continuous function $\epsilon(x)$ such that the inequality $\|z(\hat{g}(x, u)) - \hat{f}(x, u)\| \leq \epsilon(x)$ holds.

Remark 3.1: This assumption places a requirement on the approximation error between the states of the plant and the approximation model. It means that given the same initial state and the input (x, u), the state of the continuous-time system at time $\hat{g}(x, u)$ should not deviate too much from the state of the discrete-time model in (2b) that is $x^+ = \hat{f}(x, u)$. The error should be bounded a function $\epsilon(x)$. In our MPC framework it indicates that the error over $[t_k, t_{k+1})$ will be bounded by $\epsilon(x(t_k))$.

To define the terminal set $\mathcal{X_T}$, we have the following assumption:

Assumption 3.2: There exist a compact set $\mathcal{X_T} \subseteq \mathbb{R}^n$, a positive constant $d_{\tilde{x}}$, and a continuous function $h : \mathbb{R}^n \to \mathbb{R}^m$ with $h(0) = 0$ such that

$$x \in \mathcal{X_T} \Rightarrow \hat{f}(x + \tilde{x}, h(x + \tilde{x})) \in \mathcal{X_T}, \quad \forall \tilde{x} \in \mathcal{B}(d_{\tilde{x}}) \tag{4}$$

$$h(x) \in \mathcal{U}, \quad \forall x \in \mathcal{X_T} + \mathcal{B}(d_{\tilde{x}}) \tag{5}$$

$$V_f(\hat{f}(x, h(x))) - V_f(x) \leq -\kappa(x, h(x)), \forall x \in \mathcal{X_T} + \mathcal{B}(d_{\tilde{x}}). \tag{6}$$

Remark 3.2: This assumption is similar to the one in [15], which is quite standard in robust MPC. $\mathcal{X_T}$ is basically a robust positively invariant set of the system $x^+ = \hat{f}(x + \tilde{x}, h(x + \tilde{x}))$ for any disturbance $\tilde{x} \in \mathcal{B}(d_{\tilde{x}})$. The constant $d_{\tilde{x}}$ actually serves as the bound on the errors between the states predicted at the kth and $k + 1$th computation cycles.

We define $\mathcal{X}_0$ as the set of admissible initial states for the FHOCP in (2) and d_0 as the largest positive constant so that

$$\mathcal{V}_0 = \{x \in \mathbb{R}^n \mid V(x) \leq d_0\} \subseteq \mathcal{X}_0 \tag{7}$$

where V is defined in (2).

Assumption 3.3: There exist two class $\mathcal{K}$ functions α_1 and α_2 such that

$$\kappa(x, u) \geq \alpha_1(\|x\|), \quad \forall x \in \mathcal{X}_0,\ \forall u \in \mathcal{U}, \tag{8}$$

$$V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathcal{V}_0. \tag{9}$$

Let $\mathcal{X}_{N,\hat{f}}$ be the N-step reachable set of the system $z^+ = \hat{f}(z, u)$ with the initial state inside $\mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$ and $u \in \mathcal{U}$.

Assumption 3.4: There exist positive constants $L_{\hat{f}}$, L_κ, and L_{V_f} such that for any $x, y \in \mathcal{X}_{N,\hat{f}}$ and $u \in \mathcal{U}$,

$$\|\hat{f}(x, u) - \hat{f}(y, u)\| \leq L_{\hat{f}} \|x - y\| \tag{10}$$

$$\|\kappa(x, u) - \kappa(y, u)\| \leq L_\kappa \|x - y\| \tag{11}$$

$$\|V_f(x) - V_f(y)\| \leq L_{V_f} \|x - y\|. \tag{12}$$

Assumption 3.5: There exist two positive constants $\rho \in (0, 1)$ and d such that for any $x \in \mathcal{V}_0$ and $u \in \mathcal{U}$,

$$\theta\epsilon(x) \leq \rho\kappa(x, u) + d, \tag{13}$$

where $\theta = \sum_{i=1}^{N-1} L_\kappa L_{\hat{f}}^{\,i-1} + L_{V_f} L_{\hat{f}}^{\,N-1}$.

Remark 3.3: Assumption 3.5 reveals an interesting relation between the approximation error and the running cost function. It means that the discrete-time approximation model must reach certain level of accuracy (or "consistency") marked by the running cost $\kappa(x, u)$.

With these assumptions, we construct a sequence of control inputs for the FHOCP at the $(k+1)$th computation cycle:

$$\hat{u}_{k+1}^{i} = \begin{cases} \hat{u}_{k}^{i+1,*}, \ i = 0, 1, ..., N-2 \\ h(\hat{x}_{k+1}^{N-1}). \ i = N-1 \end{cases} \tag{14}$$

Then the discrete-time model in (2b) with $\hat{u}_{k+1}^{i}$ and $\hat{x}_{k+1}^{0} = x(t_{k+1})$ will generate the predicted states at the $(k+1)$st computation cycle, denoted by $\hat{x}_{k+1}^{i}$ for $i = 1, 2, \cdots, N-1$.

Lemma 3.1: If Assumption 3.3 holds, $x(t_k) \in \mathcal{V}_0$, and

$$t_{k+1} - t_k \leq \frac{\alpha_1^{-1}(d_0) - \|x(t_k)\|}{f_{\max}} \tag{15}$$

holds, where $f_{\max} = \max_{x \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right), \ u \in \mathcal{U}} f(x, u)$, then $x(t) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$ for $t \in [t_k, t_{k+1}]$.

Proof: The proof is straightforward by applying comparison principle and therefore omitted. ∎

As long as $d_0 \geq V(x_0)$ and $V(x(t_k))$ keeps decreasing, we will have $d_0 \geq V(x(t_k)) \geq \alpha_1(\|x(t_k)\|)$ and therefore the right-hand side of inequality (15) will be nonnegative. Inequality (15) basically places another requirement on $\hat{g}$, which may take a formate like $\hat{g}(x, u) = \min\left\{*, \frac{\alpha_1^{-1}(d_0) - \|x\|}{f_{\max}}\right\}$.

Lemma 3.1 implies that $t_{k+1} - t_k$ must be bounded from above so that $x(t)$ will not leave the set $\mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$. Therefore, the predicted states will stay in $\mathcal{X}_{N,\hat{f}}$. As a result, the Lipschitz constants in Assumption 3.4 will be valid in the analysis later.

The next lemma quantifies the error between the predicted states computed at the kth and $k+1$th computation cycle.

Lemma 3.2: If $x(t_k) \in \mathcal{V}_0$, $x(t_{k+1}) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$, and Assumption 3.1, 3.3, 3.4 hold, then

$$\|\hat{x}_{k+1}^{i-1} - \hat{x}_{k}^{i,*}\| \leq \epsilon_k L_{\hat{f}}^{\ i-1}, \ i = 1, 2, ..., N. \tag{16}$$

holds where $\epsilon_k = \epsilon\left(x(t_k)\right)$.

Lemma 3.2 means the error between predicted states will accumulated as the number of iterations increases. This error may affect the decrease of the Lyapunov function V, if it is large. Therefore, an upper bound is needed. This is summarized in the following lemma.

Lemma 3.3: Suppose that Assumptions 3.1–3.5 hold and $x(t_k) \in \mathcal{V}_0$, $x(t_{k+1}) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$. If inequality

$$d_{\tilde{x}} \geq \epsilon_{\max} L_{\hat{f}}^{N-1} \tag{17}$$

holds, where $\epsilon_{\max} = \max_{x \in \mathcal{V}_0} \epsilon(x)$, and $\{\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\}_{i=0}^{N-1}$ is the optimal solution to the FHOCP (2) at the kth computation cycle, then $\left\{\hat{x}_{k+1}^{i}\right\}_{i=0}^{N}$ and $\left\{\hat{u}_{k+1}^{i}\right\}_{i=0}^{N-1}$ are admissible to the FHOCP at the $k+1$st computation cycle. Moreover, the following inequality holds:

$$V(x(t_{k+1})) - V(x(t_k)) \leq d - (1-\rho)\alpha_1\left(\|x(t_k)\|\right). \tag{18}$$

Condition (17) implies that the terminal set $\mathcal{X}_{\mathcal{T}}$ must be able to guarantee set invariance in the presence of disturbances whose magnitude is less than $\epsilon_{\max} L_{\hat{f}}^{N-1}$, which is actually an upper bound on the accumulated prediction state error after N steps.

Lemma 3.3 shows the change in $V(x(t_k))$ between two consecutive sampling instants, given some assumptions on $x(t_k)$ and $x(t_{k+1})$. This result can be extended over the entire time horizon, which is presented as follows.

Theorem 3.1: Assume that the hypotheses in Lemmas 3.1–3.3 hold. If $x(t_0) \in \mathcal{V}_0$ and

$$\max_{s \in \left[0, \ \alpha_1^{-1}\left(\frac{d}{1-\rho}\right)\right]} \left[\alpha_2(s) - (1-\rho)\,\alpha_1(s)\right] \leq d_0 - d \tag{19}$$

hold, the FHOCP is always feasible and the closed-loop system is UUB.

Proof: We will be shown that

$$x(t_{k+1}) \in \mathcal{V}_0 \quad \text{and} \tag{20}$$

$$V(x(t_{k+1})) - V(x(t_k)) \leq d - (1-\rho)\,\alpha_1\left(\|x(t_k)\|\right) \tag{21}$$

hold for $k = 0, 1, 2, \cdots$, using mathematical induction.

For $k = 0$, we know $x(t_0) \in \mathcal{V}_0$. Since the hypotheses of Lemma 3.1 hold for $k = 0$, we have $x(t_1) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$. Meanwhile, because $x(t_0) \in \mathcal{V}_0 \subseteq \mathcal{X}_0$, the FHOCP in (2) admits a feasible solution at t_0. So the hypotheses of Lemma 3.3 are satisfied for $k = 0$, which implies that $\{\hat{x}_1^i\}_{i=0}^{N}$ and $\{\hat{u}_1^i\}_{i=0}^{N-1}$ are admissible to the FHOCP at t_1 with the initial condition $x(t_1)$, and

$$V(x(t_1)) - V(x(t_0)) \leq -(1-\rho)\,\alpha_1\left(\|x(t_0)\|\right) + d.$$

There are two cases to be discussed:

C1: If $\|x(t_0)\| \geq \alpha_1^{-1}\left(\frac{d}{1-\rho}\right)$, then

$$V(x(t_1)) \leq V(x(t_0)) \leq d_0,$$

which means $x(t_1) \in \mathcal{V}_0$.

C2: If $\|x(t_0)\| < \alpha_1^{-1}\left(\frac{d}{1-\rho}\right)$, then with $V(x(t_0)) \leq \alpha_2(\|x(t_0)\|)$, we have

$$V(x(t_1)) \leq \alpha_2(\|x(t_0)\|) - (1-\rho)\alpha_1\left(\|x(t_0)\|\right) + d.$$

By inequality (19), we know $V(x(t_1)) \leq d_0$.

In either case $x(t_1) \in \mathcal{V}_0$ holds. So far, we show that equation (20) and (21) hold for $k = 0$.

Assume that (20) and (21) hold for $k = p-1$, i.e.,

$$x(t_p) \in \mathcal{V}_0 \quad \text{and}$$

$$V(x(t_p)) - V(x(t_{p-1})) \leq -(1-\rho)\alpha_1\left(\|x(t_{p-1})\|\right) + d.$$

We will show that they also hold for $k = p$.

Inequality (15), together with the fact $x(t_p) \in \mathcal{V}_0 \subseteq \mathcal{X}_0$, implies $x(t_{p+1}) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$ by Lemma 3.1. Since the hypotheses of Lemma 3.3 hold for $k = p$, we know that

$$V(x(t_{p+1})) - V(x(t_p)) \leq -(1-\rho)\alpha_1\left(\|x(t_p)\|\right) + d.$$

Following the previous analysis, two cases will be discussed.

C1: If $\|x(t_p)\| \geq \alpha_1^{-1}\left(\frac{d}{1-\rho}\right)$, then

$$V(x(t_{p+1})) \leq V(x(t_p)) \leq d_0.$$

C2: If $\|x(t_p)\| < \alpha_1^{-1}\left(\frac{d}{1-\rho}\right)$, then with

$$V(x(t_p)) \leq \alpha_2(\|x(t_p)\|)$$

given by Assumption 3.3, we have

$$\begin{aligned}&V(x(t_{p+1}))\\&\leq \alpha_2(\|x(t_p)\|) - (1-\rho)\alpha_1\left(\|x(t_p)\|\right) + d.\end{aligned}$$

By inequality (19), we know $V(x(t_{p+1})) \leq d_0$. So $x(t_{p+1}) \in \mathcal{V}_0$ holds for both cases and inequalities (20) and (21) hold for all $k \in \mathbb{Z}_0^+$.

Since $x(t_k) \in \mathcal{V}_0 \subseteq \mathcal{X}_0$ for all $k \in \mathbb{Z}_0^+$, we know by Assumption 3.3 that $\alpha_1(\|x(t_k)\|) \leq V(x(t_k)) \leq \alpha_2(\|x(t_k)\|)$, which, with inequality (21), implies that $\{x(t_k)\}_{k=0}^{\infty}$ is UUB. Solving the differential inequality

$$\frac{d}{dt}\|x(t) - x(t_k)\| \leq \|f(x(t), \hat{u}_k^{0,*})\| \leq f_{\max}$$

over $[t_k, t_{k+1})$ with zero initial condition implies $\|x(t) - x(t_k)\|$ is uniformly bounded for any $t \in [t_k, t_{k+1})$. So the closed-loop system is UUB. ■

IV. An Illustrative Example

This section presents the simulation results on the proposed MPC algorithm. The system under consideration is described as follows:

$$\begin{aligned}\dot{x}_1 &= \frac{(2\|x\|+1)(0.5x_1 + 5x_2)}{0.5x^\top x + 2}\\ \dot{x}_2 &= \frac{(2\|x\|+1)(2.5x_1 + 1.25x_2 + 2u)}{0.5x^\top x + 2}.\end{aligned}$$

The input constraint is $u(t) \in [-2, 2]$.

Using Euler-forward method to approximate this nonlinear system with the time-varying sampling period $T_k = \frac{0.02x_k^\top x_k + 0.08}{(2\|x_k\|+1)}$, we obtain the state prediction model

$$\hat{x}_k^{i+1} = \hat{f}(\hat{x}_k^i, \hat{u}_k^i) = \begin{pmatrix} 1.02\hat{x}_{k,1}^i + 0.2\hat{x}_{k,2}^i \\ 0.1\hat{x}_{k,1}^i + 1.05\hat{x}_{k,2}^i + \hat{u}_k^i \end{pmatrix}.$$

Similar to the approach used in [18], we can find

$$\epsilon(x) = \|\hat{f}(x,u)\|\left(e^{L_f \hat{g}(x,u)} - 1\right)$$

with a general $\hat{g}(x,u)$ under Euler forward method, where L_f is the Lipschitz constant of $f(x,u)$ with respect to x. The running cost function and the terminal cost function are

$$\kappa(x,u) = 10\|x\| + \|u\|, \quad V_f(x) = 10\|x\|.$$

With T_k and inequality (15), we can define $\hat{g}(x,u)$

$$\hat{g}(x,u) = \min\left\{\frac{0.02x^\top x + 0.08}{(2\|x\|+1)}, \frac{20 - \|x\|}{40}\right\}.$$

Notice that the choice of $\hat{g}(x,u)$ must guarantee the satisfaction of (13). So it cannot be arbitrarily large and must follow certain formats induced by $\epsilon(x)$ and $\kappa(x,u)$ such that (13) can be verified. So given an approximation method, a possible way to define $\hat{g}$ is to find $\epsilon(x)$ with a general $\hat{g}$ first. Then based on the structure of $\epsilon(x)$ and $\kappa(x,u)$, we define the detailed expression of $\hat{g}$.

The top plot of Figure 1 shows the state trajectories that converge to the origin. The input also converges to zero, as shown in the middle plot. The bottom plot shows the history of the computation periods. It converges to 0.08, which is consistent to the theoretical result $\lim_{\|x\|\to\infty} \hat{g}(x,u) = 0.08$ in this case. During the simulation, the FHOCP only runs 30 times in total.

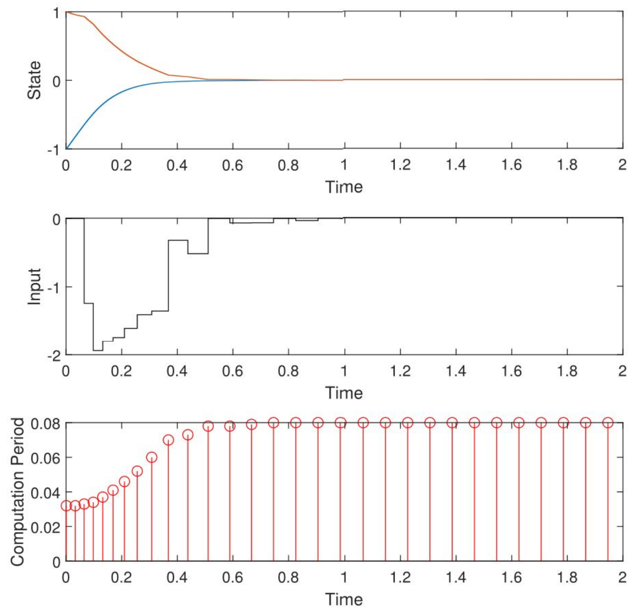

Fig. 1. The state trajectories and computation periods with $v(t) = w(t) = 0$

In the second simulation, the disturbances and measure noises are added to check system robustness. In the top plot of Figure 2 the state trajectory oscillated around the origin due to disturbances and noises. Accordingly, the inputs and the computation periods vary slightly around the steady state, as shown in the middle and bottom plots. The total number that the FHOCP runs is 31 times, which is similar the first simulation.

V. Summary

This paper provides rigorous analysis of stability of continuous-time nonlinear systems controlled by discrete-time MPC. In this framework the FHOCP is discrete-time, designed based on a sporadic approximation model of the plant that includes transitions in both state and time. Sufficient conditions are derived for the closed-loop system to be uniformly ultimately bounded. The results are applicable to most commonly used model approximation approaches, as long as the approximation error is limited over the inter-sampling time intervals.

VI. Proofs of Lemmas

A. Proof of Lemma 3.2

Proof: We prove the statement using mathematical induction. Because $x(t)$ is the solution to the system in (1) with $u(t) = \hat{u}_k^{0,*}$ over $[t_k, t_{k+1})$ starting from $x(t_k)$, by

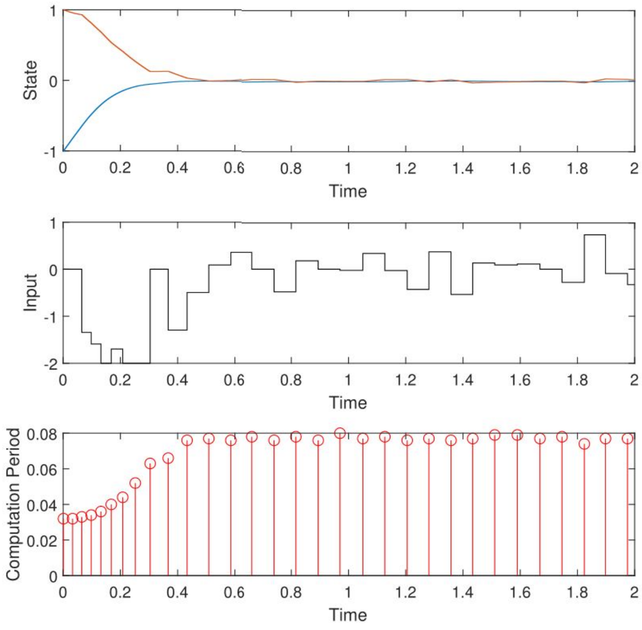

Fig. 2. The state/input trajectories and computation periods in the presence of disturbances and noises

Assumption 3.1 and $x(t_k) \in \mathcal{V}_0 \subseteq \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$,

$$\left\| x\left(t_k + \hat{g}(x(t_k), \hat{u}_k^{0,*})\right) - \hat{f}\left(x(t_k), \hat{u}_k^{0,*}\right)\right\| \leq \epsilon\left(x(t_k)\right) = \epsilon_k.$$

By equation (3), we know

$$t_{k+1} = t_k + \hat{g}(x(t_k), \hat{u}_k^{0,*}).$$

And by equation (2b), we have

$$\hat{x}_k^{1,*} = \hat{f}\left(x(t_k), \hat{u}_k^{0,*}\right).$$

So the inequality above implies for $i = 1$

$$\|\hat{x}_{k+1}^0 - \hat{x}_k^{1,*}\| = \left\| x\left(t_{k+1}\right) - \hat{x}_k^{1,*}\right\| \leq \epsilon_k.$$

Next we assume that inequality (16) holds for $i = p-1$, i.e.,

$$\|\hat{x}_{k+1}^{p-2} - \hat{x}_k^{p-1,*}\| \leq \epsilon_k L_{\hat{f}}{}^{p-2} \tag{22}$$

and prove that inequality (16) also holds for $i = p$.

According to equation (2b) and the definition of $\hat{u}_{k+1}^i$ in equation (14), we have

$$\hat{x}_k^{p,*} = \hat{f}(\hat{x}_k^{p-1,*}, \hat{u}_k^{p-1,*}) \quad \text{and}$$
$$\hat{x}_{k+1}^{p-1} = \hat{f}(\hat{x}_{k+1}^{p-2}, \hat{u}_{k+1}^{p-2}) = \hat{f}(\hat{x}_{k+1}^{p-2}, \hat{u}_k^{p-1,*}).$$

Therefore,

$$\begin{aligned}&\|\hat{x}_{k+1}^{p-1} - \hat{x}_k^{p,*}\| \\ =&\|\hat{f}(\hat{x}_{k+1}^{p-2}, \hat{u}_k^{p-1,*}) - \hat{f}(\hat{x}_k^{p-1,*}, \hat{u}_k^{p-1,*})\|.\end{aligned}$$

Since $x(t_k), x(t_{k+1}) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$, $\hat{x}_{k+1}^{p-2},\ \hat{x}_k^{p-1,*} \in \mathcal{X}_{N,\hat{f}}$ holds for any $p \leq N$. With $\hat{u}_k^{p-1,*} \in \mathcal{U}$, by inequality (10) in Assumption 3.4 and inequality (22), we have

$$\|\hat{x}_{k+1}^{p-1} - \hat{x}_k^{p,*}\| \leq L_{\hat{f}}\|\hat{x}_{k+1}^{p-2} - \hat{x}_k^{p-1,*}\| \leq \epsilon_k L_{\hat{f}}{}^{p-1},$$

which completes the proof. ■

B. Proof of Lemma 3.3

Proof: Given the assumptions in Lemma 3.2, equation (16) holds, which, together with inequality (17), implies

$$\|\hat{x}_{k+1}^{N-1} - \hat{x}_k^{N,*}\| \leq \epsilon_k L_{\hat{f}}{}^{N-1} \leq d_{\tilde{x}}. \tag{23}$$

So there exists $\tilde{x} \in \mathcal{B}(d_{\tilde{x}})$ such that

$$\hat{x}_{k+1}^{N-1} = \hat{x}_k^{N,*} + \tilde{x} \tag{24}$$

and

$$\begin{aligned}\hat{x}_{k+1}^{N} &= \hat{f}\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right) \\ &= \hat{f}\left(\hat{x}_{k+1}^{N-1}, h\left(\hat{x}_{k+1}^{N-1}\right)\right) \\ &= \hat{f}\left(\hat{x}_k^{N,*} + \tilde{x}, h\left(\hat{x}_k^{N,*} + \tilde{x}\right)\right).\end{aligned} \tag{25}$$

Because $\{\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\}_{i=0}^{N-1}$ is admissible at the kth computation cycle, $\hat{x}_k^{N,*} \in \mathcal{X}_{\mathcal{T}}$ holds and therefore

$$\hat{x}_{k+1}^{N-1} \in \mathcal{X}_{\mathcal{T}} + \mathcal{B}\left(d_{\tilde{x}}\right).$$

So by equation (5) in Assumption 3.2,

$$\hat{u}_{k+1}^{N-1} = h(\hat{x}_{k+1}^{N-1}) \in \mathcal{U}$$

holds. Meantime, by (4), $\hat{x}_{k+1}^N \in \mathcal{X}_{\mathcal{T}}$ holds. So $\{\hat{x}_{k+1}^i\}_{i=0}^N$ with $\{\hat{u}_{k+1}^i\}_{i=0}^{N-1}$ is admissible to the FHOCP at the $k+1$th computation cycle.

Let $J[\hat{\mathbf{u}}_{k+1}|x(t_{k+1})]$ be the cost of the FHOCP generated by $\hat{\mathbf{u}}_{k+1} = \left\{\hat{u}_{k+1}^i\right\}_{i=0}^{N-1}$ with the initial condition $x(t_{k+1})$.

$$\begin{aligned}&J[\hat{\mathbf{u}}_{k+1}|x(t_{k+1})] - V(x(t_k)) \\ &= \sum_{i=0}^{N-1} \kappa\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) + V_f\left(\hat{x}_{k+1}^N\right) - V(x(t_k)) \\ &= \sum_{i=0}^{N-2} \kappa\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) + \kappa\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right) + V_f\left(\hat{x}_{k+1}^N\right) \\ &\quad - V(x(t_k)) + V_f\left(\hat{x}_{k+1}^{N-1}\right) - V_f\left(\hat{x}_{k+1}^{N-1}\right) \\ &\quad + \kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) - \kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right)\end{aligned}$$

Re-arranging the terms at the right-hand side, we have

$$\begin{aligned}&J[\hat{\mathbf{u}}_{k+1}|x(t_{k+1})] - V(x(t_k)) \\ &= \underbrace{\kappa\left(\hat{x}_{k+1}^{N-1}, \hat{u}_{k+1}^{N-1}\right) + V_f(\hat{x}_{k+1}^N) - V_f(\hat{x}_{k+1}^{N-1})}_{\Psi} \\ &\quad - \kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) + \Phi\end{aligned} \tag{26}$$

where

$$\begin{aligned}\Phi = &\sum_{i=0}^{N-2} \kappa\left(\hat{x}_{k+1}^i, \hat{u}_{k+1}^i\right) + V_f\left(\hat{x}_{k+1}^{N-1}\right) + \kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) \\ &- V(x(t_k)).\end{aligned}$$

Notice that

$$\hat{x}_{k+1}^{N} = \hat{f}\left(\hat{x}_{k+1}^{N-1}, h(\hat{x}_{k+1}^{N-1})\right).$$

So, given

$$\hat{x}_{k+1}^{N-1} \in \mathcal{X}_{\mathcal{T}} + \mathcal{B}\left(d_{\tilde{x}}\right)$$

and inequality (6) in Assumption 3.2, we have $\Psi \leq 0$. Therefore, equation (26) implies

$$J[\hat{\mathbf{u}}_{k+1}|x(t_{k+1})] - V(x(t_k)) \leq \Phi - \kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right). \quad (27)$$

Consider Φ. Notice that the first term in Φ can be written as

$$\sum_{i=0}^{N-2} \kappa\left(\hat{x}_{k+1}^{i}, \hat{u}_{k+1}^{i}\right) = \sum_{i=1}^{N-1} \kappa\left(\hat{x}_{k+1}^{i-1}, \hat{u}_{k+1}^{i-1}\right).$$

According to equation (2),

$$V(x(t_k)) = V_f(\hat{x}_k^{N,*}) + \sum_{i=0}^{N-1} \kappa\left(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\right).$$

Therefore, using this equation to replace $V(x(t_k))$ in Φ yields

$$\begin{aligned}\Phi = &\sum_{i=1}^{N-1} \kappa\left(\hat{x}_{k+1}^{i-1}, \hat{u}_{k+1}^{i-1}\right) + V_f\left(\hat{x}_{k+1}^{N-1}\right) \\ &- \sum_{i=1}^{N-1} \kappa\left(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\right) - V_f\left(\hat{x}_k^{N,*}\right). \quad (28)\end{aligned}$$

By equation (14), $\hat{u}_{k+1}^{i-1} = \hat{u}_k^{i,*}$ for $i = 1, 2, \cdots, N-1$. So

$$\begin{aligned}\Phi \leq &\sum_{i=1}^{N-1} \left|\kappa\left(\hat{x}_{k+1}^{i-1}, \hat{u}_{k+1}^{i-1}\right) - \kappa\left(\hat{x}_k^{i,*}, \hat{u}_k^{i,*}\right)\right| \\ &+ \left|V_f\left(\hat{x}_{k+1}^{N-1}\right) - V_f\left(\hat{x}_k^{N,*}\right)\right| \\ \leq &\sum_{i=1}^{N-1} L_\kappa \left\|\hat{x}_{k+1}^{i-1} - \hat{x}_k^{i,*}\right\| + L_{V_f}\left\|\hat{x}_{k+1}^{N-1} - \hat{x}_k^{N,*}\right\|,\end{aligned}$$

where the last inequality comes from equation (11) and (12) in Assumption 3.4, given $x(t_k), x(t_{k+1}) \in \mathcal{B}\left(\alpha_1^{-1}(d_0)\right)$ and therefore $\hat{x}_{k+1}^{i-1}, \hat{x}_k^{i,*} \in \mathcal{X}_{N,\hat{f}}$ for $i = 1, 2\cdots, N$.

By Lemma 3.2,

$$\left\|\hat{x}_{k+1}^{i-1} - \hat{x}_k^{i,*}\right\| \leq \epsilon_k {L_{\hat{f}}}^{i-1}$$

for $i = 1, 2, \cdots, N$. Therefore,

$$\Phi \leq \epsilon_k \left(\sum_{i=1}^{N-1} L_\kappa {L_{\hat{f}}}^{i-1} + L_{V_f} {L_{\hat{f}}}^{N-1}\right) = \epsilon_k \theta.$$

With the inequality above and inequality (13) in Assumption 3.5, inequality (27) can be further simplified as

$$\begin{aligned}&J[\hat{\mathbf{u}}_{k+1}|x(t_{k+1})] - V(x(t_k)) \\ &\leq -\kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) + \epsilon_k\theta \\ &\leq -(1-\rho)\,\kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) + d.\end{aligned}$$

Therefore,

$$\begin{aligned}&V(x(t_{k+1})) - V(x(t_k)) \\ &= \min_{\hat{\mathbf{u}}} J[\hat{\mathbf{u}}|x(t_{k+1})] - V(x(t_k)) \\ &\leq -(1-\rho)\,\kappa\left(\hat{x}_k^{0,*}, \hat{u}_k^{0,*}\right) + d \\ &\leq -(1-\rho)\,\alpha_1\left(\|x(t_k)\|\right) + d,\end{aligned}$$

where the last inequality comes from Assumption 3.3 and the fact $\hat{x}_k^{0,*} = x(t_k)$. ■

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