SAV DECOUPLED ENSEMBLE ALGORITHMS FOR FAST COMPUTATION OF STOKES-DARCY FLOW ENSEMBLES

NAN JIANG* AND HUANHUAN YANG †

Abstract. Numerical modeling and simulation of complex systems is often subject to uncertainties in model parameters. Many popular uncertainty quantification (UQ) methods require repeated simulations of the underlying physical system with different samples of the uncertain model parameters. This poses great challenges to many practical engineering applications due to the high demand for computational resources. In this report we propose highly efficient ensemble simulation algorithms for fast computation of coupled flow ensembles. The proposed ensemble algorithms are based on two recently developed numerical approaches: scalar auxiliary variable (SAV) and ensemble timestepping. We introduce a new decoupling strategy using the SAV idea and incorporate the ensemble timestepping method to develop two decoupled ensemble schemes for the Stokes-Darcy system: SAV-BE-En and SAV-BDF2-En. The two ensemble algorithms are specially designed for UQ computations where a number of realizations of the underlying coupled PDE system are required for analyzing and interpreting flow statistics. Compared with traditional methods which solve for each realization independently, our proposed ensemble algorithms result in a common coefficient matrix for all realizations and efficient iterative solvers such as block CG or block GMRES can be used to solve for all realizations simultaneously reducing both computer storage and overall simulation time. We prove that both ensemble algorithms are long time stable *without* any time step conditions. We also provide a comprehensive error analysis for the fully discrete SAV-BE-En algorithm, and present a few illustrative numerical examples to demonstrate the efficiency and effectiveness of the algorithms.

Key words. Stokes-Darcy equations, ensemble algorithm, uncertainty quantification, scalar auxiliary variable

AMS subject classifications. 65C20, 65M12, 65M60, 76D07

1. Introduction.

The coupling of a free surface flow and a subsurface flow in porous media appears in many important geophysical and engineering applications. The inherent heterogeneity of the porous media and inaccurate measurement or lack of information of physical parameters lead to uncertainties in flow simulations. To quantify uncertainties and generate useful flow statistics, it is common to represent the flow parameter under consideration as a stochastic function and numerically approximate the corresponding stochastic partial differential equation (PDE) system. In ensemble-based uncertainty quantification (UQ) methods, such as the classical Monte Carlo and its variants [2, 42], stochastic collocation method [62] or the non-intrusive polynomial chaos method [54] , a number of samples of the flow parameter are first generated according to the specified probabilistic distribution and then the underlying PDE system is solved repeatedly for each sample. The main challenge in these UQ simulations is the excessive computational cost especially for complex flow problems. One way to reduce the computational cost is to reduce the sample size required to generate useful flow statistics, which has been the focus of the research direction of developing efficient UQ methods [1, 2, 24, 42, 54, 62]. Another research direction is to reduce the simulation cost of each realization by building cheap surrogate models to replace the original model [53]. Recently, Jiang and Layton proposed a different idea [31] to reduce the overall computational cost for ensemble simulations by developing an ensemble timestepping method and exploiting the structure of the corresponding linear systems. The proposed ensemble method makes use of a quantity called ensemble mean to construct linear systems sharing the same coefficient matrix for all realizations of the Navier-Stokes flows, and efficient direct and iterative solvers can be used to significantly reduce aggregate simulation cost for ensemble simulations. This ensemble method can also be combined with the aforementioned efficient UQ methods that reduce the size of parameter samples or use surrogate models leading to further reduced computational cost.

The ensemble method was originally developed for fast computation of Navier-Stokes flow ensembles corresponding to different initial conditions and/or body forces [31], and has been extended to other PDE models with different model parameters, such as Boussinesq equations [11, 13, 29], MHD equations [37, 51], heat equations [12, 49, 50], fluid-fluid interactions [8], Stokes-Darcy equations [22, 33, 35, 34, 36]. It has been extensively tested and shown to be able to significantly reduce the computational cost, see [17, 18,

*Department of Mathematics, University of Florida, Gainesville, FL 32611, jiangn@ufl.edu. This author was partially supported by the US National Science Foundation grants DMS-1720001 and DMS-2120413.

†Corresponding author. Department of Mathematics, Shantou University, Guangdong, China 515063, huan2yang@stu.edu.cn. This author was supported in part by the National Natural Science Foundation of China under grant 11801348, the key research projects of general universities in Guangdong Province (grant no. 2019KZDXM034), and the basic research and applied basic research projects in Guangdong Province (Projects of Guangdong, Hong Kong and Macao Center for Applied Mathematics, grant no. 2020B1515310018).

19, 20, 21, 27, 28, 29, 32, 30, 39, 45, 60, 61]. However, it suffers from a timestep condition that could be restrictive for some applications. For nonlinear flow problems, the timestep condition comes from bounding the nonlinear fluctuation term and can not be avoided without adding additional regularizations [32, 60], due to the way the nonlinear term is treated (linearly implicitly) in the ensemble timestepping scheme. This issue was very recently addressed for the Navier-Stokes equations [38] by adopting a scalar auxiliary variable (SAV) approach that discretizes the nonlinear term fully explicitly to avoid producing a nonlinear fluctuation term. For the linear Stokes-Darcy equations, the timestep condition usually comes from the decoupling of two subdomain flows [22, 35, 36]. In this report we propose to adopt the SAV idea to design unconditionally stable (no timestep condition) partitioned methods for simulating Stokes-Darcy coupled flow problems.

Partitioned methods [7, 40, 41, 44, 52, 56, 57] are growing popular for numerically solving the Stokes-Darcy equations as they decouple the coupling problem into two smaller subphysics problems that facilitate parallel computation of the two subdomain problems and savings in computer storage and CPU time. The main issue with partitioned methods is that the associated time step constraints can be severe for some applications, e.g., when the hydraulic conductivity tensor $\mathcal{K}$ has small eigenvalues [44]. The SAV approach was first studied in [58, 59] for gradient flows. It introduces a new scalar auxiliary variable that can be used to form a modified system of the original PDEs so that the nonlinear terms in the modified system can be canceled out in discrete schemes, leading to unconditionally stable methods for solving nonlinear systems [47, 48, 38]. Following this SAV idea, we find it is also possible to cancel out the coupling terms that usually lead to the time step constraints in a typical partitioned method . Herein we design and study unconditionally stable partitioned methods for decoupling the linear Stokes-Darcy equations and fast ensemble simulations.

Let D_f denote the surface fluid flow region and D_p the porous media flow region, where $D_f, D_p \subset R^d(d = 2,3)$ are both open, bounded domains. These two domains lie across an interface, I, from each other and $D_f \cap D_p = \emptyset, \bar{D}_f \cap \bar{D}_p = I$, see Figure 1.1. The linear Stokes-Darcy system [3, 6] that models the coupling

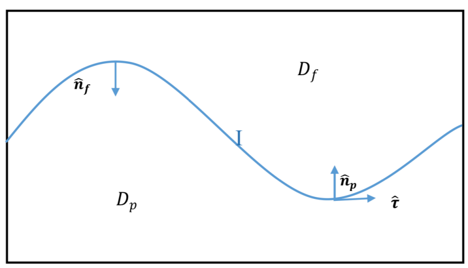


Fig. 1.1: A sketch of the porous median domain D_p, fluid domain D_f, and the interface I.

of the surface and porous media flows is: find fluid velocity $u(x,t)$, fluid pressure $p(x,t)$, and hydraulic head $\phi(x,t)$ that satisfy

$$\begin{gathered}\partial_t u - \nu\Delta u + \nabla p = f_f(x,t), \nabla\cdot u = 0 \text{ in } D_f\times(0,T],\\ S_0\partial_t\phi - \nabla\cdot(\mathcal{K}(x)\nabla\phi) = f_p(x,t) \text{ in } D_p\times(0,T],\\ \phi(x,0)=\phi^0(x) \text{ in } D_p \text{ and } u(x,0)=u^0(x) \text{ in } D_f,\\ \phi(x,t)=0 \text{ in } \partial D_p\backslash I\times(0,T] \text{ and } u(x,t)=0 \text{ in } \partial D_f\backslash I\times(0,T],\end{gathered} \tag{1.1}$$

where ν, $\mathcal{K}$, S_0, f_f, f_p, and T are the kinematic viscosity, the hydraulic conductivity tensor, specific mass storativity coefficient (positive), the external body force density, the sink/source term, and the final time, respectively. Let $\widehat{n}_{f/p}$ denote the outward unit normal vector on I associated with $D_{f/p}$, where $\widehat{n}_f = -\widehat{n}_p$. The coupling conditions across I are conservation of mass, balance of forces, and the Beavers-Joseph-Saffman condition on the tangential velocity [4, 55, 25]:

$$\begin{gathered}u\cdot\widehat{n}_f - \mathcal{K}\nabla\phi\cdot\widehat{n}_p = 0 \text{ and } p - \nu\,\widehat{n}_f\cdot\nabla u\cdot\widehat{n}_f = g\phi \text{ on } I\times(0,T],\\ -\nu\widehat{\tau}_i\cdot\nabla u\cdot\widehat{n}_f = \frac{\alpha_{\text{BJS}}}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}\widehat{\tau}_i}}u\cdot\widehat{\tau}_i \text{ on } I\times(0,T], \text{ for any tangential vector } \widehat{\tau}_i \text{ on } I.\end{gathered}$$

Here, g is the gravity constant, α_{BJS} is a dimensionless constant in the Beavers-Joseph-Saffman condition depending only on the structure of the porous medium. The conductivity $\mathcal{K}$ is assumed to be symmetric positive definite (SPD).

In this paper, we will develop two ensemble algorithms based on a scalar auxiliary variable (SAV) approach for computing an ensemble of multiple Stokes-Darcy systems to account for uncertainties in initial conditions, forcing terms, and the hydraulic conductivity tensor. Herein we consider computing an ensemble of J Stokes-Darcy systems corresponding to J different parameter sets $(u_j^0, \phi_j^0, f_{f,j}, f_{p,j}, \mathcal{K}_j)$, $j = 1, ..., J$,

$$\begin{aligned}
&\partial_t u_j - \nu \Delta u_j + \nabla p_j = f_{f,j}(x,t),\ \nabla \cdot u_j = 0 \ \text{ in } \ D_f \times (0,T],\\
&S_0 \partial_t \phi_j - \nabla \cdot (\mathcal{K}_j(x) \nabla \phi_j) = f_{p,j}(x,t) \ \text{ in } \ D_p \times (0,T],\\
&\phi_j(x,0) = \phi_j^0(x) \text{ in } D_p \text{ and } u_j(x,0) = u_j^0(x) \text{ in } D_f.
\end{aligned} \tag{1.2}$$

We have assumed there are uncertainties in initial conditions $u^0(x), \phi^0(x)$, source terms $f_f(x,t)$ and $f_p(x,t)$, and the hydraulic conductivity tensor $\mathcal{K}(x)$, then $(u_j^0, \phi_j^0, f_{f,j}, f_{p,j}, \mathcal{K}_j)$ is one of the samples drawn from the respective probabilistic distributions. For simplicity on notations in the analysis, we consider homogeneous Dirichlet boundary condition here, but the algorithms proposed in the report can be easily extended to the non-homogeneous case in the form

$$\phi_j(x,t) = b_j(x,t), \text{ in } \partial D_p \backslash I \times (0,T] \text{ and } u_j(x,t) = a_j(x,t), \text{ in } \partial D_f \backslash I \times (0,T].$$

We will next introduce the the scalar auxiliary variables and the differential equations (DEs) they satisfy. These new unknowns and the associated DEs will be added to the Stokes-Darcy system and form a new governing system for the coupled flows. The newly introduced unknowns and DEs make it possible to manipulate and cancel out the coupling terms that usually lead to the time step conditions associated with standard partitioned methods. Define the scalar auxiliary variables $r_j(t)$ by

$$r_j(t) = exp(-\frac{t}{T}). \tag{1.3}$$

Note that here the true solutions $r_j(t)$, $j = 1, 2, \cdots, J$ are all equal, but the approximate solutions r_j^n, $j = 1, 2, \cdots, J$, from the proposed numerical methods will be different. We also have

$$\frac{dr_j}{dt} = -\frac{1}{T} r_j + \frac{1}{exp(-\frac{t}{T})} \left(c_I(u_j, \phi_j) - c_I(u_j, \phi_j) \right), \tag{1.4}$$

where the interface term c_I, defined as

$$c_I(u, \phi) = g \int_I \phi u \cdot \widehat{n}_f \ ds,$$

is a coupling term that will appear in the weak formulation of the Stokes-Darcy system. The second term on the right hand side of (1.4) equals zero, but will be nonzero in the discrete schemes and plays an essential role in decoupling the computation of the free flow and the porous media flow. For example, in a standard partitioned method based on the backward Euler timestepping, one will see $c_I(u_j^n, \phi_j^{n+1}) - c_I(u_j^{n+1}, \phi_j^n)$ on the right hand side of the energy equation which can not be bounded by any positive terms on the left hand side of the energy equation without assuming a time step constraint. Now with the second term on the right hand side of (1.4), we can cancel out these problematic coupling terms and prove long time stability without any time step conditions.

We then present the following two SAV decoupled ensemble algorithms for fast computation of the Stokes-Darcy flow ensembles.

The SAV decoupled ensemble algorithm based on the Backward Euler timestepping (SAV-BE-En) reads

Algorithm 1.1 (SAV-BE-En). *Find* $(u_j^{n+1}, p_j^{n+1}, \phi_j^{n+1}) \in X_f \times Q_f \times X_p$ *and* r_j^{n+1} *satisfying for any* $(v, q, \psi) \in X_f \times Q_f \times X_p$,

$$\left(\frac{u_j^{n+1} - u_j^n}{\Delta t}, v \right)_f + \nu (\nabla u_j^{n+1}, \nabla v)_f + \sum_i \int_I \bar{\eta}_i (u_j^{n+1} \cdot \widehat{\tau}_i)(v \cdot \widehat{\tau}_i) \ ds - \left(p_j^{n+1}, \nabla \cdot v \right)_f \tag{1.5}$$

$$
+\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)(u_j^n\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds+\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(v,\phi_j^n)=(f_{f,j}^{n+1},v)_f,
$$

$$
(q,\nabla\cdot u_j^{n+1})_f=0, \tag{1.6}
$$

$$
gS_0\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t},\psi\right)_p+g(\bar{\mathcal{K}}\nabla\phi_j^{n+1},\nabla\psi)_p+g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla\phi_j^n,\nabla\psi)_p-\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(u_j^n,\psi) \tag{1.7}
$$

$$
=g(f_{p,j}^{n+1},\psi)_p,
$$

$$
\frac{r_j^{n+1}-r_j^n}{\Delta t}=-\frac{1}{T}r_j^{n+1}+\frac{1}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_j^{n+1},\phi_j^n)-c_I(u_j^n,\phi_j^{n+1})\right), \tag{1.8}
$$

where

$$
\bar{\mathcal{K}}=\frac{1}{J}\sum_{j=1}^{J}\mathcal{K}_j,\quad \eta_{i,j}=\frac{\alpha_{\mathrm{BJS}}}{\sqrt{\widehat{\tau}_i\cdot\mathcal{K}_j\widehat{\tau}_i}},\quad \textit{and}\quad \bar{\eta}_i=\frac{1}{J}\sum_{j=1}^{J}\eta_{i,j}.
$$

For the definition of function spaces X_f, Q_f, and X_p, see (2.2) in Section 2. We have denoted by $(\cdot,\cdot)_{f/p}$ the inner products in $L^2(D_{f/p})$.

The SAV decoupled ensemble algorithm based on the second order Backward Difference Formula (SAV-BDF2-En) reads

Algorithm 1.2 (SAV-BDF2-En). *Find* $(u_j^{n+1},p_j^{n+1},\phi_j^{n+1})\in X_f\times Q_f\times X_p$ *and* r_j^{n+1} *satisfying for any* $(v,q,\psi)\in X_f\times Q_f\times X_p$,

$$
\left(\frac{3u_j^{n+1}-4u_j^n+u_j^{n-1}}{2\Delta t},v\right)_f+\nu(\nabla u_j^{n+1},\nabla v)_f+\sum_i\int_I\bar{\eta}_i(u_j^{n+1}\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds-\left(p_j^{n+1},\nabla\cdot v\right)_f \tag{1.9}
$$

$$
+\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)((2u_j^n-u_j^{n-1})\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds+\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(v,2\phi_j^n-\phi_j^{n-1})=(f_{f,j}^{n+1},v)_f,
$$

$$
(q,\nabla\cdot u_j^{n+1})_f=0, \tag{1.10}
$$

$$
gS_0\left(\frac{3\phi_j^{n+1}-4\phi_j^n+\phi_j^{n-1}}{2\Delta t},\psi\right)_p+g(\bar{\mathcal{K}}\nabla\phi_j^{n+1},\nabla\psi)_p+g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(2\phi_j^n-\phi_j^{n-1}),\nabla\psi)_p \tag{1.11}
$$

$$
-\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(2u_j^n-u_j^{n-1},\psi)=g(f_{p,j}^{n+1},\psi)_p,
$$

$$
\frac{3r_j^{n+1}-4r_j^n+r_j^{n-1}}{2\Delta t}=-\frac{1}{T}r_j^{n+1}+\frac{1}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_j^{n+1},2\phi_j^n-\phi_j^{n-1})-c_I(2u_j^n-u_j^{n-1},\phi_j^{n+1})\right). \tag{1.12}
$$

The SAV-BE-En algorithm is first order convergent while SAV-BDF2-En is second order convergent. We will prove both algorithms are long time stable *without* any timestep conditions. The efficiency of the ensemble algorithms lies in the facts that (1) the coefficients of the the unknowns u_j^{n+1} and ϕ_j^{n+1} are independent of the ensemble index j so that all ensemble members share the same coefficient matrix, and (2) the SAV approach decouples the original problem into two smaller subphysics problems leading to smaller linear systems to be solved at each time step. The resulting linear systems of the proposed algorithms are in the form of

$$
A\left[\begin{array}{c|c|c|c} u_1^{n+1} & u_2^{n+1} & \cdots & u_J^{n+1}\\ p_1^{n+1} & p_2^{n+1} & \cdots & p_J^{n+1}\end{array}\right]=\left[a_1^{n+1}|a_2^{n+1}|\cdots|a_J^{n+1}\right], \tag{1.13}
$$

$$
B\left[\begin{array}{c|c|c|c}\phi_1^{n+1} & \phi_2^{n+1} & \cdots & \phi_J^{n+1}\end{array}\right]=\left[b_1^{n+1}|b_2^{n+1}|\cdots|b_J^{n+1}\right], \tag{1.14}
$$

instead of more expensive linear systems in the form of

$$
A_1\left[\begin{array}{c} u_1^{n+1}\\ p_1^{n+1}\end{array}\right]=a_1^{n+1},\quad A_2\left[\begin{array}{c} u_2^{n+1}\\ p_2^{n+1}\end{array}\right]=a_2^{n+1},\quad \cdots,\quad A_J\left[\begin{array}{c} u_J^{n+1}\\ p_J^{n+1}\end{array}\right]=a_J^{n+1}, \tag{1.15}
$$

$$B_1\phi_1^{n+1} = b_1^{n+1}, \quad B_2\phi_2^{n+1} = b_2^{n+1}, \quad \cdots, \quad B_J\phi_J^{n+1} = b_J^{n+1}, \tag{1.16}$$

or

$$C_1 \begin{bmatrix} u_1^{n+1} \\ p_1^{n+1} \\ \phi_1^{n+1} \end{bmatrix} = c_1^{n+1}, \quad C_2 \begin{bmatrix} u_2^{n+1} \\ p_2^{n+1} \\ \phi_2^{n+1} \end{bmatrix} = c_2^{n+1}, \quad \cdots, \quad C_J \begin{bmatrix} u_J^{n+1} \\ p_J^{n+1} \\ \phi_J^{n+1} \end{bmatrix} = c_J^{n+1}. \tag{1.17}$$

from using traditional methods. For linear systems in the form of (1.13) and (1.14), efficient iterative solvers as block CG [10] or block GMRES [14] can be used to significantly reduce the computational cost, compared with linear systems (1.17) obtained by a traditional nonensemble method or (1.15)-(1.16) from a nonensemble partitioned method. More details about the efficiency and implementation of the ensemble algorithms are discussed in Section 5.

The rest of the paper is organized as follows. Section 2 discusses basic notations and preliminaries. In Section 3 we prove the long time stability of both algorithms under two parameter conditions without any timestep constraints, and remark the unconditional stability while computing a single Stokes-Darcy system. In Section 4 we provide a detailed convergence analysis for the SAV-BE-En algorithm. Section 5 shows how to implement both algorithms and tests their efficiency with several numerical experiments. We conclude the paper in Section 6.

2. Notation and Preliminaries. We denote the $L^2(I)$ norm by $\|\cdot\|_I$ and the $L^2(D_{f/p})$ norms by $\|\cdot\|_{f/p}$. Further, we denote the $H^k(D_{f/p})$ norm by $\|\cdot\|_{H^k(D_{f/p})}$. The following inequalities will be used in the proofs, [44].

$$\|\phi\|_I \leq C(D_p)\sqrt{\|\phi\|_p\|\nabla\phi\|_p}, \qquad \|u\|_I \leq C(D_f)\sqrt{\|u\|_f\|\nabla u\|_f}, \tag{2.1}$$

where $C(D_{f/p}) = \mathcal{O}(\sqrt{L_{f/p}}), L_{f/p} = diameter(D_{f/p})$. Define the function spaces:

$$\begin{aligned} \text{Velocity: } & X_f := \{v \in \left(H^1(D_f)\right)^d : v = 0 \text{ on } \partial D_f \backslash I\}, \\ \text{Pressure: } & Q_f := L^2(D_f), \\ \text{Hydraulic Head: } & X_p := \{\psi \in H^1(D_p) : \psi = 0 \text{ on } \partial D_p \backslash I\}. \end{aligned} \tag{2.2}$$

To discretize the Stokes-Darcy problem in space by the finite element method, we choose conforming velocity, pressure, hydraulic head finite element spaces based on a Delaunay triangulation ($d = 2$) or tetrahedralization ($d = 3$) of the domain $D_{f/p}$ with maximum element diameter h:

$$X_f^h \subset X_f \ , \ Q_f^h \subset Q_f, \ X_p^h \subset X_p.$$

The continuity across the interface I between the finite element meshes in the two subdomains is not assumed. The finite element spaces (X_f^h, Q_f^h) are assumed to satisfy the usual discrete inf-sup $/LBB^h$ condition for stability of the discrete pressure, see [16] for more on this condition. Taylor-Hood elements, [16], are one such choice used in the numerical tests in Section 5. We will also consider the discretely divergence-free space:

$$V_f^h := \{v_h \in X_f^h : (q_h, \nabla\cdot v_h)_f = 0, \ \forall q_h \in Q_f^h\}.$$

The fully discrete SAV-BE-En approximation of (1.2) is:

ALGORITHM 2.1 (SAV-BE-En-h). *Find* $(u_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ *and* $r_{j,h}^{n+1}$ *satisfying for any* $(v_h, q_h, \psi_h) \in X_f^h \times Q_f^h \times X_p^h$,

$$\begin{aligned} &\left(\frac{u_{j,h}^{n+1} - u_{j,h}^n}{\Delta t}, v_h\right)_f + \nu(\nabla u_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds \\ &+ \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)(u_{j,h}^n\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds - \left(p_{j,h}^{n+1}, \nabla\cdot v_h\right)_f + \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(v_h, \phi_{j,h}^n) = (f_{f,j}^{n+1}, v_h)_f, \end{aligned} \tag{2.3}$$

$$(q_h, \nabla \cdot u_{j,h}^{n+1})_f = 0, \tag{2.4}$$

$$gS_0 \left(\frac{\phi_{j,h}^{n+1} - \phi_{j,h}^n}{\Delta t}, \psi_h \right)_p + g(\bar{\mathcal{K}} \nabla \phi_{j,h}^{n+1}, \nabla \psi_h)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \phi_{j,h}^n, \nabla \psi_h)_p - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(u_{j,h}^n, \psi_h) = g(f_{p,j}^{n+1}, \psi_h)_p, \tag{2.5}$$

$$\frac{r_{j,h}^{n+1} - r_{j,h}^n}{\Delta t} = -\frac{1}{T} r_{j,h}^{n+1} + \frac{1}{exp(-\frac{t^{n+1}}{T})} \left(c_I(u_{j,h}^{n+1}, \phi_{j,h}^n) - c_I(u_{j,h}^n, \phi_{j,h}^{n+1}) \right). \tag{2.6}$$

The fully discrete SAV-BDF2-En approximation of (1.2) is:

ALGORITHM 2.2 (SAV-BDF2-En-h). *Find* $(u_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_f^h \times Q_f^h \times X_p^h$ *and* $r_{j,h}^{n+1}$ *satisfying for any* $(v_h, q_h, \psi_h) \in X_f^h \times Q_f^h \times X_p^h$,

$$\left(\frac{3u_{j,h}^{n+1} - 4u_{j,h}^n + u_{j,h}^{n-1}}{2\Delta t}, v_h \right)_f + \nu(\nabla u_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i (u_{j,h}^{n+1} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i) \, ds \tag{2.7}$$

$$+ \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)((2u_{j,h}^n - u_{j,h}^{n-1}) \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i) \, ds - \left(p_{j,h}^{n+1}, \nabla \cdot v_h \right)_f + \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(v_h, 2\phi_{j,h}^n - \phi_{j,h}^{n-1})$$

$$= (f_{f,j}^{n+1}, v_h)_f,$$

$$(q_h, \nabla \cdot u_{j,h}^{n+1})_f = 0, \tag{2.8}$$

$$gS_0 \left(\frac{3\phi_{j,h}^{n+1} - 4\phi_{j,h}^n + \phi_{j,h}^{n-1}}{2\Delta t}, \psi_h \right)_p + g(\bar{\mathcal{K}} \nabla \phi_{j,h}^{n+1}, \nabla \psi_h)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla (2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \nabla \psi_h)_p \tag{2.9}$$

$$- \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \psi_h) = g(f_{p,j}^{n+1}, \psi_h)_p,$$

$$\frac{3r_{j,h}^{n+1} - 4r_{j,h}^n + r_{j,h}^{n-1}}{2\Delta t} = -\frac{1}{T} r_{j,h}^{n+1} + \frac{1}{exp(-\frac{t^{n+1}}{T})} \left(c_I(u_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \phi_{j,h}^{n+1}) \right). \tag{2.10}$$

3. Stability Analysis.

In this section we prove the proposed SAV decoupled ensemble algorithms are long time stable without any timestep conditions. Let $|\cdot|_2$ denote the 2-norm of either vectors or matrices, $k_{j,min}(x)$, $\bar{k}_{min}(x)$ be the minimum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, $\bar{\mathcal{K}}(x)$ respectively, and $\rho_j'(x)$ be the spectral radius of the fluctuation of hydraulic conductivity tensor $\mathcal{K}_j(x) - \bar{\mathcal{K}}(x)$. Since both $\mathcal{K}_j(x)$ and $\bar{\mathcal{K}}(x)$ are symmetric, $|\mathcal{K}_j(x) - \bar{\mathcal{K}}(x)|_2 = \rho_j'(x)$. We then define the following quantities that will be used in our proof.

$$\eta_{i,j}'^{max} = \max_{x \in I} |\eta_{i,j}(x) - \bar{\eta}_i(x)|, \qquad \eta_i'^{max} = \max_j \eta_{i,j}'^{max}, \qquad \bar{\eta}_i^{min} = \min_{x \in I} \bar{\eta}_i(x), \qquad \bar{\eta}_i^{max} = \max_{x \in I} \bar{\eta}_i(x),$$

$$k_{j,min} = \min_{x \in D_p} k_{j,min}(x), \qquad k_{min} = \min_j k_{j,min}, \qquad \bar{k}_{min} = \min_{x \in D_p} \bar{k}_{min}(x),$$

$$\rho_{j,max}' = \max_{x \in D_p} \rho_j'(x), \qquad \rho_{max}' = \max_j \rho_{j,max}'.$$

3.1. Long time stability of (SAV-BE-En-h).

We prove long time stability of Algorithm 2.1 under the following two parameter conditions, *without* any timestep conditions:

$$\eta_i'^{max} \le \bar{\eta}_i^{min}, \qquad \rho_{max}' < \bar{k}_{min}. \tag{3.1}$$

THEOREM 3.1 (Long time stability of Algorithm 2.1). *If the two parameter conditions in (3.1) hold, then Algorithm 2.1 is long time stable: for any* $N \ge 1$,

$$\frac{1}{2} \|u_{j,h}^N\|_f^2 + \frac{gS_0}{2} \|\phi_{j,h}^N\|_p^2 + \Delta t \sum_i \frac{\bar{\eta}_i^{min}}{2} \int_I (u_{j,h}^N \cdot \widehat{\tau}_i)^2 \, ds + \Delta t \frac{g\rho_{max}'}{2} \|\nabla \phi_{j,h}^N\|_p^2 \tag{3.2}$$

$$+ \frac{1}{2}|r_{j,h}^N|^2 + \frac{1}{2}\sum_{n=0}^{N-1}|r_{j,h}^{n+1} - r_{j,h}^n|^2 + \frac{\Delta t}{T}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}|^2$$
$$\leq \frac{1}{2}\|u_{j,h}^0\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^0\|_p^2 + \Delta t \sum_i \frac{\bar{\eta}_i^{min}}{2}\int_I (u_{j,h}^0 \cdot \widehat{\tau}_i)^2\, ds + \Delta t \frac{g\rho'_{max}}{2}\|\nabla \phi_{j,h}^0\|_p^2$$
$$+ \frac{1}{2}|r_{j,h}^0|^2 + \Delta t \sum_{n=0}^{N-1}\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t \sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{2(\bar{k}_{min} - \rho'_{max})}\|f_{p,j}^{n+1}\|_p^2.$$

Proof. Setting $v_h = u_{j,h}^{n+1}$, $q_h = p_{j,h}^{n+1}$, $\psi_h = \phi_{j,h}^{n+1}$ in Algorithm 2.1, multiplying (2.6) by $r_{j,h}^{n+1}$ and adding all four equations yields

$$\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^n\|_f^2 + \frac{1}{2\Delta t}\|u_{j,h}^{n+1} - u_{j,h}^n\|_f^2 + \nu\|\nabla u_{j,h}^{n+1}\|_f^2 + \sum_i \int_I \bar{\eta}_i (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds \tag{3.3}$$
$$+ \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^n\|_p^2 + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1} - \phi_{j,h}^n\|_p^2 + g(\bar{\mathcal{K}}\nabla\phi_{j,h}^{n+1}, \nabla\phi_{j,h}^{n+1})_p$$
$$+ \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(u_{j,h}^{n+1}, \phi_{j,h}^n) - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(u_{j,h}^n, \phi_{j,h}^{n+1})$$
$$+ \frac{1}{2\Delta t}|r_{j,h}^{n+1}|^2 - \frac{1}{2\Delta t}|r_{j,h}^n|^2 + \frac{1}{2\Delta t}|r_{j,h}^{n+1} - r_{j,h}^n|^2 + \frac{1}{T}|r_{j,h}^{n+1}|^2 - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\Big(c_I(u_{j,h}^{n+1}, \phi_{j,h}^n) - c_I(u_{j,h}^n, \phi_{j,h}^{n+1})\Big)$$
$$= (f_{f,j}^{n+1}, u_{j,h}^{n+1})_f + g(f_{p,j}^{n+1}, \phi_{j,h}^{n+1})_p - \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)(u_{j,h}^n\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\phi_{j,h}^n, \nabla\phi_{j,h}^{n+1})_p.$$

Applying Cauchy-Schwarz and Young's inequalities to the source terms, for any $\beta > 0$ we have

$$\begin{aligned}(f_{f,j}^{n+1}, u_{j,h}^{n+1})_f + g(f_{p,j}^{n+1}, \phi_{j,h}^{n+1})_p &\leq \|f_{f,j}^{n+1}\|_f\|u_{j,h}^{n+1}\|_f + g\|f_{p,j}^{n+1}\|_p\|\phi_{j,h}^{n+1}\|_p \\ &\leq C_{P,f}\|f_{f,j}^{n+1}\|_f\|\nabla u_{j,h}^{n+1}\|_f + gC_{P,p}\|f_{p,j}^{n+1}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p \\ &\leq \frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{1}{2}\nu\|\nabla u_{j,h}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta\bar{k}_{min}}\|f_{p,j}^{n+1}\|_p^2 + \beta g\bar{k}_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2.\end{aligned} \tag{3.4}$$

The other two terms on the right hand side of (3.3) can be bounded as follows.

$$-\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)(u_{j,h}^n\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\, ds \leq \sum_i \int_I |\eta_{i,j} - \bar{\eta}_i|\left|(u_{j,h}^n\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\right| ds \tag{3.5}$$
$$\leq \sum_i \eta_{i,j}^{\prime max}\int_I \left|(u_{j,h}^n\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\right|\, ds, \leq \sum_i\left[\frac{\eta_i^{\prime max}}{2}\int_I (u_{j,h}^n\cdot\widehat{\tau}_i)^2\, ds + \frac{\eta_i^{\prime max}}{2}\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\, ds\right],$$

and

$$-g\left((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\phi_{j,h}^n, \nabla\phi_{j,h}^{n+1}\right)_p \leq g\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\mathcal{K}_j - \bar{\mathcal{K}}|_2|\nabla\phi_{j,h}^n|_2\, dx \tag{3.6}$$
$$\leq g\int_{D_p}\rho'_j(x)|\nabla\phi_{j,h}^{n+1}|_2|\nabla\phi_{j,h}^n|_2\, dx \leq g\rho'_{j,max}\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\nabla\phi_{j,h}^n|_2\, dx$$
$$\leq g\rho'_{j,max}\|\nabla\phi_{j,h}^n\|_p\|\nabla\phi_{j,h}^{n+1}\|_p \leq \frac{g\rho'_{max}}{2}\|\nabla\phi_{j,h}^n\|_p^2 + \frac{g\rho'_{max}}{2}\|\nabla\phi_{j,h}^{n+1}\|_p^2.$$

Using above estimates, equation (3.3) becomes

$$\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^n\|_f^2 + \frac{1}{2}\nu\|\nabla u_{j,h}^{n+1}\|_f^2 + \sum_i\left[\frac{\bar{\eta}_i^{min}}{2} - \frac{\eta_i^{\prime max}}{2}\right]\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\, ds \tag{3.7}$$

$$+\sum_i \frac{\bar\eta_i^{min}}{2}\left[\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds - \int_I (u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right] + \sum_i \left[\frac{\bar\eta_i^{min}}{2} - \frac{\eta_i'^{max}}{2}\right]\int_I (u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds$$
$$+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2 + (1-\beta-\frac{\rho'_{max}}{\bar k_{min}})g\bar k_{min}\|\nabla\phi_{j,h}^{n+1}\|_p^2 + \frac{g\rho'_{max}}{2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2 - \|\nabla\phi_{j,h}^{n}\|_p^2\right)$$
$$+\frac{1}{2\Delta t}|r_{j,h}^{n+1}|^2 - \frac{1}{2\Delta t}|r_{j,h}^{n}|^2 + \frac{1}{2\Delta t}|r_{j,h}^{n+1}-r_{j,h}^{n}|^2 + \frac{1}{T}|r_{j,h}^{n+1}|^2 \le \frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{4\beta\bar k_{min}}\|f_{p,j}^{n+1}\|_p^2.$$

To obtain stability, we need

$$\frac{\bar\eta_i^{min}}{2} - \frac{\eta_i'^{max}}{2} \ge 0, \qquad 1-\beta-\frac{\rho'_{max}}{\bar k_{min}} \ge 0. \tag{3.8}$$

Recall that $\beta, \eta_i'^{max}, \rho'_{max}$ are all positive, we then have the following constraints on these parameters.

$$0<\beta<1, \qquad \frac{\rho'_{max}}{\bar k_{min}} < 1, \quad \eta_i'^{max} \le \bar\eta_i^{min}. \tag{3.9}$$

(3.9) leads to the two parameter conditions in (3.1) required for stability. Now if the two parameter conditions in (3.1) both hold, and taking $\beta = \frac{1}{2}(1-\frac{\rho'_{max}}{\bar k_{min}})$ (3.7) reduces to

$$\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2 + \sum_i \frac{\bar\eta_i^{min}}{2}\left[\int_I (u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds - \int_I (u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right] + \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2$$
$$+\frac{g\rho'_{max}}{2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2 - \|\nabla\phi_{j,h}^{n}\|_p^2\right) + \frac{1}{2\Delta t}|r_{j,h}^{n+1}|^2 - \frac{1}{2\Delta t}|r_{j,h}^{n}|^2 + \frac{1}{2\Delta t}|r_{j,h}^{n+1}-r_{j,h}^{n}|^2 + \frac{1}{T}|r_{j,h}^{n+1}|^2$$
$$\le \frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2 + \frac{gC_{P,p}^2}{2(\bar k_{min}-\rho'_{max})}\|f_{p,j}^{n+1}\|_p^2. \tag{3.10}$$

Sum (3.10) from $n=0$ to $N-1$ and multiply through by Δt to get

$$\frac{1}{2}\|u_{j,h}^{N}\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^{N}\|_p^2 + \Delta t\sum_i \frac{\bar\eta_i^{min}}{2}\int_I (u_{j,h}^{N}\cdot\widehat{\tau}_i)^2\ ds + \Delta t\frac{g\rho'_{max}}{2}\|\nabla\phi_{j,h}^{N}\|_p^2 \tag{3.11}$$
$$+\frac{1}{2}|r_{j,h}^{N}|^2 + \frac{1}{2}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}-r_{j,h}^{n}|^2 + \frac{\Delta t}{T}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}|^2$$
$$\le \frac{1}{2}\|u_{j,h}^{0}\|_f^2 + \frac{gS_0}{2}\|\phi_{j,h}^{0}\|_p^2 + \Delta t\sum_i \frac{\bar\eta_i^{min}}{2}\int_I (u_{j,h}^{0}\cdot\widehat{\tau}_i)^2\ ds + \Delta t\frac{g\rho'_{max}}{2}\|\nabla\phi_{j,h}^{0}\|_p^2$$
$$+\frac{1}{2}|r_{j,h}^{0}|^2 + \Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{2(\bar k_{min}-\rho'_{max})}\|f_{p,j}^{n+1}\|_p^2.$$

□

3.1.1. An alternative approach. Let $k_{j,max}(x)$ be the maximum eigenvalue of the hydraulic conductivity tensor $\mathcal{K}_j(x)$, and we define

$$\eta_{i,j}^{max} = \max_{x\in I}\eta_{i,j}(x), \qquad \eta_i^{max} = \max_j \eta_{i,j}^{max}, \qquad k_{j,max} = \max_{x\in D_p} k_{j,max}(x), \qquad k_{max} = \max_j k_{j,max}.$$

If it is easy to identify the minimum and maximum eigenvalues of the hydraulic conductivity tensor $\mathcal{K}_j(x)$ (e.g., $\mathcal{K}_j(x)$ is a diagonal matrix function), then the following algorithm can be used, which removes parameter conditions for stability, resulting in an unconditionally stable scheme.

Algorithm 3.2. *Find* $(u_{j,h}^{n+1}, p_{j,h}^{n+1}, \phi_{j,h}^{n+1}) \in X_f^h\times Q_f^h\times X_p^h$ *and* $r_{j,h}^{n+1}$ *satisfying* $\forall\ (v_h,q_h,\psi_h)\in X_f^h\times Q_f^h\times X_p^h$,

$$\left(\frac{u_{j,h}^{n+1}-u_{j,h}^{n}}{\Delta t}, v_h\right)_f + \nu(\nabla u_{j,h}^{n+1},\nabla v)_f + \sum_i\int_I \eta_i^{max}(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds \tag{3.12}$$

$$+\sum_i \int_I (\eta_{i,j}-\eta_i^{max})(u_{j,h}^n\cdot\widehat{\tau}_i)(v\cdot\widehat{\tau}_i)\ ds-\left(p_{j,h}^{n+1},\nabla\cdot v_h\right)_f+\frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi_{j,h}^n)=(f_{f,j}^{n+1},v_h)_f,$$

$$(q_h,\nabla\cdot u_{j,h}^{n+1})_f=0, \tag{3.13}$$

$$gS_0\left(\frac{\phi_{j,h}^{n+1}-\phi_{j,h}^n}{\Delta t},\psi_h\right)_p+k_{max}g(\nabla\phi_{j,h}^{n+1},\nabla\psi)_p+g((\mathcal{K}_j-k_{max}\mathcal{I})\nabla\phi_{j,h}^n,\nabla\psi)_p$$

$$-\frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(u_{j,h}^n,\psi_h)=g(f_{p,j}^{n+1},\psi_h)_p, \tag{3.14}$$

$$\frac{r_{j,h}^{n+1}-r_{j,h}^n}{\Delta t}=-\frac{1}{T}r_{j,h}^{n+1}+\frac{1}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_{j,h}^{n+1},\phi_{j,h}^n)-c_I(u_{j,h}^n,\phi_{j,h}^{n+1}\right). \tag{3.15}$$

For this approach, since $\mathcal{K}_j(x)$ and $k_{max}\mathcal{I}$ are both symmetric, we have $|\mathcal{K}_j(x)-k_{max}\mathcal{I}|_2\leq k_{max}-k_{min}$. We then prove the unconditionally long time stability of Algorithm 3.2 without any parameter conditions or timestep conditions.

Theorem 3.3 (Unconditional long time stability of Algorithm 3.2). *Algorithm 3.2 is unconditionally long time stable: for any $N\geq 1$,*

$$\frac{1}{2}\|u_{j,h}^N\|_f^2+\frac{gS_0}{2}\|\phi_{j,h}^N\|_p^2+\Delta t\sum_i\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^N\cdot\widehat{\tau}_i)^2\ ds+\Delta t\frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^N\|_p^2 \tag{3.16}$$

$$+\frac{1}{2}|r_{j,h}^N|^2+\frac{1}{2}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}-r_{j,h}^n|^2+\frac{\Delta t}{T}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}|^2$$

$$\leq\frac{1}{2}\|u_{j,h}^0\|_f^2+\frac{gS_0}{2}\|\phi_{j,h}^0\|_p^2+\Delta t\sum_i\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^0\cdot\widehat{\tau}_i)^2\ ds$$

$$+\Delta t\frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^0\|_p^2+\frac{1}{2}|r_{j,h}^0|^2+\Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2+\Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{2k_{min}}\|f_{p,j}^{n+1}\|_p^2.$$

Proof. See Appendix A. ☐

3.2. Long time stability of (SAV-BDF2-En-h). We prove long time stability of Algorithm 2.2 under two parameter conditions, *without* any timestep conditions:

$$\eta_i'^{max}\leq\frac{\bar{\eta}_i^{min}}{3},\qquad\rho'_{max}<\frac{\bar{k}_{min}}{3}. \tag{3.17}$$

Theorem 3.4 (Long time stability of Algorithm 2.2). *If the two parameter conditions in (3.17) hold, then Algorithm 2.2 is long time stable: for any $N\geq 2$,*

$$\|u_{j,h}^N\|_f^2+\|2u_{j,h}^N+u_{j,h}^{N-1}\|_f^2+\sum_{n=1}^{N-1}\|u_{j,h}^{n+1}-2u_{j,h}^n+u_{j,h}^{n-1}\|_f^2+\Delta t\sum_{n=1}^{N-1}2\nu\|\nabla u_{j,h}^{n+1}\|_f^2 \tag{3.18}$$

$$+\Delta t\sum_i 2\bar{\eta}_i^{min}\int_I(u_{j,h}^N\cdot\widehat{\tau}_i)^2\ ds+\Delta t\sum_i\frac{2\bar{\eta}_i^{min}}{3}\int_I(u_{j,h}^{N-1}\cdot\widehat{\tau}_i)^2\ ds+gS_0\|\phi_{j,h}^N\|_p^2+gS_0\|2\phi_{j,h}^N+\phi_{j,h}^{N-1}\|_p^2$$

$$+gS_0\sum_{n=1}^{N-1}\|\phi_{j,h}^{n+1}-2\phi_{j,h}^n+\phi_{j,h}^{n-1}\|_p^2+6g\rho'_{max}\Delta t\|\nabla\phi_{j,h}^N\|_p^2+2g\rho'_{max}\Delta t\|\nabla\phi_{j,h}^{N-1}\|_p^2+|r_{j,h}^N|^2$$

$$+|2r_{j,h}^N+r_{j,h}^{N-1}|^2+\sum_{n=1}^{N-1}|r_{j,h}^{n+1}-2r_{j,h}^n+r_{j,h}^{n-1}|^2+\frac{4\Delta t}{T}\sum_{n=1}^{N-1}|r_{j,h}^{n+1}|^2$$

$$\leq\|u_{j,h}^1\|_f^2+\|2u_{j,h}^1+u_{j,h}^0\|_f^2+\Delta t\sum_i 2\bar{\eta}_i^{min}\int_I(u_{j,h}^1\cdot\widehat{\tau}_i)^2\ ds+\Delta t\sum_i\frac{2\bar{\eta}_i^{min}}{3}\int_I(u_{j,h}^0\cdot\widehat{\tau}_i)^2\ ds$$

$$+ gS_0\|\phi^1_{j,h}\|^2_p + gS_0\|2\phi^1_{j,h} + \phi^0_{j,h}\|^2_p + 6g\rho'_{max}\Delta t\|\nabla\phi^1_{j,h}\|^2_p + 2g\rho'_{max}\Delta t\|\nabla\phi^0_{j,h}\|^2_p$$
$$+ |r^1_{j,h}|^2 + |2r^1_{j,h} + r^0_{j,h}|^2 + \Delta t\sum_{n=1}^{N-1}\frac{2C^2_{P,f}}{\nu}\|f^{n+1}_{f,j}\|^2_f + \Delta t\sum_{n=1}^{N-1}\frac{2gC^2_{P,p}}{\bar{k}_{min} - 3\rho'_{max}}\|f^{n+1}_{p,j}\|^2_p.$$

Proof. Setting $v_h = u^{n+1}_{j,h}$, $q_h = p^{n+1}_{j,h}$, $\psi_h = \phi^{n+1}_{j,h}$ in Algorithm 2.2, multiplying (2.10) by $r^{n+1}_{j,h}$ and adding all four equations yields

$$\begin{aligned}
&\frac{1}{4\Delta t}\|u^{n+1}_{j,h}\|^2_f + \frac{1}{4\Delta t}\|2u^{n+1}_{j,h} + u^n_{j,h}\|^2_f - \frac{1}{4\Delta t}\|u^n_{j,h}\|^2_f - \frac{1}{4\Delta t}\|2u^n_{j,h} + u^{n-1}_{j,h}\|^2_f + \frac{1}{4\Delta t}\|u^{n+1}_{j,h} - 2u^n_{j,h} + u^{n-1}_{j,h}\|^2_f \\
&+ \nu\|\nabla u^{n+1}_{j,h}\|^2_f + \sum_i\int_I \bar{\eta}_i(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds + \frac{gS_0}{4\Delta t}\|\phi^{n+1}_{j,h}\|^2_p + \frac{gS_0}{4\Delta t}\|2\phi^{n+1}_{j,h} + \phi^n_{j,h}\|^2_p - \frac{gS_0}{4\Delta t}\|\phi^n_{j,h}\|^2_p \\
&- \frac{gS_0}{4\Delta t}\|2\phi^n_{j,h} + \phi^{n-1}_{j,h}\|^2_p + \frac{gS_0}{4\Delta t}\|\phi^{n+1}_{j,h} - 2\phi^n_{j,h} + \phi^{n-1}_{j,h}\|^2_p + g(\bar{\mathcal{K}}\nabla\phi^{n+1}_{j,h}, \nabla\phi^{n+1}_{j,h})_p \\
&+ \frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(u^{n+1}_{j,h}, 2\phi^n_{j,h} - \phi^{n-1}_{j,h}) - \frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(2u^n_{j,h} - u^{n-1}_{j,h}, \phi^{n+1}_{j,h}) + \frac{1}{4\Delta t}|r^{n+1}_{j,h}|^2 \\
&+ \frac{1}{4\Delta t}|2r^{n+1}_{j,h} + r^n_{j,h}|^2 - \frac{1}{4\Delta t}|r^n_{j,h}|^2 - \frac{1}{4\Delta t}|2r^n_{j,h} + r^{n-1}_{j,h}|^2 + \frac{1}{4\Delta t}|r^{n+1}_{j,h} - 2r^n_{j,h} + r^{n-1}_{j,h}|^2 + \frac{1}{T}|r^{n+1}_{j,h}|^2 \\
&- \frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}\Big(c_I(u^{n+1}_{j,h}, 2\phi^n_{j,h} - \phi^{n-1}_{j,h}) - c_I(2u^n_{j,h} - u^{n-1}_{j,h}, \phi^{n+1}_{j,h})\Big) \qquad (3.19) \\
&= (f^{n+1}_{f,j}, u^{n+1}_{j,h})_f + g(f^{n+1}_{p,j}, \phi^{n+1}_{j,h})_p - \sum_i\int_I(\eta_{i,j} - \bar{\eta}_i)((2u^n_{j,h} - u^{n-1}_{j,h})\cdot\widehat{\tau}_i)(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \\
&\quad - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi^n_{j,h} - \phi^{n-1}_{j,h}), \nabla\phi^{n+1}_{j,h})_p.
\end{aligned}$$

Applying Cauchy-Schwarz and Young's inequalities to the source terms, for any $\beta > 0$ we have

$$\begin{aligned}
(f^{n+1}_{f,j}, u^{n+1}_{j,h})_f + g(f^{n+1}_{p,j}, \phi^{n+1}_{j,h})_p &\le \|f^{n+1}_{f,j}\|_f\|u^{n+1}_{j,h}\|_f + g\|f^{n+1}_{p,j}\|_p\|\phi^{n+1}_{j,h}\|_p \qquad (3.20)\\
&\le C_{P,f}\|f^{n+1}_{f,j}\|_f\|\nabla u^{n+1}_{j,h}\|_f + gC_{P,p}\|f^{n+1}_{p,j}\|_p\|\nabla\phi^{n+1}_{j,h}\|_p \\
&\le \frac{C^2_{P,f}}{2\nu}\|f^{n+1}_{f,j}\|^2_f + \frac{1}{2}\nu\|\nabla u^{n+1}_{j,h}\|^2_f + \frac{gC^2_{P,p}}{4\beta\bar{k}_{min}}\|f^{n+1}_{p,j}\|^2_p + \beta g\bar{k}_{min}\|\nabla\phi^{n+1}_{j,h}\|^2_p.
\end{aligned}$$

The other two terms on the right hand side of (3.19) can be bounded as follows. Using the inequality $(2a-b)^2 \le 6a^2 + 3b^2$, for any $\epsilon > 0$,

$$\begin{aligned}
-\sum_i\int_I(\eta_{i,j} - \bar{\eta}_i)((2u^n_{j,h} - u^{n-1}_{j,h})\cdot\widehat{\tau}_i)(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds &\le \sum_i\int_I|\eta_{i,j} - \bar{\eta}_i|\left|((2u^n_{j,h} - u^{n-1}_{j,h})\cdot\widehat{\tau}_i)(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)\right|ds \\
&\le \sum_i\eta'^{max}_{i,j}\int_I\left|((2u^n_{j,h} - u^{n-1}_{j,h})\cdot\widehat{\tau}_i)(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)\right|\,ds \qquad (3.21)\\
&\le \sum_i\left[\frac{\eta'^{max}_i}{2\epsilon}\int_I((2u^n_{j,h} - u^{n-1}_{j,h})\cdot\widehat{\tau}_i)^2\,ds + \frac{\epsilon\eta'^{max}_i}{2}\int_I(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right] \\
&\le \sum_i\left[\frac{3}{\epsilon}\eta'^{max}_i\int_I(u^n_{j,h}\cdot\widehat{\tau}_i)^2\,ds + \frac{3\eta'^{max}_i}{2\epsilon}\int_I(u^{n-1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds + \frac{\epsilon\eta'^{max}_i}{2}\int_I(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right]
\end{aligned}$$

and

$$-g\left((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi^n_{j,h} - \phi^{n-1}_{j,h}), \nabla\phi^{n+1}_{j,h}\right)_p \le g\int_{D_p}|\nabla\phi^{n+1}_{j,h}|_2|\mathcal{K}_j - \bar{\mathcal{K}}|_2|\nabla(2\phi^n_{j,h} - \phi^{n-1}_{j,h})|_2\,dx \qquad (3.22)$$
$$\le g\int_{D_p}\rho'_j(x)|\nabla\phi^{n+1}_{j,h}|_2|\nabla(2\phi^n_{j,h} - \phi^{n-1}_{j,h})|_2\,dx \le g\rho'_{j,max}\int_{D_p}|\nabla\phi^{n+1}_{j,h}|_2|\nabla(2\phi^n_{j,h} - \phi^{n-1}_{j,h})|_2\,dx$$

$$\leq g\rho'_{j,max}\|\nabla(2\phi^n_{j,h}-\phi^{n-1}_{j,h})\|_p\|\nabla\phi^{n+1}_{j,h}\|_p \leq \frac{3}{\epsilon}g\rho'_{max}\|\nabla\phi^n_{j,h}\|^2_p + \frac{3g\rho'_{max}}{2\epsilon}\|\nabla\phi^{n-1}_{j,h}\|^2_p + \frac{\epsilon g\rho'_{max}}{2}\|\nabla\phi^{n+1}_{j,h}\|^2_p.$$

Since all terms in (3.21) need to be bounded by $\sum_i \bar{\eta}_i^{min}\int_I(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds$, we need to minimize $(\frac{3}{\epsilon}+\frac{3}{2\epsilon}+\frac{\epsilon}{2})$ to make the time step condition sharp. This term achieves its minimum 3 when $\epsilon=3$. Similarly, we need to take $\epsilon=3$ in (3.22). Then (3.21) and (3.22) become

$$\begin{aligned} &-\sum_i\int_I(\eta_{i,j}-\bar{\eta}_i)((2u^n_{j,h}-u^{n-1}_{j,h})\cdot\widehat{\tau}_i)(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)\,ds \\ &\qquad\leq \sum_i\left[\eta_i'^{max}\int_I(u^n_{j,h}\cdot\widehat{\tau}_i)^2\,ds + \frac{\eta_i'^{max}}{2}\int_I(u^{n-1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds + \frac{3\eta_i'^{max}}{2}\int_I(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right] \end{aligned} \tag{3.23}$$

and

$$-g\left((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(2\phi^n_{j,h}-\phi^{n-1}_{j,h}),\nabla\phi^{n+1}_{j,h}\right)_p \leq g\rho'_{max}\|\nabla\phi^n_{j,h}\|^2_p + \frac{g\rho'_{max}}{2}\|\nabla\phi^{n-1}_{j,h}\|^2_p + \frac{3g\rho'_{max}}{2}\|\nabla\phi^{n+1}_{j,h}\|^2_p.$$

Using above estimates, equation (3.19) becomes

$$\begin{aligned} &\frac{1}{4\Delta t}\|u^{n+1}_{j,h}\|^2_f + \frac{1}{4\Delta t}\|2u^{n+1}_{j,h}+u^n_{j,h}\|^2_f - \frac{1}{4\Delta t}\|u^n_{j,h}\|^2_f - \frac{1}{4\Delta t}\|2u^n_{j,h}+u^{n-1}_{j,h}\|^2_f \\ &+\frac{1}{4\Delta t}\|u^{n+1}_{j,h}-2u^n_{j,h}+u^{n-1}_{j,h}\|^2_f + \frac{1}{2}\nu\|\nabla u^{n+1}_{j,h}\|^2_f + \sum_i\left[\frac{\bar{\eta}_i^{min}}{2}-\frac{3\eta_i'^{max}}{2}\right]\int_I(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\ &+\sum_i\frac{\bar{\eta}_i^{min}}{2}\left[\int_I(u^{n+1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds - \int_I(u^n_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right] + \sum_i\left[\frac{\bar{\eta}_i^{min}}{3}-\eta_i'^{max}\right]\int_I(u^n_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\ &+\sum_i\frac{\bar{\eta}_i^{min}}{6}\left[\int_I(u^n_{j,h}\cdot\widehat{\tau}_i)^2\,ds - \int_I(u^{n-1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds\right] + \sum_i\left[\frac{\bar{\eta}_i^{min}}{6}-\frac{\eta_i'^{max}}{2}\right]\int_I(u^{n-1}_{j,h}\cdot\widehat{\tau}_i)^2\,ds \\ &+\frac{gS_0}{4\Delta t}\|\phi^{n+1}_{j,h}\|^2_p + \frac{gS_0}{4\Delta t}\|2\phi^{n+1}_{j,h}+\phi^n_{j,h}\|^2_p - \frac{gS_0}{4\Delta t}\|\phi^n_{j,h}\|^2_p - \frac{gS_0}{4\Delta t}\|2\phi^n_{j,h}+\phi^{n-1}_{j,h}\|^2_p \\ &+\frac{gS_0}{4\Delta t}\|\phi^{n+1}_{j,h}-2\phi^n_{j,h}+\phi^{n-1}_{j,h}\|^2_p + \left(1-\beta-\frac{3\rho'_{max}}{\bar{k}_{min}}\right)g\bar{k}_{min}\|\nabla\phi^{n+1}_{j,h}\|^2_p + \frac{3g\rho'_{max}}{2}\left(\|\nabla\phi^{n+1}_{j,h}\|^2_p - \|\nabla\phi^n_{j,h}\|^2_p\right) \\ &+\frac{g\rho'_{max}}{2}\left(\|\nabla\phi^n_{j,h}\|^2_p - \|\nabla\phi^{n-1}_{j,h}\|^2_p\right) + \frac{1}{4\Delta t}|r^{n+1}_{j,h}|^2 + \frac{1}{4\Delta t}|2r^{n+1}_{j,h}+r^n_{j,h}|^2 \\ &-\frac{1}{4\Delta t}|r^n_{j,h}|^2 - \frac{1}{4\Delta t}|2r^n_{j,h}+r^{n-1}_{j,h}|^2 + \frac{1}{4\Delta t}|r^{n+1}_{j,h}-2r^n_{j,h}+r^{n-1}_{j,h}|^2 + \frac{1}{T}|r^{n+1}_{j,h}|^2 \\ &\leq \frac{C^2_{P,f}}{2\nu}\|f^{n+1}_{f,j}\|^2_f + \frac{gC^2_{P,p}}{4\beta\bar{k}_{min}}\|f^{n+1}_{p,j}\|^2_p. \end{aligned} \tag{3.24}$$

To obtain stability, we need

$$\frac{\bar{\eta}_i^{min}}{2}-\frac{3\eta_i'^{max}}{2}\geq 0, \qquad 1-\beta-\frac{3\rho'_{max}}{\bar{k}_{min}}\geq 0. \tag{3.25}$$

Recall that $\beta,\eta_i'^{max},\rho'_{max}$ are all positive, we then have the following constraints on these parameters:

$$0<\beta<1, \qquad \frac{\rho'_{max}}{\bar{k}_{min}}<\frac{1}{3}, \quad \eta_i'^{max}\leq\frac{1}{3}\bar{\eta}_i^{min}. \tag{3.26}$$

This leads to the two parameter conditions in (3.17) required for stability. Now if the two parameter conditions in (3.17) both hold and taking $\beta=\frac{1}{2}(1-\frac{3\rho'_{max}}{\bar{k}_{min}})$, then (3.24) reduces to

$$\frac{1}{4\Delta t}\|u^{n+1}_{j,h}\|^2_f + \frac{1}{4\Delta t}\|2u^{n+1}_{j,h}+u^n_{j,h}\|^2_f - \frac{1}{4\Delta t}\|u^n_{j,h}\|^2_f - \frac{1}{4\Delta t}\|2u^n_{j,h}+u^{n-1}_{j,h}\|^2_f \tag{3.27}$$

$$+\frac{1}{4\Delta t}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|_f^2+\frac{1}{2}\nu\|\nabla u_{j,h}^{n+1}\|_f^2+\sum_i\frac{\bar{\eta}_i^{min}}{2}\left[\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds-\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right]$$
$$+\sum_i\frac{\bar{\eta}_i^{min}}{6}\left[\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds-\int_I(u_{j,h}^{n-1}\cdot\widehat{\tau}_i)^2\ ds\right]+\frac{gS_0}{4\Delta t}\|\phi_{j,h}^{n+1}\|_p^2+\frac{gS_0}{4\Delta t}\|2\phi_{j,h}^{n+1}+\phi_{j,h}^{n}\|_p^2$$
$$-\frac{gS_0}{4\Delta t}\|\phi_{j,h}^{n}\|_p^2-\frac{gS_0}{4\Delta t}\|2\phi_{j,h}^{n}+\phi_{j,h}^{n-1}\|_p^2+\frac{gS_0}{4\Delta t}\|\phi_{j,h}^{n+1}-2\phi_{j,h}^{n}+\phi_{j,h}^{n-1}\|_p^2$$
$$+\frac{3g\rho'_{max}}{2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2-\|\nabla\phi_{j,h}^{n}\|_p^2\right)+\frac{g\rho'_{max}}{2}\left(\|\nabla\phi_{j,h}^{n}\|_p^2-\|\nabla\phi_{j,h}^{n-1}\|_p^2\right)+\frac{1}{4\Delta t}|r_{j,h}^{n+1}|^2$$
$$+\frac{1}{4\Delta t}|2r_{j,h}^{n+1}+r_{j,h}^{n}|^2-\frac{1}{4\Delta t}|r_{j,h}^{n}|^2-\frac{1}{4\Delta t}|2r_{j,h}^{n}+r_{j,h}^{n-1}|^2+\frac{1}{4\Delta t}|r_{j,h}^{n+1}-2r_{j,h}^{n}+r_{j,h}^{n-1}|^2+\frac{1}{T}|r_{j,h}^{n+1}|^2$$
$$\leq\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2+\frac{gC_{P,p}^2}{2(\bar{k}_{min}-3\rho'_{max})}\|f_{p,j}^{n+1}\|_p^2.$$

Sum (3.27) from $n=1$ to $N-1$ and multiply through by $4\Delta t$ to get

$$\|u_{j,h}^{N}\|_f^2+\|2u_{j,h}^{N}+u_{j,h}^{N-1}\|_f^2+\sum_{n=1}^{N-1}\|u_{j,h}^{n+1}-2u_{j,h}^{n}+u_{j,h}^{n-1}\|_f^2+\Delta t\sum_{n=1}^{N-1}2\nu\|\nabla u_{j,h}^{n+1}\|_f^2 \tag{3.28}$$
$$+\Delta t\sum_i 2\bar{\eta}_i^{min}\int_I(u_{j,h}^{N}\cdot\widehat{\tau}_i)^2\ ds+\Delta t\sum_i\frac{2\bar{\eta}_i^{min}}{3}\int_I(u_{j,h}^{N-1}\cdot\widehat{\tau}_i)^2\ ds+gS_0\|\phi_{j,h}^{N}\|_p^2+gS_0\|2\phi_{j,h}^{N}+\phi_{j,h}^{N-1}\|_p^2$$
$$+gS_0\sum_{n=1}^{N-1}\|\phi_{j,h}^{n+1}-2\phi_{j,h}^{n}+\phi_{j,h}^{n-1}\|_p^2+6g\rho'_{max}\Delta t\|\nabla\phi_{j,h}^{N}\|_p^2+2g\rho'_{max}\Delta t\|\nabla\phi_{j,h}^{N-1}\|_p^2+|r_{j,h}^{N}|^2$$
$$+|2r_{j,h}^{N}+r_{j,h}^{N-1}|^2+\sum_{n=1}^{N-1}|r_{j,h}^{n+1}-2r_{j,h}^{n}+r_{j,h}^{n-1}|^2+\frac{4\Delta t}{T}\sum_{n=1}^{N-1}|r_{j,h}^{n+1}|^2$$
$$\leq\|u_{j,h}^{1}\|_f^2+\|2u_{j,h}^{1}+u_{j,h}^{0}\|_f^2+\Delta t\sum_i 2\bar{\eta}_i^{min}\int_I(u_{j,h}^{1}\cdot\widehat{\tau}_i)^2\ ds+\Delta t\sum_i\frac{2\bar{\eta}_i^{min}}{3}\int_I(u_{j,h}^{0}\cdot\widehat{\tau}_i)^2\ ds$$
$$+gS_0\|\phi_{j,h}^{1}\|_p^2+gS_0\|2\phi_{j,h}^{1}+\phi_{j,h}^{0}\|_p^2+6g\rho'_{max}\Delta t\|\nabla\phi_{j,h}^{1}\|_p^2+2g\rho'_{max}\Delta t\|\nabla\phi_{j,h}^{0}\|_p^2$$
$$+|r_{j,h}^{1}|^2+|2r_{j,h}^{1}+r_{j,h}^{0}|^2+\Delta t\sum_{n=1}^{N-1}\frac{2C_{P,f}^2}{\nu}\|f_{f,j}^{n+1}\|_f^2+\Delta t\sum_{n=1}^{N-1}\frac{2gC_{P,p}^2}{\bar{k}_{min}-3\rho'_{max}}\|f_{p,j}^{n+1}\|_p^2.$$

□

REMARK 3.5. *For the first order method (SAV-BE-En-h), the parameter conditions in (3.1) indicate that the fluctuation needs to be smaller than the mean. They are usually easy to fulfill in UQ applications where the magnitude of the fluctuation is generally much smaller than the magnitude of the uncertain parameter. The parameter conditions in (3.17) for the second order method (SAV-BDF2-En-h) are stricter than the ones for the first order method, but still be expected to be satisfied easily in many applications. If they are not satisfied for a large ensemble, one can split the large ensemble into smaller ensembles to make these conditions satisfied and apply the ensemble algorithm to each smaller ensemble.*

REMARK 3.6 (Unconditionally long time stability when solving a single system). *Parameter conditions are only needed in the ensemble algorithms to bound the fluctuations of the parameters for stability. For the case of computing a single Stokes-Darcy system (setting $J=1$ in our ensemble algorithm), the fluctuations are equal to zero, and thus the parameter conditions are automatically satisfied. In other words, the proposed algorithms are unconditionally long time stable for solving a single Stokes-Darcy system.*

4. Error Analysis.

In this section, we analyze the error of Algorithm 2.1. We assume the finite element spaces satisfy the approximation properties of piecewise polynomials on quasiuniform meshes

$$\inf_{v_h\in X_f^h}\|\nabla(v-v_h)\|_f\leq Ch^k\|v\|_{H^{k+1}(D_f)}\qquad\forall v\in[H^{k+1}(D_f)]^d, \tag{4.1}$$
$$\inf_{q_h\in Q_f^h}\|q-q_h\|_f\leq Ch^{s+1}\|q\|_{H^{s+1}(D_f)}\qquad\forall q\in H^{s+1}(D_f), \tag{4.2}$$

$$\inf_{\psi_h \in X_p^h} \|\nabla(\psi - \psi_h)\|_p \le Ch^m \|\psi\|_{H^{m+1}(D_p)} \qquad \forall \psi \in H^{m+1}(D_p), \tag{4.3}$$

where the generic constant $C > 0$ is independent of the mesh size h. An example for which both the LBB^h stability condition and the approximation properties are satisfied is the finite elements (P_{l+1}–P_l–P_{l+1}), $l \ge 1$, see [16, 15, 43] for more details.

We also assume the following regularity on the true solution of the Stokes-Darcy equations.

$$\begin{aligned}
&u_j \in L^\infty(0,T;H^{k+1}(D_f)), \partial_t u_j \in L^2(0,T;H^{k+1}(D_f)), \partial_{tt} u_j \in L^2(0,T;L^2(D_f)),\\
&\phi_j \in L^\infty(0,T;H^{m+1}(D_p)), \partial_t \phi_j \in L^2(0,T;H^{m+1}(D_p)), \partial_{tt} \phi_j \in L^2(0,T;L^2(D_p)),\\
&p_j \in L^2(0,T;H^{s+1}(D_f)).
\end{aligned}$$

For functions $v(x,t)$ defined on $(0,T)$, we define the continuous norm

$$\|v\|_{m,k,r} := \|v\|_{L^m(0,T;H^k(D_r))},\ r \in \{f,p\}.$$

Given a time step Δt, let $t_n = n\Delta t, T = N\Delta t$, $v^n = v(x,t_n)$ and define the discrete norms

$$|||v|||_{\infty,k,s} = \max_{0\le n\le N} \|v^n\|_{H^k(D_s)} \quad \text{and} \quad |||v|||_{m,k,s} := \left(\sum_{n=0}^{N} \|v^n\|^m_{H^k(D_s)} \Delta t\right)^{1/m}, \ \ s \in \{f,p\}.$$

We will use the discrete Gronwall inequality (Lemma 4.1 below) in the error analysis, see [23] for proof.

Lemma 4.1. *Let $D \ge 0$ and $\kappa_n, A_n, B_n, C_n \ge 0$ for any integer $n \ge 0$ and satisfy*

$$A_{\tilde N} + \Delta t \sum_{n=0}^{\tilde N} B_n \le \Delta t \sum_{n=0}^{\tilde N} \kappa_n A_n + \Delta t \sum_{n=0}^{\tilde N} C_n + D \ \textit{for}\ \tilde N \ge 0.$$

Suppose that for all n, $\Delta t \kappa_n < 1$, and set $g_n = (1 - \Delta t \kappa_n)^{-1}$. Then,

$$A_{\tilde N} + \Delta t \sum_{n=0}^{\tilde N} B_n \le exp(\Delta t \sum_{n=0}^{\tilde N} g_n \kappa_n)[\Delta t \sum_{n=0}^{\tilde N} C_n + D] \ \textit{for}\ \tilde N \ge 0.$$

Let

$$u_j^n = u_j(x,t_n), \quad p_j^n = p_j(x,t_n), \quad \phi_j^n = \phi_j(x,t_n), \quad r_j^n = r_j(t_n).$$

Denote the errors by

$$e_{j,u}^n := u_j^n - u_{j,h}^n, \qquad e_{j,\phi}^n := \phi_j^n - \phi_{j,h}^n, \qquad e_{j,r}^n := r_j^n - r_{j,h}^n.$$

We prove the convergence of Algorithm 2.1 under two parameter conditions, without any timestep conditions:

$$\eta_i'^{max} < \bar\eta_i^{min}, \qquad \rho'_{max} < \bar k_{min}. \tag{4.4}$$

Theorem 4.2 (Error Estimate). *For any $j = 1, \ldots, J$, if the two parameter conditions in (4.4) hold, and the timestep Δt is sufficiently small, i.e., $\Delta t \le 1/C_0$, where C_0 is a constant independent of h, then there is a positive constant C independent of the time step Δt and mesh size h such that*

$$\|e_{j,u}^N\|_f^2 + \Delta t \sum_{n=0}^{N-1} \nu \|\nabla e_{j,u}^n\|_f^2 + \frac{\nu}{5}\Delta t \|\nabla e_{j,u}^N\|_f^2 + \Delta t \sum_i \eta_i'^{max} \int_I (e_{j,u}^N \cdot \widehat{\tau}_i)^2 \, ds \tag{4.5}$$

$$+ gS_0 \|e_{j,\phi}^N\|_p^2 + \Delta t \left(\frac{7}{9} g\rho'_{max} + \frac{2}{9} g\bar k_{min}\right) \|\nabla e_{j,\phi}^N\|_p^2 + |e_{j,r}^N|^2 + \sum_{n=0}^{N-1} |e_{j,r}^{n+1} - e_{j,r}^n|^2 + \Delta t \sum_{n=0}^{N-1} \frac{1}{T} |e_{j,r}^{n+1}|^2$$

$$\leq exp(\frac{CT}{1-C\Delta t})\Bigg(C\Delta t^2\|\partial_t u_j\|_{2,1,f} + Ch^{2k}\|\|u_j\|\|^2_{2,k+1,f} + C\Delta t^2\|\partial_t\phi_j\|_{2,1,p} + Ch^{2m}\|\|\phi_j\|\|^2_{2,m+1,p}$$
$$+ Ch^{2k+2}\|\partial_t u_j\|^2_{2,k+1,f} + Ch^{2m+2}\|\partial_t\phi_j\|^2_{2,m+1,p} + C\Delta t^2\|\partial_{tt}u_j\|_{2,0,f} + C\Delta t^2\|\partial_{tt}\phi_j\|_{2,0,p} + C\Delta t^2$$
$$+ Ch^{2s+2}\|\|p_j\|\|^2_{2,s+1,f} + Ch^{2k+2}\|\|u_j\|\|^2_{\infty,k+1,f} + +Ch^{2m+2}\|\|\phi_j\|\|^2_{\infty,m+1,p}\Bigg).$$

Proof. See Appendix B. ◻

The error estimate indicates the numerical errors $e^N_{j,u}$, $e^N_{j,\phi}$, and $e^N_{j,r}$ of the free fluid velocity, the hydraulic head, and the auxiliary variable, respectively, are all first order convergent in time. In particular, if Taylor-Hood elements ($k=2$, $s=1$) are used for approximating (u_j, p_j), i.e., the C^0 piecewise-quadratic velocity space X^h_f and the C^0 piecewise-linear pressure space Q^h_f, and P_2 element ($m=2$) is used for X^h_p, we have the following estimate.

COROLLARY 4.3. *If (X^h_f, Q^h_f, X^h_p) are chosen as the (P_2, P_1, P_2) elements, we have*

$$\|e^N_{j,u}\|^2_f + \Delta t\sum_{n=0}^{N-1}\nu\|\nabla e^n_{j,u}\|^2_f + \frac{\nu}{5}\Delta t\|\nabla e^N_{j,u}\|^2_f + \Delta t\sum_i \eta'^{max}_i\int_I(e^N_{j,u}\cdot\widehat{\tau}_i)^2\,ds \tag{4.6}$$
$$+ gS_0\|e^N_{j,\phi}\|^2_p + \Delta t\left(\frac{7}{9}g\rho'_{max} + \frac{2}{9}g\bar{k}_{min}\right)\|\nabla e^N_{j,\phi}\|^2_p + |e^N_{j,r}|^2 + \sum_{n=0}^{N-1}|e^{n+1}_{j,r} - e^n_{j,r}|^2 + \Delta t\sum_{n=0}^{N-1}\frac{1}{T}|e^{n+1}_{j,r}|^2$$
$$\leq C(h^4 + \Delta t^2)\ .$$

5. Numerical Tests. In this section, we present the numerical implementation of the SAV approaches, and run simulations to confirm the convergence rates of the proposed SAV-BE-En and SAV-BDF2-En algorithms. Numerical tests are also presented to show the efficiency and effectiveness of SAV-BE-En (Due to page limit we will only report results of SAV-BE-En). In all simulations, the spatial discretization is based on the Taylor-Hood elements (P2-P1) for the Stokes problem and piecewise quadratic finite elements (P2) for the Darcy problem.

5.1. Numerical Implementation. In the SAV-BE-En-h and SAV-BDF2-En-h algorithms given by (2.3)-(2.6) and (2.7)-(2.10), although the u and ϕ are decoupled, they are still coupled with r and additional decoupling is needed to fully decouple the PDE system for desired efficiency. In this section we present the implementation algorithms for both the SAV-BE-En-h and SAV-BDF2-En-h algorithms following the decoupling strategies in [47, 48, 38]. Let

$$S^{n+1}_j = \frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}, \qquad \Longrightarrow \qquad r^{n+1}_{j,h} = exp(-\frac{t^{n+1}}{T})\,S^{n+1}_j, \tag{5.1}$$
$$u^{n+1}_{j,h} = \hat{u}^{n+1}_{j,h} + S^{n+1}_j\breve{u}^{n+1}_{j,h}, \qquad p^{n+1}_{j,h} = \hat{p}^{n+1}_{j,h} + S^{n+1}_j\breve{p}^{n+1}_{j,h}, \qquad \phi^{n+1}_{j,h} = \hat{\phi}^{n+1}_{j,h} + S^{n+1}_j\breve{\phi}^{n+1}_{j,h}. \tag{5.2}$$

5.1.1. SAV-BE-En-h. Instead of solving (2.3)-(2.6), we solve the following four subproblems for $(\hat{u}^{n+1}_{j,h}, \hat{p}^{n+1}_{j,h})$, $\hat{\phi}^{n+1}_{j,h}$, $(\breve{u}^{n+1}_{j,h}, \breve{p}^{n+1}_{j,h})$, $\breve{\phi}^{n+1}_{j,h}$ respectively.

(BE sub-problem 1): Find $(\hat{u}^{n+1}_{j,h}, \hat{p}^{n+1}_{j,h}) \in X^h_f \times Q^h_f$ satisfying $\forall\ (v_h, q_h) \in X^h_f \times Q^h_f$,

$$\begin{cases}\frac{1}{\Delta t}\left(\hat{u}^{n+1}_{j,h}, v_h\right)_f + \nu(\nabla\hat{u}^{n+1}_{j,h}, \nabla v_h)_f + \sum_i\int_I\bar{\eta}_i(\hat{u}^{n+1}_{j,h}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds - \left(\hat{p}^{n+1}_{j,h}, \nabla\cdot v_h\right)_f \\ \qquad = (f^{n+1}_{f,j}, v_h)_f + \frac{1}{\Delta t}\left(u^n_{j,h}, v_h\right)_f - \sum_i\int_I(\eta_{i,j} - \bar{\eta}_i)(u^n_{j,h}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds, \quad \text{in } D_f \\ (q_h, \nabla\cdot\hat{u}^{n+1}_{j,h})_f = 0, \quad \text{in } D_f \\ \hat{u}^{n+1}_{j,h} = a^{n+1}_{j,h}, \quad \text{on } \partial D_f\backslash I.\end{cases}$$

(BE sub-problem 2): Find $\hat{\phi}_{j,h}^{n+1} \in X_p^h$ satisfying $\forall\ \psi_h \in X_p^h$,

$$\begin{cases} \dfrac{gS_0}{\Delta t}\left(\hat{\phi}_{j,h}^{n+1}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\hat{\phi}_{j,h}^{n+1}, \nabla\psi_h)_p \\ \qquad = g(f_{p,j}^{n+1}, \psi_h)_p + \dfrac{gS_0}{\Delta t}\left(\phi_{j,h}^{n}, \psi_h\right)_p - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla\phi_{j,h}^{n}, \nabla\psi_h)_p, \quad \text{in } D_p \\ \hat{\phi}_{j,h}^{n+1} = b_{j,h}^{n+1}, \quad \text{on } \partial D_p \backslash I. \end{cases}$$

(BE sub-problem 3): Find $(\breve{u}_{j,h}^{n+1}, \breve{p}_{j,h}^{n+1}) \in X_f^h \times Q_f^h$ satisfying $\forall\ (v_h, q_h) \in X_f^h \times Q_f^h$,

$$\begin{cases} \dfrac{1}{\Delta t}\left(\breve{u}_{j,h}^{n+1}, v_h\right)_f + \nu(\nabla\breve{u}_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i(\breve{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\ ds - \left(\breve{p}_{j,h}^{n+1}, \nabla\cdot v_h\right)_f \\ \qquad = -c_I(v_h, \phi_{j,h}^{n}), \quad \text{in } D_f \\ (q_h, \nabla\cdot\breve{u}_{j,h}^{n+1})_f = 0, \quad \text{in } D_f \\ \breve{u}_{j,h}^{n+1} = 0, \quad \text{on } \partial D_f \backslash I. \end{cases}$$

(BE sub-problem 4): Find $\breve{\phi}_{j,h}^{n+1} \in X_p^h$ satisfying $\forall\ \psi_h \in X_p^h$,

$$\begin{cases} \dfrac{gS_0}{\Delta t}\left(\breve{\phi}_{j,h}^{n+1}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\breve{\phi}_{j,h}^{n+1}, \nabla\psi_h)_p = c_I(u_{j,h}^{n}, \psi_h), \quad \text{in } D_p \\ \breve{\phi}_{j,h}^{n+1} = 0, \quad \text{on } \partial D_p \backslash I. \end{cases}$$

Now we need to derive an equation for S_j^{n+1}. Multiplying (2.6) by $r_{j,h}^{n+1}$ gives

$$\frac{r_{j,h}^{n+1} - r_{j,h}^{n}}{\Delta t}\cdot r_{j,h}^{n+1} + \frac{1}{T}|r_{j,h}^{n+1}|^2 - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_{j,h}^{n+1}, \phi_{j,h}^{n}) - c_I(u_{j,h}^{n}, \phi_{j,h}^{n+1})\right) = 0. \tag{5.3}$$

Plugging (5.1) into (5.3) gives

$$\begin{aligned} &(\frac{1}{\Delta t} + \frac{1}{T})(r_{j,h}^{n+1})^2 - \frac{1}{\Delta t}r_{j,h}^{n}r_{j,h}^{n+1} - S_j^{n+1}\left(c_I(u_{j,h}^{n+1}, \phi_{j,h}^{n}) - c_I(u_{j,h}^{n}, \phi_{j,h}^{n+1})\right) = 0 \\ \Longrightarrow\ &(\frac{1}{\Delta t} + \frac{1}{T})exp(-\frac{2t^{n+1}}{T})\,(S_j^{n+1})^2 - \frac{1}{\Delta t}r_{j,h}^{n}exp(-\frac{t^{n+1}}{T})\,S_j^{n+1} \\ &\quad - S_j^{n+1}\left(c_I(\hat{u}_{j,h}^{n+1} + S_j^{n+1}\breve{u}_{j,h}^{n+1}, \phi_{j,h}^{n}) - c_I(u_{j,h}^{n}, \hat{\phi}_{j,h}^{n+1} + S_j^{n+1}\breve{\phi}_{j,h}^{n+1})\right) = 0. \end{aligned}$$

At last, we obtain the equation for S_j^{n+1} as

$$S_j^{n+1}\left(A_j^{n+1}S_j^{n+1} + B_j^{n+1}\right) = 0 \qquad \Longrightarrow \qquad S_j^{n+1} = -\frac{B_j^{n+1}}{A_j^{n+1}}, \tag{5.4}$$

where

$$\begin{aligned} A_j^{n+1} &= (\frac{1}{\Delta t} + \frac{1}{T})exp(-\frac{2t^{n+1}}{T}) - c_I(\breve{u}_{j,h}^{n+1}, \phi_{j,h}^{n}) + c_I(u_{j,h}^{n}, \breve{\phi}_{j,h}^{n+1}), \\ B_j^{n+1} &= -\frac{1}{\Delta t}r_{j,h}^{n}exp(-\frac{t^{n+1}}{T}) - c_I(\hat{u}_{j,h}^{n+1}, \phi_{j,h}^{n}) + c_I(u_{j,h}^{n}, \hat{\phi}_{j,h}^{n+1}). \end{aligned}$$

After getting $\hat{u}_j^{n+1}, \breve{u}_j^{n+1}$, $\hat{\phi}_j^{n+1}, \breve{\phi}_j^{n+1}$, S_j^{n+1} can be computed directly using formula (5.4), and then we have $u_j^{n+1} = \hat{u}_j^{n+1} + S_j^{n+1}\breve{u}_j^{n+1}$, $p_j^{n+1} = \hat{p}_j^{n+1} + S_j^{n+1}\breve{p}_j^{n+1}$ and $\phi_j^{n+1} = \hat{\phi}_j^{n+1} + S_j^{n+1}\breve{\phi}_j^{n+1}$.

5.1.2. SAV-BDF2-En-h. Instead of solving (2.7)-(2.10), we solve the following four subproblems for $(\hat{u}_{j,h}^{n+1}, \hat{p}_{j,h}^{n+1})$, $\hat{\phi}_{j,h}^{n+1}$, $(\breve{u}_{j,h}^{n+1}, \breve{p}_{j,h}^{n+1})$, $\breve{\phi}_{j,h}^{n+1}$ respectively.

(BDF2 sub-problem 1): Find $(\hat{u}_{j,h}^{n+1}, \hat{p}_{j,h}^{n+1}) \in X_f^h \times Q_f^h$ satisfying $\forall$ $(v_h, q_h) \in X_f^h \times Q_f^h$,

$$\begin{cases} \frac{3}{2\Delta t}\left(\hat{u}_{j,h}^{n+1}, v_h\right)_f + \nu(\nabla \hat{u}_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i(\hat{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds - \left(\hat{p}_{j,h}^{n+1}, \nabla\cdot v_h\right)_f \\ \qquad = (f_{f,j}^{n+1}, v_h)_f + \frac{2}{\Delta t}\left(u_{j,h}^{n}, v_h\right)_f - \frac{1}{2\Delta t}\left(u_{j,h}^{n-1}, v_h\right)_f \\ \qquad - \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)((2u_{j,h}^n - u_{j,h}^{n-1})\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds, \quad \text{in } D_f \\ (q_h, \nabla\cdot\hat{u}_{j,h}^{n+1})_f = 0, \quad \text{in } D_f \\ \hat{u}_{j,h}^{n+1} = a_{j,h}^{n+1}, \quad \text{on } \partial D_f\backslash I. \end{cases}$$

(BDF2 sub-problem 2): Find $\hat{\phi}_{j,h}^{n+1} \in X_p^h$ satisfying $\forall$ $\psi_h \in X_p^h$,

$$\begin{cases} \frac{3gS_0}{2\Delta t}\left(\hat{\phi}_{j,h}^{n+1}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\hat{\phi}_{j,h}^{n+1}, \nabla\psi_h)_p = g(f_{p,j}^{n+1}, \psi_h)_p + \frac{2gS_0}{\Delta t}\left(\phi_{j,h}^n, \psi_h\right)_p - \frac{gS_0}{2\Delta t}\left(\phi_{j,h}^{n-1}, \psi_h\right)_p \\ \qquad - g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(2\phi_{j,h}^n - \phi_{j,h}^{n-1}, \nabla\psi_h)_p, \quad \text{in } D_p \\ \hat{\phi}_{j,h}^{n+1} = b_{j,h}^{n+1}, \quad \text{on } \partial D_p\backslash I. \end{cases}$$

(BDF2 sub-problem 3): Find $(\breve{u}_{j,h}^{n+1}, \breve{p}_{j,h}^{n+1}) \in X_f^h \times Q_f^h$ satisfying $\forall$ $(v_h, q_h) \in X_f^h \times Q_f^h$,

$$\begin{cases} \frac{3}{2\Delta t}\left(\breve{u}_{j,h}^{n+1}, v_h\right)_f + \nu(\nabla \breve{u}_{j,h}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i(\breve{u}_{j,h}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\, ds - \left(\breve{p}_{j,h}^{n+1}, \nabla\cdot v_h\right)_f \\ \qquad = -c_I(v_h, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}), \quad \text{in } D_f \\ (q_h, \nabla\cdot\breve{u}_{j,h}^{n+1})_f = 0, \quad \text{in } D_f \\ \breve{u}_{j,h}^{n+1} = 0, \quad \text{on } \partial D_f\backslash I. \end{cases}$$

(BDF2 sub-problem 4): Find $\breve{\phi}_{j,h}^{n+1} \in X_p^h$ satisfying $\forall$ $\psi_h \in X_p^h$,

$$\begin{cases} \frac{3gS_0}{2\Delta t}\left(\breve{\phi}_{j,h}^{n+1}, \psi_h\right)_p + g(\bar{\mathcal{K}}\nabla\breve{\phi}_{j,h}^{n+1}, \nabla\psi_h)_p = c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \psi_h), \quad \text{in } D_p \\ \breve{\phi}_{j,h}^{n+1} = 0, \quad \text{on } \partial D_p\backslash I. \end{cases}$$

Now we need to derive an equation for S_j^{n+1}. Multiplying (2.10) by $r_{j,h}^{n+1}$ gives

$$\begin{aligned} &\frac{3r_{j,h}^{n+1} - 4r_{j,h}^n + r_{j,h}^{n-1}}{2\Delta t}\cdot r_{j,h}^{n+1} + \frac{1}{T}|r_{j,h}^{n+1}|^2 \\ &\qquad - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \phi_{j,h}^{n+1})\right) = 0. \end{aligned} \tag{5.5}$$

Plugging (5.1) into (5.5) gives

$$\begin{aligned} &(\frac{3}{2\Delta t} + \frac{1}{T})(r_{j,h}^{n+1})^2 - \frac{2}{\Delta t}r_{j,h}^n r_{j,h}^{n+1} + \frac{1}{2\Delta t}r_{j,h}^{n-1}r_{j,h}^{n+1} \\ &\qquad - S_j^{n+1}\left(c_I(u_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \phi_{j,h}^{n+1})\right) = 0 \\ &\Longrightarrow (\frac{3}{2\Delta t} + \frac{1}{T})exp(-\frac{2t^{n+1}}{T})(S_j^{n+1})^2 - \frac{2}{\Delta t}r_{j,h}^n exp(-\frac{t^{n+1}}{T})S_j^{n+1} + \frac{1}{2\Delta t}r_{j,h}^{n-1}exp(-\frac{t^{n+1}}{T})S_j^{n+1} \\ &\qquad - S_j^{n+1}\left(c_I(\hat{u}_{j,h}^{n+1} + S_j^{n+1}\breve{u}_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) - c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \hat{\phi}_{j,h}^{n+1} + S_j^{n+1}\breve{\phi}_{j,h}^{n+1})\right) = 0. \end{aligned}$$

Table 5.1: Convergence rates of the SAV-BE-En algorithm for u, p, ϕ with $J = 3$, $\Delta t = h$.

Δt	$\|u_h - u\|_{H^1}^{E,1}$	Rate	$\|u_h - u\|_{H^1}^{E,2}$	Rate	$\|u_h - u\|_{H^1}^{E,3}$	Rate
1/8	2.346×10^{-2}	—	2.342×10^{-2}	—	2.516×10^{-2}	—
1/16	1.138×10^{-2}	1.04	1.145×10^{-2}	1.03	1.246×10^{-2}	1.01
1/32	5.567×10^{-3}	1.03	5.619×10^{-3}	1.03	6.162×10^{-3}	1.02
1/64	2.748×10^{-3}	1.02	2.778×10^{-3}	1.02	3.060×10^{-3}	1.01
1/128	1.365×10^{-3}	1.01	1.381×10^{-3}	1.01	1.524×10^{-3}	1.01
Δt	$\|p_h - p\|_{L^2}^{E,1}$	Rate	$\|p_h - p\|_{L^2}^{E,2}$	Rate	$\|p_h - p\|_{L^2}^{E,3}$	Rate
1/8	2.883×10^{-2}	—	3.942×10^{-2}	—	5.719×10^{-2}	—
1/16	1.538×10^{-2}	0.91	2.083×10^{-2}	0.92	2.995×10^{-2}	0.93
1/32	7.938×10^{-3}	0.95	1.069×10^{-2}	0.96	1.528×10^{-2}	0.97
1/64	4.033×10^{-3}	0.98	5.412×10^{-3}	0.98	7.712×10^{-3}	0.99
1/128	2.033×10^{-3}	0.99	2.726×10^{-3}	0.99	3.877×10^{-3}	0.99
Δt	$\|\phi_h - \phi\|_{H^1}^{E,1}$	Rate	$\|\phi_h - \phi\|_{H^1}^{E,2}$	Rate	$\|\phi_h - \phi\|_{H^1}^{E,3}$	Rate
1/8	7.815×10^{-2}	—	4.686×10^{-2}	—	4.758×10^{-2}	—
1/16	3.875×10^{-2}	1.01	2.308×10^{-2}	1.02	2.358×10^{-2}	1.01
1/32	1.917×10^{-2}	1.02	1.132×10^{-2}	1.03	1.169×10^{-2}	1.01
1/64	9.519×10^{-3}	1.01	5.590×10^{-3}	1.02	5.810×10^{-3}	1.01
1/128	4.741×10^{-3}	1.01	2.775×10^{-3}	1.01	2.895×10^{-3}	1.00

At last, we obtain the equation for S_j^{n+1} as

$$S_j^{n+1}\left(A_j^{n+1}S_j^{n+1} + B_j^{n+1}\right) = 0 \qquad \Longrightarrow \qquad S_j^{n+1} = -\frac{B_j^{n+1}}{A_j^{n+1}}, \tag{5.6}$$

where

$$\begin{aligned}
A_j^{n+1} &= (\frac{3}{2\Delta t} + \frac{1}{T})exp(-\frac{2t^{n+1}}{T}) - c_I(\breve{u}_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) + c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \breve{\phi}_{j,h}^{n+1}),\\
B_j^{n+1} &= -\frac{2}{\Delta t}r_{j,h}^n exp(-\frac{t^{n+1}}{T}) + \frac{1}{2\Delta t}r_{j,h}^{n-1}exp(-\frac{t^{n+1}}{T}) - c_I(\hat{u}_{j,h}^{n+1}, 2\phi_{j,h}^n - \phi_{j,h}^{n-1}) + c_I(2u_{j,h}^n - u_{j,h}^{n-1}, \hat{\phi}_{j,h}^{n+1}).
\end{aligned}$$

After getting $\hat{u}_j^{n+1}, \breve{u}_j^{n+1}$, $\hat{\phi}_j^{n+1}, \breve{\phi}_j^{n+1}$, S_j^{n+1} can be computed directly using formula (5.6), and then we have $u_j^{n+1} = \hat{u}_j^{n+1} + S_j^{n+1}\breve{u}_j^{n+1}$, $p_j^{n+1} = \hat{p}_j^{n+1} + S_j^{n+1}\breve{p}_j^{n+1}$ and $\phi_j^{n+1} = \hat{\phi}_j^{n+1} + S_j^{n+1}\breve{\phi}_j^{n+1}$.

5.2. Convergence test. The domains considered in this test are $D_f = (0,1) \times (1,2)$ and $D_p = (0,1) \times (0,1)$ with interface $I = [0,1] \times \{1\}$. The model parameters, g, ν, S_0, and α_{BJS} are set to be one, and the hydraulic conductivity tensor is set as a diagonal matrix diag(k_{11}, k_{22}) with k_{11} and k_{22} being constants. We construct the exact solution as follows while ensuring all the boundary conditions and initial conditions are compatible:

$$\begin{aligned}
u(x,y,t) &= (u_1(x,y,t),\ u_2(x,y,t)),\\
u_1(x,y,t) &= \left(x^2(y-1)^2 + \exp(y/\sqrt{k_{11}})\right)\cos(t),\\
u_2(x,y,t) &= \left(\frac{2}{3}x(1-y)^3 + k_{22}(2 - \pi\sin(\pi x)\right)\cos(t),\\
p(x,y,t) &= (2 - \pi\sin(\pi x))\sin(0.5\pi y)\cos(t),\\
\phi(x,y,t) &= (2 - \pi\sin(\pi x))(1 - y - \cos(\pi y))\cos(t).
\end{aligned}$$

Table 5.2: Convergence rates of the SAV-BDF2-En algorithm for u, p, ϕ with $J = 3$, $\Delta t = h$.

Δt	$\|u_h - u\|_{H^1}^{E,1}$	Rate	$\|u_h - u\|_{H^1}^{E,2}$	Rate	$\|u_h - u\|_{H^1}^{E,3}$	Rate
1/8	2.357×10^{-3}	-	2.561×10^{-3}	-	2.805×10^{-3}	-
1/16	4.731×10^{-4}	2.32	5.194×10^{-4}	2.30	5.789×10^{-4}	2.28
1/32	1.051×10^{-4}	2.17	1.160×10^{-4}	2.16	1.308×10^{-4}	2.15
1/64	2.479×10^{-5}	2.08	2.744×10^{-5}	2.08	3.116×10^{-5}	2.07
1/128	6.029×10^{-6}	2.04	6.680×10^{-6}	2.04	7.614×10^{-6}	2.03
Δt	$\|p_h - p\|_{L^2}^{E,1}$	Rate	$\|p_h - p\|_{L^2}^{E,2}$	Rate	$\|p_h - p\|_{L^2}^{E,3}$	Rate
1/8	6.875×10^{-3}	-	7.310×10^{-3}	-	7.894×10^{-3}	-
1/16	1.721×10^{-3}	2.00	1.853×10^{-3}	1.98	2.024×10^{-3}	1.96
1/32	4.305×10^{-4}	2.00	4.665×10^{-4}	1.99	5.125×10^{-4}	1.98
1/64	1.076×10^{-4}	2.00	1.170×10^{-4}	2.00	1.289×10^{-4}	1.99
1/128	2.692×10^{-5}	2.00	2.930×10^{-5}	2.00	3.234×10^{-5}	2.00
Δt	$\|\phi_h - \phi\|_{H^1}^{E,1}$	Rate	$\|\phi_h - \phi\|_{H^1}^{E,2}$	Rate	$\|\phi_h - \phi\|_{H^1}^{E,3}$	Rate
1/8	4.734×10^{-3}	-	4.771×10^{-3}	-	4.954×10^{-3}	-
1/16	8.526×10^{-4}	2.47	8.070×10^{-4}	2.56	8.253×10^{-4}	2.59
1/32	1.799×10^{-4}	2.25	1.538×10^{-4}	2.39	1.521×10^{-4}	2.44
1/64	4.203×10^{-5}	2.10	3.300×10^{-5}	2.22	3.146×10^{-5}	2.27
1/128	1.025×10^{-5}	2.04	7.641×10^{-6}	2.11	7.087×10^{-6}	2.15

The initial and boundary conditions and the forcing terms are then chosen from this exact solution. A group of simulations with $J = 3$ members are performed for this convergence test. The three members are chosen by setting J different hydraulic conductivity tensors, i.e. the j-th sample of k_{11} and k_{22} are

$$k_{11}^j = 1 - 0.1(j-1), \quad k_{22}^j = 1 + 0.1(j-1), \quad j = 1, 2, 3.$$

To check the temporal convergence rate, we uniformly refine the mesh size h and time step size $\Delta t = h$ simutaneously, from initial time step size $\Delta t = 1/8$ to final size $\Delta t = 1/128$. In this setup, the expected errors are $O(h^2 + \Delta t) = O(\Delta t)$ for SAV-BE-En and $O(h^2 + \Delta t^2) = O(\Delta t^2)$ for SAV-BDF2-En. The approximation errors at the final time $T = 5$ by the SAV-BE-En scheme are listed in Table 5.1 for the fluid velocity u, fluid pressure p, and hydraulic head ϕ, illustrating that the SAV-BE-En algorithm is first order in time convergent. We also list the results by the SAV-BDF2-En scheme in Table 5.2, from which we observe the expected second order convergence.

5.3. Stochastic example. We then apply the SAV-BE-En algorithm to the computation of ensemble flows by setting a random hydraulic conductivity tensor $\mathcal{K}(x, y, \omega)$ in the Stokes-Darcy equations. Here $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space. The conductivity $\mathcal{K}(x, y, \omega)$ is assumed to be a diagonal stochastic tensor $\mathrm{diag}(k_{11}(x, y, \omega), k_{22}(x, y, \omega))$ that has a continuous and bounded correlation function. Specifically, the entries are given by the Karhunen-Loève expansion

$$k_{11}(x, y, \omega) = k_{22}(x, y, \omega) = a_0 + \sigma\sqrt{\lambda_0}Y_0(\omega) + \sum_{i=1}^{n_f} \sigma\sqrt{\lambda_i}[Y_i(\omega)\cos(i\pi x) + Y_{n_f+i}(\omega)\sin(i\pi x))], \tag{5.7}$$

where $\lambda_0 = \frac{1}{2}\sqrt{\pi L_c}$, $\lambda_i = \sqrt{\pi}L_c\exp(-\frac{1}{4}(i\pi L_c)^2)$ for $i = 1, \cdots, n_f$, and $Y_0, \cdots, Y_{2n_f}$ are indepdendent and identically uniformly distributed in the interval $[-\sqrt{3}, \sqrt{3}]$, so they have zero mean and unit variance. In the simulation, we take $n_f = 2$, so there are totally 5 random variables $Y_0, Y_1, \cdots, Y_4$. The other values are taken as $L_c = 0.25, a_0 = 1, \sigma = 0.15$.

The computational domain and physical parameters are mostly the same as those in the convergence test, except that we set the initial condition and Dirichlet boundary condition as

$$u(x, y, t, \omega) = (u_1(x, y, t, \omega),\ u_2(x, y, t, \omega)),$$

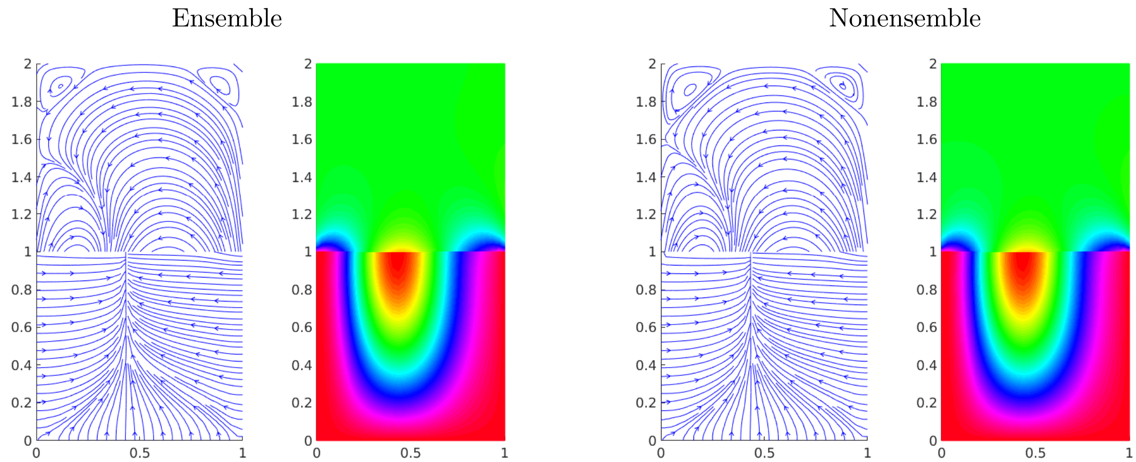


Fig. 5.1: Simulations at $T = 1.6$ by SAV-BE ensemble and nonensemble algorithms using sparse-grid method with $J = 241$ collocation points, $h = 1/50, \Delta t = 1/100$. From left to right: streamlines of the expectations of fluid flow velocity u and porous media flow velocity $v = -\mathcal{K}\nabla\phi$ by the ensemble scheme; expectations of fluid flow pressure p and hydraulic head ϕ by the ensemble scheme; streamlines of the expectations of u and $v = -\mathcal{K}\nabla\phi$ by the nonensemble scheme; expectations of p and ϕ by the nonensemble scheme.

$$
\begin{aligned}
u_1(x, y, t, \omega) &= Y_0(\omega)\big(y^2 - 2y + 1\big)\cos(t),\\
u_2(x, y, t, \omega) &= Y_1(\omega)\big(x^2 - x\big)\cos(t),\\
\phi(x, y, t, \omega) &= Y_2(\omega) y \cos(t).
\end{aligned}
$$

The problem is associated with the forcing terms $f_f = (Y_3(\omega)xy, Y_3(\omega)xy),\ f_p = Y_4(\omega)xy$.

We solve the stochastic Stokes-Darcy problem by a sparse-grid collocation method utilizing univariate Guassian quadrature. Taking $h = 1/50$ and $\Delta t = 1/100$, the numerical solutions at $T = 1.6$ using SAV-BE ensemble and nonensemble algorithms with $J = 241$ collocation points are illustrated in Figure 5.1. The streamlines of the expectations of fluid flow velocity u and porous media flow velocity $v = -\mathcal{K}\nabla\phi$ are plotted in the first and third pictures, for the ensemble and nonensemble schemes respectively. The expectations of fluid flow pressure p and hydraulic head ϕ are also plotted in the second and forth pictures. Figure 5.1 shows that the ensemble and nonensemble schemes obtain almost identical numerical results.

The computational times using SAV-BE ensemble and nonensemble algorithms are listed in Table 5.3. It is apparent that the SAV-BE ensemble method takes much less CPU time than the nonensemble algorithm. The ensemble algorithm reduces the computational time of the nonensemble algorithm to 6.83% in this test, thanks to the design that the algebraic matrix in the ensemble scheme is a constant matrix and shared by all realizations, such that the 241 linear systems in each time step can be simultaneously solved by block iterative solvers. To be specific, the symmetric positive definite system from BE sub-problem 2 and 4 (Darcy part) with multiple right hand sides can be solved by the block conjugate gradient (CG) solver; the indefinite system from BE sub-problem 1 and 3 (Stokes part) with multiple right hand sides can be efficiently solved by the block generalized minimal residual (GMRES) method. In this test, we use the breakdown-free block CG solver developed in [26], which addressed the rank deficiency issue, and the block GMRES algorithm with deflation [5, BFGMRESD(m)] to remove redundant information due to linear dependence of multiple residuals. The preconditioners used are the multigrid preconditioner for block CG and the block triangular preconditioner, which is tested to be faster than the block diagonal preconditioner, for block GMRES. For details on preconditioners the readers are referred to [9]. Our MATLAB implementation is based on the data structure of the iFEM package.

5.4. Realistic application. We then apply the proposed SAV-BE-En method to a more realistic simulation of the subsurface flow in a karst aquifer, inspired by example 4 in [46]. As shown in Figure 5.2, the free flow domain D_f with a curvy boundary $\overline{ABCDEFGH}$ is a Y-shape conduit which has a curvy

Table 5.3: CPU time using sparse-grid method with $J = 241$ collocation points, $h = 1/50, \Delta t = 1/100, T = 1.6$. The average CPU time per time step is denoted by $\bar{t}_{cpu}$.

	SAV-BE ensemble	SAV-BE nonensemble
Average time $\bar{t}_{cpu}$ per step	57 s	3.5×241 s
Total CPU time	9274 s	135711 s
CPU time percentage	6.83%	100%

interface with the porous media flow domain D_p, both of which form a unit square. Specifically, $A = (0, 0.8)$, $B = (0, 0.55)$, $C = (0.55, 0.4)$, $D = (0.7, 0)$, $E = (0.85, 0)$, $F = (0.75, 0.45)$, $G = (1, 0.5)$, and $H = (1, 0.7)$. The physical parameters g, ν, and S_0 are set to be equal to one, and $\alpha_{\text{BJS}} = 0.1$. In the simulation, we set the source terms to be zero and $\phi = 0$ on $\partial D_p \backslash I$. The hydraulic conductivity $\mathcal{K}(x, y, \omega)$ is assumed to be a diagonal stochastic tensor

$$\mathcal{K}(x, y, \omega) = \begin{bmatrix} mk_{11}(x, y, \omega) & 0 \\ 0 & mk_{22}(x, y, \omega) \end{bmatrix},$$

with $k_{11}(x, y, \omega)$ and $k_{22}(x, y, \omega)$ expressed as in (5.7). Here m is the conductivity magnitude, which varies in our experiments. The inflow/outflow boundary condition for u is

$$u = \begin{cases} (s_1, 0) & on \quad \overline{AB} \\ (0, s_2) & on \quad \overline{DE} \\ (s_3, 0) & on \quad \overline{GH} \end{cases},$$

where s_1, s_2 and s_3 are constants.

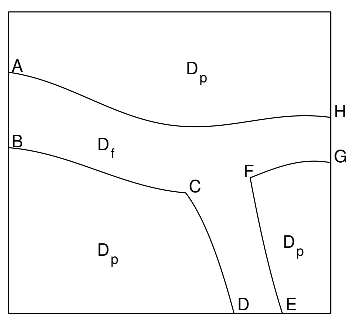
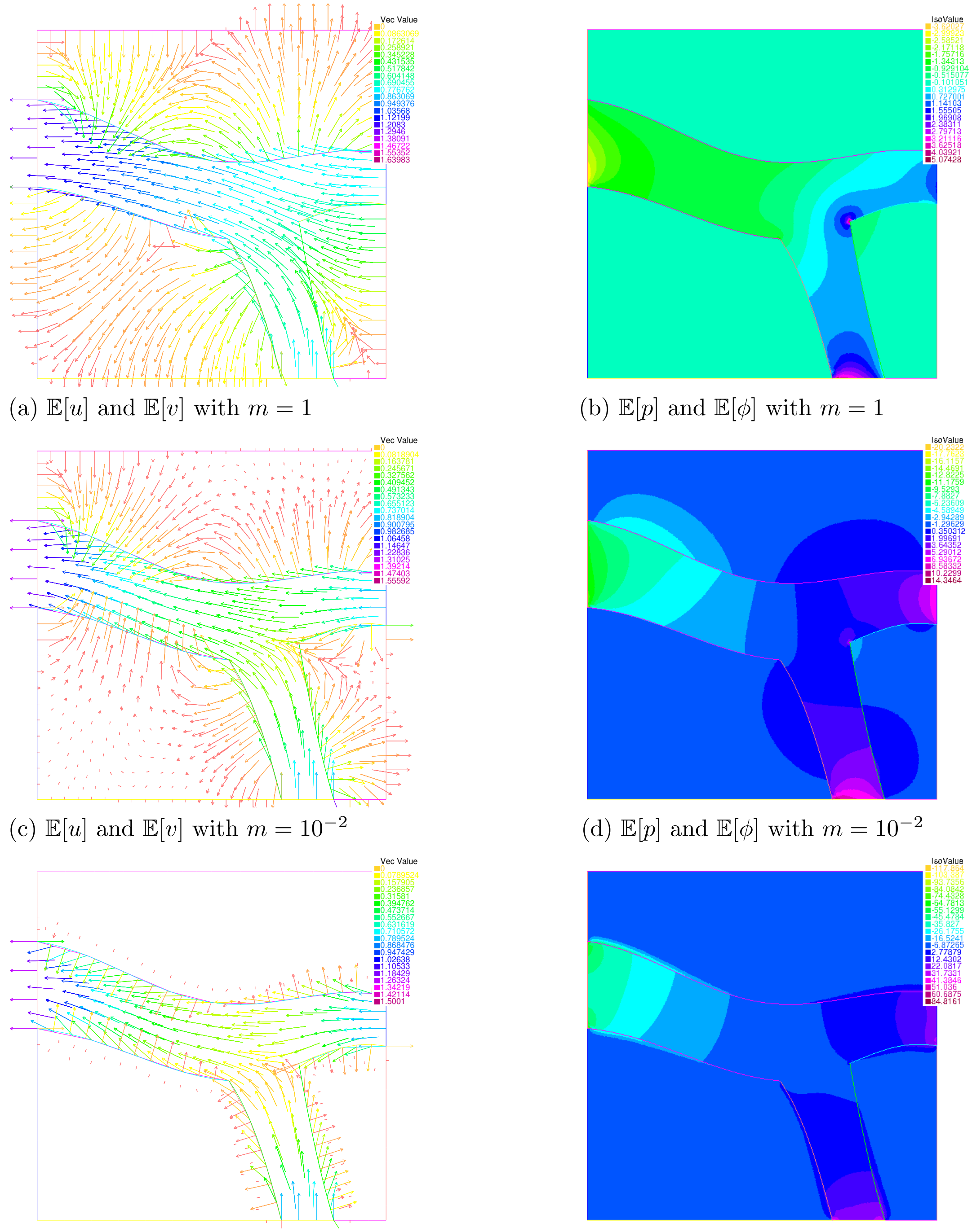


Fig. 5.2: Domains with curvy interface for simulating the subsurface flow in a karst aquifer.

Taking $h = 0.022$ and $\Delta t = 0.01$, the numerical solutions at $T = 1.0$ solved by SAV-BE-En with a sparse grid method ($J = 241$ collocation points) are illustrated in Figure 5.3, where the computed expectations of u, $v = -\mathcal{K}\nabla\phi$, p, and ϕ for different scenarioes are presented. In all cases, the inflow conditions are fixed by setting $s_2 = 1$ and $s_3 = -1$, and the outflow condition is given by $s_1 = -1.5$. To test the effect of the hydraulic conductivity on the solution, we set the magnitude m to be $1, 10^{-2}$, and 10^{-4}. The corresponding simulations are plotted from top to bottom. It is obvious that when m decreases, the flow speed in porous media is significantly reduced.

6. Conclusions. We have presented a new strategy to decouple the Stokes-Darcy system using a recently developed SAV idea, for fast computation of coupled flow ensembles. The proposed two SAV decoupled ensemble algorithms: SAV-BE-En and SAV-BDF2-En are extremely efficient as all ensemble members share the same constant coefficient matrix and the computation of the free flow and the porous



(a) $\mathbb{E}[u]$ and $\mathbb{E}[v]$ with $m = 1$

(b) $\mathbb{E}[p]$ and $\mathbb{E}[\phi]$ with $m = 1$

(c) $\mathbb{E}[u]$ and $\mathbb{E}[v]$ with $m = 10^{-2}$

(d) $\mathbb{E}[p]$ and $\mathbb{E}[\phi]$ with $m = 10^{-2}$

(e) $\mathbb{E}[u]$ and $\mathbb{E}[v]$ with $m = 10^{-4}$

(f) $\mathbb{E}[p]$ and $\mathbb{E}[\phi]$ with $m = 10^{-4}$

Fig. 5.3: Simulations with different conductivity magnitude m. From top to bottom: $m = 1,\ 10^{-2},\ 10^{-4}$. Left: expectations of fluid flow velocity u and porous media flow velocity $v = -\mathcal{K}\nabla\phi$; right: expectations of fluid flow pressure p and hydraulic head ϕ.

media flow are fully decoupled resulting in smaller linear systems to be solved. We proved both ensemble algorithms are long time stable under two parameter conditions, *without* any time step conditions. In particular, for a single simulation both algorithms are unconditionally long time stable. We have also provided a comprehensive error analysis for the first order SAV-BE-En algorithm. Numerical experiments are presented to confirm the convergence order and demonstrate the efficiency and effectiveness of the proposed ensemble algorithms in UQ and realistic coupled flow simulations.

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Appendix A. Proof of unconditional long time stability of Algorithm 3.2 .

Proof. Setting $v_h = u_{j,h}^{n+1}$, $q_h = p_{j,h}^{n+1}$, $\psi_h = \phi_{j,h}^{n+1}$ in Algorithm 3.2, multiplying (3.15) by $r_{j,h}^{n+1}$ and adding all four equations yields

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2-\frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2+\frac{1}{2\Delta t}\|u_{j,h}^{n+1}-u_{j,h}^{n}\|_f^2+\nu\|\nabla u_{j,h}^{n+1}\|_f^2 \\
&+\sum_i\int_I \eta_i^{max}(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2-\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2 \\
&+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}-\phi_{j,h}^{n}\|_p^2+gk_{max}(\nabla\phi_{j,h}^{n+1},\nabla\phi_{j,h}^{n+1})_p+\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(u_{j,h}^{n+1},\phi_{j,h}^{n})-\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}c_I(u_{j,h}^{n},\phi_{j,h}^{n+1}) \\
&+\frac{1}{2\Delta t}|r_{j,h}^{n+1}|^2-\frac{1}{2\Delta t}|r_{j,h}^{n}|^2+\frac{1}{2\Delta t}|r_{j,h}^{n+1}-r_{j,h}^{n}|^2+\frac{1}{T}|r_{j,h}^{n+1}|^2-\frac{r_j^{n+1}}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_{j,h}^{n+1},\phi_{j,h}^{n})-c_I(u_{j,h}^{n},\phi_{j,h}^{n+1})\right) \\
&=(f_{f,j}^{n+1},u_{j,h}^{n+1})_f+g(f_{p,j}^{n+1},\phi_{j,h}^{n+1})_p-\sum_i\int_I(\eta_{i,j}-\eta_i^{max})(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds \\
&\quad -g((\mathcal{K}_j-k_{max}\mathcal{I})\nabla\phi_{j,h}^{n},\nabla\phi_{j,h}^{n+1})_p.
\end{aligned}
\tag{A.1}
$$

The main difference from the proof of Theorem (3.1) is on the estimates of the following two terms.

$$
\begin{aligned}
&-\sum_i\int_I(\eta_{i,j}-\eta_i^{max})(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds\le\sum_i\int_I|\eta_{i,j}-\eta_i^{max}|\,(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds \\
&\le\sum_i\eta_i^{max}\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)\ ds\le\sum_i\left[\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds+\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&-g\left((\mathcal{K}_j-k_{max}\mathcal{I})\nabla\phi_{j,h}^{n},\nabla\phi_{j,h}^{n+1}\right)_p\le g\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\mathcal{K}_j-k_{max}\mathcal{I}|_2|\nabla\phi_{j,h}^{n}|_2\,dx \\
&\le g(k_{max}-k_{min})\int_{D_p}|\nabla\phi_{j,h}^{n+1}|_2|\nabla\phi_{j,h}^{n}|_2\,dx\le g(k_{max}-k_{min})\|\nabla\phi_{j,h}^{n}\|_p\|\nabla\phi_{j,h}^{n+1}\|_p \\
&\le\frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^{n}\|_p^2+\frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^{n+1}\|_p^2.
\end{aligned}
$$

Then we have the following inequality

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2-\frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2+\frac{1}{2}\nu\|\nabla u_{j,h}^{n+1}\|_f^2+\sum_i\frac{\eta_i^{max}}{2}\left[\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds-\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right] \\
&+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2-\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2+(1-\beta-\frac{k_{max}-k_{min}}{k_{max}})gk_{max}\|\nabla\phi_{j,h}^{n+1}\|_p^2 \\
&+\frac{g(k_{max}-k_{min})}{2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2-\|\nabla\phi_{j,h}^{n}\|_p^2\right) \\
&+\frac{1}{2\Delta t}|r_{j,h}^{n+1}|^2-\frac{1}{2\Delta t}|r_{j,h}^{n}|^2+\frac{1}{2\Delta t}|r_{j,h}^{n+1}-r_{j,h}^{n}|^2+\frac{1}{T}|r_{j,h}^{n+1}|^2 \\
&\le\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2+\frac{gC_{P,p}^2}{4\beta k_{max}}\|f_{p,j}^{n+1}\|_p^2.
\end{aligned}
\tag{A.2}
$$

Since we assume $\mathcal{K}_j$ is SPD, and any two ensemble members have different hydraulic conductivity tensor $\mathcal{K}$, we have $k_{max}>k_{min}>0$ and thus $0<\frac{k_{max}-k_{min}}{k_{max}}<1$. So we do not need any constraints on these parameters. Now taking $\beta=\frac{1}{2}(1-\frac{k_{max}-k_{min}}{k_{max}})=\frac{k_{min}}{2k_{max}}$, (A.2) reduces to

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|u_{j,h}^{n+1}\|_f^2-\frac{1}{2\Delta t}\|u_{j,h}^{n}\|_f^2+\sum_i\frac{\eta_i^{max}}{2}\left[\int_I(u_{j,h}^{n+1}\cdot\widehat{\tau}_i)^2\ ds-\int_I(u_{j,h}^{n}\cdot\widehat{\tau}_i)^2\ ds\right] \\
&+\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n+1}\|_p^2-\frac{gS_0}{2\Delta t}\|\phi_{j,h}^{n}\|_p^2+\frac{g(k_{max}-k_{min})}{2}\left(\|\nabla\phi_{j,h}^{n+1}\|_p^2-\|\nabla\phi_{j,h}^{n}\|_p^2\right)
\end{aligned}
\tag{A.3}
$$

$$+\frac{1}{2\Delta t}|r_{j,h}^{n+1}|^2-\frac{1}{2\Delta t}|r_{j,h}^{n}|^2+\frac{1}{2\Delta t}|r_{j,h}^{n+1}-r_{j,h}^{n}|^2+\frac{1}{T}|r_{j,h}^{n+1}|^2$$
$$\le\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2+\frac{gC_{P,p}^2}{2k_{min}}\|f_{p,j}^{n+1}\|_p^2.$$

Sum (A.3) from $n=0$ to $N-1$ and multiply through by Δt to get

$$\frac{1}{2}\|u_{j,h}^N\|_f^2+\frac{gS_0}{2}\|\phi_{j,h}^N\|_p^2+\Delta t\sum_i\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^N\cdot\widehat{\tau}_i)^2\,ds+\Delta t\frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^N\|_p^2 \tag{A.4}$$
$$+\frac{1}{2}|r_{j,h}^N|^2+\frac{1}{2}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}-r_{j,h}^n|^2+\frac{\Delta t}{T}\sum_{n=0}^{N-1}|r_{j,h}^{n+1}|^2$$
$$\le\frac{1}{2}\|u_{j,h}^0\|_f^2+\frac{gS_0}{2}\|\phi_{j,h}^0\|_p^2+\Delta t\sum_i\frac{\eta_i^{max}}{2}\int_I(u_{j,h}^0\cdot\widehat{\tau}_i)^2\,ds$$
$$+\Delta t\frac{g(k_{max}-k_{min})}{2}\|\nabla\phi_{j,h}^0\|_p^2+\frac{1}{2}|r_{j,h}^0|^2+\Delta t\sum_{n=0}^{N-1}\frac{C_{P,f}^2}{2\nu}\|f_{f,j}^{n+1}\|_f^2+\Delta t\sum_{n=0}^{N-1}\frac{gC_{P,p}^2}{2k_{min}}\|f_{p,j}^{n+1}\|_p^2.$$

□

Appendix B. Proof of Theorem 4.2.

Proof. Step 1: formulate the energy equations of the solution errors $e_{j,u}^{n+1}$, $e_{j,\phi}^{n+1}$, *and* $e_{j,r}^{n+1}$.

For $\forall v_h\in V_f^h,\forall\psi_h\in X_p^h,\forall\lambda_h^{n+1}\in Q_f^h$, the true solution (u_j,p_j,ϕ_j) satisfies

$$\left(\frac{u_j^{n+1}-u_j^n}{\Delta t},v_h\right)_f+\nu(\nabla u_j^{n+1},\nabla v_h)_f+\sum_i\int_I\eta_{i,j}(u_j^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds \tag{B.1}$$
$$-\left(p_j^{n+1}-\lambda_h^{n+1},\nabla\cdot v_h\right)_f+c_I(v_h,\phi_j^n)=(f_{f,j}^{n+1},v_h)_f+\epsilon_{j,f}^{n+1}(v_h),$$
$$gS_0\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t},\psi_h\right)_p+g(\mathcal{K}_j\nabla\phi_j^{n+1},\nabla\psi_h)_p-c_I(u_j^n,\psi_h) \tag{B.2}$$
$$=g(f_{p,j}^{n+1},\psi_h)_p+\epsilon_{j,p}^{n+1}(\psi_h),$$

and the true solution r_j satisfies

$$\frac{r_j^{n+1}-r_j^n}{\Delta t}=-\frac{1}{T}r_j^{n+1}+\frac{1}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_j^{n+1},\phi_j^n)-c_I(u_j^n,\phi_j^{n+1})+\epsilon_j^{n+1}(r)\right). \tag{B.3}$$

The consistency errors $\epsilon_{j,f}^{n+1}(v_h),\epsilon_{j,p}^{n+1}(\psi_h),\epsilon_j^{n+1}(r)$ are defined by

$$\epsilon_{j,f}^{n+1}(v_h):=\left(\frac{u_j^{n+1}-u_j^n}{\Delta t}-\partial_t u_j(t_{n+1}),v_h\right)_f-c_I(v_h,\phi_j^{n+1}-\phi_j^n),$$
$$\epsilon_{j,p}^{n+1}(\psi_h):=gS_0\left(\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t}-\partial_t\phi_j(t_{n+1}),\psi_h\right)_p+c_I(u_j^{n+1}-u_j^n,\psi_h),$$
$$\epsilon_j^{n+1}(r):=\frac{r_j^{n+1}-r_j^n}{\Delta t}-\dot{r}_j(t_{n+1})+\frac{1}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_j^{n+1},\phi_j^{n+1}-\phi_j^n)-c_I(u_j^{n+1}-u_j^n,\phi_j^{n+1})\right).$$

Subtracting (2.3)-(2.4) from (B.1), (2.5) from (B.2), (2.6) from (B.3) gives, for $\forall v_h\in V_f^h,\forall\psi_h\in X_p^h,\forall\lambda_h^{n+1}\in Q_f^h$,

$$\left(\frac{e_{j,u}^{n+1}-e_{j,u}^n}{\Delta t},v_h\right)_f+\nu(\nabla e_{j,u}^{n+1},\nabla v_h)_f+\sum_i\int_I\bar{\eta}_i(e_{j,u}^{n+1}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\,ds \tag{B.4}$$

$$+\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,(e^n_{j,u}\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\;ds - \left(p^{n+1}_j-\lambda^{n+1}_h,\nabla\cdot v_h\right)_f + c_I(v_h,\phi^n_j) - \frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_{j,h})$$

$$=-\sum_i \int_I (\eta_{i,j}-\bar{\eta}_i)\,((u^{n+1}_j-u^n_j)\cdot\widehat{\tau}_i)(v_h\cdot\widehat{\tau}_i)\;ds+\epsilon^{n+1}_{j,f}(v_h),$$

$$gS_0\left(\frac{e^{n+1}_{j,\phi}-e^n_{j,\phi}}{\Delta t},\psi_h\right)_p+g(\bar{\mathcal{K}}\nabla e^{n+1}_{j,\phi},\nabla\psi_h)_p+g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla e^n_{j,\phi},\nabla\psi_h)_p \tag{B.5}$$

$$-c_I(u^n_j,\psi_h)+\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_{j,h},\psi_h)=-g((\mathcal{K}_j-\bar{\mathcal{K}})\nabla(\phi^{n+1}_j-\phi^n_j),\nabla\psi_h)_p+\epsilon^{n+1}_{j,p}(\psi_h),$$

$$\frac{e^{n+1}_{j,r}-e^n_{j,r}}{\Delta t}=-\frac{1}{T}e^{n+1}_{j,r}+\frac{1}{exp(-\frac{t^{n+1}}{T})}\Bigg(c_I(u^{n+1}_j,\phi^n_j)-c_I(u^{n+1}_{j,h},\phi^n_{j,h}) \tag{B.6}$$

$$-c_I(u^n_j,\phi^{n+1}_j)+c_I(u^n_{j,h},\phi^{n+1}_{j,h})\Bigg)+\epsilon^{n+1}_j(r).$$

The coupling terms in (B.4) and (B.5) can be rewritten as

$$c_I(v_h,\phi^n_j)-\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_{j,h}) \tag{B.7}$$

$$=\frac{exp(-\frac{t^{n+1}}{T})}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_j)-\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_j)+\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_j)-\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_{j,h})$$

$$=\frac{e^{n+1}_{j,r}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,\phi^n_j)+\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(v_h,e^n_{j,\phi}),$$

and

$$c_I(u^n_j,\psi_h)-\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_{j,h},\psi_h) \tag{B.8}$$

$$=\frac{exp(-\frac{t^{n+1}}{T})}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_j,\psi_h)-\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_j,\psi_h)+\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_j,\psi_h)-\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_{j,h},\psi_h)$$

$$=\frac{e^{n+1}_{j,r}}{exp(-\frac{t^{n+1}}{T})}c_I(u^n_j,\psi_h)+\frac{r^{n+1}_{j,h}}{exp(-\frac{t^{n+1}}{T})}c_I(e^n_{j,u},\psi_h).$$

The coupling terms in (B.6) can be rewritten as

$$c_I(u^{n+1}_j,\phi^n_j)-c_I(u^{n+1}_{j,h},\phi^n_{j,h})=c_I(u^{n+1}_j,\phi^n_j)-c_I(u^{n+1}_{j,h},\phi^n_j)+c_I(u^{n+1}_{j,h},\phi^n_j)-c_I(u^{n+1}_{j,h},\phi^n_{j,h}) \tag{B.9}$$

$$=c_I(e^{n+1}_{j,u},\phi^n_j)+c_I(u^{n+1}_{j,h},e^n_{j,\phi}),$$

and

$$c_I(u^n_j,\phi^{n+1}_j)-c_I(u^n_{j,h},\phi^{n+1}_{j,h})=c_I(u^n_j,\phi^{n+1}_j)-c_I(u^n_j,\phi^{n+1}_{j,h})+c_I(u^n_j,\phi^{n+1}_{j,h})-c_I(u^n_{j,h},\phi^{n+1}_{j,h}) \tag{B.10}$$

$$=c_I(u^n_j,e^{n+1}_{j,\phi})+c_I(e^n_{j,u},\phi^{n+1}_{j,h}).$$

Multiplying (B.6) by $e^{n+1}_{j,r}$ gives

$$\frac{1}{2\Delta t}\left(|e^{n+1}_{j,r}|^2-|e^n_{j,r}|^2+|e^{n+1}_{j,r}-e^n_{j,r}|^2\right)+\frac{1}{T}|e^{n+1}_{j,r}|^2 \tag{B.11}$$

$$=\frac{e^{n+1}_{j,r}}{exp(-\frac{t^{n+1}}{T})}\Bigg(c_I(e^{n+1}_{j,u},\phi^n_j)+c_I(u^{n+1}_{j,h},e^n_{j,\phi})-c_I(u^n_j,e^{n+1}_{j,\phi})-c_I(e^n_{j,u},\phi^{n+1}_{j,h})\Bigg)+\epsilon^{n+1}_j(r)e^{n+1}_{j,r}.$$

Step 2: split the solution errors, formulate the energy equation of errors in finite element spaces. Let U_j^{n+1}, Φ_j^{n+1} be any interpolation of u_j^{n+1} and ϕ_j^{n+1} in V_f^h and X_p^h correspondingly. Denote

$$
\begin{aligned}
e_{j,u}^{n+1} &= (u_j^{n+1} - U_j^{n+1}) + \left(U_j^{n+1} - u_{j,h}^{n+1}\right) =: \mu_{j,u}^{n+1} + \xi_{j,u}^{n+1}, \\
e_{j,\phi}^{n+1} &= (\phi_j^{n+1} - \Phi_j^{n+1}) + \left(\Phi_j^{n+1} - \phi_{j,h}^{n+1}\right) =: \mu_{j,\phi}^{n+1} + \xi_{j,\phi}^{n+1}.
\end{aligned}
$$

Then (B.4), (B.5) and (B.11) can be rewritten as

$$
\begin{aligned}
&\left(\frac{\xi_{j,u}^{n+1} - \xi_{j,u}^n}{\Delta t}, v_h\right)_f + \nu(\nabla \xi_{j,u}^{n+1}, \nabla v_h)_f + \sum_i \int_I \bar{\eta}_i (\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds \qquad \text{(B.12)} \\
&\quad + \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\xi_{j,u}^n \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds - \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla \cdot v_h\right)_f + \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(v_h, \xi_{j,\phi}^n) \\
&= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,((u_j^{n+1} - u_j^n) \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds + \epsilon_{j,f}^{n+1}(v_h) - \left(\frac{\mu_{j,u}^{n+1} - \mu_{j,u}^n}{\Delta t}, v_h\right)_f - \nu(\nabla \mu_{j,u}^{n+1}, \nabla v_h)_f \\
&\quad - \sum_i \int_I \bar{\eta}_i (\mu_{j,u}^{n+1} \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds - \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\mu_{j,u}^n \cdot \widehat{\tau}_i)(v_h \cdot \widehat{\tau}_i)\, ds \\
&\quad - \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(v_h, \phi_j^n) - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(v_h, \mu_{j,\phi}^n),
\end{aligned}
$$

$$
\begin{aligned}
&gS_0 \left(\frac{\xi_{j,\phi}^{n+1} - \xi_{j,\phi}^n}{\Delta t}, \psi_h\right)_p + g(\bar{\mathcal{K}} \nabla \xi_{j,\phi}^{n+1}, \nabla \psi_h)_p + g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \xi_{j,\phi}^n, \nabla \psi_h)_p - \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(\xi_{j,u}^n, \psi_h) \qquad \text{(B.13)} \\
&= -g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla (\phi_j^{n+1} - \phi_j^n), \nabla \psi_h)_p + \epsilon_{j,p}^{n+1}(\psi_h) - gS_0 \left(\frac{\mu_{j,\phi}^{n+1} - \mu_{j,\phi}^n}{\Delta t}, \psi_h\right)_p \\
&\quad - g(\bar{\mathcal{K}} \nabla \mu_{j,\phi}^{n+1}, \nabla \psi_h)_p - g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \mu_{j,\phi}^n, \nabla \psi_h)_p + \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(u_j^n, \psi_h) + \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} c_I(\mu_{j,u}^n, \psi_h),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(|e_{j,r}^{n+1}|^2 - |e_{j,r}^n|^2 + |e_{j,r}^{n+1} - e_{j,r}^n|^2\right) + \frac{1}{T}|e_{j,r}^{n+1}|^2 \qquad \text{(B.14)} \\
&= \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \Bigg(c_I(\xi_{j,u}^{n+1}, \phi_j^n) + c_I(\mu_{j,u}^{n+1}, \phi_j^n) + c_I(u_{j,h}^{n+1}, \xi_{j,\phi}^n) + c_I(u_{j,h}^{n+1}, \mu_{j,\phi}^n) - c_I(u_j^n, \xi_{j,\phi}^{n+1}) \\
&\quad - c_I(u_j^n, \mu_{j,\phi}^{n+1}) - c_I(\xi_{j,u}^n, \phi_{j,h}^{n+1}) - c_I(\mu_{j,u}^n, \phi_{j,h}^{n+1}) \Bigg) + \epsilon_j^{n+1}(r) \cdot e_{j,r}^{n+1}.
\end{aligned}
$$

Setting $v_h = \xi_{j,u}^{n+1}$ in (B.12), $\psi_h = \xi_{j,\phi}^{n+1}$ in (B.13) and adding these two equations and (B.14) yields

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,u}^n\|_f^2 + \frac{1}{2\Delta t}\|\xi_{j,u}^{n+1} - \xi_{j,u}^n\|_f^2 + \nu\|\nabla \xi_{j,u}^{n+1}\|_f^2 + \sum_i \int_I \bar{\eta}_i (\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)^2\, ds \qquad \text{(B.15)} \\
&+ \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^n\|_p^2 + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1} - \xi_{j,\phi}^n\|_p^2 + g(\bar{\mathcal{K}} \nabla \xi_{j,\phi}^{n+1}, \nabla \xi_{j,\phi}^{n+1})_p \\
&+ \frac{1}{2\Delta t}\left(|e_{j,r}^{n+1}|^2 - |e_{j,r}^n|^2 + |e_{j,r}^{n+1} - e_{j,r}^n|^2\right) + \frac{1}{T}|e_{j,r}^{n+1}|^2 \\
&= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\xi_{j,u}^n \cdot \widehat{\tau}_i)(\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)\, ds - \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,((u_j^{n+1} - u_j^n) \cdot \widehat{\tau}_i)(\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)\, ds \\
&\quad - \sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\mu_{j,u}^n \cdot \widehat{\tau}_i)(\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)\, ds - \sum_i \int_I \bar{\eta}_i (\mu_{j,u}^{n+1} \cdot \widehat{\tau}_i)(\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)\, ds \\
&\quad - g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla (\phi_j^{n+1} - \phi_j^n), \nabla \xi_{j,\phi}^{n+1})_p - g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \mu_{j,\phi}^n, \nabla \xi_{j,\phi}^{n+1})_p - g((\mathcal{K}_j - \bar{\mathcal{K}}) \nabla \xi_{j,\phi}^n, \nabla \xi_{j,\phi}^{n+1})_p
\end{aligned}
$$

$$
\begin{aligned}
&- g(\bar{\mathcal{K}}\nabla\mu_{j,\phi}^{n+1}, \nabla\xi_{j,\phi}^{n+1})_p - \nu(\nabla\mu_{j,u}^{n+1}, \nabla\xi_{j,u}^{n+1})_f - \left(\frac{\mu_{j,u}^{n+1}-\mu_{j,u}^{n}}{\Delta t}, \xi_{j,u}^{n+1}\right)_f - gS_0\left(\frac{\mu_{j,\phi}^{n+1}-\mu_{j,\phi}^{n}}{\Delta t}, \xi_{j,\phi}^{n+1}\right)_p \\
&- \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\Big(c_I(\xi_{j,u}^{n+1}, \xi_{j,\phi}^{n}) - c_I(\xi_{j,u}^{n}, \xi_{j,\phi}^{n+1})\Big) + \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\Big(c_I(\mu_{j,u}^{n}, \xi_{j,\phi}^{n+1}) - c_I(\xi_{j,u}^{n+1}, \mu_{j,\phi}^{n})\Big) \\
&+ \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\Bigg(c_I(\mu_{j,u}^{n+1}, \phi_j^n) + c_I(u_{j,h}^{n+1}, \xi_{j,\phi}^{n}) + c_I(u_{j,h}^{n+1}, \mu_{j,\phi}^{n}) - c_I(u_j^n, \mu_{j,\phi}^{n+1}) - c_I(\xi_{j,u}^{n}, \phi_{j,h}^{n+1}) \\
&- c_I(\mu_{j,u}^{n}, \phi_{j,h}^{n+1})\Bigg) + \epsilon_{j,f}^{n+1}(\xi_{j,u}^{n+1}) + \epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) + \epsilon_j^{n+1}(r)e_{j,r}^{n+1} + \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot\xi_{j,u}^{n+1}\right)_f \\
&:= \mathcal{R}_1 + \cdots + \mathcal{R}_{14} + \epsilon_{j,f}^{n+1}(\xi_{j,u}^{n+1}) + \epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) + \epsilon_j^{n+1}(r)e_{j,r}^{n+1} + \left(p_j^{n+1} - \lambda_h^{n+1}, \nabla\cdot\xi_{j,u}^{n+1}\right)_f .
\end{aligned}
$$

Step 3: bound the right hand side of (B.15)*, i.e. the energy equation of errors in finite element spaces.* Next we bound the terms on the right hand side of (B.15).

$$
\begin{aligned}
\mathcal{R}_1 &= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\xi_{j,u}^{n}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \le \sum_i \eta_{i,j}^{\prime max}\int_I |(\xi_{j,u}^{n}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\; ds \qquad \text{(B.16)}\\
&\le \sum_i \left[\frac{\eta_i^{\prime max}}{2}\int_I (\xi_{j,u}^{n}\cdot\widehat{\tau}_i)^2\; ds + \frac{\eta_i^{\prime max}}{2}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right].
\end{aligned}
$$

By (2.1) and Poincaré inequality, we have, for any $\sigma_1 > 0$

$$
\begin{aligned}
\mathcal{R}_2 &= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \le \sum_i \eta_{i,j}^{\prime max}\int_I |((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\; ds \\
&\le \sum_i \eta_i^{\prime max}\left[\frac{1}{2\sigma_1}\int_I \left((u_j^{n+1}-u_j^n)\cdot\widehat{\tau}_i\right)^2\; ds + \frac{\sigma_1}{2}\int_I \left(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i\right)^2\; ds\right] \qquad \text{(B.17)}\\
&\le \sum_i \left[\frac{\eta_i^{\prime max}}{2\sigma_1}\|u_j^{n+1}-u_j^n\|_I^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[\frac{C_{P,f}C^2(D_f)}{2\sigma_1}\eta_i^{\prime max}\|\nabla(u_j^{n+1}-u_j^n)\|_f^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t u_j)\|_f^2\; dt + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right].
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_3 &= -\sum_i \int_I (\eta_{i,j} - \bar{\eta}_i)\,(\mu_{j,u}^{n}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \le \sum_i \eta_{i,j}^{\prime max}\int_I |(\mu_{j,u}^{n}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)|\; ds \qquad \text{(B.18)}\\
&\le \sum_i \eta_i^{\prime max}\left[\frac{1}{2\sigma_1}\int_I (\mu_{j,u}^{n}\cdot\widehat{\tau}_i)^2\; ds + \frac{\sigma_1}{2}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\right]\; ds \\
&\le \sum_i \left[\frac{1}{2\sigma_1}\eta_i^{\prime max}\|\mu_{j,u}^{n}\|_I^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[C\|\nabla\mu_{j,u}^{n}\|_f^2 + \frac{\sigma_1}{2}\eta_i^{\prime max}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right].
\end{aligned}
$$

Similarly, for any $\sigma_2 > 0$

$$
\begin{aligned}
\mathcal{R}_4 &= -\sum_i \int_I \bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)\; ds \le \sum_i \left[\frac{\bar{\eta}_i^{max}}{4\sigma_2\bar{\eta}_i^{min}}\int_I \bar{\eta}_i(\mu_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds + \frac{\sigma_2\bar{\eta}_i^{min}}{\bar{\eta}_i^{max}}\int_I \bar{\eta}_i(\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right] \\
&\le \sum_i \left[\frac{(\bar{\eta}_i^{max})^2}{4\sigma_2\bar{\eta}_i^{min}}\|\mu_{j,u}^{n+1}\|_I^2 + \sigma_2\bar{\eta}_i^{min}\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\; ds\right]
\end{aligned}
$$

$$\leq \sum_i \left[C\|\nabla \mu_{j,u}^{n+1}\|_f^2 + \sigma_2 \bar{\eta}_i^{min} \int_I (\xi_{j,u}^{n+1} \cdot \widehat{\tau}_i)^2 \, ds \right]. \tag{B.19}$$

The hydraulic conductivity tensor terms are estimated as follows.

$$\begin{aligned}
\mathcal{R}_5 = -g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla \xi_{j,\phi}^n, \nabla \xi_{j,\phi}^{n+1})_p &\leq g \int_{D_p} |\nabla \xi_{j,\phi}^n|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\
&\leq g \int_{D_p} \rho_j'(x) |\nabla \xi_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \leq g\rho_{j,max}' \int_{D_p} |\nabla \xi_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\
&\leq g\rho_{max}' \|\nabla \xi_{j,\phi}^n\|_p \|\nabla \xi_{j,\phi}^{n+1}\|_p \leq \frac{g\rho_{max}'}{2} \|\nabla \xi_{j,\phi}^n\|_p^2 + \frac{g\rho_{max}'}{2} \|\nabla \xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned} \tag{B.20}$$

For any $\sigma_3 > 0$, we have

$$\begin{aligned}
\mathcal{R}_6 = -g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla(\phi_j^{n+1} - \phi_j^n), \nabla \xi_{j,\phi}^{n+1})_p &\leq g \int_{D_p} |\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\
&\leq g \int_{D_p} \rho_j'(x) |\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \leq g\rho_{j,max}' \int_{D_p} |\nabla(\phi_j^{n+1} - \phi_j^n)|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\
&\leq g\rho_{max}' \|\nabla(\phi_j^{n+1} - \phi_j^n)\|_p \|\nabla \xi_{j,\phi}^{n+1}\|_p \leq \frac{g\rho_{max}'}{2\sigma_3} \|\nabla(\phi_j^{n+1} - \phi_j^n)\|_p^2 + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \\
&\leq \frac{g\rho_{max}'}{2\sigma_3} \| \int_{t^n}^{t^{n+1}} \nabla(\partial_t \phi_j) \, dt\|_p^2 + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \\
&\leq \frac{g\rho_{max}'}{2\sigma_3} \Delta t \int_{t^n}^{t^{n+1}} \|\nabla(\partial_t \phi_j)\|_p^2 \, dt + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned} \tag{B.21}$$

Similarly,

$$\begin{aligned}
\mathcal{R}_7 = -g((\mathcal{K}_j - \bar{\mathcal{K}})\nabla \mu_{j,\phi}^n, \nabla \xi_{j,\phi}^{n+1})_p &\leq g \int_{D_p} |\nabla \mu_{j,\phi}^n|_2 |\mathcal{K}_j - \bar{\mathcal{K}}|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\
&\leq g \int_{D_p} \rho_j'(x) |\nabla \mu_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \leq g\rho_{j,max}' \int_{D_p} |\nabla \mu_{j,\phi}^n|_2 |\nabla \xi_{j,\phi}^{n+1}|_2 \, dx \\
&\leq g\rho_{max}' \|\nabla \mu_{j,\phi}^n\|_p \|\nabla \xi_{j,\phi}^{n+1}\|_p \leq \frac{g\rho_{max}'}{2\sigma_3} \|\nabla \mu_{j,\phi}^n\|_p^2 + \frac{\sigma_3}{2} g\rho_{max}' \|\nabla \xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned} \tag{B.22}$$

For any $\beta_1 > 0, \beta_2 > 0$,

$$\begin{aligned}
\mathcal{R}_8 + \mathcal{R}_9 &= -g(\bar{\mathcal{K}}\nabla \mu_{j,\phi}^{n+1}, \nabla \xi_{j,\phi}^{n+1})_p - \nu(\nabla \mu_{j,u}^{n+1}, \nabla \xi_{j,u}^{n+1})_f \\
&\leq C \left(\|\nabla \mu_{j,u}^{n+1}\|_f^2 + \|\nabla \mu_{j,\phi}^{n+1}\|_p^2 \right) + \beta_1 \nu \|\nabla \xi_{j,u}^{n+1}\|_f^2 + \beta_2 g \bar{k}_{min} \|\nabla \xi_{j,\phi}^{n+1}\|_p^2,
\end{aligned} \tag{B.23}$$

and

$$\begin{aligned}
&\mathcal{R}_{10} + \mathcal{R}_{11} = -\left(\frac{\mu_{j,u}^{n+1} - \mu_{j,u}^n}{\Delta t}, \xi_{j,u}^{n+1} \right)_f - gS_0 \left(\frac{\mu_{j,\phi}^{n+1} - \mu_{j,\phi}^n}{\Delta t}, \xi_{j,\phi}^{n+1} \right)_p \\
&\leq \frac{C_{P,f}^2}{4\nu\beta_1} \| \frac{\mu_{j,u}^{n+1} - \mu_{j,u}^n}{\Delta t} \|_f^2 + \beta_1 \nu \|\nabla \xi_{j,u}^{n+1}\|_f^2 + \frac{C_{P,p}^2 g S_0^2}{4\beta_2 \bar{k}_{min}} \| \frac{\mu_{j,\phi}^{n+1} - \mu_{j,\phi}^n}{\Delta t} \|_p^2 + \beta_2 g \bar{k}_{min} \|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \\
&\leq \frac{C_{P,f}^2}{4\nu\beta_1} \left\| \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \partial_t \mu_{j,u} \, dt \right\|_f^2 + \beta_1 \nu \|\nabla \xi_{j,u}^{n+1}\|_f^2 + \frac{C_{P,p}^2 g S_0^2}{4\beta_2 \bar{k}_{min}} \left\| \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \partial_t \mu_{j,\phi} \, dt \right\|_p^2 + \beta_2 g \bar{k}_{min} \|\nabla \xi_{j,\phi}^{n+1}\|_p^2 \\
&\leq \frac{C}{\Delta t} \int_{t^n}^{t^{n+1}} \|\partial_t \mu_{j,u}\|_f^2 \, dt + \beta_1 \nu \|\nabla \xi_{j,u}^{n+1}\|_f^2 + \frac{C}{\Delta t} \int_{t^n}^{t^{n+1}} \|\partial_t \mu_{j,\phi}\|_p^2 \, dt + \beta_2 g \bar{k}_{min} \|\nabla \xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned} \tag{B.24}$$

From the stability result (3.2), we have $|r_{j,h}^{n+1}| \le C_3$ and thus $\frac{|r_{j,h}^{n+1}|}{exp(-\frac{t^{n+1}}{T})} = |r_{j,h}^{n+1}| exp(\frac{t^{n+1}}{T}) \le C$. For any $\beta_1 > 0, \beta_2 > 0$,

$$\begin{aligned}
|\mathcal{R}_{12}| &= \left| \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \big(c_I(\xi_{j,u}^{n+1}, \xi_{j,\phi}^{n}) - c_I(\xi_{j,u}^{n}, \xi_{j,\phi}^{n+1}) \big) \right| \le C|c_I(\xi_{j,u}^{n+1}, \xi_{j,\phi}^{n})| + C|c_I(\xi_{j,u}^{n}, \xi_{j,\phi}^{n+1})| \quad \text{(B.25)}\\
&\le C\|\xi_{j,u}^{n+1}\|_f^{1/2}\|\nabla\xi_{j,u}^{n+1}\|_f^{1/2}\|\xi_{j,\phi}^{n}\|_p^{1/2}\|\nabla\xi_{j,\phi}^{n}\|_p^{1/2} + C\|\xi_{j,u}^{n}\|_f^{1/2}\|\nabla\xi_{j,u}^{n}\|_f^{1/2}\|\xi_{j,\phi}^{n+1}\|_p^{1/2}\|\nabla\xi_{j,\phi}^{n+1}\|_p^{1/2}\\
&\le C\|\xi_{j,u}^{n+1}\|_f\|\nabla\xi_{j,u}^{n+1}\|_f + C\|\xi_{j,\phi}^{n}\|_p\|\nabla\xi_{j,\phi}^{n}\|_p + C\|\xi_{j,u}^{n}\|_f\|\nabla\xi_{j,u}^{n}\|_f + C\|\xi_{j,\phi}^{n+1}\|_p\|\nabla\xi_{j,\phi}^{n+1}\|_p\\
&\le C\|\xi_{j,u}^{n+1}\|_f^2 + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + C\|\xi_{j,\phi}^{n}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n}\|_p^2\\
&\quad + C\|\xi_{j,u}^{n}\|_f^2 + \beta_1\nu\|\nabla\xi_{j,u}^{n}\|_f^2 + C\|\xi_{j,\phi}^{n+1}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

By trace theorem, we have the following estimates

$$\begin{aligned}
\mathcal{R}_{13} &= \frac{r_{j,h}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \Big(c_I(\mu_{j,u}^{n}, \xi_{j,\phi}^{n+1}) - c_I(\xi_{j,u}^{n+1}, \mu_{j,\phi}^{n}) \Big) \quad \text{(B.26)}\\
&\le C\left(\|\nabla\mu_{j,u}^{n}\|_f^2 + \|\nabla\mu_{j,\phi}^{n}\|_p^2\right) + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

To bound $\mathcal{R}_{14}$, we split it into three terms and bound each. Since $u_j \in L^\infty(0,T;H^{k+1}(D_f)), \phi_j \in L^\infty(0,T;H^{m+1}(D_p))$, we have

$$\begin{aligned}
&\frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \left(c_I(\mu_{j,u}^{n+1}, \phi_j^{n}) - c_I(u_j^{n}, \mu_{j,\phi}^{n+1}) \right) \le e|e_{j,r}^{n+1}||c_I(\mu_{j,u}^{n+1}, \phi_j^{n})| + e|e_{j,r}^{n+1}||c_I(u_j^{n}, \mu_{j,\phi}^{n+1})| \quad \text{(B.27)}\\
&\le C|e_{j,r}^{n+1}|\|\nabla\mu_{j,u}^{n+1}\|_f\|\nabla\phi_j^{n}\|_p + C|e_{j,r}^{n+1}|\|\nabla u_j^{n}\|_f\|\nabla\mu_{j,\phi}^{n+1}\|_p \le C|e_{j,r}^{n+1}|\|\nabla\mu_{j,u}^{n+1}\|_f + C|e_{j,r}^{n+1}|\|\nabla\mu_{j,\phi}^{n+1}\|_p\\
&\le \frac{1}{8T}|e_{j,r}^{n+1}|^2 + C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right).
\end{aligned}$$

Since $u_j \in L^\infty(0,T;H^{k+1}(D_f)), \phi_j \in L^\infty(0,T;H^{m+1}(D_p))$, and $|e_{j,r}^{n+1}| = |exp(-\frac{t^{n+1}}{T}) - r_{j,h}^{n+1}| \le e + |r_{j,h}^{n+1}| \le C$, we have

$$\begin{aligned}
&\frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \left(c_I(u_{j,h}^{n+1}, \mu_{j,\phi}^{n}) - c_I(\mu_{j,u}^{n}, \phi_{j,h}^{n+1}) \right)\\
&= \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \left(c_I(u_j^{n+1} - e_{j,u}^{n+1}, \mu_{j,\phi}^{n}) - c_I(\mu_{j,u}^{n}, \phi_j^{n+1} - e_{j,\phi}^{n+1}) \right)\\
&= \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})} \left(c_I(u_j^{n+1}, \mu_{j,\phi}^{n}) - c_I(e_{j,u}^{n+1}, \mu_{j,\phi}^{n}) - c_I(\mu_{j,u}^{n}, \phi_j^{n+1}) + c_I(\mu_{j,u}^{n}, e_{j,\phi}^{n+1}) \right)\\
&\le e|e_{j,r}^{n+1}| \left(|c_I(u_j^{n+1}, \mu_{j,\phi}^{n})| + |c_I(\mu_{j,u}^{n}, \phi_j^{n+1})| \right)\\
&\quad + e|e_{j,r}^{n+1}| \Big(|c_I(\xi_{j,u}^{n+1}, \mu_{j,\phi}^{n})| + |c_I(\mu_{j,u}^{n+1}, \mu_{j,\phi}^{n})| + |c_I(\mu_{j,u}^{n}, \xi_{j,\phi}^{n+1})| + |c_I(\mu_{j,u}^{n}, \mu_{j,\phi}^{n+1})| \Big)\\
&\le e|e_{j,r}^{n+1}| \left(\|\nabla u_j^{n+1}\|_f\|\nabla\mu_{j,\phi}^{n}\|_p + \|\nabla\mu_{j,u}^{n}\|_f\|\nabla\phi_j^{n+1}\|_p \right)\\
&\quad + e|e_{j,r}^{n+1}| \Big(\|\xi_{j,u}^{n+1}\|_f^{1/2}\|\nabla\xi_{j,u}^{n+1}\|_f^{1/2}\|\nabla\mu_{j,\phi}^{n}\|_p + \|\nabla\mu_{j,u}^{n}\|_f\|\xi_{j,\phi}^{n+1}\|_p^{1/2}\|\nabla\xi_{j,\phi}^{n+1}\|_p^{1/2} \Big)\\
&\quad + e|e_{j,r}^{n+1}| \Big(\|\nabla\mu_{j,u}^{n+1}\|_f\|\nabla\mu_{j,\phi}^{n}\|_p + \|\nabla\mu_{j,u}^{n}\|_f\|\nabla\mu_{j,\phi}^{n+1}\|_p \Big)\\
&\le \frac{1}{8T}|e_{j,r}^{n+1}|^2 + C\|\nabla\mu_{j,\phi}^{n}\|_p^2 + C\|\nabla\mu_{j,u}^{n}\|_f^2\\
&\quad + C\|\xi_{j,u}^{n+1}\|_f\|\nabla\xi_{j,u}^{n+1}\|_f + C\|\nabla\mu_{j,\phi}^{n}\|_p^2 + C\|\nabla\mu_{j,u}^{n}\|_f^2 + C\|\xi_{j,\phi}^{n+1}\|_p\|\nabla\xi_{j,\phi}^{n+1}\|_p\\
&\quad + C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\|\nabla\mu_{j,\phi}^{n}\|_p^2 + C\|\nabla\mu_{j,u}^{n}\|_f^2 + C\|\nabla\mu_{j,\phi}^{n+1}\|_p^2\\
&\le C\|\xi_{j,u}^{n+1}\|_f^2 + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 + C\|\xi_{j,\phi}^{n+1}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2
\end{aligned}$$

$$+ C\|\nabla \mu_{j,u}^{n+1}\|_f^2 + C\|\nabla \mu_{j,\phi}^{n}\|_p^2 + C\|\nabla \mu_{j,u}^{n}\|_f^2 + C\|\nabla \mu_{j,\phi}^{n+1}\|_p^2 + \frac{1}{8T}|e_{j,r}^{n+1}|^2.$$

Since $u_j \in L^\infty(0,T;H^{k+1}(D_f)), \phi_j \in L^\infty(0,T;H^{m+1}(D_p))$, and $|e_{j,r}^{n+1}| = |exp(-\frac{t^{n+1}}{T}) - r_{j,h}^{n+1}| \leq e + |r_{j,h}^{n+1}| \leq C$, we have

$$\begin{aligned}
&\frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_{j,h}^{n+1},\xi_{j,\phi}^n) - c_I(\xi_{j,u}^n,\phi_{j,h}^{n+1})\right) \qquad \text{(B.28)}\\
&= \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_j^{n+1} - e_{j,u}^{n+1},\xi_{j,\phi}^n) - c_I(\xi_{j,u}^n,\phi_j^{n+1} - e_{j,\phi}^{n+1})\right)\\
&= \frac{e_{j,r}^{n+1}}{exp(-\frac{t^{n+1}}{T})}\left(c_I(u_j^{n+1},\xi_{j,\phi}^n) - c_I(e_{j,u}^{n+1},\xi_{j,\phi}^n) - c_I(\xi_{j,u}^n,\phi_j^{n+1}) + c_I(\xi_{j,u}^n,e_{j,\phi}^{n+1})\right)\\
&\leq e|e_{j,r}^{n+1}|\left(|c_I(u_j^{n+1},\xi_{j,\phi}^n)| + |c_I(\xi_{j,u}^n,\phi_j^{n+1})|\right)\\
&\quad + e|e_{j,r}^{n+1}|\left(|c_I(\xi_{j,u}^{n+1},\xi_{j,\phi}^n)| + |c_I(\mu_{j,u}^{n+1},\xi_{j,\phi}^n)| + |c_I(\xi_{j,u}^n,\xi_{j,\phi}^{n+1})| + |c_I(\xi_{j,u}^n,\mu_{j,\phi}^{n+1})|\right)\\
&\leq C|e_{j,r}^{n+1}|\left(\|\nabla u_j^{n+1}\|_f\|\xi_{j,\phi}^n\|_p^{1/2}\|\nabla\xi_{j,\phi}^n\|_p^{1/2} + \|\xi_{j,u}^n\|_f^{1/2}\|\nabla\xi_{j,u}^n\|_f^{1/2}\|\nabla\phi_j^{n+1}\|_p\right)\\
&+ C|e_{j,r}^{n+1}|\left(\|\xi_{j,u}^{n+1}\|_f^{1/2}\|\nabla\xi_{j,u}^{n+1}\|_f^{1/2}\|\xi_{j,\phi}^n\|_p^{1/2}\|\nabla\xi_{j,\phi}^n\|_p^{1/2} + \|\xi_{j,u}^n\|_f^{1/2}\|\nabla\xi_{j,u}^n\|_f^{1/2}\|\xi_{j,\phi}^{n+1}\|_p^{1/2}\|\nabla\xi_{j,\phi}^{n+1}\|_p^{1/2}\right)\\
&+ C|e_{j,r}^{n+1}|\left(\|\nabla\mu_{j,u}^{n+1}\|_f\|\xi_{j,\phi}^n\|_p^{1/2}\|\nabla\xi_{j,\phi}^n\|_p^{1/2} + \|\xi_{j,u}^n\|_f^{1/2}\|\nabla\xi_{j,u}^n\|_f^{1/2}\|\nabla\mu_{j,\phi}^{n+1}\|_p\right)\\
&\leq \frac{1}{8T}|e_{j,r}^{n+1}|^2 + \left(C\|\xi_{j,\phi}^n\|_p\|\nabla\xi_{j,\phi}^n\|_p + C\|\xi_{j,u}^n\|_f\|\nabla\xi_{j,u}^n\|_f\right)\\
&+ C\|\xi_{j,u}^{n+1}\|_f\|\nabla\xi_{j,u}^{n+1}\|_f + C\|\xi_{j,\phi}^n\|_p\|\nabla\xi_{j,\phi}^n\|_p + C\|\xi_{j,u}^n\|_f\|\nabla\xi_{j,u}^n\|_f + C\|\xi_{j,\phi}^{n+1}\|_p\|\nabla\xi_{j,\phi}^{n+1}\|_p\\
&+ C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\|\xi_{j,\phi}^n\|_p\|\nabla\xi_{j,\phi}^n\|_p + C\|\xi_{j,u}^n\|_f\|\nabla\xi_{j,u}^n\|_f + C\|\nabla\mu_{j,\phi}^{n+1}\|_p^2\\
&\leq C\|\xi_{j,u}^{n+1}\|_f\|\nabla\xi_{j,u}^{n+1}\|_f + C\|\xi_{j,u}^n\|_f\|\nabla\xi_{j,u}^n\|_f + C\|\xi_{j,\phi}^{n+1}\|_p\|\nabla\xi_{j,\phi}^{n+1}\|_p + C\|\xi_{j,\phi}^n\|_p\|\nabla\xi_{j,\phi}^n\|_p\\
&\quad + C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\|\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \frac{1}{8T}|e_{j,r}^{n+1}|^2\\
&\leq C\|\xi_{j,u}^{n+1}\|_f^2 + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f + C\|\xi_{j,u}^n\|_f^2 + \beta_1\nu\|\nabla\xi_{j,u}^n\|_f^2 + C\|\xi_{j,\phi}^{n+1}\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p\\
&\quad + C\|\xi_{j,\phi}^n\|_p^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^n\|_p + C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\|\nabla\mu_{j,\phi}^{n+1}\|_p^2 + \frac{1}{8T}|e_{j,r}^{n+1}|^2.
\end{aligned}$$

Next we bound the consistency errors.

$$\begin{aligned}
\epsilon_{j,f}^{n+1}(\xi_{j,u}^{n+1}) &\leq C\left\|\frac{u_j^{n+1} - u_j^n}{\Delta t} - \partial_t u_j(t_{n+1})\right\|_f^2 + C\|\nabla(\phi_j^{n+1} - \phi_j^n)\|_p^2 + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2 \qquad \text{(B.29)}\\
&\leq C\Delta t\int_{t^n}^{t^{n+1}}\|\partial_{tt}u_j\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla(\partial_t\phi_j)\|_p^2\,dt + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2.
\end{aligned}$$

$$\begin{aligned}
\epsilon_{j,p}^{n+1}(\xi_{j,\phi}^{n+1}) &\leq C\left\|\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} - \partial_t\phi_j(t_{n+1})\right\|_p^2 + C\|\nabla(u_j^{n+1} - u_j^n)\|_f^2 + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 \qquad \text{(B.30)}\\
&\leq C\Delta t\int_{t^n}^{t^{n+1}}\|\partial_{tt}\phi_j\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla(\partial_t u_j)\|_f^2\,dt + \beta_2 g\bar{k}_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2.
\end{aligned}$$

Since $u_j \in L^\infty(0,T;H^{k+1}(D_f)), \phi_j \in L^\infty(0,T;H^{m+1}(D_p))$, we have

$$\begin{aligned}
&\epsilon_j^{n+1}(r)\cdot e_{j,r}^{n+1} \qquad \text{(B.31)}\\
&\leq \left|\frac{r_j^{n+1} - r_j^n}{\Delta t} - \dot{r}_j(t_{n+1})\right||e_{j,r}^{n+1}| + e|c_I(u_j^{n+1},\phi_j^{n+1} - \phi_j^n)||e_{j,r}^{n+1}| + e|c_I(u_j^{n+1} - u_j^n,\phi_j^{n+1})||e_{j,r}^{n+1}|\\
&\leq \left|\frac{r_j^{n+1} - r_j^n}{\Delta t} - \dot{r}_j(t_{n+1})\right||e_{j,r}^{n+1}| + C\|\nabla u_j^{n+1}\|\|\nabla(\phi_j^{n+1} - \phi_j^n)\||e_{j,r}^{n+1}| + C\|\nabla(u_j^{n+1} - u_j^n)\|\|\nabla\phi_j^{n+1}\||e_{j,r}^{n+1}|
\end{aligned}$$

$$
\begin{aligned}
&\le \Big|\frac{r_j^{n+1}-r_j^n}{\Delta t} - \dot r_j(t_{n+1})\Big||e_{j,r}^{n+1}| + C\|\nabla(\phi_j^{n+1}-\phi_j^n)\||e_{j,r}^{n+1}| + C\|\nabla(u_j^{n+1}-u_j^n)\||e_{j,r}^{n+1}| \\
&\le C\Big|\frac{r_j^{n+1}-r_j^n}{\Delta t} - \dot r_j(t_{n+1})\Big|^2 + C\|\nabla(\phi_j^{n+1}-\phi_j^n)\|^2 + C\|\nabla(u_j^{n+1}-u_j^n)\|^2 + \frac{1}{8T}|e_{j,r}^{n+1}|^2 \\
&\le C\Delta t\int_{t^n}^{t^{n+1}} |\ddot r_j|^2 dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t\phi_j)\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t u_j)\|_f^2\,dt + \frac{1}{8T}|e_{j,r}^{n+1}|^2.
\end{aligned}
$$

The pressure term can be bounded as follows.

$$
\left(p_j^{n+1}-\lambda_h^{n+1}, \nabla\cdot\xi_{j,u}^{n+1}\right)_f \le C\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2 + \beta_1\nu\|\nabla\xi_{j,u}^{n+1}\|_f^2. \tag{B.32}
$$

Step 4: finalize the estimate for errors in finite element spaces, then the estimate for solution errors $e_{j,u}^{n+1}$, $e_{j,\phi}^{n+1}$, *and* $e_{j,r}^{n+1}$.

Combining all these estimates, taking $\beta_1 = \frac{1}{20}$, we have the following inequality

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,u}^{n}\|_f^2 + \frac{\nu}{2}\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{\nu}{10}\left(\|\nabla\xi_{j,u}^{n+1}\|_f^2 - \|\nabla\xi_{j,u}^{n}\|_f^2\right) \\
&+\sum_i\left((1-\sigma_2)\bar\eta_i^{min} - (1+\sigma_1)\eta_i'^{max}\right)\int_I(\xi_{j,u}^{n+1}\cdot\widehat\tau_i)^2\,ds \\
&+\sum_i\frac12\eta_i'^{max}\left(\int_I(\xi_{j,u}^{n+1}\cdot\widehat\tau_i)^2\,ds - \int_I(\xi_{j,u}^{n}\cdot\widehat\tau_i)^2\,ds\right) + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n}\|_p^2 \\
&+\left(1-9\beta_2-(1+\sigma_3)\frac{\rho'_{max}}{\bar k_{min}}\right)g\bar k_{min}\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 + \left(\frac12 g\rho'_{max} + 2\beta_2 g\bar k_{min}\right)\left(\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 - \|\nabla\xi_{j,\phi}^{n}\|_p^2\right) \\
&+\frac{1}{2\Delta t}\left(|e_{j,r}^{n+1}|^2 - |e_{j,r}^{n}|^2 + |e_{j,r}^{n+1}-e_{j,r}^{n}|^2\right) + \frac{1}{2T}|e_{j,r}^{n+1}|^2 \\
&\le C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla(\partial_t u_j)\|_f^2\,dt + C\|\nabla\mu_{j,u}^n\|_f^2 + C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla(\partial_t\phi_j)\|_p^2\,dt + C\|\nabla\mu_{j,\phi}^n\|_p^2 \\
&\quad + C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right) + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}}\|\partial_t\mu_{j,u}\|_f^2\,dt + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}}\|\partial_t\mu_{j,\phi}\|_p^2\,dt \\
&\quad + C\left(\|\xi_{j,u}^{n+1}\|_f^2 + \|\xi_{j,u}^{n}\|_f^2\right) + C\left(\|\xi_{j,\phi}^{n+1}\|_p^2 + \|\xi_{j,\phi}^{n}\|_p^2\right) + C\left(\|\nabla\mu_{j,u}^n\|_f^2 + \|\nabla\mu_{j,\phi}^n\|_p^2\right) \\
&\quad + C\Delta t\int_{t^n}^{t^{n+1}}\|\partial_{tt}u_j\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\partial_{tt}\phi_j\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}|\ddot r_j|^2\,dt \\
&\quad + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla(\partial_t u_j)\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}}\|\nabla(\partial_t\phi_j)\|_p^2\,dt + C\|p_j^{n+1}-\lambda_h^{n+1}\|_f^2.
\end{aligned} \tag{B.33}
$$

To make sure the fifth and ninth term on the left hand side are non-negative, we need $0<\sigma_2<1$, $0<\beta_2<1/9$, and

$$
\frac{\eta_i'^{max}}{\bar\eta_i^{min}} \le \frac{1-\sigma_2}{1+\sigma_1}, \qquad \frac{\rho'_{max}}{\bar k_{min}} < \frac{1}{1+\sigma_3}. \tag{B.34}
$$

For $\forall\sigma_2\in(0,1), \forall\sigma_1>0, \forall\sigma_3>0$, we can derive that $\frac{1-\sigma_2}{1+\sigma_1}, \frac{1}{1+\sigma_3}\in(0,1)$. Now if the two parameter conditions in (4.4) are satisfied, we have $\frac{\eta_i'^{max}}{\bar\eta_i^{min}}, \frac{\rho'_{max}}{\bar k_{min}}\in(0,1)$. Then we can easily find $\sigma_2\in(0,1), \sigma_1>0$ such that $\frac{\eta_i'^{max}}{\bar\eta_i^{min}} = \frac{1-\sigma_2}{1+\sigma_1}$, and $\sigma_3>0$ such that $\frac{\rho'_{max}}{\bar k_{min}} < \frac{1}{1+\sigma_3}$. Here we take

$$
\sigma_1=\sigma_2=\frac{\bar\eta_i^{min}-\eta_i'^{max}}{\bar\eta_i^{min}+\eta_i'^{max}},\ \sigma_3=\frac12\left(\frac{\bar k_{min}}{\rho'_{max}}-1\right),\ \beta_2=\frac{1}{18}\left(1-\frac{\rho'_{max}}{\bar k_{min}}\right),
$$

then $(1-\sigma_2)\bar\eta_i^{min}-(1+\sigma_1)\eta_i'^{max}=0,\ 1-9\beta_2-(1+\sigma_3)\frac{\rho'_{max}}{\bar k_{min}}=0$.

Under the two parameter conditions in (4.4), (B.33) reduces to

$$
\begin{aligned}
&\frac{1}{2\Delta t}\|\xi_{j,u}^{n+1}\|_f^2 - \frac{1}{2\Delta t}\|\xi_{j,u}^{n}\|_f^2 + \frac{\nu}{2}\|\nabla\xi_{j,u}^{n+1}\|_f^2 + \frac{\nu}{10}\left(\|\nabla\xi_{j,u}^{n+1}\|_f^2 - \|\nabla\xi_{j,u}^{n}\|_f^2\right) \\
&+\sum_i \frac{1}{2}\eta_i'^{max}\left(\int_I (\xi_{j,u}^{n+1}\cdot\widehat{\tau}_i)^2\,ds - \int_I (\xi_{j,u}^{n}\cdot\widehat{\tau}_i)^2\,ds\right) + \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n+1}\|_p^2 - \frac{gS_0}{2\Delta t}\|\xi_{j,\phi}^{n}\|_p^2 \\
&+\left(\frac{7}{18}g\rho'_{max} + \frac{1}{9}g\bar{k}_{min}\right)\left(\|\nabla\xi_{j,\phi}^{n+1}\|_p^2 - \|\nabla\xi_{j,\phi}^{n}\|_p^2\right) + \frac{1}{2\Delta t}\left(|e_{j,r}^{n+1}|^2 - |e_{j,r}^{n}|^2 + |e_{j,r}^{n+1} - e_{j,r}^{n}|^2\right) + \frac{1}{2T}|e_{j,r}^{n+1}|^2 \\
&\le C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t u_j)\|_f^2\,dt + C\|\nabla\mu_{j,u}^n\|_f^2 + C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t\phi_j)\|_p^2\,dt + C\|\nabla\mu_{j,\phi}^n\|_p^2 \\
&\quad + C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right) + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} \|\partial_t\mu_{j,u}\|_f^2\,dt + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} \|\partial_t\mu_{j,\phi}\|_p^2\,dt \\
&\quad + C\left(\|\xi_{j,u}^{n+1}\|_f^2 + \|\xi_{j,u}^{n}\|_f^2\right) + C\left(\|\xi_{j,\phi}^{n+1}\|_p^2 + \|\xi_{j,\phi}^{n}\|_p^2\right) + C\left(\|\nabla\mu_{j,u}^n\|_f^2 + \|\nabla\mu_{j,\phi}^n\|_p^2\right) \\
&\quad + C\Delta t\int_{t^n}^{t^{n+1}} \|\partial_{tt}u_j\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\partial_{tt}\phi_j\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} |\ddot{r}_j|^2\,dt \\
&\quad + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t u_j)\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t\phi_j)\|_p^2\,dt + C\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2.
\end{aligned}
\tag{B.35}
$$

Since $\xi_{j,u}^0 = 0$, $\xi_{j,\phi}^0 = 0$, and $e_{j,r}^0 = 0$, summing up from $n = 0$ to $n = N-1$ and multiplying through by $2\Delta t$ yields

$$
\begin{aligned}
&\|\xi_{j,u}^{N}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\nu\|\nabla\xi_{j,u}^{n}\|_f^2 + \frac{\nu}{5}\Delta t\|\nabla\xi_{j,u}^{N}\|_f^2 + \Delta t\sum_i \eta_i'^{max}\int_I (\xi_{j,u}^{N}\cdot\widehat{\tau}_i)^2\,ds \\
&+ gS_0\|\xi_{j,\phi}^{N}\|_p^2 + \Delta t\left(\frac{7}{9}g\rho'_{max} + \frac{2}{9}g\bar{k}_{min}\right)\|\nabla\xi_{j,\phi}^{N}\|_p^2 + |e_{j,r}^N|^2 + \sum_{n=0}^{N-1}|e_{j,r}^{n+1} - e_{j,r}^{n}|^2 + \Delta t\sum_{n=0}^{N-1}\frac{1}{T}|e_{j,r}^{n+1}|^2 \\
&\le 2\Delta t\sum_{n=0}^{N-1}\Bigg\{C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t u_j)\|_f^2\,dt + C\|\nabla\mu_{j,u}^n\|_f^2 + C\|\nabla\mu_{j,u}^{n+1}\|_f^2 + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t\phi_j)\|_p^2\,dt \\
&\quad + C\|\nabla\mu_{j,\phi}^n\|_p^2 + C\left(\|\nabla\mu_{j,u}^{n+1}\|_f^2 + \|\nabla\mu_{j,\phi}^{n+1}\|_p^2\right) + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} \|\partial_t\mu_{j,u}\|_f^2\,dt + \frac{C}{\Delta t}\int_{t^n}^{t^{n+1}} \|\partial_t\mu_{j,\phi}\|_p^2\,dt \\
&\quad + C\left(\|\xi_{j,u}^{n+1}\|_f^2 + \|\xi_{j,u}^{n}\|_f^2\right) + C\left(\|\xi_{j,\phi}^{n+1}\|_p^2 + \|\xi_{j,\phi}^{n}\|_p^2\right) + C\left(\|\nabla\mu_{j,u}^n\|_f^2 + \|\nabla\mu_{j,\phi}^n\|_p^2\right) \\
&\quad + C\Delta t\int_{t^n}^{t^{n+1}} \|\partial_{tt}u_j\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\partial_{tt}\phi_j\|_p^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} |\ddot{r}_j|^2\,dt \\
&\quad + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t u_j)\|_f^2\,dt + C\Delta t\int_{t^n}^{t^{n+1}} \|\nabla(\partial_t\phi_j)\|_p^2\,dt + C\|p_j^{n+1} - \lambda_h^{n+1}\|_f^2\Bigg\}.
\end{aligned}
\tag{B.36}
$$

Taking infimum of $\|\nabla\mu_{j,u}\|$ over the space X_f^h, $\|\nabla\mu_{j,\phi}\|$ over the space X_p^h, $\|\nabla p_j^{n+1} - \lambda_h^{n+1}\|$ over the space Q_f^h, and using interpolation inequalities, we obtain

$$
\begin{aligned}
&\|\xi_{j,u}^{N}\|_f^2 + \Delta t\sum_{n=0}^{N-1}\nu\|\nabla\xi_{j,u}^{n}\|_f^2 + \frac{\nu}{5}\Delta t\|\nabla\xi_{j,u}^{N}\|_f^2 + \Delta t\sum_i \eta_i'^{max}\int_I (\xi_{j,u}^{N}\cdot\widehat{\tau}_i)^2\,ds \\
&+ gS_0\|\xi_{j,\phi}^{N}\|_p^2 + \Delta t\left(\frac{7}{9}g\rho'_{max} + \frac{2}{9}g\bar{k}_{min}\right)\|\nabla\xi_{j,\phi}^{N}\|_p^2 + |e_{j,r}^N|^2 + \sum_{n=0}^{N-1}|e_{j,r}^{n+1} - e_{j,r}^{n}|^2 + \Delta t\sum_{n=0}^{N-1}\frac{1}{T}|e_{j,r}^{n+1}|^2 \\
&\le \Delta t\sum_{n=0}^{N} C\left(\|\xi_{j,u}^{n}\|_f^2 + \|\xi_{j,\phi}^{n}\|_p^2\right)
\end{aligned}
\tag{B.37}
$$

$$+ C\Delta t^2 \|\partial_t u_j\|_{2,1,f} + Ch^{2k} \|\!|u_j|\!\|^2_{2,k+1,f} + C\Delta t^2 \|\partial_t \phi_j\|_{2,1,p} + Ch^{2m} \|\!|\phi_j|\!\|^2_{2,m+1,p}$$
$$+ Ch^{2k+2} \|\partial_t u_j\|^2_{2,k+1,f} + Ch^{2m+2} \|\partial_t \phi_j\|^2_{2,m+1,p} + C\Delta t^2 \|\partial_{tt} u_j\|_{2,0,f} + C\Delta t^2 \|\partial_{tt} \phi_j\|_{2,0,p} + C\Delta t^2$$
$$+ Ch^{2s+2} \|\!|p_j|\!\|^2_{2,s+1,f}. \tag{B.38}$$

Next we apply the discrete Gronwall inequality. Assuming Δt is sufficiently small, i.e., $\Delta t < 1/C$, we have

$$\|\xi^N_{j,u}\|^2_f + \Delta t \sum_{n=0}^{N-1} \nu \|\nabla \xi^n_{j,u}\|^2_f + \frac{\nu}{5} \Delta t \|\nabla \xi^N_{j,u}\|^2_f + \Delta t \sum_i \eta_i'^{max} \int_I (\xi^N_{j,u} \cdot \widehat{\tau}_i)^2 \, ds \tag{B.39}$$
$$+ gS_0 \|\xi^N_{j,\phi}\|^2_p + \Delta t \left(\frac{7}{9} g\rho'_{max} + \frac{2}{9} g\bar{k}_{min} \right) \|\nabla \xi^N_{j,\phi}\|^2_p + |e^N_{j,r}|^2 + \sum_{n=0}^{N-1} |e^{n+1}_{j,r} - e^n_{j,r}|^2 + \Delta t \sum_{n=0}^{N-1} \frac{1}{T} |e^{n+1}_{j,r}|^2$$
$$\le exp(\frac{CT}{1 - C\Delta t}) \Bigg(C\Delta t^2 \|\partial_t u_j\|_{2,1,f} + Ch^{2k} \|\!|u_j|\!\|^2_{2,k+1,f} + C\Delta t^2 \|\partial_t \phi_j\|_{2,1,p} + Ch^{2m} \|\!|\phi_j|\!\|^2_{2,m+1,p}$$
$$+ Ch^{2k+2} \|\partial_t u_j\|^2_{2,k+1,f} + Ch^{2m+2} \|\partial_t \phi_j\|^2_{2,m+1,p} + C\Delta t^2 \|\partial_{tt} u_j\|_{2,0,f} + C\Delta t^2 \|\partial_{tt} \phi_j\|_{2,0,p} + C\Delta t^2$$
$$+ Ch^{2s+2} \|\!|p_j|\!\|^2_{2,s+1,f} \Bigg).$$

Since

$$\|\mu^N_{j,u}\|^2_f + \Delta t \sum_{n=0}^{N-1} \nu \|\nabla \mu^n_{j,u}\|^2_f + \frac{\nu}{5} \Delta t \|\nabla \mu^N_{j,u}\|^2_f + \Delta t \sum_i \eta_i'^{max} \int_I (\mu^N_{j,u} \cdot \widehat{\tau}_i)^2 \, ds \tag{B.40}$$
$$+ gS_0 \|\mu^N_{j,\phi}\|^2_p + \Delta t \left(\frac{7}{9} g\rho'_{max} + \frac{2}{9} g\bar{k}_{min} \right) \|\nabla \mu^N_{j,\phi}\|^2_p$$
$$\le h^{2k+2} \|\!|u_j|\!\|^2_{\infty,k+1,f} + Ch^{2k} \|\!|u_j|\!\|^2_{2,k+1,f} + Ch^{2m+2} \|\!|\phi_j|\!\|^2_{\infty,m+1,p} + Ch^{2m} \|\!|\phi_j|\!\|^2_{2,m+1,p},$$

using triangle inequality on the error equations yields

$$\|e^N_{j,u}\|^2_f + \Delta t \sum_{n=0}^{N-1} \nu \|\nabla e^n_{j,u}\|^2_f + \frac{\nu}{5} \Delta t \|\nabla e^N_{j,u}\|^2_f + \Delta t \sum_i \eta_i'^{max} \int_I (e^N_{j,u} \cdot \widehat{\tau}_i)^2 \, ds \tag{B.41}$$
$$+ gS_0 \|e^N_{j,\phi}\|^2_p + \Delta t \left(\frac{7}{9} g\rho'_{max} + \frac{2}{9} g\bar{k}_{min} \right) \|\nabla e^N_{j,\phi}\|^2_p + |e^N_{j,r}|^2 + \sum_{n=0}^{N-1} |e^{n+1}_{j,r} - e^n_{j,r}|^2 + \Delta t \sum_{n=0}^{N-1} \frac{1}{T} |e^{n+1}_{j,r}|^2$$
$$\le exp(\frac{CT}{1 - C\Delta t}) \Bigg(C\Delta t^2 \|\partial_t u_j\|_{2,1,f} + Ch^{2k} \|\!|u_j|\!\|^2_{2,k+1,f} + C\Delta t^2 \|\partial_t \phi_j\|_{2,1,p} + Ch^{2m} \|\!|\phi_j|\!\|^2_{2,m+1,p}$$
$$+ Ch^{2k+2} \|\partial_t u_j\|^2_{2,k+1,f} + Ch^{2m+2} \|\partial_t \phi_j\|^2_{2,m+1,p} + C\Delta t^2 \|\partial_{tt} u_j\|_{2,0,f} + C\Delta t^2 \|\partial_{tt} \phi_j\|_{2,0,p} + C\Delta t^2$$
$$+ Ch^{2s+2} \|\!|p_j|\!\|^2_{2,s+1,f} + Ch^{2k+2} \|\!|u_j|\!\|^2_{\infty,k+1,f} + Ch^{2m+2} \|\!|\phi_j|\!\|^2_{\infty,m+1,p} \Bigg),$$

and completes the proof. □