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Representation Homology of Topological Spaces

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In this paper, we introduce and study representation homology of topological spaces, which is a natural homological extension of representation varieties of fundamental groups. We give an elementary construction of representation homology parallel to the Loday–Pirashvili construction of higher Hochschild homology; in fact, we establish a direct geometric relation between the two theories by proving that the representation homology of the suspension of a (pointed connected) space is isomorphic to its higher Hochschild homology. We also construct some natural maps and spectral sequences relating representation homology to other homology theories associated with spaces (such as Pontryagin algebras, $\mathbb{S}^1$-equivariant homology of the free loop space, and stable homology of automorphism groups of f.g. free groups). We compute representation homology explicitly (in terms of known invariants) in a number of interesting cases, including spheres, suspensions, complex projective spaces, Riemann surfaces, and some 3-dimensional manifolds, such as link complements in $\mathbb{R}^3$ and the lens spaces $L(p, q)$. In the case of link complements, we identify the representation homology in terms of ordinary Hochschild homology, which gives a new algebraic invariant of links in $\mathbb{R}^3$.

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1 Introduction

Representation homology is an algebraic homology theory associated with derived representation schemes, which are natural ("derived") extensions of classical representation varieties. The subject may be plainly viewed as part of derived algebraic geometry (see, e.g., [75]); however, somewhat surprisingly, there are more elementary constructions. What makes representation homology interesting (it seems) are the relations between these different constructions and interpretations coming from different parts of mathematics.

In the present paper, we give two (equivalent) definitions of representation homology of topological spaces: one in terms of (non-abelian) derived functors on simplicial groups and the other in terms of classical homological algebra in functor categories. The first definition is inspired by our earlier work on representation homology of algebras (see [4–6]) while the second by the Loday–Pirashvili approach to higher Hochschild homology [60]. Both definitions are conceptually very simple and accessible to computations: in this paper, we will use them in a complementary way to establish basic properties of representation homology and do some examples; in our subsequent paper [10], we will look at applications. We begin with some motivation for studying representation homology.

1.1 Representation varieties and representation homology

Let G be a finite-dimensional affine algebraic group defined over a field k of characteristic zero. For any (discrete) group Γ, the set of all representations of Γ in G has a natural structure of an affine k-scheme called the *representation scheme* $\mathrm{Rep}_G(\Gamma)$. Representation schemes and associated varieties play an important role in many areas of mathematics, most notably in representation theory and low-dimensional topology. In representation theory, the fundamental problem is to understand the structure of representations of Γ in G. One can approach this problem geometrically by studying the natural (adjoint) action of the group G on the variety $\mathrm{Rep}_G(\Gamma)$. When k is algebraically closed and Γ is finitely generated, the equivariant geometry of $\mathrm{Rep}_G(\Gamma)$ is closely related to the representation theory of Γ: the equivalence classes of representations of Γ in G are in bijection with the G-orbits in $\mathrm{Rep}_G(\Gamma)$, and the geometry of G-orbits determines the algebraic structure of representations. This relation has been extensively studied since the late '70s, and the representation varieties have become a standard tool in representation theory of groups (see, e.g., [51, 70]).

In topology, one is usually interested in global algebro-geometric invariants of spaces defined in terms of representation varieties of fundamental groups. For example, if K is a knot in $\mathbb{S}^3$, many classical invariants of K arise from its character variety $\chi_G(K) := \mathrm{Rep}_G[\pi_1(X_K)]/\!/G$, which is the (categorical) quotient of the representation variety of the fundamental group of the knot complement $X_K := \mathbb{S}^3 \setminus K$. These invariants include, in particular, the classical Alexander polynomial $\Delta_K(t)$ (in the simplest case when $G = \mathbb{C}^*$, see, e.g., [55]), the so-called A-polynomial $A_K(m,l)$ (see [19]), the Casson invariant [20], and the famous Chern–Simons invariant [47], all of which are defined for $G = \mathrm{SL}_2(\mathbb{C})$. In fact, for $G = \mathrm{SL}_2(\mathbb{C})$, the entire character variety, or rather its coordinate ring $\mathcal{O}[\chi_G(K)]$, has a purely topological interpretation as a Kauffman bracket skein module of X_K (see [63]).

Despite being useful tools, the representation varieties have some intrinsic deficiencies. First of all, these varieties are usually very singular, which makes it hard to understand their geometry. Thus, in representation theory, one faces the problem of resolving singularities of $\mathrm{Rep}_G(\Gamma)$. In topology, the use of representation varieties is mostly limited to (compact orientable) surfaces, hyperbolic 3-manifolds, and knot complements in $\mathbb{S}^3$, all of which are known to be aspherical spaces. The homotopy type of such a space is completely determined by the isomorphism type of its fundamental group, which makes representation varieties of these groups very strong and efficient invariants. For more general spaces, however, one needs to take into account a higher homotopy information, and looking at representation varieties of fundamental groups (or even, higher homotopy groups) is not enough.

A natural way to remedy these problems is to replace the representation functor Rep_G with its (non-abelian) derived functor DRep_G much in the same way as one replaces non-exact additive functors in classical homological algebra (such as "$\otimes$" and "Hom") with corresponding derived functors ("$\otimes^{\boldsymbol{L}}$" and "$\mathbf{R}$Hom"). Geometrically, passing from the representation scheme $\mathrm{Rep}_G(\Gamma)$ to the derived representation scheme $\mathrm{DRep}_G(\Gamma)$ amounts to desingularizing $\mathrm{Rep}_G(\Gamma)$, while topologically, this yields a new homology theory of spaces that captures a good deal of homotopy information and refines the classical representation varieties of fundamental groups in an interesting and nontrivial way.

To explain this idea in more precise terms, we recall that the representation scheme $\mathrm{Rep}_G(\Gamma)$ is defined as the functor on the category of commutative k-algebras:

$$\mathrm{Rep}_G(\Gamma) : \mathtt{Comm\ Alg}_k \to \mathtt{Set} \quad A \mapsto \mathrm{Hom}_{\mathtt{Gr}}(\Gamma, G(A)), \tag{1.1}$$

assigning to a k-algebra A the set of families of representations of Γ in G parametrized by the k-scheme $\mathrm{Spec}(A)$. It is well known that the functor (1.1) is representable, and we denote the corresponding commutative algebra by $\Gamma_G = \mathcal{O}[\mathrm{Rep}_G(\Gamma)]$: this is the coordinate ring of the affine k-scheme $\mathrm{Rep}_G(\Gamma)$. Varying Γ (while keeping G fixed), we can now regard Γ_G as a functor on the category of groups:

$$(-)_G : \mathtt{Gr} \to \mathtt{Comm\ Alg}_k \quad \Gamma \mapsto \Gamma_G, \tag{1.2}$$

which we call the *representation functor* in G. The functor (1.2) extends naturally to the category $\mathtt{sGr}$ of simplicial groups, taking values in the category $\mathtt{sComm\,Alg}_k$ of simplicial commutative algebras. Both categories $\mathtt{sGr}$ and $\mathtt{sComm\,Alg}_k$ carry standard (simplicial) model structures, with weak equivalences being the weak homotopy equivalences of underlying simplicial sets. The functor $(-)_G : \mathtt{sGr} \to \mathtt{sComm\,Alg}_k$ is not homotopy invariant: in general, it does not preserve weak equivalences and hence does not descend to a functor between the homotopy categories $\mathtt{Ho}(\mathtt{sGr})$ and $\mathtt{Ho}(\mathtt{sComm\,Alg}_k)$. However, it is easy to check that $(-)_G$ takes weak equivalences between cofibrant objects in $\mathtt{sGr}$ to weak equivalences in $\mathtt{sComm\,Alg}_k$ (see Lemma 3.1). Hence, by standard homotopical algebra, it has a (total) left derived functor

$$\boldsymbol{L}(-)_G : \mathtt{Ho}(\mathtt{sGr}) \to \mathtt{Ho}(\mathtt{sComm\ Alg}_k). \tag{1.3}$$

We call (1.3) the *derived representation functor* in G. Heuristically, $\boldsymbol{L}(-)_G$ may be thought of as the "best possible" approximation of the representation functor (1.2) at the level of homotopy categories. When applied to a simplicial group Γ, the functor (1.3) is represented by a simplicial commutative algebra that we denote by $\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$. The derived representation scheme $\mathrm{DRep}_G(\Gamma)$ is then defined formally as the "**Spec**" of $\mathcal{O}[\mathrm{DRep}_G(\Gamma)$, that is, the simplicial algebra $\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ viewed as an object of the opposite category $\mathtt{Ho}(\mathtt{sComm\,Alg}_k)^{\mathrm{op}}$. The homotopy groups of $\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ depend only on Γ and G, with $\pi_0\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ being canonically isomorphic to $\pi_0(\Gamma)_G$. In particular, if Γ is a discrete simplicial group, then $\pi_0\mathcal{O}[\mathrm{DRep}_G(\Gamma)] \cong \Gamma_G$. Extending our terminology from [4, 6], we will refer to $\pi_*\mathcal{O}[\mathrm{DRep}_G(\Gamma)]$ as the *representation homology* of Γ in G and denote it $\mathrm{HR}_*(\Gamma, G)$. We should mention that representation homology of associative and Lie algebras was introduced and studied in [4–6]. The idea of deriving the representation functor was motivated by noncommutative geometry, where the representation functor plays an important role (see [38, 49] and also [7]).

Next, we recall that the model category $\mathtt{sGr}$ of simplicial groups is Quillen equivalent to the category of reduced simplicial sets, $\mathtt{sSet}_0$, which is, in turn, Quillen equivalent to the category $\mathtt{Top}_{0,*}$ of pointed connected topological spaces. These classical equivalences are given by two pairs of adjoint functors:

$$\mathbb{G} : \mathtt{sSet}_0 \rightleftarrows \mathtt{sGr} : \overline{W}\,, \qquad |-| : \mathtt{sSet}_0 \rightleftarrows \mathtt{Top}_{0,*} : \overline{S},$$

the construction of which will be briefly reviewed in Section 2.2. Here, we only recall that $\mathbb{G}$ is the Kan loop group functor that assigns to a reduced simplicial set $X \in \mathtt{sSet}_0$ a semi-free simplicial group $\mathbb{G}X$, which is a simplicial model of the based loop space $\Omega|X|$ (see [44]). The Kan loop group functor preserves weak equivalences and hence induces a functor between the homotopy categories: $\mathbb{G} : \mathrm{Ho}(\mathtt{sSet}_0) \to \mathrm{Ho}(\mathtt{sGr})$. Combining this last functor with (1.3), we set $\mathcal{O}[\mathrm{DRep}_G(X)] := \boldsymbol{L}(\mathbb{G}X)_G$ and define the *representation homology* of $X \in \mathtt{sSet}_0$ by

$$\mathrm{HR}_*(X, G) := \pi_*\mathcal{O}[\mathrm{DRep}_G(X)]. \tag{1.4}$$

By definition, $\mathrm{HR}_*(X, G)$ is a graded commutative algebra that depends only on the homotopy type of X and hence is a homotopy invariant of the corresponding space $|X|$. In degree zero, we have $\mathrm{HR}_0(X, G) \cong (\pi_1(X))_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$, where $\pi_1(X)$ is the fundamental group of X. To avoid confusion, we emphasize that $\mathrm{HR}_*(X, G) \ncong \mathrm{HR}_*(\pi_1(X), G)$ in general; however, if Γ is a discrete group and X is a $K(\Gamma, 1)$-space (e.g., $X = \mathrm{B}\Gamma$), then we do have a natural isomorphism $\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(\Gamma, G)$, so there is no ambiguity in our notation.

The goal of the present paper is three-fold. First, we establish basic properties of the derived representation functor (1.3). Second, we give an elementary construction of representation homology in terms of classical (abelian) homological algebra. Our construction is analogous to Pirashvili's construction of higher order Hochschild homology, and it provides a natural interpretation of representation homology as functor homology. This opens up the way to efficient computations and places representation homology in one row with other classical invariants such as Hochschild and cyclic homology. Third, we construct some spectral sequences and natural maps relating representation homology to other homology theories associated with spaces (including the Pontryagin algebra $\mathrm{H}_*(\Omega X)$, higher Hochschild homology, and stable homology of the automorphism groups of f.g. free groups $\mathbb{F}_n$). We also compute representation homology explicitly in a number of interesting cases, including the spheres $\mathbb{S}^n$, suspensions ΣX, co-H-spaces, closed surfaces of arbitrary genus, and some classical 3-dimensional spaces, such as the link complements in $\mathbb{R}^3$ and the lens spaces $L(p,q)$. In our subsequent paper, [10], we will extend these computations to arbitrary simply connected topological spaces by expressing the representation homology of a 1-connected space of finite rational type in terms of its Quillen and Sullivan models and give some applications to representation theory.

1.2 Main results

We now proceed with a summary of the main results of the paper. Recall that an affine algebraic group G is defined by its functor of points, which is a group-valued representable functor on commutative algebras. This functor extends in the natural way to simplicial commutative algebras:

$$G: \mathtt{sCommAlg}_k \to \mathtt{sGr} \quad A_* \mapsto G(A_*). \tag{1.5}$$

By definition, the representation functor (1.2) is left adjoint to the functor of points of G; hence, its simplicial extension is left adjoint to (1.5). Thus, for any affine algebraic group, we have the adjunction

$$(-)_G : \mathtt{sGr} \rightleftarrows \mathtt{sCommAlg}_k : G. \tag{1.6}$$

Our 1st main result reads

Theorem 1.1. The functor (1.5) has a total right derived functor $\boldsymbol{R}G : \mathtt{Ho}(\mathtt{sCommAlg}_k) \to \mathtt{Ho}(\mathtt{sGr})$, which is right adjoint to the derived representation functor (1.3); thus, (1.6) induces the derived adjunction

$$\boldsymbol{L}(-)_G : \mathtt{Ho}(\mathtt{sGr}) \rightleftarrows \mathtt{Ho}(\mathtt{sCommAlg}_k) : \boldsymbol{R}G. \tag{1.7}$$

Note that the categories $\mathtt{sGr}$ and $\mathtt{sCommAlg}_k$ have natural (simplicial) model structures, and the above result would be immediate from the well-known adjunction theorem of Quillen [65] if (1.6) were Quillen functors. However, it is easy to see that the functors (1.6) do *not* form a Quillen pair of model categories, nor even do they form a deformable adjunction of homotopical categories in the sense of [23]. Theorem 1.1 is therefore an interesting and fairly nontrivial result, which is—to the best of our knowledge—new.

In the special case when $G = \mathrm{GL}_n$, the derived functor $\boldsymbol{R}G$ can be described explicitly as the composite of two well-known functors:

$$\boldsymbol{R}\mathrm{GL}_n \cong \boldsymbol{L}l \circ \widehat{\mathrm{GL}}_n. \tag{1.8}$$

The functor $\widehat{\mathrm{GL}}_n : \mathtt{sCommAlg}_k \to \mathtt{sMon}$ takes values in the category of simplicial monoids, assigning to a simplicial algebra A_* the simplicial monoid of $(n \times n)$-matrices over A_* "invertible up to homotopy": more precisely, $\widehat{\mathrm{GL}}_n(A_*)$ is defined by the pull-back diagram in the category $\mathtt{sMon}$:

$$\begin{tikzcd} \widehat{\mathrm{GL}}_n(A_*) \arrow[r] \arrow[d] & \mathrm{GL}_n(\pi_0 A_*) \arrow[d] \\ M_n(A_*) \arrow[r] & M_n(\pi_0 A_*). \end{tikzcd} \tag{1.9}$$

This functor was originally introduced by Waldhausen [77] to define a homotopy invariant version of algebraic K-theory of simplicial rings. The second functor in (1.8) is the total left derived functor $\boldsymbol{L}l : \mathtt{Ho(sMon)} \to \mathtt{Ho(sGr)}$ of the group completion (localization) of simplicial monoids: it can be viewed as a special case of the classical Dwyer–Kan localization of simplicial categories studied in [24]. Formula (1.8) is rather unusual as it expresses a right derived functor in terms of a left derived one.

For an arbitrary algebraic group G, we construct an explicit model for $\boldsymbol{R}G$ using the recent work of Galatius and Venkatesh [33]. In this model, instead of simplicial monoids, we factor $\boldsymbol{R}G$ through the reduced simplicial spaces (or reduced Segal precategories) in the sense of Bergner [12]. This construction leads to a more general definition of representation homology that applies to simplicial spaces and does not use the Kan loop group equivalence (see Section 3.4).

The second main result of this paper is an interpretation of representation homology in terms of classical homological algebra in functor categories. To this end we consider the category $\mathfrak{G}$ of finitely generated free groups with objects $\langle n \rangle := \mathbb{F}\langle x_1, x_2, \ldots, x_n \rangle$ (for $n \geq 0$) and morphisms being the arbitrary group homomorphisms. This category is a PROP (i.e., a small permutative category) with monoidal structure $\langle n \rangle \boxtimes \langle m \rangle = \langle n + m \rangle$, and it is known that the category of k-algebras over $\mathfrak{G}$ (i.e., the category of strict monoidal functors from $\mathfrak{G}$ to the category $\mathtt{Vect}_k$ of k-vector spaces) is equivalent to the category of commutative Hopf k-algebras (see, e.g., [40].). Under this equivalence, a commutative Hopf algebra $\mathcal{H}$ corresponds to the functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{Vect}_k$, $\langle n \rangle \mapsto \mathcal{H}^{\otimes n}$, which actually takes its values in the category of commutative algebras. Note that any functor $F : \mathfrak{G} \to \mathtt{CommAlg}_k$ extends naturally (by taking the left Kan extension along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{FGr}$) to the category of all (based) free groups, $\mathtt{FGr}$, whose objects are the free groups $\mathbb{F}\langle S\rangle$ given with a prescribed generating set S and morphisms are the arbitrary group homomorphisms; we denote this extension by $\tilde{F} : \mathtt{FGr} \to \mathtt{Comm\,Alg}_k$.

Now, mimicking the Pirashvili construction of higher Hochschild homology (cf. [60] and Section 4.1 below), for a reduced simplicial set $X \in \mathtt{sSet}_0$ and a commutative Hopf algebra $\mathcal{H}$, we consider the composition of functors

$$\Delta^{\mathrm{op}} \xrightarrow{\mathbb{G}X} \mathtt{FGr} \xrightarrow{\tilde{\underline{\mathcal{H}}}} \mathtt{Comm\,Alg}_k,$$

where $\mathbb{G}X$ is the Kan loop group construction of X and $\tilde{\underline{\mathcal{H}}}$ is the (left Kan) extension of the strict monoidal functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{Vect}_k$ corresponding to $\mathcal{H}$. This defines a simplicial commutative algebra $\underline{\mathcal{H}}(\mathbb{G}X)$, whose homotopy groups we denote by

$$\mathrm{HR}_*(X, \mathcal{H}) := \pi_* \underline{\mathcal{H}}(\mathbb{G}X) = \mathrm{H}_*[N(\underline{\mathcal{H}}(\mathbb{G}X))]. \tag{1.10}$$

It turns out that this definition is *equivalent* to our original definition of representation homology (1.4) given in terms of the derived representation functor $\boldsymbol{L}(-)_G$. Precisely (cf. Proposition 4.1), we have

Proposition 1.1. Let G be an affine group scheme over k with coordinate ring $\mathcal{H} = \mathcal{O}(G)$. Then, for any $X \in \mathtt{sSet}_0$, there is a natural isomorphism of graded commutative algebras

$$\mathrm{HR}_*(X, \mathcal{O}(G)) \cong \mathrm{HR}_*(X, G).$$

Thanks to Proposition 1.1, we may (and will) use the notation $\mathrm{HR}_*(X, G)$ and $\mathrm{HR}_*(X, \mathcal{H})$ interchangeably, without causing confusion. Although its proof is almost immediate, Proposition 1.1 has a number of important implications. First, we state the following theorem, which is the main result of Section 4 (see Theorem 4.3).

Theorem 1.2. For any $X \in \mathtt{sSet}_0$, there is a natural 1st quadrant spectral sequence

$$E^2_{pq} = \mathrm{Tor}^{\mathfrak{G}}_p(\underline{\mathrm{H}}_q(\Omega X; k), \underline{\mathcal{H}}) \underset{p}{\Longrightarrow} \mathrm{HR}_n(X, \mathcal{H}) \tag{1.11}$$

converging to the representation homology of X.

The spectral sequence (1.11) relates the representation homology $\mathrm{HR}_*(X,\mathcal{H})$ of a space X to its Pontryagin algebra $\mathrm{H}_*(\Omega X;k)$. To describe the E_2-term of (1.11) we recall that $\mathrm{H}_*(\Omega X;k)$ has a natural structure of a graded cocommutative Hopf algebra with coproduct induced by the Alexander–Whitney diagonal and the product by the Eilenberg–Zilber map. For each $q\in\mathbb{Z}$, the assignment $\langle n\rangle\mapsto[\mathrm{H}^{\otimes n}]_q$, where $[\mathrm{H}^{\otimes n}]_q$ is the q-th graded component of the n-th tensor power of $\mathrm{H}=\mathrm{H}_*(\Omega X;k)$, defines a functor $\underline{\mathrm{H}}_q:\mathfrak{G}^{\mathrm{op}}\to\mathtt{Vect}_k$, which is the 1st argument of the "Tor" in (1.11). The "Tor" itself is the (abelian) derived functor of the tensor product $\otimes_{\mathfrak{G}}$ between covariant and contravariant $\mathtt{Vect}_k$-valued functors over the (small) category $\mathfrak{G}$. The spectral sequence (1.11) is a counterpart of Pirashvili's fundamental spectral sequence for higher Hochschild homology (cf. [60,Theorem 2.4]); however, in the case of representation homology it takes a more geometric form.

Theorem 1.2 has several interesting implications. First of all, it shows that the representation homology $\mathrm{HR}_*(X,\mathcal{H})$ is stable under Pontryagin equivalences (i.e., maps of spaces $X\to Y$ inducing isomorphisms of Pontryagin algebras $\mathrm{H}_*(\Omega X;k)\xrightarrow{\sim}\mathrm{H}_*(\Omega Y;k)$), and hence, if X is simply connected, $\mathrm{HR}_*(X,\mathcal{H})$ is actually a *rational* homotopy invariant of X (see Proposition 4.2). Next, if X is a $K(\Gamma,1)$-space, the spectral sequence (1.11) degenerates giving an isomorphism (cf. Corollary 4.3)

$$\mathrm{HR}_*(\Gamma,G)\cong\mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma],\mathcal{O}(G)),\tag{1.12}$$

where $k[\Gamma]$ is the group algebra of Γ equipped with the natural (cocommutative) Hopf algebra structure. The isomorphism (1.12) shows that the representation homology has a natural "Tor" interpretation, similar to the classical (Connes) interpretation of the Hochschild and cyclic homology (see [50, Chap. 6]). It is also interesting to compare (1.12) with another natural isomorphism

$$\mathrm{H}_{*+1}(\Gamma,k)\cong\mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma],\mathrm{lin}_k),\tag{1.13}$$

which provides a "Tor" interpretation (over $\mathfrak{G}$) for the ordinary homology of Γ as a discrete group. Here lin_k stands for the linearization functor $\mathfrak{G}\to\mathtt{Vect}_k$ that takes the free group $\langle n\rangle$ to the vector space k^n (see (4.9)). Note that if $G=\mathbb{G}_a$ is the additive group over k, then we have a natural isomorphism of functors $\mathcal{O}(\mathbb{G}_a)\cong\mathrm{Sym}_k(\mathrm{lin}_k)$, which implies $\mathrm{HR}_*(\Gamma,\mathbb{G}_a)\cong\Lambda_k[\mathrm{H}_{*+1}(\Gamma,k)]$. More generally, for any (pointed connected) space X, we have an isomorphism of graded commutative algebras (see Example 3.1)

$$\mathrm{HR}_*(X, \mathbb{G}_a) \cong \Lambda_k[\mathrm{H}_{*+1}(X, k)], \tag{1.14}$$

where $\Lambda_k[\mathrm{H}_{*+1}(X,k)]$ is the symmetric algebra of the graded vector space $\mathrm{H}_{*+1}(X,k) = \oplus_{i\geq 0}\mathrm{H}_{i+1}(X,k)$. Thus, we may think of representation homology as a generalization of the ordinary (singular) homology of spaces.

In Section 5, we show that representation homology can be also viewed as a generalization of higher Hochschild homology of spaces. The main result of this section reads (cf. Theorems 5.1 and 5.2).

Theorem 1.3. Let $\mathcal{H}$ be a commutative Hopf algebra.

(a) For any simplicial set $X \in \mathtt{sSet}$, there is a natural isomorphism

$$\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HH}_*(X, \mathcal{H}), \tag{1.15}$$

where $X_+ = X \sqcup \{*\}$ is a pointed simplicial set obtained from X by adjoining functorially a basepoint, and Σ is the (reduced) suspension functor on the category of pointed simplicial sets.

(b) For any pointed simplicial set $X \in \mathtt{sSet}_*$, there is a natural isomorphism

$$\mathrm{HR}_*(\Sigma X, \mathcal{H}) \cong \mathrm{HH}_*(X, \mathcal{H}; k), \tag{1.16}$$

where $\mathrm{HH}_*(X, \mathcal{H}; k)$ is the Pirashvili–Hochschild homology of the commutative algebra $\mathcal{H}$ with coefficients in k viewed as an $\mathcal{H}$-module via the Hopf algebra counit $\varepsilon : \mathcal{H} \to k$.

The proof of Theorem 1.3 is based on Milnor's classical *FK*-construction [56] that gives a simple simplicial group model for the space $\Omega\Sigma|X|$.

Theorem 1.3 has strong implications: in particular, it allows one to compute the representation homology of suspensions in a completely explicit way. It is known that ΣX for any pointed connected space X is *rationally* homotopy equivalent to a bouquet of spheres of dimension ≥ 2. Since representation homology depends only on the rational homotopy type of a space, the isomorphism (1.16), together with Pirashvili's computations [60] of higher Hochschild homology of spheres, implies (cf. Proposition 5.3).

Proposition 1.2. For any pointed connected space X of finite type, there is an isomorphism

$$\mathrm{HR}_*(\Sigma X, G) \cong \Lambda_k[\,\overline{\mathrm{H}}_*(X;\mathfrak{g}^*)],$$

where $\Lambda_k[\,\overline{\mathrm{H}}_*(X;\mathfrak{g}^*)]$ is the graded symmetric algebra of the reduced (singular) homology of X with coefficients in the dual Lie algebra of the group G.

By induction, Proposition 1.2 implies $\mathrm{HR}_*(\Sigma^n X, G) \cong \Lambda_k\left(\overline{\mathrm{H}}_*(X;\mathfrak{g}^*)[n-1]\right)$ for all $n \geq 1$. In particular, for $\mathbb{S}^n \cong \Sigma\mathbb{S}^{n-1}$, we have

$$\mathrm{HR}_*(\mathbb{S}^n, G) \cong \Lambda_k(\mathfrak{g}^*[n-1]) \;\; n \geq 2. \tag{1.17}$$

In Section 6, we compute representation homology of some classical non-simply connected spaces. Our examples include closed surfaces of arbitrary genus (both orientable and non-orientable) as well as some three-dimensional spaces (the link complements in $\mathbb{R}^3$ and $\mathbb{S}^3$, the lens spaces $L(p,q)$, and a general closed orientable 3-manifold). The representation homology of surfaces and link complements is expressed in terms of classical Hochschild homology of $\mathcal{O}(G)$ and related commutative algebras. For example, for the link complements in $\mathbb{R}^3$, we prove the following (cf. Theorem 6.1).

Theorem 1.4. Let L be a link in $\mathbb{R}^3$ obtained as the Alexander closure of a braid $\beta \in B_n$. Then the representation homology of the complement of its (regular) neighborhood in $\mathbb{R}^3$ is given by

$$\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G) \cong \mathrm{HH}_*(\mathcal{O}(G^n),\, \mathcal{O}(G^n)_\beta). \tag{1.18}$$

The right-hand side of (1.18) is the (ordinary) Hochschild homology of the associative algebra $\mathcal{O}(G^n)$ with bimodule coefficients. The bimodule $\mathcal{O}(G^n)_\beta$ is isomorphic to $\mathcal{O}(G^n) = \mathcal{O}(G)^{\otimes n}$ as a left module, while the right action of $\mathcal{O}(G^n)$ is twisted by an element β viewed as an automorphism of $\mathcal{O}(G)^{\otimes n}$ via the Artin representation of the braid group B_n.

Theorem 1.4 shows that the Hochschild homology groups $\mathrm{HH}_*(\mathcal{O}(G^n),\, \mathcal{O}(G^n)_\beta)$ are algebraic invariants of links in $\mathbb{R}^3$, which, to the best of our knowledge, have not appeared in the earlier literature. We should mention, however, that the representation homology of link complements bears a striking resemblance to *knot contact homology*, which is a new geometric homology theory of knots and links in $\mathbb{R}^3$ defined in [58] and studied extensively in recent years (see the remark after Theorem 6.1). We will discuss the relation between representation homology and knot contact homology in our subsequent paper.

In Section 6, we also discuss a multiplicative version of the derived Harish–Chandra conjecture proposed in [6]. If G is a connected reductive group with a maximal torus $T \subset G$ and W is the associated Weyl group, then for any space X, there is a natural map

$$\mathrm{HR}_*(X, G)^G \to \mathrm{HR}_*(X, T)^W, \tag{1.19}$$

which we call the *derived Harish–Chandra homomorphism* (cf. [6, Section 7]). In view of (1.17), by the classical Chevalley restriction theorem [18], the map (1.19) is an isomorphism for any odd-dimensional sphere $X = \mathbb{S}^{2p+1}$. We conjecture that (1.19) is also an isomorphism for the two-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, which gives the following explicit formula for the representation homology of $\mathbb{T}^2$ (see Section 6.1.1, Conjecture 1):

$$\mathrm{HR}_*(\mathbb{T}^2, G)^G \cong [\mathcal{O}(T \times T) \otimes \Lambda_k^*(\mathfrak{h}^*)]^W,$$

where $\mathfrak{h}$ is the Lie algebra of T (i.e., a Cartan subalgebra of $\mathfrak{g}$) and $\mathfrak{h}^*$ is its linear dual. As for the Drinfeld homomorphism, it would be interesting to find more examples of spaces, for which the map (1.19) is an isomorphism, and/or give an abstract characterization of all such spaces.

In the last section of the paper, we give another interpretation of representation homology as the Hochschild–Mitchell homology of a certain bifunctor on the category of finitely generated free groups $\mathfrak{G}$. Such an interpretation is useful for several reasons. First, it allows us to define representation *cohomology* in a natural way (by simply replacing the Hochschild–Mitchell homology with the Hochschild–Mitchell cohomology of the same bifunctor). Second, it suggests that it is natural to extend the definition of representation (co)homology by taking the Hochschild–Mitchell (co)homology of $\mathfrak{G}$ with coefficients in an arbitrary bifunctor D: that is, $\mathrm{HR}(D) := \mathrm{HH}(\mathfrak{G}, D)$. Third and most important, it exhibits a close analogy with *topological* Hochschild homology, which is known to be isomorphic to the Hochschild–Mitchell homology of the category $\mathfrak{G}_{\mathrm{ab}}$ of finitely generated free *abelian* groups (see [62]). Motivated by this analogy, we construct functorial trace maps

$$\mathrm{DTr}_n^{\mathfrak{G}}(D):\ \mathrm{H}_*(\mathrm{Aut}(\mathbb{F}_n), D_n) \to \mathrm{HR}_*(D) \quad \forall\, n \geq 1,$$

relating homology of the automorphism groups of f.g. free groups with appropriate coefficients to representation homology. These maps are compatible with natural inclusions $\mathrm{Aut}(\mathbb{F}_n) \hookrightarrow \mathrm{Aut}(\mathbb{F}_{n+1})$ and hence have an stable limit as $n \to \infty$. The corresponding stable map $\mathrm{DTr}^{\mathfrak{G}}_{\infty}(D) : \mathrm{H}_*(\mathrm{Aut}_{\infty}, D_{\infty}) \to \mathrm{HR}_*(D)$ can be viewed as a non-abelian analogue of the classical Dennis trace relating topological Hochschild homology to stable homology of general linear groups. We conjecture that the map $\mathrm{DTr}^{\mathfrak{G}}_{\infty}(D)$ is actually an isomorphism, whenever D is a *polynomial* bifunctor (cf. Conjecture 2). This is a non-abelian analogue of a theorem of Scorichenko [28].

1.3 Relation to derived algebraic geometry

The derived representation schemes $\mathrm{DRep}_G(X)$ are basic objects of derived algebraic geometry. To the best of our knowledge, the first construction of this kind—the derived moduli space $\boldsymbol{R}\mathrm{Loc}_G(X)$ of G-local systems over a finite, pointed, connected CW complex X—was proposed by Kapranov in [45]. He defined $\boldsymbol{R}\mathrm{Loc}_G(X)$ using a simplicial DG scheme $\boldsymbol{R}BG$ that played the role of a canonical "injective resolution" of the classifying space BG of the algebraic group G in the category of simplicial DG schemes. A more refined construction $\mathbf{Map}(X, BG)$—called the derived mapping stack of flat G-bundles on X—was developed by Toën and Vezzosi in [76] (see also [59]), using local homotopy theory of simplicial presheaves on the category of (derived) affine schemes. For a detailed comparison of these two constructions with our construction of $\mathrm{DRep}_G(X)$, we refer the reader to the appendix of [9], where we showed that—despite different frameworks—all three constructions are essentially equivalent.

We would like to conclude this introduction by mentioning some interesting topological generalizations of higher Hochschild homology that appeared in recent years, such as factorization homology (see, e.g., [36, 37]) and higher topological Hochschild homology [15]. Our results show that representation homology, while closely related to Hochschild homology, is a richer and somewhat more geometric theory that blends topology and representation theory in a very natural way. It would therefore be interesting to see if representation homology admits topological refinements similar to those of Hochschild homology.

1.4 Appendix

The paper contains an appendix, where we collect basic facts and prove some new results in abstract homotopy theory concerning derived functors. The main result of the appendix—Theorem A.2—arises from our attempt to abstract the situation of Theorem 3.1: it is a version of Quillen's derived adjunction theorem for homotopical categories. This theorem as well as Theorem A.3 and Lemma A.1 are of independent interest.

1.5 Outline of the paper

The paper is organized as follows. In Section 2, we introduce notation and recall some basic facts about simplicial sets and spaces. In Section 3, we study basic properties of the derived representation functor and define representation homology. In Section 4, we give our second construction of representation homology in terms of functor homology and derive its implications. In Section 4, we establish the isomorphism between the representation homology of suspensions and higher Hochschild homology. In Section 6, we give examples computing representation homology explicitly for some geometrically interesting spaces. In Section 7, we identify representation homology in terms of Hochschild–Mitchell homology and construct a non-abelian analogue of the Dennis trace map relating representation homology to the stable homology of automorphism groups of finitely generated free groups. The paper ends with an appendix where we recall basic definitions and prove a few results from abstract homotopy theory used in Section 3.

2 Preliminaries

In this section, we introduce notation and recall some basic definitions related to simplicial sets. Standard references for this material are [54], [39], and [79, Chapter 8].

2.1 Simplicial objects

Let Δ denote the simplicial category. Recall that the objects of Δ are the finite-ordered sets $[n] := \{0, 1, \dots, n\}$, $n \geq 0$, and the morphisms are the (weakly) order preserving maps $[n] \to [m]$. A *simplicial object* in a category $\mathscr{C}$ is a contravariant functor from Δ to $\mathscr{C}$: that is, $\Delta^{\mathrm{op}} \to \mathscr{C}$. The simplicial objects in $\mathscr{C}$ form a category, with morphisms being the natural transformations of functors. We denote this category by $s\mathscr{C}$. If $X \in \mathrm{Ob}(s\mathscr{C})$, we write $X_n := X([n])$.

The category Δ is generated by two distinguished classes of morphisms $\{\delta^i\}_{0\leq i\leq n}^{n\geq 1}$ and $\{\sigma^j\}_{0\leq j\leq n}^{n\geq 0}$, whose images under $X \in s\mathscr{C}$ are called the *face* and *degeneracy maps* of X, respectively. The map $\delta^i : [n-1] \to [n]$ is the (unique) injection that does not contain "i" in its image; the corresponding face map is denoted by $d_i := X(\delta^i) : X_n \to X_{n-1}$. Similarly, for $n \geq 0$, the map $\sigma^i : [n+1] \to [n]$ is the (unique) surjection in Δ that takes value "i" twice. The image of σ^i under X is the degeneracy map $s_i := X(\sigma^i) : X_n \to X_{n+1}$. The face and degeneracy maps of a simplicial object satisfy the following *simplicial relations*:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \quad \text{if } i < j \\
d_i s_j &= s_{j-1} d_i \quad \text{if } i < j \\
d_i s_j &= s_j d_{i-1} \quad \text{if } i > j+1 \\
s_i s_j &= s_{j+1} s_i \quad \text{if } i \le j \\
d_i s_j &= \mathrm{Id} \qquad \text{if } i = j\,, j+1.
\end{aligned} \tag{2.1}$$

Thus, a simplicial object in $s\mathscr{C}$ is determined by a family $X = \{X_n\}_{n\ge 0}$ of objects in $\mathscr{C}$ together with morphisms $d_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ satisfying the relations (2.1). The object X_n is usually called the "set" of n-*simplices* of X, and the 0-simplices are usually called the *vertices* of X.

We let $\mathtt{sSet}$ denote the category of simplicial sets (i.e., simplicial objects in the category $\mathtt{Set}$). A simplicial set X is called *reduced* if it has a single vertex, that is, $X_0 = \{*\}$. The full subcategory of $\mathtt{sSet}$ consisting of reduced simplicial sets will be denoted $\mathtt{sSet}_0$. A simplicial set X is called *pointed* if there are distinguished simplices $x_n \in X_n$, one in each degree, such that $x_n = s_0(x_{n-1})$ for all $n \ge 1$. The sequence $(x_0, x_1, x_2, \ldots) \in \prod_{n\ge 0} X_n$ is called a basepoint of X. The category of pointed simplicial sets will be denoted $\mathtt{sSet}_*$. Note that $\mathtt{sSet}_0$ can also be viewed as a full subcategory of $\mathtt{sSet}_*$ as every reduced simplicial set has a canonical (unique) basepoint.

Given $X \in \mathtt{sSet}$, the set of *nondegenerate* n-simplices of X is defined to be

$$\overline{X}_n := X_n \setminus \bigcup_{i=0}^{n-1} s_i(X_{n-1}).$$

Every element of X_n can be uniquely expressed in terms of the nondegenerate elements of X (see [35, Lemma 11] for a precise statement). In particular, a simplicial set can be defined by specifiying its nondegenerate simplices together with the restriction of each face map to the set of nondegenerate simplices.

We give a few basic examples of simplicial sets that will be used in this paper.

2.1.1 *Discrete simplicial objects*

To any object $A \in \mathscr{C}$ one can associate a simplicial object $A_* \in s\mathscr{C}$, with $A_n = A$ and d_i, s_j being the identity map of A for all n, i, j. This gives a fully faithful embedding $\mathscr{C} \hookrightarrow s\mathscr{C}$. The objects of $s\mathscr{C}$ arising this way are called *discrete simplicial objects*.

2.1.2 *Geometric simplices*

The n-dimensional geometric simplex is the topological space

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1\,,\, x_i \geq 0\,\}.$$

Let e_i denote the vertex of Δ^n with i-th coordinate 1. For any morphism $f : [m] \to [n]$ in Δ, there is a (unique) linear map $\mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ sending e_i to $e_{f(i)}$ that restricts to a map of topological spaces $f^* : \Delta^m \to \Delta^n$. The collection $\Delta^* := \{\Delta^n\}_{n\geq 0}$ forms a *cosimplicial space*, that is, a (covariant) functor $\Delta \to \mathtt{Top}$, where $\mathtt{Top}$ denotes the category of (compactly generated weakly Hausdorff) topological spaces. This functor is faithful: it gives a topological realization of the simplicial category, which was historically the first definition of Δ.

2.1.3 *Standard simplices*

Let $Y : \Delta \hookrightarrow \mathtt{sSet}$ denote the Yoneda embedding. The functor Y assigns to $[n]$ a simplicial set $\Delta[n]_*$ called the *standard n-simplex*. Explicitly, $\Delta[n]_*$ is given by

$$\Delta[n]_k := \mathrm{Hom}_{\Delta}([k],[n]) \cong \{(n_0, \ldots, n_k) \,|\, 0 \leq n_0 \leq \ldots \leq n_k \leq n\},$$

where a function $f : [k] \to [n]$ is identified with the sequence of its values $(f(0), \ldots, f(k))$. Under this identification, the nondegenerate simplices correspond to *strictly* increasing functions, and the face and degeneracy maps in $\Delta[n]_*$ are given by

$$d_i(n_0, \ldots, n_k) = (n_0, \ldots, \hat{n}_i, \ldots, n_k)\,, \quad s_j(n_0, \ldots, n_k) = (n_0, \ldots, n_j, n_j, \ldots, n_k).$$

By Yoneda lemma, for any simplicial set X, there is a natural bijection

$$\mathrm{Hom}_{\mathtt{sSet}}(\Delta[n]_*, X) \cong X_n,$$

which shows that $\Delta[n]_*$ (co)represents the functor: $\mathtt{sSet} \to \mathtt{Set}\,,\ X \mapsto X_n$.

2.1.4 *Simplicial spheres*

The Yoneda functor $Y : \Delta \to$ `sSet` can be also regarded as a cosimplicial object in the category of simplicial sets. In particular, for any $n \geq 1$, there are $n+1$ coface maps $d^i : \Delta[n-1]_* \to \Delta[n]_*$, $0 \leq i \leq n$. Using these maps, we define the *boundary* of $\Delta[n]_*$ to be the simplicial subset

$$\partial\Delta[n]_* := \bigcup_{0 \leq i \leq n} d^i(\Delta[n-1]_*) \subset \Delta[n]_*,$$

The *simplicial n-sphere* is then defined to be the corresponding quotient set $\mathbb{S}^n_* := \Delta[n]_*/\partial\Delta[n]_*$. It is easy to see that the only nondegenerate simplices in $\mathbb{S}^n_*$ are in degree 0 and n, with $\overline{\mathbb{S}}^n_0 = \{*\}$ and $\overline{\mathbb{S}}^n_n = \{S\}$, where S is the image of the map Id $\in \Delta[n]_n$ in $\mathbb{S}^n_n$. Note that $d_i(S) = s_0^{n-1}(*)$ for all i. Thus, the simplicial structure of $\mathbb{S}^n_*$ reflects the standard CW decomposition of the n-sphere $\mathbb{S}^n$ with one cell in dimension 0 and one cell in dimension n.

The simplicial 1-sphere $\mathbb{S}^1_*$ is called the *simplicial circle*. By Example 2.1.3, we have $\Delta[1]_k \cong \{(\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{k+1-i}) \mid i = 0,1,\ldots,k+1\}$ and $\partial\Delta[1]_k = \{(0,\ldots 0),(1,\ldots,1)\}$. Hence, $\mathbb{S}^1_*$ is given explicitly by

$$\mathbb{S}^1_k \cong \{(\underbrace{0,\ldots,0}_{i},\underbrace{1,\ldots,1}_{k+1-i}) \mid i = 1,\ldots,k+1\},$$

with $(0,\ldots,0)$ corresponding to the basepoint $*$.

There is an important functor $|-| :$ `sSet` $\to$ `Top` assigning to each simplicial set X a topological space $|X|$ called the *geometric realization* of X. Explicitly, the space $|X|$ is defined by

$$|X| := \bigsqcup_{n \geq 0} (X_n \times \Delta^n)/\sim,$$

where each set X_n is equipped with discrete topology and the equivalence relation is given by

$$(d_i x, p) \sim (x, d^i p) \text{ for } (x,p) \in X_n \times \Delta^{n-1}$$
$$(s_j x, p) \sim (x, s^j p) \text{ for } (x,p) \in X_{n-1} \times \Delta^n .$$

More formally (see, e.g., [66, Section 1.3]), the functor $|-| :$ `sSet` $\to$ `Top` can be defined as the (left) Kan extension $|-| = \mathrm{Lan}_Y(\Delta^*)$ of the geometric simplex Δ^* along the Yoneda embedding $Y : \Delta \to \mathtt{sSet}$. It follows from this definition that $|\Delta[n]_*| \cong \Delta^n$ for all $n \geq 0$, and in general, $|X| \cong \operatorname{colim} \Delta^n$, where the colimit is taken over all morphisms of $\Delta[n]_* \to X$, $n \geq 0$. If $X \in \mathtt{sSet}$ is a simplicial set and $x_0 \in X_0$, we write $\pi_n(X, x_0)$ for the *n-th homotopy group* of X at x_0, which is, by definition, the n-th homotopy group $\pi_n(|X|, x_0)$ of the geometric realization of X.

The category $\mathtt{sSet}$ has a standard model structure, where the weak equivalences are the morphisms inducing weak homotopy equivalences of the corresponding geometric realizations. The cofibrations are levelwise injective maps and the fibrations are the Kan fibrations (see [54, §7]). This structure gives a model structure on $\mathtt{sSet}_0$.

Let $(X, *)$ be a pointed topological space. The *(total) singular complex* of X is a simplicial set $S_*(X)$ defined by $S_n(X) := \operatorname{Hom}_{\mathtt{Top}}(\Delta^n, X)$. The *Eilenberg subcomplex* of $S_*(X)$ is

$$\overline{S}_n(X) := \{f : \Delta^n \to X : f(v_i) = * \text{ for all vertices } v_i \in \Delta^n\}.$$

If X is connected, the natural inclusion $\overline{S}_*(X) \hookrightarrow S_*(X)$ is a weak equivalence of simplicial sets. Further, if we restrict $\overline{S}$ to the category $\mathtt{Top}_{0,*}$ of connected pointed spaces, we get the pair of adjoint functors

$$|-| : \mathtt{sSet}_0 \rightleftarrows \mathtt{Top}_{0,*} : \overline{S}, \tag{2.2}$$

which induce mutually inverse equivalences of the homotopy categories: $\mathtt{Ho}(\mathtt{sSet}_0) \simeq \mathtt{Ho}(\mathtt{Top}_{0,*})$. This equivalence justifies the following standard convention that we will follow throughout the paper.

Convention. We shall not notationally distinguish between a reduced simplicial set X and its geometric realization $|X|$. Nor shall we distinguish notationally between a topological space and a (reduced) simplicial model of that space.

2.2 The Kan loop group construction

We will briefly review the classical construction of Kan [44] that provides a functorial simplicial group model of the based loop space ΩX. For details and proofs we refer the reader to [54, Chapter VI] and [39, Chapter V]). Let $\mathtt{sGr}$ denote the category of simplicial groups. It has a standard model structure, where the weak equivalences and fibrations of simplicial groups are the weak equivalences and fibrations of the underlying simplicial sets. We note that, unlike $\mathtt{sSet}$, the model category $\mathtt{sGr}$ is fibrant: by a classical theorem of Moore, every simplicial group is a Kan complex (see [54, Theorem 17.1]).

Definition 2.1. A simplicial group $\Gamma = \{\Gamma_n\}_{n\geq 0}$ is called *semi-free* if there is a sequence of subsets $B_n \subset \Gamma_n$, one in each degree, such that Γ_n is freely generated by B_n, and the set $B = \bigcup_{n\geq 0} B_n$ is closed under degeneracies of Γ, that is, $s_j(B_{n-1}) \subseteq B_n$ for all $0 \leq j \leq n-1$ and $n \geq 1$. The subset $\overline{B}_n := B_n \setminus \bigcup_{i=0}^{n-1} s_i(B_{n-1})$ is called the set of *nondegenerate generators* of Γ of degree n.

One can show that every element in $\overline{B}_n$ is nondegenerate (when considered as an element of the underlying simplicial set), and a semi-free simplicial group is determined by specifying the sets of nondegenerate generators $\overline{B}_n$ and the face elements of these generators.

Semi-free simplicial groups are cofibrant objects in the model category $\mathtt{sGr}$. The Kan loop group construction provides an important class of semi-free simplicial groups that arise naturally from reduced simplicial sets. To be precise, the Kan construction defines a pair of adjoint functors:

$$\mathbb{G} : \mathtt{sSet}_0 \rightleftarrows \mathtt{sGr} : \overline{W}, \tag{2.3}$$

where $\mathbb{G}$ is called the *Kan loop group functor* and $\overline{W}$ is the *classifying simplicial complex*. The functor $\mathbb{G}$ preserves weak equivalences and cofibrations, while $\overline{W}$ preserves weak equivalences and fibrations (see [39, Proposition V.6.3]). Hence, (2.3) is a Quillen pair, which is actually a Quillen equivalence: that is, the functors $\mathbb{G}$ and $\overline{W}$ induce mutually inverse equivalences between the homotopy categories of $\mathtt{sSet}_0$ and $\mathtt{sGr}$ (see [39, Corollary V.6.4]). Combining this with the classical Quillen equivalence (2.2) between topological spaces and simplicial sets:

$$\mathtt{Top}_{0,*} \xrightarrow{\bar{S}} \mathtt{sSet}_0 \xrightarrow{\mathbb{G}} \mathtt{sGr}$$

we get equivalences of the homotopy categories:

$$\mathtt{Ho}(\mathtt{Top}_{0,*}) \cong \mathtt{Ho}(\mathtt{sSet}_0) \cong \mathtt{Ho}(\mathtt{sGr}).$$

For further use, we recall the explicit construction of the functor $\mathbb{G}$. Given a reduced simplicial set $X = \{X_n\}_{n\geq 0}$, the set of n-simplices of $\mathbb{G}X$ is defined by

$$\mathbb{G}X_n \;=\; \langle X_{n+1}\rangle / \langle s_0(x) = 1\,,\ \forall\, x \in X_n\rangle \;\cong\; \langle B_n\rangle,$$

where $B_n := X_{n+1} \setminus s_0(X_n)$ and the isomorphism is induced by the inclusion $B_n \hookrightarrow X_{n+1}$. The degeneracy maps $s_j^{\mathbb{G}X} : \mathbb{G}X_n \to \mathbb{G}X_{n+1}$ are induced by the degeneracy maps $s_{j+1} : X_{n+1} \to X_{n+2}$ of the simplicial set X, and the face maps $d_i^{\mathbb{G}X} : \mathbb{G}X_n \to \mathbb{G}X_{n-1}$ are given by

$$d_0^{\mathbb{G}X}(x) := (d_1 x) \cdot (d_0 x)^{-1} \quad \text{and} \quad d_i^{\mathbb{G}X}(x) := d_{i+1}(x)\,,\ \forall\, i > 0.$$

Conversely, given a simplicial group $\Gamma = \{\Gamma_n\}_{n\geq 0}$, the simplicial set $\overline{W}\Gamma$ is defined by $\overline{W}\Gamma_0 := \{*\}$ and $\overline{W}\Gamma_n := \Gamma_{n-1} \times \Gamma_{n-2} \times \ldots \times \Gamma_0$ for $n \geq 0$. The degeneracy and face maps of $\overline{W}\Gamma$ are given explicitly in [54, §21]. We note that when restricted to *discrete* simplicial groups, the functor $\overline{W}$ coincides with the usual nerve construction, that is, $\overline{W}\Gamma = \mathrm{B}\Gamma$ for any discrete group Γ.

Proposition 2.1. The Kan loop group $\mathbb{G}X$ of any reduced simplicial set X is semi-free. More precisely, for each $n > 0$, the composite map $\tau : X_n \to \langle X_n \rangle \twoheadrightarrow \mathbb{G}X_{n-1}$ is injective when restricted to the subset $\overline{X}_n \subset X_n$, and the image $\tau(\overline{X}_n) \subset \mathbb{G}X_{n-1}$ forms the set of nondegenerate generators $\overline{B}_{n-1} = \tau(\overline{X}_n)$ in degree $(n-1)$ of the semi-free basis $\{B_n\}_{n\geq 0}$ of $\mathbb{G}X$.

The following fundamental theorem clarifies the meaning of the Kan loop group construction.

Theorem 2.1 (Kan [44]). For any reduced simplicial set X, there is a weak homotopy equivalence

$$|\mathbb{G}X| \simeq \Omega|X|,$$

where $\Omega|X|$ is the (Moore) based loop space of $|X|$.

A detailed proof of Theorem 2.1 can be found in [54, § 26]. Its significance becomes clear from the following considerations. Given any path-connected CW complex Y one can choose a pointed connected simplicial set X' such that $|X'| \simeq Y$. If X is the path-connected component of X' containing the basepoint, then X is a reduced simplicial set such that $|X| \simeq |X'| \simeq Y$ because Y is connected. Hence, applying the Kan loop group construction to X, we get $|\mathbb{G}X| \simeq \Omega Y$. Thus, $\mathbb{G}X$ is a semi-free simplicial group model of the based loop space of Y. In this way, the based loop space of any path-connected CW complex admits a simplicial group model.

3 Representation Homology

In this section, we define representation homology as the homotopy groups of the (non-abelian) derived representation functor associated with an affine algebraic group. We establish the existence and basic properties of this functor as well as indicate some generalizations. Our construction follows the approach of our earlier papers [4–6] where we studied the representation homology of associative and Lie algebras.

3.1 Definition of representation homology

Fix an affine algebraic group scheme G over a field k of characteristic 0. Recall that G is given by a representable functor on the category of commutative k-algebras with values in the category of groups:

$$G : \mathtt{Comm\,Alg}_k \to \mathtt{Gr}\,, \quad A \mapsto G(A). \tag{3.1}$$

A commutative algebra that represents (3.1) is called the coordinate ring of G and denoted $\mathcal{O}(G)$. This algebra is equipped with a coproduct $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$, $f \mapsto f^{(1)} \otimes f^{(2)}$, which is dual to the multiplication in G and makes $\mathcal{O}(G)$ a commutative Hopf algebra.

Lemma/Definition. The functor (3.1) has a left adjoint

$$(-)_G \,:\, \mathtt{Gr} \to \mathtt{Comm\,Alg}_k \quad \Gamma \mapsto \Gamma_G, \tag{3.2}$$

which we call the *representation functor* in G.

Proof. Given a group $\Gamma \in \mathtt{Gr}$ define the algebra Γ_G by the following canonical presentation:

$$\Gamma_G = \mathrm{Sym}_k(\, k[\Gamma] \otimes_k \mathcal{O}(G)\,)/I,$$

where the ideal I of relations is generated by

$$\begin{aligned} &\gamma \otimes f_1 f_2 - (\gamma \otimes f_1) \cdot (\gamma \otimes f_2), \\ &\gamma_1\gamma_2 \otimes f - (\gamma_1 \otimes f^{(1)}) \cdot (\gamma_2 \otimes f^{(2)}), \\ &e_\Gamma \otimes f - f(e_G) \cdot 1 \quad \gamma \otimes 1 - 1 \end{aligned} \tag{3.3}$$

for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $f, f_1, f_2 \in \mathcal{O}(G)$. If $A \in \mathtt{CommAlg}_k$ is a commutative algebra, a group homomorphism $\varphi : \Gamma \to G(A) = \mathrm{Hom}(\mathcal{O}(G), A)$ determines a linear map $k[\Gamma] \otimes \mathcal{O}(G) \to A$, which, in turn, induces—modulo the relations (3.3)—an algebra homomorphism $\varphi^{\#} : \Gamma_G \to A$. It is straightforward to check that $\varphi \mapsto \varphi^{\#}$ gives the required bijection $\mathrm{Hom}_{\mathtt{Gr}}(\Gamma, G(A)) \cong \mathrm{Hom}_{\mathtt{CommAlg}_k}(\Gamma_G, A)$. ■

We remark that, for a fixed group Γ, the algebra Γ_G represents the functor

$$\mathrm{Rep}_G(\Gamma) : \mathtt{CommAlg}_k \to \mathtt{Set} \quad A \mapsto \mathrm{Hom}_{\mathtt{Gr}}(\Gamma, G(A)),$$

which is the functor of points of an affine k-scheme $\mathrm{Rep}_G(\Gamma)$ parametrizing the representations of Γ in G; hence, geometrically, we can think of Γ_G as the coordinate ring $\mathcal{O}[\mathrm{Rep}_G(\Gamma)]$ of $\mathrm{Rep}_G(\Gamma)$.

Next, we embed the category of groups into the category $\mathtt{sGr}$ of simplicial groups and extend the functor (3.2) to $\mathtt{sGr}$ in the natural way, assigning to a simplicial group $\Gamma_* : \Delta^{\mathrm{op}} \to \mathtt{Gr}$ the simplicial commutative algebra $(\Gamma_*)_G : \Delta^{\mathrm{op}} \to \mathtt{Gr} \to \mathtt{CommAlg}_k$. We will keep the notation $(-)_G$ for this extended representation functor:

$$(-)_G : \mathtt{sGr} \to \mathtt{sCommAlg}_k. \tag{3.4}$$

Both categories $\mathtt{sGr}$ and $\mathtt{sCommAlg}_k$ have natural (simplicial) model structures, with weak equivalence being the weak homotopy equivalence of the underlying simplicial sets. However, the representation functor (3.4) is not homotopy invariant—it does not preserve weak equivalences—hence, in order to work in a homotopical context we should replace or approximate (3.4) with a derived functor (see [65], [25]). The existence of this derived functor is easy to establish.

Lemma 3.1. The functor (3.4) maps the weak equivalences between cofibrant objects in $\mathtt{sGr}$ to weak equivalences in $\mathtt{sCommAlg}_k$, and hence has a total left derived functor

$$\boldsymbol{L}(-)_G : \mathrm{Ho}(\mathtt{sGr}) \to \mathrm{Ho}(\mathtt{sCommAlg}_k). \tag{3.5}$$

Proof. Suppose that $f : \Gamma \to \Gamma'$ is a weak equivalence between cofibrant simplicial groups. Since sGr is a fibrant model category, Γ and Γ' are both fibrant–cofibrant objects. By Whitehead's theorem, the map f has then a homotopy inverse $g : \Gamma' \to \Gamma$, such that $fg \sim \mathrm{Id}$ and $gf \sim \mathrm{Id}$. Now, any homotopy between fibrant–cofibrant objects can be realized using a good cylinder object in $\mathtt{sGr}$. Since sGr is a simplicial model category, there is a natural choice of good cylinder objects for Γ and Γ': namely, $\Gamma \sqcup \Gamma \to \Gamma \times \Delta[1] \to \Gamma$, and similarly for Γ'. For such cylinder objects, the simplicial homotopies (see [54, Def. 5.1]) can be defined by explicit combinatorial relations that are preserved by the functor $(-)_G$. Thus, we conclude that $g_G : \Gamma'_G \to \Gamma_G$ is a homotopy inverse of $f_G : \Gamma_G \to \Gamma'_G$ in $\mathtt{sCommAlg}_k$ and hence f_G and g_G are mutually inverse isomorphisms in $\mathtt{Ho}(\mathtt{sCommAlg}_k)$. The existence of the derived functor (3.5) follows now from [25, Prop. 9.3]. ∎

Now, for a fixed simplicial group $\Gamma \in \mathtt{sGr}$, we formally define the *derived representation scheme* $\mathrm{DRep}_G(\Gamma)$ as $\mathrm{Spec}\,\boldsymbol{L}(\Gamma)_G$, that is, the simplicial algebra $\boldsymbol{L}(\Gamma)_G$ viewed as an object of the opposite category $\mathtt{Ho}(\mathtt{sCommAlg}_k)^{\mathrm{op}}$. We call the homotopy groups of $\boldsymbol{L}(\Gamma)_G$ the *representation homology of* Γ *in* G and write

$$\mathrm{HR}_*(\Gamma, G) := \pi_* \boldsymbol{L}(\Gamma)_G .$$

By comparing the universal mapping properties, it is easy to check that the functor (3.4) commutes with π_0; hence, for any $\Gamma \in \mathtt{sGr}$, there is a natural isomorphism in $\mathtt{CommAlg}_k$:

$$\mathrm{HR}_0(\Gamma, G) \cong [\pi_0(\Gamma)]_G . \tag{3.6}$$

In particular, if $\Gamma \in \mathtt{Gr}$ is a constant simplicial group, we have $\mathrm{HR}_0(\Gamma, G) \cong \Gamma_G$, which justifies our notation and terminology for $\mathrm{DRep}_G(\Gamma)$.

Next, recall the fundamental theorem of Kan [44] that identifies the homotopy types of simplicial groups with those of pointed connected spaces. To be precise, the Kan theorem asserts that the category of simplicial groups is Quillen equivalent to the category $\mathtt{sSet}_0$ of reduced simplicial sets, which is, in turn, Quillen equivalent to the category $\mathtt{Top}_{0,*}$ of pointed connected spaces. As a result, we have natural equivalences of homotopy categories

$$\mathtt{Ho}(\mathtt{Top}_{0,*}) \cong \mathtt{Ho}(\mathtt{sSet}_0) \cong \mathtt{Ho}(\mathtt{sGr}). \tag{3.7}$$

This leads us to the main definition.

Definition 3.1. For a space $X \in \mathtt{Top}_{0,*}$, we define the *derived representation scheme* $\mathrm{DRep}_G(X)$ to be $\mathrm{DRep}_G(\Gamma X)$, where ΓX is a(ny) simplicial group model of X (i.e., a simplicial group that corresponds to X under the Kan equivalence). The *representation* *homology of X in G* is then defined by

$$\mathrm{HR}_*(X, G) := \pi_* \boldsymbol{L}(\Gamma X)_G. \tag{3.8}$$

By definition, $\mathrm{HR}_*(X, G)$ is a graded commutative algebra, with $\mathrm{HR}_0(X, G)$ naturally isomorphic to $[\pi_1(X)]_G = \mathcal{O}[\mathrm{Rep}_G(\pi_1(X))]$, the coordinate ring of the representation scheme $\mathrm{Rep}_G[\pi_1(X)]$. The last isomorphism is the composition of (3.6) with the natural isomorphism $\pi_0(\Gamma X) \cong \pi_1(X)$.

For a reduced simplicial set $X \in \mathtt{sSet}_0$, the Kan loop group $\mathbb{G}X$ provides a canonical (functorial) simplicial group model for $|X|$. Since this simplicial group is semi-free (see Section 2.2), we have

$$\mathrm{HR}_*(X, G) \cong \pi_* \, (\mathbb{G}X)_G. \tag{3.9}$$

This formula can be used to compute representation homology in some simple cases.

Example 3.1. Let $\mathbb{G}_a$ be the additive group over k, that is, the affine algebraic group defined by the functor $\mathbb{G}_a : \mathtt{Comm\,Alg}_k \to \mathtt{Gr}, A \mapsto (A, +)$, where $(A, +)$ denotes the underlying abelian group of the algebra A. It is easy to see that, for any $\Gamma \in \mathtt{Gr}$, there is a natural bijection $\mathrm{Hom}_{\mathtt{Gr}}(\Gamma, \mathbb{G}_a(A)) \cong \mathrm{Hom}_{\mathtt{Comm\,Alg}_k}(\mathrm{Sym}_k(\mathrm{lin}_k\Gamma), A)$, where $\mathrm{lin}_k(\Gamma) := \Gamma_{\mathrm{ab}} \otimes_{\mathbb{Z}} k$. Hence, the representation functor in $\mathbb{G}_a$ is given by the composition $(-)_{\mathbb{G}_a} = \mathrm{Sym}_k \circ \mathrm{lin}_k : \mathtt{Gr} \to \mathtt{Vect}_k \to \mathtt{Comm\,Alg}_k$. Using formula (3.9), for an arbitrary $X \in \mathtt{sSet}_0$, we can now compute

$$\begin{aligned} \mathrm{HR}_*(X, \mathbb{G}_a) &\cong \pi_* \, \Lambda_k \, [(\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k] \\ &\cong \Lambda_k \left[\pi_*((\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k)\right] \\ &\cong \Lambda_k \left[\pi_*(\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k\right] \\ &\cong \Lambda_k \left[\mathrm{H}_{*+1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} k\right] \\ &\cong \Lambda_k \left[\mathrm{H}_{*+1}(X, k)\right], \end{aligned}$$

where Λ_k is the graded symmetric algebra functor over k and $\mathrm{H}_{*+1}(X, \mathbb{Z}) := \oplus_{i \geq 0} \mathrm{H}_{i+1}(X, \mathbb{Z})$ is the singular homology of X. Note that, besides (3.9), we used here the classical isomorphism of Kan: $\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong \mathrm{H}_{*+1}(X, \mathbb{Z})$ (see, e.g., [54, Theorem 26.9]) and the well-known fact that the functor Λ_k commutes with homology when k has characteristic zero (see, e.g., [65, Part I, Prop. 4.5]). This example shows that representation homology may be viewed as a generalization of the ordinary singular homology.

3.2 The derived representation adjunction

By definition, the representation functor (3.2) is left adjoint to (the functor of points of) the algebraic group G. This adjunction extends automatically to simplicial categories:

$$(-)_G : \mathtt{sGr} \rightleftarrows \mathtt{sCommAlg}_k : G, \tag{3.10}$$

and the natural question is whether (3.10) induces an adjunction between derived functors on the corresponding homotopy categories. The (affirmative) answer to this question would be immediate from Quillen's fundamental theorem [65, I.4.5, Theorem 3] if (3.10) were a pair of Quillen functors between model categories. However, this is not the case. By definition, any left Quillen functor preserves cofibrations, which means, in particular, that it maps cofibrant objects in one model category to cofibrant ones in the other. Unfortunately, the representation functor (3.4) lacks this property even in simplest cases. Take, for example, $G = \mathbb{G}_m$, the multiplicative group, and apply (3.4) to the free group on one generator $\Gamma = \mathbb{F}_1$, which is obviously a cofibrant object in $\mathtt{sGr}$. The result is $\Gamma_G \cong k[x, x^{-1}]$, which is not a cofibrant simplicial algebra in $\mathtt{sCommAlg}_k$. Another problem is that the right adjoint functor in (3.10) is not homotopical and hence should be replaced by a right derived functor $\boldsymbol{R}G$. But the existence of $\boldsymbol{R}G$ is not clear because the standard (projective) model structure on $\mathtt{sCommAlg}_k$ is fibrant. Nevertheless, somewhat surprisingly, we still have the following.

Theorem 3.1. The algebraic group functor $G : \mathtt{sCommAlg}_k \to \mathtt{sGr}$ has a total right derived functor, which is right adjoint to the derived representation functor (3.5):

$$\boldsymbol{L}(-)_G : \mathrm{Ho}(\mathtt{sGr}) \rightleftarrows \mathrm{Ho}(\mathtt{sCommAlg}_k) : \boldsymbol{R}G.$$

Moreover, both $\boldsymbol{L}(-)_G$ and $\boldsymbol{R}G$ are absolute derived functors in the sense of Deligne–Maltsiniotis [53].

The 1st statement of Theorem 3.1 shows that the simplicial adjunction (3.10) behaves like a Quillen adjunction, and the last statement shows that the corresponding derived functors are as "good" (well-behaved) as derived functors of Quillen functors. In particular, like the derived functor of a left Quillen functor, the derived representation functor has the following important property that plays a crucial role in computations of representation homology in Section 6.

Theorem 3.2. The derived representation functor (3.5) preserves arbitrary (small) homotopy colimits.

The main idea behind our proof of Theorem 3.1 is to "forget" the model structure on sGr, thinking of this category simply as a homotopical category in the sense of [23], and then "approximate" it with another model category, which is "almost" Quillen equivalent to sGr in the sense of [17]. For reader's convenience, we recall basic definitions and the necessary results from [23] and [17] in the Appendix, where we also prove abstract versions of Theorems 3.1 and 3.2 (see Theorems A.2 and A.3, respectively). The proof of Theorem 3.1 will consist of verifying the conditions of Theorem A.2; we will divide it into two cases: $G = \mathrm{GL}_n$ and the general case: G is an arbitrary algebraic group. For GL_n, we will provide detailed arguments, while in the general case, we will only sketch the proof leaving technical details for our subsequent paper.

We begin with the following observation refining the result of Lemma 3.1.

Proposition 3.1. The representation functor (3.4) is left deformable on sGr, and hence its total left derived functor (3.5) is an absolute derived functor in the sense of [53].

Proof. Write $\mathcal{C}$ for sGr viewed as a homotopical category, and let $\mathcal{C}_Q$ denote its full subcategory consisting of semi-free simplicial groups (see Definition 2.1). Then $\mathcal{C}_Q \hookrightarrow \mathcal{C}$ is a left deformation retract of $\mathcal{C}$, with retraction functor $Q : \mathcal{C} \to \mathcal{C}$ being the composition $Q := \mathbb{G}\,\overline{W}$ and the morphism $q : Q \to \mathrm{Id}_{\mathcal{C}}$ given by the counit of the Kan loop group adjunction (2.3). Indeed, by Kan's theorem, the morphism q is a natural weak equivalence (see [39, Prop. V.6.3.]), and its image is contained in $\mathcal{C}_Q$. The proof of Lemma 3.1 shows that (3.4) is homotopical on $\mathcal{C}_Q$ and hence, by definition, left deformable. The result now follows from Proposition A.1. ∎

3.3 Proof of Theorem 3.1 for $G = \mathrm{GL}_n$

Let sMon denote the category of simplicial monoids equipped with the standard (projective) model structure. Consider the natural adjunction

$$l : \mathtt{sMon} \rightleftarrows \mathtt{sGr} : r, \tag{3.11}$$

where r is the inclusion functor, and l is the group completion (localization) functor. The next observation is a consequence of a known theorem of Dwyer and Kan [24].

Lemma 3.2. The adjunction (3.11) is a left model approximation of $\mathtt{sGr}$ in the sense of [17].

Proof. We need to verify the three conditions of Definition A.1 (see Appendix). Condition (1) is obvious, and (2) follows from the fact that l is a left Quillen functor (when $\mathtt{sMon}$ and $\mathtt{sGr}$ are regarded as model categories). It suffices only to check (3). For this, observe that any map from a simplicial monoid to a simplicial group, say $f : M \to r(\Gamma)$, can be factored as $M \xrightarrow{\eta_M} rl(M) \xrightarrow{rf^{\#}} r(\Gamma)$, where η_M is the group completion (localization) map and $f^{\#} : l(M) \to \Gamma$ is the map adjoint to f under (3.11). If f is a weak equivalence in $\mathtt{sMon}$, then M is a group-like simplicial monoid (i.e., $\pi_0(M) \cong \pi_0(\Gamma)$ is a group), and hence, if M is also cofibrant, by [24, Proposition 10.4], η_M is a weak equivalence. By 2-of-3 property, the map $rf^{\#} : rl(M) \to r(\Gamma)$ is then a weak equivalence, and since r reflects weak equivalences, $f^{\#} : l(M) \to \Gamma$ is a weak equivalence as well. ■

We will apply Theorem A.2 to the representation adjunction (3.10) with $G = \mathrm{GL}_n$ using the left model approximation (3.11). Note that this model approximation is good for the representation functor (3.4) (for any algebraic group G), since the group completion functor maps cofibrant objects in $\mathtt{sMon}$, which are (retracts of) semi-free simplicial monoids, to (retracts of) semi-free simplical groups, on which the functor (3.4) is homotopical by Lemma 3.1.

From now on, we assume that $G = \mathrm{GL}_n$ and write $F := (\text{—})_{\mathrm{GL}_n} : \mathtt{sGr} \to \mathtt{sCommAlg}_k$ for the corresponding representation functor. Let $\mathtt{sMon}_0$ denote the full subcategory of $\mathtt{sMon}$ consisting of group-like simplicial monoids—by the Dwyer–Kan theorem [24], the essential image of the functor $\bar{r} : \mathtt{Ho(sGr)} \hookrightarrow \mathtt{Ho(sMon)}$ is precisely $\mathtt{Ho}(\mathtt{sMon}_0)$.

The inclusion functor $i : \mathtt{sMon}_0 \hookrightarrow \mathtt{sMon}$ has a right adjoint $\hat{U} : \mathtt{sMon} \to \mathtt{sMon}_0$ defined by the pull-back diagram in $\mathtt{sMon}$:

$$\begin{tikzcd} \hat{U}(M_*) \arrow[r] \arrow[d] & U[\pi_0(M_*)] \arrow[d] \\ M_* \arrow[r] & \pi_0(M_*), \end{tikzcd}$$

where $U : \mathtt{Mon} \to \mathtt{Gr}$ is the functor assigning to a monoid its subgroup of units. To construct the functors $\hat{F}$ and $\hat{G}$ we start with the natural adjunction $(k[-])_n : \mathtt{sMon} \rightleftarrows$

$\mathtt{sCommAlg}_k : M_n$ and compose it with $i : \mathtt{sMon}_0 \rightleftarrows \mathtt{sMon} : \hat{U}$, that is, define

$$\hat{F} := (k[-])_n \circ i : \ \mathtt{sMon}_0 \rightleftarrows \mathtt{sCommAlg}_k \ : \ \hat{G} := \hat{U} \circ M_n. \tag{3.12}$$

Note that the right adjoint $\hat{G}$ in (3.12) is precisely the Waldhausen functor $\widehat{\mathrm{GL}}_n$ defined in the introduction (see (1.9)). In particular, it is a homotopical functor that takes its values in group-like simplicial monoids: thus, we have $\boldsymbol{R}\hat{G} = \hat{G}$ and $\mathrm{Im}(\hat{G}) \subseteq \mathrm{Im}(\bar{r})$, that is, condition *(iii)* of Theorem A.2 holds.

The left adjoint $\hat{F}$ is obtained by restricting to $\mathtt{sMon}_0$ the functor $(k[-])_n$, which is left Quillen on $\mathtt{sMon}$: hence, $\hat{F}$ is homotopical on cofibrant objects in $\mathtt{sMon}_0$, and therefore Theorem A.2(i) holds.

Next, factor $r := r_0 \circ i : \mathtt{sGr} \hookrightarrow \mathtt{sMon}_0 \hookrightarrow \mathtt{sMon}$ and observe that $\hat{F} \circ r_0 = (k[-])_n \circ r$ is left adjoint to $U \circ M_n = \mathrm{GL}_n$. Hence, there is a canonical isomorphism of functors $\hat{F} \circ r_0 \cong F$. It remains only to show that $\mathtt{sGr} \xrightarrow{r_0} \mathtt{sMon}_0 \xrightarrow{\hat{F}} \mathtt{sCommAlg}_k$ is a left deformable pair. For this, in the notation of Proposition A.2, we take $\mathcal{C}_Q$ to be the full subcategory of semi-free simplicial groups in $\mathcal{C} := \mathtt{sGr}$ and $\mathcal{D}_Q$ the full subcategory of $\mathcal{D} := \mathtt{sMon}_0$ consisting of monoids M such that $k[M]$ is a simplicial k-algebra that is degreewise a direct limit of formally smooth k-algebras having semifree DG resolutions with finitely many generators in each homological degree. Both $\mathcal{C}_Q$ and $\mathcal{D}_Q$ are left deformation retracts of the corresponding homotopical categories: $\mathcal{C}_Q$ contains the image of the deformation functor $Q = \mathbb{G}\overline{W}$ associated with the Kan loop group adjunction (see Proposition 3.1), while $\mathcal{D}_Q$ contains the image of $Q_0 : \mathtt{sMon}_0 \to \mathtt{sMon}_0$, which is the restriction of the cofibrant reprelacement functor Q on $\mathtt{sMon}$. Since r_0 is homotopical and $r_0(\mathcal{C}_Q) \subseteq \mathcal{D}_Q$, we need only to check that $\hat{F}$ is homotopical on $\mathcal{D}_Q$. Since $\hat{F} = (-)_n \circ k[-]$ and $k[-]$ is homotopical on $\mathtt{sMon}_0$, it suffices to check that $(-)_n : \mathtt{sAlg}_k \to \mathtt{sCommAlg}_k$ is homotopical on simplicial k-algebras that are degreewise (direct limits of) formally smooth k-algebras having semifree DG resolutions with finitely many generators in each homological degree. Now, this last fact follows from [7, Theorem 21], saying that such associative algebras are adapted for the representation functor $(-)_n : \mathtt{Alg}_k \to \mathtt{CommAlg}_k$ (in the sense that $\boldsymbol{L}(A)_n \cong A_n$ for such A's) and the well-known abstract result from homotopical algebra saying that the simplicial objects, which are *degreewise* adapted for a functor F, are actually adapted for F (see, e.g., [78, Theorem 9.2.2]).

Summing up, we showed that all three conditions of Theorem A.2 hold for the adjoint pair (3.12), except that the left adjoint $\hat{F}$ is not defined on the entire model category $\mathtt{sMon}$ but rather on its full subcategory $\mathtt{sMon}_0$. However, this last subcategory is closed under weak equivalences and coincides with $\mathrm{Im}(\bar{r})$; hence, the result of Theorem A.2 still holds (see Remark 2 after the proof of Theorem A.2). This completes the proof of Theorem 3.1 for $G = \mathrm{GL}_n$.

Theorem A.2 gives an explicit formula for the total derived functor $\boldsymbol{R}G$: namely,

$$\boldsymbol{R}\mathrm{GL}_n = \boldsymbol{L}l \circ \widehat{\mathrm{GL}_n}. \tag{3.13}$$

This allows us to compute the homotopy groups $\boldsymbol{R}^i\mathrm{GL}_n := \pi_i \boldsymbol{R}\mathrm{GL}_n$ for all $i \geq 0$.

Proposition 3.2. For any $A_* \in \mathtt{sCommAlg}_k$,

$$\boldsymbol{R}^i\mathrm{GL}_n(A_*) \cong \begin{cases} \mathrm{GL}_n[\pi_0(A_*)] & \text{for} \quad i = 0 \\ \mathrm{M}_n[\pi_i(A_*)] & \text{for} \quad i \geq 1. \end{cases}$$

Proof. Let $Q\widehat{\mathrm{GL}}_n(A_*) \xrightarrow{\sim} \widehat{\mathrm{GL}}_n(A_*)$ be a cofibrant resolution of $\widehat{\mathrm{GL}}_n(A_*)$ in the (model) category $\mathtt{sMon}$. By (3.13), we have $\boldsymbol{R}\mathrm{GL}_n(A_*) \cong lQ\widehat{\mathrm{GL}}_n(A_*)$. On the other hand, $Q\widehat{\mathrm{GL}}_n(A_*)$ is a group-like simplicial monoid, since so is $\widehat{\mathrm{GL}}_n(A_*)$. Hence, by the Dwyer–Kan theorem, the group completion map $Q\widehat{\mathrm{GL}}_n(A_*) \xrightarrow{\sim} lQ\widehat{\mathrm{GL}}_n(A_*)$ is a weak equivalence. Thus, we have a zigzag of weak equivalences

$$\widehat{\mathrm{GL}}_n(A_*) \xleftarrow{\sim} Q\widehat{\mathrm{GL}}_n(A_*) \xrightarrow{\sim} lQ\widehat{\mathrm{GL}}_n(A_*),$$

from which we conclude that $\boldsymbol{R}^i\mathrm{GL}_n(A_*) \cong \pi_i\widehat{\mathrm{GL}}_n(A_*)$. The result now is immediate from the definition of $\widehat{\mathrm{GL}}_n$ (see [77]). ■

3.4 (Sketch of) Proof of Theorem 3.1 in the general case

For a general algebraic group G, we will use a different model approximation of $\mathtt{sGr}$ given by reduced simplicial spaces. By a *simplicial space* we mean a bisimplicial set of which we think as a functor $X_* : \Delta^{\mathrm{op}} \to \mathtt{sSet}$, $[n] \mapsto X_n$, with simplicial components X_n viewed as "vertical" simplicial sets. We call X_* *reduced* if $X_0 = \Delta[0]$ is the one-point (discrete) simplicial set. We write $\mathtt{sSp} = \mathtt{sSet}^{\Delta^{\mathrm{op}}}$ for the category of all simplicial spaces and $\mathtt{sSp}_*$ for its full subcategory consisting of reduced ones. The category $\mathtt{sSp}$ is known to carry several interesting model structures. We will use two of these: the projective model structure in which the weak equivalences and fibrations are the levelwise weak equivalences (resp., fibrations) of simplicial sets and its (left Bousfield) localization with respect to Segal maps introduced in [67]. We denote the projective model structure simply by $\mathtt{sSp}$ and its localization by $\mathfrak{L}\mathtt{sSp}$. As shown in [12], both model structures "restrict" to reduced simplicial spaces, and we denote the corresponding model categories by $\mathtt{sSp}_*$ and $\mathfrak{L}\mathtt{sSp}_*$, respectively.

The reduced simplicial spaces are related to simplicial groups by the pair of adjoint functors

$$\underline{\pi}_1 : \mathfrak{L}\mathtt{sSp}_* \rightleftarrows \mathtt{sGr} : \underline{N}, \tag{3.14}$$

where $\underline{N}$ is the nerve functor applied degreewise to components of simplicial groups: i.e., for $\Gamma_* \in \mathrm{Ob}(\mathtt{sGr})$,

$$\underline{N}_\bullet(\Gamma_*) : \Delta^{\mathrm{op}} \to \mathtt{sSet}, \quad [n] \mapsto N_n(\Gamma_*) = \Gamma_*^n,$$

and $\underline{\pi}_1$ is the fundamental group functor applied degreewise to bisimplicial sets: i.e., for $X = \{X_{p,q}\}_{p,q\geq 0}$,

$$\underline{\pi}_1(X) : \Delta^{\mathrm{op}} \to \mathtt{Gr} \quad [q] \mapsto \pi_1(X_{*,q}).$$

The fact that $\underline{\pi}_1$ is left adjoint to $\underline{N}$ follows from the well-known fact that the fundamental group functor $\pi_1 : \mathtt{sSet}_0 \to \mathtt{Gr}$ on reduced simplicial sets is left adjoint to the simplicial nerve $N : \mathtt{Gr} \to \mathtt{sSet}_0$ on the category of groups. Now, in place of Lemma 3.2, we have the following.

Lemma 3.3. The adjunction (3.14) is a left model approximation of $\mathtt{sGr}$.

Proof. This follows from a theorem of Bergner (see [12, Theorem 1.6]) that asserts that the model category $\mathfrak{L}\mathtt{sSp}_*$ is Quillen equivalent to the category of simplicial monoids, $\mathtt{sMon}$, equipped with the standard (projective) model structure. In fact, one can check that (3.14) factors as $\mathfrak{L}\mathtt{sSp}_* \rightleftarrows \mathtt{sMon} \rightleftarrows \mathtt{sGr}$, where the 1st adjunction is Bergner's Quillen equivalence, with its right adjoint being a homotopical functor, and the 2nd adjunction is (3.11), which is, by Lemma 3.2, a left model approximation of $\mathtt{sGr}$. It follows that (3.14) is a left model approximation of $\mathtt{sGr}$ as well. ■

Next, to define the functor $\hat{G} : \mathtt{sCommAlg}_k \to \mathfrak{L}\mathtt{sSp}_*$ we will use a construction of Galatius and Venkatesh (see [33, Section 5]). We start with the cosimplicial commutative algebra

$$\mathcal{O}(N_\bullet G) : \Delta \to \mathtt{CommAlg}_k \quad [n] \mapsto \mathcal{O}(N_n G) = \mathcal{O}(G)^{\otimes n}.$$

Taking the cofibrant replacement of $\mathcal{O}(N_\bullet G)$ in $\mathtt{sCommAlg}_k$ in each cosimplicial degree, we get a cosimplicial simplicial algebra $c\,\mathcal{O}(N_\bullet G) \in \mathtt{sCommAlg}_k^{\Delta}$ and then define $\hat{G}$ by

$$\hat{G} : \mathtt{sCommAlg}_k \to \mathfrak{L}\mathtt{sSp}_* \quad A_* \mapsto \mathrm{Map}\,(c\,\mathcal{O}(N_\bullet G),\, A_*), \tag{3.15}$$

where "Map" stands for the standard (simplicial) function complex in $\mathtt{sCommAlg}_k$. Note that $\hat{G}(A_*)_0 := \mathrm{Map}\,(c\,\mathcal{O}(N_0 G),\, A_*) = \mathrm{Map}\,(k,\, A_*) \cong \Delta[0]$ for any A_*, so $\hat{G}$ indeed takes its values in the category of *reduced* simplicial spaces.

By formal properties of function complexes, the functor $\hat{G}$ has a left adjoint given by

$$\hat{F} :\ \mathfrak{L}\,\mathtt{sSp}_* \to \mathtt{sCommAlg}_k \quad X \mapsto X \otimes_\Delta c\,\mathcal{O}(N_\bullet G), \tag{3.16}$$

where $\otimes_\Delta$ is the functor tensor product over the category Δ in the simplicial category $\mathtt{sCommAlg}_k$ (see, e.g., [66, (4.1.1)]).

Proposition 3.3. The adjoint functors $\hat{F} : \mathfrak{L}\mathtt{sSp}_* \rightleftarrows \mathtt{sCommAlg}_k : \hat{G}$ form a Quillen pair.

Proof. Sketch of proof One proves this in two steps. First, one checks that the functors $(\hat{F}, \hat{G})$ form a Quillen pair $\hat{F} : \mathtt{sSp}_* \rightleftarrows \mathtt{sCommAlg}_k : \hat{G}$ for the projective model structure on $\mathtt{sSp}_*$. Then, one shows that this Quillen pair "localizes" to a Quillen pair on $\mathfrak{L}\mathtt{sSp}_*$ by checking that the left derived functor $\mathbb{L}F : \mathtt{sSp}_* \to \mathtt{Ho}(\mathtt{sCommAlg}_k)$ maps the Segal morphisms in $\mathtt{sSp}_*$ to isomorphisms in $\mathtt{Ho}(\mathtt{sCommAlg}_k)$. ■

We have now defined all ingredients of Theorem A.2. To show that this theorem applies to the representation adjunction (3.10) we need to verify its assumptions (*i*), (*ii*), and (*iii*). Condition (*i*)—$(\hat{F}, \hat{G})$ being a deformable adjunction—is immediate from Proposition 3.3. Condition (*iii*) is not difficult to check since $\hat{G}$ is a homotopical functor (and therefore $\boldsymbol{R}\hat{G} = \hat{G}$). The main work is to verify condition (*ii*): in particular, to prove that $\boldsymbol{L}F \cong \boldsymbol{L}\hat{F} \circ \underline{N}$. The details of this verification will appear in our subsequent paper. Here we only mention that, as an intermediate step, we prove the following lemma that provides an alternative way to define representation homology of spaces, without using the Kan loop group construction.

Lemma 3.4. For any $X \in \mathtt{sSet}_0$, there is a natural isomorphism in $\mathtt{Ho}(\mathtt{sCommAlg}_k)$:

$$\boldsymbol{L}F(\mathbb{G}X) \cong \boldsymbol{L}\hat{F}(X^t),$$

where $(-)^t : \mathtt{sSet}_0 \hookrightarrow \mathtt{sSp}_*$ is the "transpose" inclusion functor identifying a simplicial set X with the simplicial space $X^t = \{(X^t)_n\}_{n\geq 0}$ with discrete components $(X^t)_n = X_n$.

We conclude by pointing out that, once the conditions of Theorem A.2 are verified and Theorem 3.1 is proved, Theorem 3.2 follows immediately from Theorem A.3, since in our situation both $\mathcal{C} = \mathtt{sGr}$ and $\mathcal{D} = \mathtt{sCommAlg}_k$ carry model category structures and hence the colimits on these categories exist and are left deformable by results of [17] (see Theorem A.4(3)).

4 Functor Homology Interpretation

In this section, we give our second definition of representation homology parallel to Pirashvili's definition of higher Hochschild homology [60]. We begin by reviewing the construction of [60].

4.1 Higher Hochschild homology

Let $\mathfrak{F}_*$ denote the category of finite pointed sets with objects $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$, and morphisms $f : [n] \to [m]$ being arbitrary set maps such that $f(0) = 0$. Let $F : \mathfrak{F}_* \to \mathtt{Vect}_k$ be a covariant functor. We extend F to the category $\mathtt{Set}_*$ of *all* pointed sets in a natural way, using the left Kan extension along the inclusion $\mathfrak{F}_* \hookrightarrow \mathtt{Set}_*$. We keep the notation F for the extended functor: explicitly, $F : \mathtt{Set}_* \to \mathtt{Vect}_k$ is given by $F(X) = \operatorname{colim} F([n])$, where the colimit is taken over all pointed inclusions $[n] \hookrightarrow X$.

Given a pointed simplicial set $X \in \mathtt{sSet}_*$, we define a simplicial k-vector space $F(X)$ as the composition of functors

$$F(X) : \ \Delta^{\mathrm{op}} \xrightarrow{X} \mathtt{Set}_* \xrightarrow{F} \mathtt{Vect}_k. \tag{4.1}$$

We denote the homotopy groups of $F(X)$ by $\pi_* F(X)$ and recall that $\pi_* F(X) := \mathrm{H}_*[N(F(X))]$, where N is the Dold–Kan normalization functor.

Now, any commutative k-algebra A and an A-module M (viewed as a symmetric bimodule) give rise to a functor $\mathfrak{F}_* \to \mathtt{Vect}_k$ that assigns to the set $[n]$ the vector space $M \otimes A^{\otimes n}$ and to a pointed map $f : [n] \to [m]$, the action of f on $M \otimes A^{\otimes n}$ given by

$$f_*(a_0 \otimes a_1 \otimes \ldots \otimes a_n) := b_0 \otimes b_1 \otimes \ldots \otimes b_m,$$

where $b_j := \prod_{i \in f^{-1}(j)} a_i$ for $j = 0, 1, \dots m$. Following [60], we denote this functor by $\mathcal{L}(A, M)$, and for a pointed simplicial set $X \in \mathtt{sSet}_*$, define

$$\mathrm{HH}_*(X, A, M) := \pi_* \mathcal{L}(A, M)(X).$$

Thus, $\mathrm{HH}_*(X, A, M)$ is the homology of the complex $\mathrm{C}_*(X, A, M) := N[\mathcal{L}(A, M)(X)]$, which we call the *Pirashvili–Hochschild complex* of A with coefficients in M associated to X.

Example 4.1. Let $X = \mathbb{S}^1_*$ be the simplicial circle. Recall that the set of n-simplices $\mathbb{S}^1_n$ can be identified with the set of monotone sequences of 0's and 1's of length $n+1$ modulo the identification $(0, 0, \dots, 0) \sim (1, 1, \dots, 1)$ (see Section 2.1.4). For a nonzero sequence $x \in \mathbb{S}^1_n$, let $n(x)$ denote the position of the first 1. The map $x \mapsto n(x) - 1$ identifies $\mathbb{S}^1_n$ with $[n]$. Under this identification, the degeneracy map $s_i : [n] \to [n+1]$ corresponds to the unique monotone injection skipping $i+1$ in its image and the face map $d_i : [n] \to [n-1]$ is given by $d_i(j) = j$ for $j < i$, $d_i(i) = i$ for $i < n$, $d_n(n) = 0$, and $d_i(j) = j - 1$ for $j > i$. From this description of $\mathbb{S}^1_*$, it is easy to see that the Pirashvili complex $\mathrm{C}_*(\mathbb{S}^1, A, M)$ for $\mathbb{S}^1$ is precisely the classical Hochschild complex $\mathrm{C}_*(A, M)$. Thus, $\mathrm{HH}_*(\mathbb{S}^1, A, M) = \mathrm{HH}_*(A, M)$ for any commutative algebra A and A-module M. In a similar way, one can explicitly describe the Pirashvili complex $\mathrm{C}_*(\mathbb{S}^n, A, M)$ for the n-dimensional simplicial sphere $\mathbb{S}^n_*$. The corresponding homology groups $\mathrm{HH}_*(\mathbb{S}^n, A, M)$ are denoted $\mathrm{HH}^{[n]}_*(A, M)$ and called *the Hochschild homology of* (A, M) *of order* n.

In the present paper, we will mostly deal with two cases: $M = A$ and $M = k$, where in the last case the module structure on k comes from an augmentation $A \to k$. To simplify the notation we will write $\mathrm{HH}_*(X, A)$ for $\mathrm{HH}_*(X, A, A)$ and regard $X \mapsto \mathrm{HH}_*(X, A)$ as a functor on the category of (pointed) simplicial sets assuming A to be fixed. We will refer to this functor as a *higher Hochschild homology* of spaces.

There is another, more conceptual way to define higher Hochschild homology, using homological algebra of functor categories over PROPs. Recall that a PROP is a permutative category $(\mathcal{P}, \boxtimes)$ whose set of objects is indexed by (or identified with) the natural numbers $\mathcal{N}$ and whose monoidal structure $\boxtimes$ is given by addition in $\mathcal{N}$ (see [52]). A k-algebra over a PROP $\mathcal{P}$ is a strict symmetric monoidal functor from $\mathcal{P}$ to the tensor category $\mathtt{Vect}_k$.

To define Hochschild homology we take $\mathcal{P}$ to be a category $\mathfrak{F}$ of finite sets with monoidal structure given by disjoint union. More precisely, we let $\mathfrak{F}$ denote the full subcategory of $\mathtt{Set}$ whose objects are the sets $\underline{n} := \{1, 2, \dots, n\}$ for $n \geq 0$ (where, by convention, $\underline{0} = \varnothing$) and morphisms are arbitrary set maps. The monoidal structure on $\mathfrak{F}$ is given by $\underline{n} \boxtimes \underline{m} = \underline{n+m}$. It is well known and easy to prove (see, e.g., [61, Section 2]) that the category of k-algebras over $\mathfrak{F}$ is equivalent to the category $\mathtt{CommAlg}_k$, the equivalence being given by the functor $A \mapsto [(-\otimes A) : \underline{n} \mapsto A^{\otimes n}]$. We will write $\underline{A}$ for the algebra over $\mathfrak{F}$ corresponding to the commutative algebra $A \in \mathtt{CommAlg}_k$.

Now, let $\mathfrak{F}$-$\mathtt{Mod}$ (resp., $\mathtt{Mod}$-$\mathfrak{F}$) denote the category of all covariant (resp., contravariant) functors from $\mathfrak{F}$ to the category of vector spaces. The notation suggests that one should think of the objects of $\mathfrak{F}$-$\mathtt{Mod}$ and $\mathtt{Mod}$-$\mathfrak{F}$ as left and right $\mathfrak{F}$-modules, respectively. These categories are both abelian with enough projective and injective objects. Furthermore, they are related by a bifunctor

$$- \otimes_{\mathfrak{F}} - : \mathtt{Mod}\text{-}\mathfrak{F} \times \mathfrak{F}\text{-}\mathtt{Mod} \to \mathtt{Vect}_k$$

that is right exact with respect to each argument, preserves sums, and is left balanced (see, e.g., [60, Sect. 1.5]). Explicitly, for a right $\mathfrak{F}$-module $\mathcal{N}$ and a left $\mathfrak{F}$-module $\mathcal{M}$,

$$\mathcal{N} \otimes_{\mathfrak{F}} \mathcal{M} = \Big[\bigoplus_{n \geq 0} \mathcal{N}(\underline{n}) \otimes_k \mathcal{M}(\underline{n})\Big] / R\,, \tag{4.2}$$

where R is the subspace spanned by the vectors of the form $\mathcal{N}(f)x \otimes y - x \otimes \mathcal{M}(f)y$ with $x \in \mathcal{N}(\underline{n})$ and $y \in \mathcal{M}(\underline{m})$ and f running over all maps in $\mathrm{Hom}_{\mathtt{Set}}(\underline{m}, \underline{n})$.

Next, we consider the functor

$$h : \mathfrak{F} \to \mathtt{Mod}\text{-}\mathfrak{F}\,, \quad \underline{n} \mapsto k[\mathrm{Hom}_{\mathfrak{F}}(-, \underline{n})], \tag{4.3}$$

where $k[S]$ denotes the vector space generated by a set S, and extend (4.3) to the category simplicial sets in two steps. First, we define a functor $\mathtt{Set} \to \mathtt{Mod}$-$\mathfrak{F}$ by taking the left Kan extension of (4.3) along the natural inclusion $\mathfrak{F} \hookrightarrow \mathtt{Set}$, and then we extend this degreewise to simplicial sets. Abusing notation, we will continue to denote the resulting functor by $h : \mathtt{sSet} \to \mathtt{sMod}$-$\mathfrak{F}$. Composing h with the normalization functor $\underline{N} : \mathtt{sMod}\text{-}\mathfrak{F} \to \mathtt{Ch}_{\geq 0}(\mathtt{Mod}\text{-}\mathfrak{F})$ assigns to every simplicial set X a chain complex and hence an object in the derived category $\mathscr{D}(\mathtt{Mod}\text{-}\mathfrak{F})$ that we denote by $\underline{N}(h(X))$.

Now, recall that any commutative algebra A defines an algebra $\underline{A}$ over the PROP $\mathfrak{F}$ that can be viewed as an object in $\mathfrak{F}$-$\mathtt{Mod}$. With this interpretation of A, we have the following result.

Theorem 4.1. For any $X \in \mathtt{sSet}$ and $A \in \mathtt{Comm\,Alg}_k$, there is a natural isomorphism

$$\mathrm{HH}_*(X, A) \cong \mathrm{H}_*[\underline{N}(h(X)) \otimes^{\boldsymbol{L}}_{\mathfrak{F}} \underline{A}\,].$$

Although Theorem 4.1 is not explicitly stated in [60], it can be deduced from results of this paper. We do not give a proof of Theorem 4.1 here as in the next section, we prove the analogous theorem for representation homology (see Theorem 4.2).

4.2 Representation homology as functor homology

We now define the representation homology of a (reduced) simplicial set by mimicking Pirashvili's definition of higher Hochshild homology. Our starting point is the known fact that the category of commutative Hopf algebras over a field k is equivalent to the category of k-algebras of the PROP of finitely generated free groups (see, e.g., [61, Sect. 5] and [40] for a detailed proof). To be precise, let $\mathfrak{G}$ denote the full subcategory of $\mathtt{Gr}$ whose objects are the free groups based on the sets $\underline{n} = \{1, 2, \ldots, n\}$ for $n \geq 0$. We denote such groups by $\langle n \rangle := \mathbb{F}\langle \underline{n} \rangle$ (where, by convention, $\langle 0 \rangle$ is the identity group) and write $k\langle n \rangle$ for the corresponding group algebras over k. The category $\mathfrak{G}$ is a PROP, with monoidal product $\boxtimes$ being the free product of groups, so that $\langle n \rangle \boxtimes \langle m \rangle = \langle n + m \rangle$. A commutative Hopf algebra $\mathcal{H}$ over k defines the (strong monoidal) covariant functor $\mathfrak{G} \to \mathtt{Vect}_k$, $\langle n \rangle \mapsto \mathcal{H}^{\otimes n}$, which we denote by $\underline{\mathcal{H}}$. The assignment $\mathcal{H} \mapsto \underline{\mathcal{H}}$ gives an equivalence between the category of commutative Hopf algebras over k and the category of k-algebras over the PROP $\mathfrak{G}$. Dually, the category of *cocommutative* Hopf algebras is equivalent to the category of k-algebras over the opposite PROP $\mathfrak{G}^{\mathrm{op}}$.

Now, observe that for any commutative Hopf algebra $\mathcal{H}$, the functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{Vect}_k$ takes values in the category of commutative algebras, that is, it can be viewed as a functor $\underline{\mathcal{H}} : \mathfrak{G} \to \mathtt{Comm\,Alg}_k$. We extend this last functor to the category $\mathtt{FGr}$ of all free groups by taking the left Kan extension along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{FGr}$. To be precise, let $\mathtt{FGr}$ denote the category of *based* free groups whose objects are pairs (Γ, S), where $\Gamma = \langle S \rangle$ is a free group with a specified generating set S, and morphisms are arbitrary group homomorphisms $\Gamma \to \Gamma'$ (not necessarily, preserving the generating sets). We have the natural inclusion functor $i : \mathfrak{G} \hookrightarrow \mathtt{FGr}$ that takes $\langle n \rangle$ to $(\langle n \rangle, \underline{n})$. The Kan extension of $\underline{\mathcal{H}}$ along i then defines a functor $\mathtt{FGr} \to \mathtt{Comm\,Alg}_k$ that assigns to the free group $\langle S \rangle$ on a set S the commutative algebra $S \otimes \mathcal{H} = \otimes_{s \in S} \mathcal{H}_s$. We continue to denote this functor by $\underline{\mathcal{H}}$.

Let X be a reduced simplicial set (or equivalently, a pointed connected topological space). Recall that the Kan loop group construction gives a functor $\mathbb{G}X : \Delta^{\mathrm{op}} \to \mathtt{FGr}$

that takes $[n] \in \Delta^{\mathrm{op}}$ to the free group $\mathbb{G}X_n = \langle B_n \rangle$ based on the set $B_n = X_{n+1} \setminus s_0(X_n)$. Now, given a commutative Hopf algebra $\mathcal{H}$, we consider the composition of functors

$$\Delta^{\mathrm{op}} \xrightarrow{\mathbb{G}X} \mathtt{FGr} \xrightarrow{\mathcal{H}} \mathtt{Comm\,Alg}_k,$$

which defines a simplicial commutative algebra $\underline{\mathcal{H}}(\mathbb{G}X)$.

Definition 4.1. The *representation homology* of X in $\mathcal{H}$ is defined by

$$\mathrm{HR}_*(X, \mathcal{H}) := \pi_*[\underline{\mathcal{H}}(\mathbb{G}X)] = \mathrm{H}_*[N(\underline{\mathcal{H}}(\mathbb{G}X))]. \tag{4.4}$$

Clearly, a morphism $f : X \to Y$ of reduced simplicial sets induces a map of graded commutative algebras $\mathrm{HR}_*(f, \mathcal{H}) : \mathrm{HR}_*(X, \mathcal{H}) \to \mathrm{HR}_*(Y, \mathcal{H})$. Thus, representation homology defines a covariant functor $\mathrm{HR}(-, \mathcal{H}) : \mathtt{sSet}_0 \to \mathtt{grComm\,Alg}_k$. The following proposition justifies the above definition of representation homology.

Proposition 4.1. Let G be an affine group scheme defined over k with coordinate ring $\mathcal{H} = \mathcal{O}(G)$. Then, for any $X \in \mathtt{sSet}_0$, there is a natural isomorphism of graded commutative algebras

$$\mathrm{HR}_*(X, \mathcal{O}(G)) \cong \mathrm{HR}_*(X, G). \tag{4.5}$$

In particular, $\mathrm{HR}_0(X, \mathcal{O}(G)) \cong \pi_1(X)_G$, where $\pi_1(X)$ is the fundamental group of X.

Proof. If $\mathcal{H} = \mathcal{O}(G)$, we have natural isomorphisms $\underline{\mathcal{H}}(\langle S \rangle) \cong \bigotimes_{s \in S} \mathcal{O}(G)_s \cong (\langle S \rangle)_G$ for any set S. This implies that $\underline{\mathcal{H}}(\mathbb{G}X) \cong (\mathbb{G}X)_G$ in $s\mathtt{Comm\,Alg}_k$. On the other hand, by Proposition 2.1, the simplicial group $\mathbb{G}X$ is semi-free, and hence a cofibrant object in $\mathtt{sGr}$. This implies that $(\mathbb{G}X)_G \cong \boldsymbol{L}(\mathbb{G}X)_G$ in $\mathtt{Ho}(s\mathtt{Comm\,Alg}_k)$, which, in turn, implies the isomorphism (4.5) in homology. The isomorphism for $\mathrm{HR}_0(X, G)$ is the composition of (4.5) with (3.6) and the natural isomorphism of groups $\pi_0(\mathbb{G}X) \cong \pi_1(X)$. ■

Let Γ be a discrete group, and let $X = \mathrm{B}\Gamma$ be the classifying space (i.e., the simplicial nerve) of Γ. As a simple application of Proposition 4.1, we get the following.

Corollary 4.1. $\mathrm{HR}_*(\mathrm{B}\Gamma, \mathcal{O}(G)) \cong \mathrm{HR}_*(\Gamma, G)$. In particular, $\mathrm{HR}_0(\mathrm{B}\Gamma, G) \cong \Gamma_G$.

Proof. The Kan adjunction (2.3) gives the canonical cofibrant resolution $\mathbb{G}\overline{W}\Gamma \xrightarrow{\sim} \Gamma$ in $\mathtt{sGr}$. Since Γ is discrete, we have $\overline{W}\Gamma = \mathrm{B}\Gamma$, and the result follows from Proposition 4.1. ■

Corollary 4.2. For any $X, Y \in \mathtt{sSet}_0$, there is a natural isomorphism

$$\mathrm{HR}_*(X \vee Y, G) \cong \mathrm{HR}_*(X, G) \otimes \mathrm{HR}_*(Y, G).$$

Proof. Recall that the wedge sum is a (categorical) coproduct in $\mathtt{sSet}_0$. Since $\mathbb{G}$ is a left adjoint functor, we have $\mathbb{G}(X \vee Y) \cong \mathbb{G}X * \mathbb{G}Y$. By Theorem 3.2, it follows that

$$\boldsymbol{L}(\mathbb{G}(X \vee Y))_G \cong \boldsymbol{L}(\mathbb{G}X)_G \otimes \boldsymbol{L}(\mathbb{G}Y)_G.$$

The desired result is now immediate from Künneth's theorem and Proposition 4.1. ■

4.2.1 *The fundamental spectral sequence*

Now, we introduce the functor categories $\mathfrak{G}$-$\mathtt{Mod}$ and $\mathtt{Mod}$-$\mathfrak{G}$, whose objects are all covariant (resp., contravariant) functors from $\mathfrak{G}$ to the category of vector spaces. We regard these objects as left and right modules over $\mathfrak{G}$, respectively. Both categories are abelian with sufficiently many projective and injective objects. There is a natural bifunctor

$$- \otimes_{\mathfrak{G}} - : \mathtt{Mod}\text{-}\mathfrak{G} \times \mathfrak{G}\text{-}\mathtt{Mod} \to \mathtt{Vect}_k,$$

which is right exact with respect to each argument, preserves sums, and is left balanced in the sense of [16]. Explicitly, this bifunctor can be defined by formula (4.2) with $\mathfrak{F}$ replaced by $\mathfrak{G}$.

Since $- \otimes_{\mathfrak{G}} -$ is left balanced, the derived functors with respect to each argument are naturally isomorphic, and we denote their common value by $\mathrm{Tor}_*^{\mathfrak{G}}(-,-)$. Note that for any left $\mathfrak{G}$-module $\mathcal{M}$, the functor $- \otimes_{\mathfrak{G}} \mathcal{M} : \mathtt{Mod}\text{-}\mathfrak{G} \to \mathtt{Vect}_k$ is left adjoint to the functor $\underline{\mathrm{Hom}}(\mathcal{M}, -) : \mathtt{Vect}_k \to \mathtt{Mod}\text{-}\mathfrak{G}$, where $\underline{\mathrm{Hom}}(\mathcal{M}, V)$ is the right $\mathfrak{G}$-module $\langle n \rangle \mapsto \mathrm{Hom}_k(\mathcal{M}(\langle n \rangle), V)$ for any vector space V. Similarly, for any right $\mathfrak{G}$-module $\mathcal{N}$, the functor $\mathcal{N} \otimes_{\mathfrak{G}} -$ is left adjoint to the functor $\underline{\mathrm{Hom}}(\mathcal{N}, -) : \mathtt{Vect}_k \to \mathfrak{G}\text{-}\mathtt{Mod}$. Hence, both functors $- \otimes_{\mathfrak{G}} \mathcal{M}$ and $\mathcal{N} \otimes_{\mathfrak{G}} -$ commute with colimits.

To state our 1st theorem we need some notation. First, we recall that if Γ is any group, $k[\Gamma]$ is a cocommutative Hopf algebra; thus, $k[\Gamma]$ defines a right $\mathfrak{G}$-module in $\mathtt{Mod}$-$\mathfrak{G}$. Now, if X is a reduced simplicial set, $k[\mathbb{G}X]$ defines a simplicial right $\mathfrak{G}$-module in $\mathtt{sMod}$-$\mathfrak{G}$. Applying the normalization functor $\underline{N} : \mathtt{sMod}\text{-}\mathfrak{G} \to \mathtt{Ch}_{\geq 0}(\mathtt{Mod}\text{-}\mathfrak{G})$ to this simplicial module, we get a chain complex of $\mathfrak{G}$-modules and hence an object in the derived category $\mathscr{D}(\mathtt{Mod}\text{-}\mathfrak{G})$. Abusing notation, we will denote this object by $\underline{N}(k[\mathbb{G}X])$.

Theorem 4.2. For any $X \in \mathtt{sSet}_0$ and any commutative Hopf algebra $\mathcal{H}$, there is a natural isomorphism of graded commutative algebras

$$\mathrm{HR}_*(X,\mathcal{H}) \cong \mathrm{H}_*[\underline{N}(k[\mathbb{G}X]) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \underline{\mathcal{H}}].$$

To prove Theorem 4.2 we need a simple lemma. Recall that for $n \geq 0$, we denote by $k\langle n\rangle$ the group algebra of the free group based on the set $\underline{n} = \{1, 2, \dots, n\}$. Regarding it as a cocommutative Hopf algebra, we get a right $\mathfrak{G}$-module that (to simplify the notation) we also denote by $k\langle n\rangle$.

Lemma 4.1. For each $n \geq 0$, the $\mathfrak{G}$-module $k\langle n\rangle$ is a projective object in $\mathtt{Mod}\text{-}\mathfrak{G}$.

Proof. For a fixed $n \geq 0$, let $h^n := k[\mathrm{Hom}_{\mathfrak{G}}(-, \langle n\rangle)]$ denote the standard right $\mathfrak{G}$-module associated to the object $\langle n\rangle \in \mathfrak{G}$. By Yoneda lemma, there is a natural isomorphism $\mathrm{Hom}_{\mathtt{Mod}\text{-}\mathfrak{G}}(h^n, \mathcal{N}) \cong \mathcal{N}(\langle n\rangle)$ for any $\mathcal{N} \in \mathtt{Mod}\text{-}\mathfrak{G}$. The sequence of $\mathfrak{G}$-modules $0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$ is exact in $\mathtt{Mod}\text{-}\mathfrak{G}$ if and only if the sequence of k-vector spaces $0 \to \mathcal{N}'(\langle n\rangle) \to \mathcal{N}(\langle n\rangle) \to \mathcal{N}''(\langle n\rangle) \to 0$ is exact for all $n \geq 0$. It follows that $\mathrm{Hom}_{\mathtt{Mod}\text{-}\mathfrak{G}}(h^n, -) : \mathtt{Mod}\text{-}\mathfrak{F} \to \mathtt{Vect}_k$ is an exact functor, and hence h^n is a projective object in $\mathtt{Mod}\text{-}\mathfrak{G}$. On the other hand, for any $m \geq 0$, we have

$$h^n(\langle m\rangle) = k[\mathrm{Hom}_{\mathfrak{G}}(\langle m\rangle, \langle n\rangle)] \cong k[\langle n\rangle^{\times m}] \cong [k\langle n\rangle]^{\otimes m} = k\langle n\rangle(\langle m\rangle),$$

which shows that $k\langle n\rangle \cong h^n$ as right $\mathfrak{G}$-modules. This finishes the proof of the lemma. ■

Proof of Theorem 4.2. By Lemma 4.1, for any $n \geq 0$, $k\langle n\rangle$ is a projective right $\mathfrak{G}$-module such that $k\langle n\rangle \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \cong \underline{\mathcal{H}}(\langle n\rangle)$. Since colimits of projective modules are flat and commute with left Kan extensions, this implies that $k\langle S\rangle$ is a *flat* right $\mathfrak{G}$-module and $k\langle S\rangle \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \cong \underline{\mathcal{H}}(\langle S\rangle)$ for any set S. Extending the last isomorphism levelwise to simplicial sets, we get an isomorphism of simplicial vector spaces $k[\mathbb{G}X] \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \cong \underline{\mathcal{H}}(\mathbb{G}X)$. Further, since each $k[\mathbb{G}X_n]$ is a flat right $\mathfrak{G}$-module, the normalized chain complex $\underline{N}(k[\mathbb{G}X])$ is a complex of flat $\mathfrak{G}$-modules; hence, we have a natural isomorphism in the derived category $\mathscr{D}(\mathtt{Mod}\text{-}\mathfrak{G})$:

$$\underline{N}(k[\mathbb{G}X]) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \underline{\mathcal{H}} \cong \underline{N}(k[\mathbb{G}X]) \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \cong N(\underline{\mathcal{H}}(\mathbb{G}X)).$$

At the homology level, this induces the desired isomorphism of Theorem 4.2. ■

For our next theorem we recall that the singular chain complex $C_*(\Omega X; k)$ of the (Moore) loop space ΩX of a pointed topological space X has a natural structure of a DG Hopf algebra. The coproduct on $C_*(\Omega X; k)$ is induced by the Alexander–Whitney diagonal, while the product comes from the structure of a topological monoid on ΩX via the Eilenberg–Zilber map (see, e.g., [26, Section 26]). Thus, the homology $\mathrm{H}_*(\Omega X; k)$ of ΩX is a graded *cocommutative* Hopf algebra called the *Pontryagin algebra* of X.

Now, any graded cocommutative Hopf algebra H defines a graded right $\mathfrak{G}$-module $\underline{\mathrm{H}}$ (i.e., a contravariant functor from $\mathfrak{G}$ to the category of graded vector spaces). For $q \in \mathbb{Z}$, we let $\underline{\mathrm{H}}_q$ denote the graded component of $\underline{\mathrm{H}}$ of degree q; thus, $\underline{\mathrm{H}}_q : \mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$ is a right $\mathfrak{G}$-module that assigns $\langle n \rangle \mapsto [\mathrm{H}^{\otimes n}]_q$, the q-th graded component of the graded vector space $\mathrm{H}^{\otimes n}$. Note that the $\mathfrak{G}$-module $\underline{\mathrm{H}}_q$ depends on *all* graded components of the Hopf algebra H, and not solely on H_q. With this notation, we can now state our second theorem, which is an analogue of [60, Theorem 2.4] for representation homology.

Theorem 4.3. There is a natural 1st quadrant spectral sequence

$$E^2_{pq} = \mathrm{Tor}_p^{\mathfrak{G}}(\underline{\mathrm{H}}_q(\Omega X; k), \underline{\mathcal{H}}) \underset{p}{\Longrightarrow} \mathrm{HR}_n(X, \mathcal{H}) \tag{4.6}$$

converging to the representation homology of X.

Proof. Recall from the proof of Theorem 4.2 that $\underline{N}(k[\langle X \rangle])$ is a nonnegatively graded chain complex of flat right $\mathfrak{G}$-modules. Hence, for any left $\mathfrak{G}$-module $\underline{\mathcal{H}}$, there is a standard "Hypertor" spectral sequence (see, e.g., [79,Application 5.7.8]):

$$E^2_{pq} = \mathrm{Tor}_p^{\mathfrak{G}}(\mathrm{H}_q[\underline{N}(k[\mathbb{G}X])], \underline{\mathcal{H}}) \underset{p}{\Longrightarrow} \mathrm{H}_{p+q}[\underline{N}(k[\mathbb{G}X]) \otimes_{\mathfrak{G}} \underline{\mathcal{H}}].$$

By Theorem 4.2, the limit of this spectral sequence is isomorphic to $\mathrm{HR}_*(X, \mathcal{H})$. To prove the theorem we need only to show that $\mathrm{H}_*[\underline{N}(k[\mathbb{G}X])] \cong \underline{\mathrm{H}}_*(\Omega X; k)$ as graded right $\mathfrak{G}$-modules.

By Kan's Theorem 2.1, $|\mathbb{G}X|$ is weakly equivalent to the based loop space ΩX. In fact, both $|\mathbb{G}X|$ and ΩX have natural structures of topological monoids, and they are known to be weakly equivalent as an H-spaces (see, e.g., [11, Sect. 2 and Prop. 3.3(c)]). This implies, in particular, that $\mathrm{H}_*[N(k[\mathbb{G}X])] \cong \mathrm{H}_*(\Omega X; k)$ as graded Hopf algebras, and hence $\underline{\mathrm{H}}_*[N(k[\mathbb{G}X])] \cong \underline{\mathrm{H}}_*(\Omega X; k)$ as graded $\mathfrak{G}$-modules. Note that $N(k[\mathbb{G}X])$ stands here for the normalized chain complex of the simplicial Hopf algebra $k[\mathbb{G}X]$, while $\underline{N}(k[\mathbb{G}X])$

in the above spectral sequence denotes the normalized chain complex of the simplicial $\mathfrak{G}$-module $k[\mathbb{G}X]$. We need to check that $\mathrm{H}_*[\underline{N}(k[\mathbb{G}X])] \cong \underline{\mathrm{H}}_*[N(k[\mathbb{G}X])]$ as graded $\mathfrak{G}$-modules. Now, the simplicial $\mathfrak{G}$-module $k[\mathbb{G}X]$ assigns to $\langle m\rangle \in \mathfrak{G}$ the simplicial vector space $k[\mathbb{G}X_*]^{\otimes m} = \{k[\mathbb{G}X_n]^{\otimes m}\}_{n\geq 0}$. By the Eilenberg–Zilber theorem, the normalized chain complex of this simplicial vector space is homotopy equivalent to $N(k[\mathbb{G}X])^{\otimes m}$, while, by Kunneth's formula, the homology of $N(k[\mathbb{G}X])^{\otimes m}$ is naturally isomorphic to $\mathrm{H}_*[N(k[\mathbb{G}X])]^{\otimes m}$. This shows that $\mathrm{H}_*(\underline{N}(k[\mathbb{G}X]))(\langle m\rangle) \cong \mathrm{H}_*[N(k[\mathbb{G}X])]^{\otimes m}$ for any $m \geq 0$, completing the proof of the theorem. ■

Theorem 4.3 has several interesting implications. First, we consider one important special case when the spectral sequence (4.6) collapses at E^2-term.

Corollary 4.3. Let Γ be a discrete group. Then, for any affine algebraic group G, there is a natural isomorphism

$$\mathrm{HR}_*(\mathrm{B}\Gamma, G) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)).$$

In particular, $\mathrm{HR}_0(\mathrm{B}\Gamma, G) \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G)$.

Proof. The classifying space $X = \mathrm{B}\Gamma$ is an Eilenberg–MacLane space of type $K(\Gamma, 1)$. Its loop space ΩX is homotopy equivalent to Γ, where Γ is considered as a discrete topological space. Hence, $\mathrm{H}_q(\Omega X; k) = 0$ for all $q > 0$, while $\mathrm{H}_0(\Omega X; k) \cong k[\Gamma]$ as a Hopf algebra. Thus, for $X = \mathrm{B}\Gamma$, the spectral sequence (4.6) collapses on the p-axis, giving the required isomorphism. ■

Remark. Combining the isomorphisms of Corollaries 4.1 and 4.3, we can express the representation homology of Γ (originally defined as a non-abelian derived functor) in terms of classical abelian homological algebra:

$$\mathrm{HR}_*(\Gamma, G) \cong \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], \mathcal{O}(G)).$$

In degree 0, we have a natural isomorphism expressing the coordinate ring of the representation variety $\mathrm{Rep}_G(\Gamma)$ as a functor tensor product:

$$\mathcal{O}[\mathrm{Rep}_G(\Gamma)] \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G).$$

This last isomorphism was found in [46], and it was one of the starting points for the present paper.

Remark. The result of Theorem 4.3 holds for any (not necessarily, monoidal) left $\mathfrak{G}$-module. In particular, if we take a *reductive* affine algebraic group G and define a left $\mathfrak{G}$-module $\mathcal{O}(G)^G \in \mathfrak{G}\text{-}\mathtt{Mod}$ by the formula $\langle n\rangle \mapsto [\mathcal{O}(G)^{\otimes n}]^G = \mathcal{O}(G\times \overset{n}{\ldots} \times G)^G$, then, for any $X \in \mathtt{sSet}_0$, we obtain a homology spectral sequence

$$E^2_{pq} = \mathrm{Tor}^{\mathfrak{G}}_p(\underline{\mathrm{H}}_q(\Omega X; k),\, \mathcal{O}(G)^G) \Longrightarrow \mathrm{HR}_n(X,\, G)^G \tag{4.7}$$

converging to the G-invariant part of representation homology of X. The proof of Corollary 4.3 shows that, for $X = \mathrm{B}\Gamma$, the spectral sequence (4.7) collapses on the p-axis, giving an isomorphism

$$\mathrm{HR}_*(\mathrm{B}\Gamma, G)^G \cong \mathrm{Tor}^{\mathfrak{G}}_*(k[\Gamma], \mathcal{O}(G)^G).$$

In degree 0, we therefore have $\mathcal{O}[\mathrm{Rep}_G(\Gamma)]^G \cong k[\Gamma] \otimes_{\mathfrak{G}} \mathcal{O}(G)^G$.

Remark. Using Corollary 4.3, we can write the 5-term exact sequence associated to the spectral sequence (4.6) in the form

$$\mathrm{HR}_2(X, G) \to \mathrm{HR}_2(\pi_1(X),\, G) \to \underline{\mathrm{H}}_1(\Omega X; k) \otimes_{\mathfrak{G}} \mathcal{O}(G) \to \mathrm{HR}_1(X, G) \to \mathrm{HR}_1(\pi_1(X),\, G) \to 0.$$

If the fundamental group $\pi_1(X)$ is f.g. virtually free (in particular, finite or f.g. free), then, by [9, Theorem 5.1], $\mathrm{HR}_i(\pi_1(X),\, G)$ vanishes for all $i > 0$, and hence in this case, we get

$$\mathrm{HR}_1(X, G) \cong \underline{\mathrm{H}}_1(\Omega X; k) \otimes_{\mathfrak{G}} \mathcal{O}(G).$$

To state further consequences of Theorem 4.3 we introduce some terminology. We will say that a map $f : X \to Y$ of pointed topological spaces is a *Pontryagin equivalence* (over k) if it induces an isomorphism $\mathrm{H}_*(\Omega X; k) \cong \mathrm{H}_*(\Omega Y; k)$ of Pontryagin algebras (or equivalently, a quasi-isomorphism $C_*(\Omega X; k) \xrightarrow{\sim} C_*(\Omega Y; k)$ of DG Hopf algebras). The next result is obtained by applying to (4.6) a standard comparison theorem for homology spectral sequences (see [79, Theorem 5.1.12]).

Corollary 4.4. If $f : X \to Y$ is a Pontryagin equivalence, the induced map on representation homology $f_* : \mathrm{HR}_*(X, \mathcal{H}) \xrightarrow{\sim} \mathrm{HR}_*(Y, \mathcal{H})$ is an isomorphism for any Hopf algebra $\mathcal{H}$.

We remark that Corollary 4.4 does not say that an *arbitrary* isomorphism of Hopf algebras $\mathrm{H}_*(\Omega X; k) \cong \mathrm{H}_*(\Omega Y; k)$ gives an isomorphism $\mathrm{HR}_*(X, \mathcal{H}) \cong \mathrm{HR}_*(Y, \mathcal{H})$. (Indeed, an abstract isomorphism of Pontryagin algebras need not even induce a map on representation homology.) Still, Corollary 4.3 shows that if both X and Y are aspherical spaces, then any isomorphism of Pontryagin algebras induces an isomorphism on representation homology.

Next, we recall that the singular chain complex $C_*(X; k)$ of any space X is naturally a DG coalgebra with comultiplication defined by the Alexander–Whitney diagonal. Moreover, if X is path-connected, there is a quasi-isomorphism of DG coalgebras (see [27, Theorem 6.3])

$$C_*(X; k) \simeq \boldsymbol{B}[C_*(\Omega X; k)],$$

where $\boldsymbol{B}$ is the classical bar construction. Since $\boldsymbol{B}$ preserves quasi-isomorphisms, any Pontryagin equivalence $f : X \to Y$ of path-connected spaces is necessarily a homology equivalence, that is, it induces an isomorphism on singular homology $\mathrm{H}_*(X; k) \xrightarrow{\sim} \mathrm{H}_*(Y; k)$. The converse is not always true unless X and Y are simply connected. In the latter case, we have the following well-known result (cf. [65, Part I, Prop. 1.1]).

Lemma 4.2. Let $f : X \to Y$ be a map of simply connected pointed topological spaces. The following conditions are equivalent:

(1) f is a rational homology equivalence: that is, $f_* : \mathrm{H}_*(X; \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(Y; \mathbb{Q})$;
(2) f is a rational Pontryagin equivalence: that is, $f_* : \mathrm{H}_*(\Omega X; \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(\Omega Y; \mathbb{Q})$;
(3) f is a rational homotopy equivalence: that is, $f_* : \pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \pi_*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows a classical theorem of Adams [1] that asserts that, for any simply-connected space X, there is a quasi-isomorphism of DG algebras: $C_*(\Omega X; k) \simeq \boldsymbol{\Omega}[C_*(X; k)]$, where $\boldsymbol{\Omega}$ is the cobar construction.

To prove that (2) $\Leftrightarrow$ (3) we first recall that, for any simply connected X, the $\mathbb{Q}$-vector space $L_X := \pi_*(\Omega X)_{\mathbb{Q}} \cong \pi_{*+1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ carries a natural bracket (called the Whitehead product) making it a graded Lie algebra (called the *homotopy Lie algebra* of X.). Thus, a map $f : X \to Y$ is a rational homotopy equivalence if and only if it induces an isomorphism of Lie algebras $f_* : L_X \to L_Y$. Then, a classical theorem of Milnor and

Moore (see [26, Theorem 21.5]) implies that the Hurewicz homomorphism $\pi_*(\Omega X) \to \mathrm{H}_*(\Omega X;\mathbb{Q})$ induces an isomorphism of graded Hopf algebras $UL_X \xrightarrow{\sim} \mathrm{H}_*(\Omega X;\mathbb{Q})$, where $U(L_X)$ is the universal enveloping algebra of L_X. This yields the equivalence (2) ⇔ (3). ■

We say that a map $f : X \to Y$ of simply connected spaces is a *rational homotopy equivalence* if the equivalent conditions of Lemma 4.2 hold.

Proposition 4.2. A rational homotopy equivalence induces an isomorphism on representation homology. Thus, $\mathrm{HR}_*(X,\mathcal{H})$ depends only on the rational homotopy type of X.

Proof. By Lemma 4.2(2), a rational homotopy equivalence $X \to Y$ induces an isomorphism $\mathrm{H}_*(\Omega X;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(\Omega Y;\mathbb{Q})$. Since $\mathrm{char}(k) = 0$, we have $\mathbb{Q} \subseteq k$, and the universal coefficient theorem implies that $\mathrm{H}_*(\Omega X;k) \cong \mathrm{H}_*(\Omega Y;k)$. The claim then follows from Corollary 4.4. ■

Next, we look at higher connected spaces. Recall that a space X is called n-*connected* if X is path-connected and its 1st n homotopy groups vanish, that is, $\pi_i(X) = 0$ for $1 \le i \le n$.

Proposition 4.3. Let X be an n-connected space for some $n \ge 1$, and let $\mathcal{H} = \mathcal{O}(G)$. Then

$$\mathrm{HR}_q(X,G) = \begin{cases} k & \text{for} \quad q = 0 \\ 0 & \text{for} \quad 1 \le q < n \\ \mathrm{H}_{q+1}(X;\mathfrak{g}^*) & \text{for} \quad n \le q \le 2n-1, \end{cases} \tag{4.8}$$

where $\mathfrak{g} := \mathrm{Lie}(G)$ is the Lie algebra of G and $\mathfrak{g}^*$ is its k-linear dual.

Proof. If a space X is n-connected, its homotopy Lie algebra $L_X = \pi_*(\Omega X)_{\mathbb{Q}} \cong \pi_{*+1}(X)_{\mathbb{Q}}$ is n-reduced, that is, $(L_X)_q = 0$ for $0 \le q \le n-1$. Since $\mathrm{H}_*(\Omega X;\mathbb{Q}) \cong UL_X$ and $\mathbb{Q} \subseteq k$, we have $\mathrm{H}_0(\Omega X;k) \cong k$, $\mathrm{H}_q(\Omega X;k) = 0$ for $1 \le q \le n-1$, and

$$\mathrm{H}_q(\Omega X;k) \cong (L_X)_q \otimes_{\mathbb{Q}} k \cong \pi_{q+1}(X)_k \cong \mathrm{H}_{q+1}(X;k) \quad \text{for} \quad n \le q \le 2n-1,$$

where the last isomorphism is a consequence of the rational Hurewicz theorem (see, e.g., [48]).

Now, recall that for a fixed $q \geq 0$, the right $\mathfrak{G}$-module $\underline{\mathrm{H}}_q(\Omega X; k)$ is defined as the functor $\mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$, $\langle m \rangle \mapsto [\mathrm{H}_*(\Omega X; k)^{\otimes m}]_q$. It follows from this definition that

$$\underline{\mathrm{H}}_q(\Omega X; k) = \begin{cases} \underline{k} & \text{for} \quad q = 0 \\ 0 & \text{for} \quad 1 \leq q \leq n-1 \\ \mathrm{lin}_k^* \otimes \mathrm{H}_{q+1}(X; k) & \text{for} \quad n \leq q \leq 2n-1, \end{cases}$$

where lin_k is the linearization functor:

$$\mathrm{lin}_k : \mathfrak{G} \to \mathtt{Vect}_k \quad \langle m \rangle \mapsto \langle m \rangle_{\mathrm{ab}} \otimes_{\mathbb{Z}} k = k^{\oplus m}, \tag{4.9}$$

and $\mathrm{lin}_k^* : \mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$ denotes its composition with linear duality. Thus, for X n-connected, the E^2-terms of the spectral sequence (4.6) can be identified as

$$E^2_{pq} \cong \begin{cases} k & \text{for} \quad p = 0 \;\; q = 0 \\ \mathrm{Tor}_p^{\mathfrak{G}}(\underline{k},\, \underline{\mathcal{H}}) & \text{for} \quad p > 0 \;\; q = 0 \\ 0 & \text{for} \quad p \geq 0 \;\; 1 \leq q < n \\ \mathrm{Tor}_p^{\mathfrak{G}}(\mathrm{lin}_k^*,\, \underline{\mathcal{H}}) \otimes \mathrm{H}_{q+1}(X; k) & \text{for} \quad p \geq 0 \;\; n \leq q \leq n-1. \end{cases} \tag{4.10}$$

By Lemma 4.1, the right $\mathfrak{G}$-module $\underline{k} = k\langle 0 \rangle$ is projective. Hence, $E^2_{p,0} = 0$ for $p > 0$. On the other hand, $\mathrm{lin}_k^* \otimes_{\mathfrak{G}} \underline{\mathcal{H}} \cong \mathfrak{g}^*$, while $\mathrm{Tor}_p^{\mathfrak{G}}(\mathrm{lin}_k^*,\, \underline{\mathcal{H}}) = 0$ for $p > 0$. Hence, for $n \leq q \leq 2n-1$, we have

$$E^2_{0,q} = \mathfrak{g}^* \otimes \mathrm{H}_{q+1}(X; k) \cong \mathrm{H}_{q+1}(X; \mathfrak{g}^*), \qquad E^2_{pq} = 0 \quad \text{for } p > 0. \tag{4.11}$$

The vanishing of E^2_{pq} for all $p > 0$ in the range $0 \leq q \leq 2n-1$ shows that the spectral sequence (4.6) collapses on the q-axis for these values of q. Thus, we have $\mathrm{HR}_q(X, \underline{\mathcal{H}}) \cong E^2_{0,q}$ for $0 \leq q \leq 2n-1$. By (4.10) and (4.11), these are the desired isomorphisms (4.8). ■

Remark. Proposition 4.3 shows that the representation homology of an n-connected space in sufficiently low degrees ($q \leq 2n-1$) depends only on the Lie algebra $\mathfrak{g}$. The main theorem of [10] implies a much stronger result: the *whole* $\mathrm{HR}_*(X, G)$ is determined by $\mathfrak{g}$ if X is (at least) 1-connected. Thus, for simply connected spaces, the representation homology with coefficients in an algebraic group G depends only on the connected component G_0 of the identity element in G: that is, $\mathrm{HR}_*(X, G) \cong \mathrm{HR}_*(X, G_0)$. This last statement is *not* true in general, for non-simply connected spaces: indeed, already in the simplest example $X = \mathbb{S}^1$, we have $\mathrm{HR}_*(\mathbb{S}^1, G) \cong \mathcal{O}(G)$.

5 Representation Homology and Higher Hochschild Homology

In Section 4.2, we defined representation homology by analogy with Hochschild homology, using Kan's simplicial loop group construction. In this section, we establish a direct relation between these two homology theories using another classical construction in simplicial homotopy theory due to Milnor [56].

5.1 Main theorems

We begin by recalling a standard simplicial model for a (reduced) suspension ΣX of a space X. The *suspension functor* on pointed simplicial sets is defined by

$$\Sigma : \mathtt{sSet}_* \to \mathtt{sSet}_0 , \quad X \mapsto C(X)/X ,$$

where $C(X) \in \mathtt{sSet}_*$ is the reduced cone over X. For a pointed simplicial set $X = \{X_n\}_{n\geq 0}$, the set of n-simplices in $C(X)$ is given by

$$C(X)_n := \{(x, m) : x \in X_{n-m} , 0 \leq m \leq n\} ,$$

with all $(*, m)$ being identified to $*$. The face and degeneracy maps in $C(X)$ are defined by

$$d_i : C(X)_n \to C(X)_{n-1} , \qquad (x, m) \mapsto \begin{cases} (x, m-1) & \text{if } 0 \leq i < m \\ (d^X_{i-m}(x), m) & \text{if } m \leq i \leq n \end{cases}$$

$$s_j : C(X)_n \to C(X)_{n+1} , \qquad (x, m) \mapsto \begin{cases} (x, m+1) & \text{if } 0 \leq j < m \\ (s^X_{j-m}(x), m) & \text{if } m \leq j \leq n, \end{cases}$$

where $d_1(x, 1) = *$ for all $x \in X_0$.

The embedding $X \hookrightarrow C(X)$ is given by $x \mapsto (x, 0)$, and ΣX is defined to be the corresponding quotient set. Note that, unlike $C(X)$, the simplicial set ΣX is reduced, since $(x, 0) = *$ in ΣX for all $x \in X$ (in particular, we have $C(X)_0 = \{(x, 0) : x \in X_0\} \sim \{*\}$). Now, for any pointed simplicial set X, there is a homotopy equivalence $|\Sigma X| \simeq \Sigma|X|$, where $\Sigma|X|$ is reduced suspension of the geometric realization of X in the usual topological sense.

The next two theorems constitute the main result of this section.

Theorem 5.1. For any commutative Hopf algebra $\mathcal{H}$ and any *pointed* simplicial set X, there is a natural isomorphism of graded commutative algebras

$$\mathrm{HR}_*(\Sigma X, \mathcal{H}) \cong \mathrm{HH}_*(X, \mathcal{H}; k)\,.$$

To state the next theorem, we recall that there is a natural way to make an arbitrary simplicial set pointed by adding to it a disjoint basepoint. To be precise, the forgetful functor $\mathtt{sSet}_* \to \mathtt{sSet}$ has a left adjoint $(-)_+ : \mathtt{sSet} \to \mathtt{sSet}_*$ obtained by extending to simplicial sets the obvious functor $X \mapsto X \sqcup \{*\}$ on the category of sets. Explicitly, if $\{X_n\}_{n\geq 0}$ is a simplicial set, then $(X_+)_n = X_n \sqcup \{*\}$ for all n, and the face and degeneracy maps of X_+ are the (unique) basepoint-preserving extensions of the corresponding maps of X. Being a left adjoint, the functor $(-)_+$ commutes with colimits; in particular, we have

$$|X_+| \cong |X|_+\,,$$

where $|X|_+$ is the space obtained from $|X|$ by adjoining a basepoint.

Theorem 5.2. For any commutative Hopf algebra $\mathcal{H}$ and any simplicial set X, there is an isomorphism of graded commutative algebras

$$\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HH}_*(X, \mathcal{H})\,.$$

The proofs of Theorems 5.1 and 5.2 are based on a classical simplicial group model of the spaces $\Omega\Sigma X$, which we now briefly review.

5.2 Milnor's *FK*-construction

For a pointed simplicial set $K \in \mathtt{sSet}_*$, we define $FK := \mathbb{G}\Sigma K$. Then, by Kan's Theorem 2.1, there is a homotopy equivalence of spaces

$$|FK| \simeq \Omega\Sigma|K|\,.$$

The following observation is due to Milnor [56] (see also [39, Theorem V.6.15]).

Lemma 5.1 (Milnor). For any $K \in \mathtt{sSet}_*$, FK is a semi-free simplicial group generated by the simplicial set K with basepoint identified with 1, that is,

$$FK_n = (\mathbb{G}\Sigma K)_n \cong \langle K_n \rangle / \langle s_0^n(*) = 1 \rangle \cong \langle K_n \setminus s_0^n(*) \rangle\,.$$

The face and degeneracy maps are induced by the face and degeneracy maps of K.

Proof. By definition of the reduced suspension, we have $(x, 0) = *$ for all $x \in K$ and $s_0(x, m) = (x, m+1)$ for all $m > 0$. Hence, $(\Sigma K)_{n+1}/s_0(\Sigma K_n) = \{(x, 1) \,|\, x \in K_n\}$, with $(*, 1)$ being the basepoint. It follows that

$$(\mathbb{G}\Sigma K)_n = \langle (\Sigma K)_{n+1}/s_0(\Sigma K_n)\rangle \cong \langle K_n\rangle/(* = 1)\,.$$

To calculate the face and degeneracy maps, we recall from Section 2.2 that

$$d_0^{\mathbb{G}\Sigma K}(x, 1) = d_1(x, 1)\, d_0(x, 1)^{-1} = (d_0 x, 1)\,(x, 0)^{-1} = (d_0 x, 1)\,,$$

and $d_i^{\mathbb{G}\Sigma K}(x, 1) = d_{i+1}(x, 1) = (d_i x, 1)$ for $i > 0$. Similarly, $s_j^{\mathbb{G}\Sigma K}(x, 1) = s_{j+1}(x, 1) = (s_j x, 1)$ for all $j \geq 0$. This proves the desired lemma. ∎

5.2.1 *Proofs of Theorems 5.1 and 5.2*

Recall that, for a commutative Hopf algebra $\mathcal{H}$, we denote by $\underline{\mathcal{H}}$ the functor $\mathtt{FGr} \to \mathtt{Comm\,Alg}_k$ on the category of based free groups obtained from the $\mathfrak{G}$-module $\langle n\rangle \mapsto \mathcal{H}^{\otimes n}$ by taking its left Kan extension along the inclusion $\mathfrak{G} \hookrightarrow \mathtt{FGr}$ (see Section 4.2).

Proposition 5.1. There is an isomorphism of functors from $\mathtt{sSet}$ to $\mathtt{sComm\,Alg}_k$:

$$\underline{\mathcal{H}} \circ \mathbb{G} \circ \Sigma \circ (-)_+ \cong (- \otimes \mathcal{H})\,,$$

where $\mathcal{H}$ in the right-hand side is regarded as a commutative k-algebra.

Proof. By Lemma 5.1, for any simplicial set $X = \{X_n\}_{n \geq 0}$, there are natural isomorphisms of groups $[\mathbb{G}\Sigma(X_+)]_n \cong \langle X_n\rangle$, $n \geq 0$, with structure maps on $\mathbb{G}\Sigma(X_+)$ being compatible with those of X. By applying the functor $\mathcal{H}$, we thus get isomorphisms of simplicial commutative algebras

$$\underline{\mathcal{H}}([\mathbb{G}\Sigma(X_+)]_*) \cong \underline{\mathcal{H}}[\langle X_*\rangle] \cong X_* \otimes \mathcal{H},$$

which are obviously functorial in X. This proves the proposition. ∎

Theorem 5.2 is an immediate consequence of the above proposition. To prove Theorem 5.1, we first note that, although the unreduced cone on a space X coincides with the reduced cone on X_+, the corresponding suspensions differ. Instead, for any pointed space X, there is a homotopy equivalence (see [54, p. 106])

$$\Sigma(X_+) \simeq \Sigma X \vee \mathbb{S}^1\,. \tag{5.1}$$

From this we can deduce the following.

Lemma 5.2. For a pointed topological space X, there is a natural isomorphism

$$\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HR}_*(\Sigma X, \mathcal{H}) \otimes \mathcal{H}.$$

Proof. Applying Corollary 4.2 to (5.1), we have $\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HR}_*(\Sigma X, \mathcal{H}) \otimes \mathrm{HR}_*(\mathbb{S}^1, \mathcal{H})$. Now, since $\mathbb{S}^1 \cong \Sigma(\mathrm{pt}_+)$, Theorem 5.2 implies $\mathrm{HR}_*(\mathbb{S}^1, \mathcal{H}) \cong \mathrm{HH}_*(\mathrm{pt}, \mathcal{H}) \cong \mathcal{H}$, where $\mathcal{H}$ is concentrated in degree 0. It follows that $\mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \cong \mathrm{HR}_*(\Sigma X, \mathcal{H}) \otimes \mathcal{H}$ as desired. ∎

Lemma 5.2 shows that $\mathrm{HR}_*(\Sigma X, \mathcal{H}) \cong \mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \otimes_{\mathcal{H}} k$. Combining this last isomorphism with that of Theorem 5.2, we now conclude

$$\mathrm{HR}_*(\Sigma X, \mathcal{H}) \cong \mathrm{HR}_*(\Sigma(X_+), \mathcal{H}) \otimes_{\mathcal{H}} k \cong \mathrm{HH}_*(X, \mathcal{H}) \otimes_{\mathcal{H}} k \cong \mathrm{HH}_*(X, \mathcal{H}; k).$$

This proves Theorem 5.1.

5.3 Examples

We conclude this section with a few simple examples illustrating the use of Theorems 5.1 and 5.2. More examples will be given in the next two sections. In what follows, G denotes an arbitrary affine algebraic group and $\mathfrak{g} = \mathrm{Lie}(G)$ stands for its Lie algebra.

5.3.1 *Spheres*

The representation homology of the circle $\mathbb{S}^1$ is given by $\mathrm{HR}_0(\mathbb{S}^1, G) \cong \mathcal{O}(G)$ and $\mathrm{HR}_i(\mathbb{S}^1, G) = 0$ for $i > 0$. This follows, for example, from Lemma 4.1 and Corollary 4.3 (since $\mathbb{S}^1 \cong B\mathbb{Z}$). Now, for higher dimensional spheres, we have the following.

Proposition 5.2. $\mathrm{HR}_*(\mathbb{S}^n, G) \cong \Lambda_k(\mathfrak{g}^*[n-1])$ for all $n \geq 2$.

Proof. Note that $\mathbb{S}^n \simeq \Sigma\, \mathbb{S}^{n-1}$ for all $n \geq 2$. By Theorem 5.1, we conclude

$$\mathrm{HR}_*(\mathbb{S}^n, G) \;\cong\; \mathrm{HH}_*(\mathbb{S}^{n-1}, \mathcal{O}(G); k) \;\cong\; \Lambda_{\mathcal{O}(G)}(\Omega^1(G)[n-1]) \otimes_{\mathcal{O}(G)} k \;\cong\; \Lambda_k(\mathfrak{g}^*[n-1])\,,$$

where the 2nd isomorphism follows from [60, Section 5.5]. ■

5.3.2 *Suspensions*

We now generalize the previous example to arbitrary suspensions.

Proposition 5.3. Let ΣX be the suspension of a pointed connected space X of finite type. Then

$$\mathrm{HR}_*(\Sigma X, G) \;\cong\; \Lambda_k[\overline{\mathrm{H}}_*(X; \mathfrak{g}^*)], \tag{5.2}$$

where $\overline{\mathrm{H}}_*(X; \mathfrak{g}^*)$ stands for the reduced (singular) homology of X with constant coefficients in $\mathfrak{g}^*$.

Consequently, by induction,

$$\mathrm{HR}_*(\Sigma^n X, G) \;\cong\; \Lambda_k\big(\overline{\mathrm{H}}_*(X; \mathfrak{g}^*)[n-1]\big) \quad \forall\, n \geq 1.$$

Proof. It is known (see [26, Theorem 24.5]) that ΣX is rationally homotopy equivalent to a bouquet of spheres: $\Sigma X \simeq_{\mathbb{Q}} \bigvee_{i\in I} \mathbb{S}^{n_i}$, where each $\mathbb{S}^{n_i}$ have dimension $n_i \geq 2$. By Proposition 4.2, it thus suffices to compute $\mathrm{HR}_*(S, G)$ for $S := \bigvee_{i\in I} \mathbb{S}^{n_i}$. Note that the reduced homology $\overline{\mathrm{H}}_*(S; k)$ of S is isomorphic to $\oplus_{i\in I}\, k \cdot v_i$ with trivial coproduct, where v_i is a basis element of homological degree $\deg(v_i) = n_i$. Now, by Corollary 4.2 and Proposition 5.2, we have

$$\begin{aligned}
\mathrm{HR}_*(\Sigma X, G) \;&\cong\; \mathrm{HR}_*(S, G) \cong \bigoplus_{i\,\in\, I} \mathrm{HR}_*(\mathbb{S}^{n_i}, G) \cong \bigoplus_{i\,\in\, I} \Lambda_k(\mathfrak{g}^*[n_i - 1]) \\
&\cong\; \Lambda_k \bigoplus_{i\,\in\, I} \mathfrak{g}^*[n_i - 1]\big) \cong \Lambda_k\Big(\bigoplus_{n\,\geq\, 2} \mathfrak{g}^* \otimes \mathrm{H}_n(\Sigma X; k)[n-1]\Big) \\
&\cong\; \Lambda_k\Big(\bigoplus_{n\,\geq\, 1} \mathfrak{g}^* \otimes \mathrm{H}_n(X; k)[n]\Big) \cong \Lambda_k\Big[\,\mathfrak{g}^* \otimes \overline{\mathrm{H}}_*(X; k)\Big] \cong \Lambda_k[\overline{\mathrm{H}}_*(X; \mathfrak{g}^*)],
\end{aligned}$$

where the last isomorphism is a consequence of the Universal Coefficient Theorem. ■

As a consequence of Theorem 5.2, Lemma 5.2, and Proposition 5.3, we have the following general formula for the higher Hochschild homology of X with coefficients in a commutative Hopf algebra.

Corollary 5.1. For any pointed connected topological space X of finite type,

$$\mathrm{HH}_*(X, \mathcal{O}(G)) \cong \Lambda_{\mathcal{O}(G)}\big[\,\overline{\mathrm{H}}_*(X; k) \otimes \Omega^1(G)\,\big]. \tag{5.3}$$

The isomorphism (5.3) is a refinement of Pirashvili's generalization of the classical HKR theorem that (in our notation) asserts that $\mathrm{HH}_*(X, A) \cong \underline{\mathrm{H}}_*(X; k) \otimes_{\mathfrak{F}} \underline{A}$ for any smooth commutative algebra A (cf. [60, Theorem 4.6]).

5.3.3 *Co-H-spaces*

The result of Proposition 5.3 can be seen in a more conceptual way. The key fact is that the suspension ΣX of any pointed connected space X is a cogroup object in the homotopy category of pointed spaces, with coproduct $\Sigma X \to \Sigma X \vee \Sigma X$ given by the natural "pinching" map (see, e.g., [73, p. 41]). The functor $\mathrm{HR}_*(-, G) : \mathtt{Ho}(\mathtt{Top}_{0,*}) \to \mathtt{grCommAlg}_k$ preserves coproducts and hence maps cogroup objects in $\mathtt{Ho}(\mathtt{Top}_{0,*})$ to cogroup objects in $\mathtt{grCommAlg}_k$. The latter are precisely the graded commutative Hopf algebras; thus, the representation homology of ΣX carries a natural Hopf algebra structure for any space X. Since ΣX is 1-connected, $\mathrm{HR}_*(\Sigma X, G)$ is actually a *connected* graded commutative Hopf algebra, and hence, by the (dual) Milnor–Moore theorem (see [29, Theorem 0.2]), its underlying algebra structure is free: that is, $\mathrm{HR}_*(\Sigma X, G) \cong \Lambda_k V$ for some graded vector space V. As shown in the proof of Proposition 5.3, the rational equivalence $\Sigma X \simeq_{\mathbb{Q}} \bigvee_{i\in I} \mathbb{S}^{n_i}$ implies $V \cong \overline{\mathrm{H}}_*(X; \mathfrak{g}^*)$, and it is easy to see that (5.2) is actually an isomorphism of graded Hopf algebras.

The above argument is similar to Berstein's "categorical" proof of the classical Bott–Samelson theorem describing the Pontryagin algebra of the suspension ΣX (see [13]). This formal argument works actually for any (simply connected associative) co-H-space, provided one replaces the homology $\overline{\mathrm{H}}_*(X, k)$ with the so-called Berstein–Scheerer coalgebra $B_*(X, k)$ of X (see, e.g., [2]). In this way, we have the following generalization of Proposition 5.3.

Proposition 5.4. Let X be a 1-connected, associative co-H-space. Then

$$\mathrm{HR}_*(X, G) \cong \Lambda_k\big[B_*(X, k) \otimes \mathfrak{g}^*\big],$$

where $B_*(X, k)$ is the Berstein–Scheerer coalgebra of X.

We remark that $B_*(\Sigma X, k) \cong \overline{\mathrm{H}}_*(X, k)$ as coalgebras (see [2, p. 1150]), so in the case of suspensions, Proposition 5.4 indeed reduces to Proposition 5.3.

6 Examples: Surfaces and 3-Manifolds

In this section, using standard topological decompositions, we compute representation homology of some classical non-simply connected spaces. Our examples include closed surfaces (both orientable and non-orientable) as well as some three-dimensional spaces (link complements in $\mathbb{R}^3$, lens spaces, and general closed orientable 3-manifolds). The representation homology of surfaces and link complements is given in terms of classical Hochschild homology of $\mathcal{O}(G)$ (or $\mathcal{O}(G^n)$ for some $n \geq 2$) with twisted coefficients. The representation homology of a closed 3-manifold M is expressed in terms of a differential "Tor", which gives rise to an (Eilenberg–Moore) spectral sequence converging to $\mathrm{HR}_*(M, G)$.

6.1 Surfaces

6.1.1 *The torus*

As a cell complex, the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ can be constructed as the homotopy cofibre (the mapping cone) of the map $\alpha : \mathbb{S}^1_c \to \mathbb{S}^1_a \vee \mathbb{S}^1_b$, where the subscripts on the circles indicate the generators of the respective fundamental groups, and the map itself is specified, up to homotopy, by its effect on these generators:

$$\alpha(c) = [a, b] := aba^{-1}b^{-1}. \tag{6.1}$$

Thus, $\mathbb{T}^2 \simeq \mathrm{hocolim}[\, * \leftarrow \mathbb{S}^1_c \xrightarrow{\alpha} \mathbb{S}^1_a \vee \mathbb{S}^1_b]$, where the homotopy colimit is taken in the category $\mathtt{Top}_{0,*}$ of connected pointed spaces. Applying to this the Kan loop group functor $\mathbb{G}$ (more precisely, the composition of $\mathbb{G}$ with the Eilenberg subcomplex functor $\overline{S}$, see Section 2.2), we get a simplicial group model for $\mathbb{T}^2$:

$$\mathbb{G}(\mathbb{T}^2) \cong \mathrm{hocolim}[\, 1 \leftarrow \mathbb{F}_1 \xrightarrow{\alpha} \mathbb{F}_2\,]. \tag{6.2}$$

Here $\mathbb{F}_1$ and $\mathbb{F}_2$ are the free groups on the generators c and $\{a, b\}$ respectively; the map α is given by (6.1), and the homotopy colimit is taken in the category $\mathtt{sGr}$ of simplicial groups.

Now, by Theorem 3.2, the derived representation functor preserves homotopy pushouts for any algebraic group G. Hence, it follows from (6.2) that

$$\mathcal{O}[\mathrm{DRep}_G(\mathbb{T}^2)] \cong \mathrm{hocolim}[\, k \leftarrow \mathcal{O}(G) \xrightarrow{\alpha_*} \mathcal{O}(G \times G)\,], \tag{6.3}$$

where the homotopy colimit is taken in $s\mathtt{CommAlg}_k$, and the map $\alpha_* : \mathcal{O}(G) \to \mathcal{O}(G \times G)$ is induced by (6.1) (explicitly, $\alpha_*(f)(x, y) = f([x, y])$ for $f \in \mathcal{O}(G)$). Since

$$\mathrm{hocolim}\,[\,k \leftarrow \mathcal{O}(G) \xrightarrow{\alpha_*} \mathcal{O}(G \times G)\,] \,\cong\, \mathcal{O}(G \times G) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k,$$

by Proposition 4.1, we conclude that

$$\mathrm{HR}_*(\mathbb{T}^2, G) \,\cong\, \mathrm{Tor}_*^{\mathcal{O}(G)}(\mathcal{O}(G \times G),\, k), \tag{6.4}$$

where $\mathcal{O}(G \times G)$ is viewed as a (right) $\mathcal{O}(G)$-module via the algebra map α_*.

By standard homological algebra (see [16, Theorem 2.1, p. 185]), we can identify the Tor-groups in (6.4) as the classical Hochschild homology of $\mathcal{O}(G)$ with coefficients in the bimodule $\mathcal{O}(G \times G)$, where the right $\mathcal{O}(G)$-module structure is given via the map α_* and the left module structure via the augmentation map $\varepsilon : \mathcal{O}(G) \to k$:

$$\mathrm{HR}_*(\mathbb{T}^2, G) \,\cong\, \mathrm{HH}_*(\mathcal{O}(G),\, {}_{\varepsilon}\mathcal{O}(G \times G)_{\alpha}). \tag{6.5}$$

Alternatively, for classical (matrix) groups G, we can give an explicit "small" DG algebra model for the representation homology $\mathrm{HR}_*(\mathbb{T}^2, G)$. Specifically, let $\mathfrak{m} := \mathrm{Ker}(\varepsilon)$ denote the maximal (augmentation) ideal of $\mathcal{O}(G)$ corresponding to the identity element $e \in G$. Assume that $\mathfrak{m}$ is generated by a *regular* sequence of elements $(r_1, r_2, \ldots, r_d)$ in $\mathcal{O}(G)$, so that $d = \dim G$. Consider the free module $E := \mathcal{O}(G)^{\oplus d}$ and define the $\mathcal{O}$-module map $\pi : E \to \mathcal{O}(G)$ by $\pi(f_1, f_2, \ldots, f_d) := \sum_{i=1}^{d} r_i f_i$. Then, associated to (E, π) is the (global) Koszul complex $K_*(G) := (\Lambda^*_{\mathcal{O}(G)}(E), \delta_K)$ with differential

$$\delta_K(e_0 \wedge e_1 \wedge \ldots \wedge e_n) = \sum_{i=0}^{n} (-1)^i\, \pi(e_i)\; e_0 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_n.$$

Since $\mathfrak{m}$ is generated by a regular sequence, the canonical projection $K_*(G) \twoheadrightarrow \mathcal{O}(G)/\mathfrak{m} \cong k$ is a quasi-isomorphism of complexes, and therefore $K_*(G)$ is a free resolution of k over $\mathcal{O}(G)$. It follows from (6.4) that

$$\mathrm{Tor}_*^{\mathcal{O}(G)}(\mathcal{O}(G \times G),\, k) \,\cong\, \mathrm{H}_*[\mathcal{A}(\mathbb{T}^2, G)], \tag{6.6}$$

where $\mathcal{A}(\mathbb{T}^2, G) := \mathcal{O}(G \times G) \otimes_{\mathcal{O}(G)} K_*(G)$ is a commutative DG algebra with differential $d = \mathrm{Id} \otimes \delta_K$. In particular, $\mathrm{HR}_i(\mathbb{T}^2, G) = 0$ for all $i > \dim G$.

We conclude this example with a conjectural description of the G-invariant part of representation homology $\mathrm{HR}_*(\mathbb{T}^2, G)^G$. Our conjecture can be viewed as a multiplicative analogue of the derived Harish–Chandra conjecture proposed in [6].

Assume that G is a connected reductive algebraic group of rank $l \geq 1$ defined over an algebraically closed field k of characteristic zero. Let $T \subset G$ be a Cartan subgroup (i.e., a maximal torus) in G, and let W be the corresponding Weyl group. Note that, since T is commutative, the map $\alpha_* : \mathcal{O}(T) \to \mathcal{O}(T \times T)$ associated to T factors through the augmentation $\varepsilon : \mathcal{O}(T) \to k$. Hence, by (6.4), we have canonical isomorphisms

$$\begin{aligned} \mathrm{HR}_*(\mathbb{T}^2, T) &\cong \mathrm{Tor}_*^{\mathcal{O}(T)}(\mathcal{O}(T \times T), k) \\ &\cong \mathcal{O}(T \times T) \otimes \mathrm{Tor}_*^{\mathcal{O}(T)}(k, k) \\ &\cong \mathcal{O}(T \times T) \otimes \Lambda_k^*(\mathfrak{m}_T/\mathfrak{m}_T^2) \\ &\cong \mathcal{O}(T \times T) \otimes \Lambda_k^*(\mathfrak{h}^*), \end{aligned} \tag{6.7}$$

where $\mathfrak{m}_T := \mathrm{Ker}(\varepsilon)$ is the augmentation ideal, $\mathfrak{h} = (\mathfrak{m}_T/\mathfrak{m}_T^2)^*$ is the Lie algebra of T (i.e., a Cartan subalgebra of $\mathfrak{g}$), and $\Lambda_k^*(\mathfrak{h}^*)$ is the (homologically) graded exterior algebra with $\mathfrak{h}^*$ placed in degree one.

Now, by functoriality, the natural inclusion $T \hookrightarrow G$ induces a map of simplicial commutative algebras

$$\Phi_G(\mathbb{T}^2) : \mathcal{O}[\mathrm{DRep}_G(\mathbb{T}^2)]^G \to \mathcal{O}[\mathrm{DRep}_T(\mathbb{T}^2)]^W, \tag{6.8}$$

which is (a multiplicative analogue of) the derived Harish–Chandra homomorphism constructed in [6]. Then, the multiplicative version of the derived Harish–Chandra conjecture states the following.

Conjecture 1. Assume that G is one of the classical groups $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{Sp}_{2n}(k)$, $n \geq 1$, or any simply connected, semi-simple(It is known that every simply connected reductive affine algebraic group is automatically semi-simple. This follows from two classical facts: (1) every reductive Lie algebra is a product of a semi-simple one and an abelian one; (2) there are no nontrivial simply connected abelian reductive algebraic groups.) affine algebraic group. Then the derived Harish–Chandra homomorphism (6.8) is a weak equivalence in $s\mathtt{Comm\,Alg}_k$. Hence, by (6.7), there is an isomorphism of graded commutative algebras

$$\mathrm{HR}_*(\mathbb{T}^2, G)^G \cong [\mathcal{O}(T \times T) \otimes \Lambda_k^*(\mathfrak{h}^*)]^W. \tag{6.9}$$

We illustrate Conjecture 1 for $G = \mathrm{GL}_n$. Since $\mathcal{O}(\mathrm{GL}_n) \cong k[x_{ij}, \det(x_{ij})^{-1}]_{1\le i,j\le n}$, the elements $\{x_{ij} - \delta_{ij}\}_{1\le i,j\le n}$ form a regular sequence in $\mathcal{O}(\mathrm{GL}_n)$ generating the maximal ideal $\mathfrak{m}$, so we have a canonical commutative DG algebra representing $\mathrm{HR}_*(\mathbb{T}^2, \mathrm{GL}_n)$:

$$\mathcal{A}(\mathbb{T}^2, \mathrm{GL}_n) \cong k[x_{ij},\, y_{ij},\, \theta_{ij};\ \det(X)^{-1},\ \det(Y)^{-1}]_{1\le i,j\le n}.$$

Here the variables x_{ij} and y_{ij} have homological degree 0, θ_{ij} have homological degree 1, and $\det(X)$ and $\det(Y)$ denote the determinants of the generic matrices $X := \|x_{ij}\|$ and $Y := \|y_{ij}\|$. The differential on $\mathcal{A}(\mathbb{T}^2, \mathrm{GL}_n)$ can be written in matrix terms as

$$d\,\Theta = XYX^{-1}Y^{-1} - I_n,$$

where $\Theta := \|\theta_{ij}\|$ and I_n is the identity $n\times n$-matrix. The Harish–Chandra homomorphism

$$\Phi_{\mathrm{GL}_n}(\mathbb{T}^2):\ \mathcal{A}(\mathbb{T}^2, \mathrm{GL}_n)^{\mathrm{GL}_n} \to k\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}, \theta_1, \ldots, \theta_n\right]^{S_n}$$

is given explicitly (on generators) by the following map:

$$x_{ij} \mapsto \delta_{ij} x_i \quad y_{ij} \mapsto \delta_{ij} y_i, \quad \theta_{ij} \mapsto \delta_{ij}\theta_i,$$

and the derived Harish–Chandra conjecture asserts that $\Phi_{\mathrm{GL}_n}(\mathbb{T}^2)$ induces an isomorphism (cf. (6.9))

$$\mathrm{HR}_*(\mathbb{T}^2, \mathrm{GL}_n)^{\mathrm{GL}_n} \xrightarrow{\sim} k\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1^{\pm 1}, \ldots, y_n^{\pm 1}, \theta_1, \ldots, \theta_n\right]^{S_n}, \tag{6.10}$$

where $\theta_1, \ldots, \theta_n$ have homological degree 1 and the symmetric group S_n acts diagonally by permuting the variables. Note that, in the case of $\mathrm{GL}_n(k)$, unlike for other algebraic groups, Conjecture 1 follows from the derived Harish–Chandra conjecture for the corresponding Lie algebra $\mathfrak{gl}_n(k)$ stated in [6]. This is because the Harish–Chandra map $\Phi_{\mathrm{GL}_n}(\mathbb{T}^2)$ can be obtained by formally localizing the derived Harish–Chandra map for the Lie algebra $\mathfrak{gl}_n(k)$ (cf. [6, Sect. 4]). In particular, the evidence collected in [6] for $\mathfrak{gl}_n(k)$ also supports Conjecture 1 for $\mathrm{GL}_n(k)$. we list some of this evidence here.

(1) Conjecture 1 holds for $\mathrm{GL}_2(k)$ and $\mathrm{GL}_\infty(k)$. This follows from [6, Theorems 4.1 and 4.2(ii)].

(2) For all $n \geq 1$, the map (6.10) is degreewise surjective. This follows from [6, Theorem 4.2(i)].

3) For all $n \geq 1$, $\mathrm{HR}_i(\mathbb{T}^2, \mathrm{GL}_n)^{\mathrm{GL}_n} = 0$ for $i > n$. This follows from [7, Theorem 27].

4) For any G as in Conjecture 1, the map (6.10) is an isomorphism in homological degree *zero*, that is, $\mathrm{HR}_0(\mathbb{T}^2, G)^G \cong \mathcal{O}(T \times T)^W$. This follows from a theorem of Thaddeus [74] (see also [71]).

Finally, we remark that, for $G = \mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, and $\mathrm{Sp}_{2n}(k)$, the Harish–Chandra map is known to be an isomorphism in homological degree 0: $\mathrm{HR}_0(\mathbb{T}^N, G)^G \cong \mathcal{O}(T^N)^W$ for all tori $\mathbb{T}^N$, $N \geq 2$ (see [71]). However, by results of [6, Sect. 5.2], the above isomorphism does not extend to higher homological degrees when $N \geq 3$. In other words, the derived Harish Chandra homomorphism $\Phi_{\mathrm{GL}_n}(\mathbb{T}^N)$ is not a week equivalence for higher dimensional tori $\mathbb{T}^N$, $N \geq 3$.

6.1.2 *Riemann surfaces*

The above computation of representation homology of the 2-torus naturally generalizes to Riemann surfaces of an arbitrary genus. To be precise, let Σ_g denote a closed connected orientable surface of genus $g \geq 1$. As a 2-dimensional cell complex, Σ_g can be described as the homotopy cofibre of the map $\alpha^g : \mathbb{S}^1_c \to \vee_{i=1}^g \left(\mathbb{S}^1_{a_i} \vee \mathbb{S}^1_{b_i}\right)$ defined by

$$\alpha^g(c) = [a_1, b_1]\,[a_2, b_2]\,\ldots\,[a_g, b_g], \tag{6.11}$$

where $a_1, b_1, \ldots, a_g, b_g$ denote the a- and b-cycles on Σ_g generating the fundamental group $\pi_1(\Sigma_g, *)$. This gives the simplicial group model $\mathbb{G}(\Sigma_g) \cong \mathrm{hocolim}[\,1 \leftarrow \mathbb{F}_1 \xrightarrow{\alpha^g} \mathbb{F}_{2g}\,]$ of Σ_g, which, in turn, implies

$$\mathcal{O}[\mathrm{DRep}_G(\Sigma_g)] \cong \mathrm{hocolim}\,[\,k \leftarrow \mathcal{O}(G) \xrightarrow{\alpha^g_*} \mathcal{O}(G^{2g})\,] \cong \mathcal{O}(G^{2g}) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k,$$

where the map $\alpha^g_* : \mathcal{O}(G) \to \mathcal{O}(G^{2g})$ is defined by

$$\alpha^g_*(f)(x_1, y_1, \ldots, x_g, y_g) := f([x_1, y_1]\,[x_2, y_2]\,\ldots\,[x_g, y_g]) \quad f \in \mathcal{O}(G).$$

By Proposition 4.1, we conclude

$$\mathrm{HR}_*(\Sigma_g, G) \cong \mathrm{Tor}_*^{\mathcal{O}(G)}(\mathcal{O}(G^{2g}), k) \cong \mathrm{HH}_*(\mathcal{O}(G), {}_{\varepsilon}\mathcal{O}(G^{2g})_{\alpha}), \tag{6.12}$$

where ${}_{\varepsilon}\mathcal{O}(G^{2g})_{\alpha}$ is the bimodule with left and and right $\mathcal{O}(G)$-module structure given by the maps ε and α_*^g, respectively.

In case when $\mathfrak{m} \subset \mathcal{O}(G)$ is generated by a regular sequence, we can also express the representation homology of Σ_g as the homology of the commutative DG algebra $\mathcal{A}(\Sigma_g, G) := \mathcal{O}(G^{2g}) \otimes_{\mathcal{O}(G)} K_*(G)$, where $K_*(G)$ is the global Koszul complex constructed in Section 6.1.1:

$$\mathrm{HR}_*(\Sigma_g, G) \cong \mathrm{H}_*[\mathcal{A}(\Sigma_g, G)].$$

Like in the torus case, for a reductive group G with a Cartan subgroup T, there is an algebra map induced by the derived Harish–Chandra homomorpism $\Phi_G(\Sigma_G)$:

$$\mathrm{HR}_*(\Sigma_g, G)^G \to [\mathcal{O}(T^{2g}) \otimes \Lambda_k^*(\mathfrak{h}^*)]^W,$$

where W operates diagonally on the target. However, in contrast to the torus case, this map seems far from being an isomorphism in general. In fact, for $g \geq 2$, it is conjectured in [9] that $\mathrm{HR}_i(\Sigma_g, G) = 0$ if $i > \dim \mathcal{Z}(G)$, where $\mathcal{Z}(G)$ denotes the center of G; in particular, this implies that $\mathrm{HR}_i(\Sigma_g, G) = 0$ for all $i > 0$ if G is semisimple.

6.2 3-Manifolds

6.2.1 *Link complements in* $\mathbb{R}^3$

By a link L in $\mathbb{R}^3$ we mean a smooth (oriented) embedding of the disjoint union $\mathbb{S}^1 \sqcup \ldots \sqcup \mathbb{S}^1$ of (a finite number of) copies of $\mathbb{S}^1$ into $\mathbb{R}^3$. The link complement $X := \mathbb{R}^3 \setminus L$ is then defined to be the complement of an (open) tubular neighborhood of the image of L in $\mathbb{R}^3$. To describe a simplicial group model for X we recall two classical facts from geometric topology (cf. [14]). First, by a well-known theorem of Alexander, every link L in $\mathbb{R}^3$ can be obtained geometrically as the closure of a braid β in $\mathbb{R}^3$ (we write $L = \hat{\beta}$ to indicate this relation). Second, for each $n \geq 1$, the braids on n strands in $\mathbb{R}^3$ form a group B_n (the Artin braid group), which admits a faithful representation by automorphisms of the free group $\mathbb{F}_n$ (the Artin representation). Specifically, the group B_n is generated by $n-1$ elements ("flips") $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad (\text{if } |i-j| > 1), \qquad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad (\text{if } |i-j| = 1),$$

and in terms of these generators, the Artin representation $B_n \to \mathrm{Aut}(\mathbb{F}_n)$ is given by

$$\sigma_i : \begin{cases} x_i & \mapsto \quad x_i\, x_{i+1}\, x_i^{-1} \\ x_{i+1} & \mapsto \quad x_i \\ x_j & \mapsto \quad x_j \quad (j \neq i, i+1). \end{cases} \tag{6.13}$$

To simplify the notation we will identify B_n with its image in $\mathrm{Aut}(\mathbb{F}_n)$ under (6.13).

The next proposition can be viewed as a refinement of a classical theorem of Artin and Birman [14, Theorem 2.2] describing the fundamental group of the link complement $\mathbb{R}^3\backslash L$ in terms of the Artin representation (see the remark below).

Proposition 6.1. Let $L = \hat{\beta}$ be a link in $\mathbb{R}^3$ given by the closure of a braid $\beta \in B_n$. Then

$$\mathbb{G}(\mathbb{R}^3\backslash L) \;\cong\; \mathrm{hocolim}\,[\,\mathbb{F}_n \xleftarrow{(\beta,\mathrm{Id})} \mathbb{F}_n \amalg \mathbb{F}_n \xrightarrow{(\mathrm{Id},\mathrm{Id})} \mathbb{F}_n\,], \tag{6.14}$$

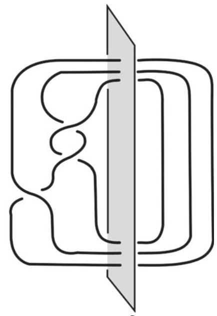
where β acts on $\mathbb{F}_n$ via the Artin representation (6.13).

Remark. Note that the homotopy pushout in (6.14) coincides with the homotopy coequalizer [*FIXGRAPHICS*] of the two endomorphisms Id and β of $\mathbb{F}_n$. Hence, (6.14) implies

$$\begin{aligned} \pi_1(\mathbb{R}^3\backslash L, *) &\cong \pi_0[\mathbb{G}(\mathbb{R}^3\backslash L)] \cong \mathrm{coeq}\; \boldsymbol{L}\mathrm{coeq}[\;\mathbb{F}_n \; \overset{\mathrm{Id}}{\underset{\beta}{\rightrightarrows}} \; \mathbb{F}_n\;] \\ &\cong \langle x_1, \ldots, x_n \mid \beta(x_1) = x_1, \ldots, \beta(x_n) = x_n \rangle, \end{aligned}$$

which is the Artin–Birman presentation of the link group $\pi(L) := \pi_1(\mathbb{R}^3\backslash L, *)$.

Proof. The proof is based on a simple van Kampen type argument (cf. [14]). Let us place the n-braid β in a regular position in the region $x < 0$ in $\mathbb{R}^3$, so that its starting points $\{p_1, p_2, \ldots, p_n\}$ and end points $\{q_1, q_2, \ldots, q_n\}$ are located on the z-axis with coordinates $q_1 < q_2 < \ldots < q_n < p_n < p_{n-1} < \ldots < p_1$. The link L is the closure of β obtained by joining the points p_i to q_i ($i = 1, 2, \ldots, n$) by simple arcs in the region $x > 0$, as shown in the picture



Now, let $X := \mathbb{R}^3 \backslash L$ denote the complement of L. Define

$$X_{\geq 0} := \{(x, y, z) \in X \,:\, x \geq 0\} \quad X_{\leq 0} := \{(x, y, z) \in X \,:\, x \leq 0\} \quad X_0 := X_{\geq 0} \cap X_{\leq 0},$$

with a (common) basepoint $*$ in X_0. It is easy to see that $X_{\geq 0}$ is homeomorphic to the cylinder over $\mathbb{R}^2 \backslash \{p_1, \ldots, p_n\}$, which is, in turn, homotopic to $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$, where $\mathbb{D}^2$ is a two-dimensional disk in (the yz-plane) $\mathbb{R}^2$ encompassing the points $\{p_1, \ldots, p_n\}$. Similarly, we have $X_{\leq 0} \cong (\mathbb{R}^2 \backslash \{q_1, \ldots, q_n\}) \times [0, 1] \simeq \mathbb{D}^2 \backslash \{q_1, \ldots, q_n\}$, and

$$X_0 \simeq \mathbb{D}^2 \backslash \{p_1, \ldots, p_n, q_1, \ldots, q_n\} \simeq \mathbb{D}^2 \backslash \{p_1, \ldots, p_n\} \vee \mathbb{D}^2 \backslash \{q_1, \ldots, q_n\}.$$

Under these identifications, the natural inclusions $X_{\leq 0} \hookleftarrow X_0 \hookrightarrow X_{\geq 0}$ can be identified with

$$\mathbb{D}^2 \backslash \{q_1, \ldots, q_n\} \xleftarrow{(f_\beta, \mathrm{Id})} \mathbb{D}^2 \backslash \{p_1, \ldots, p_n\} \vee \mathbb{D}^2 \backslash \{q_1, \ldots, q_n\} \xrightarrow{(\mathrm{Id}, f_e)} \mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}, \quad (6.15)$$

where the map f_β is determined (uniquely up to homotopy) by the braid β and the map f_e is determined by the trivial braid connecting the points p_i and q_i. Thus, we can represent X in $\mathtt{Ho}(\mathtt{Top}_{0,*})$ as the homotopy pushout of the diagram (6.15).

Next, recall that B_n can be identified with the mapping class group of $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$ comprising (the isotopy classes of) orientation-preserving homeomorphisms that fix pointwise the boundary of $\mathbb{D}^2$. As a mapping class group, B_n acts naturally on the fundamental group $\pi_1(\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}, *)$ and the latter can be identified with the free group $\mathbb{F}_n$ on generators $x_1, \ldots, x_n$ represented by small loops in $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$ around the points p_i. It is well known (see [14]) that the action of B_n on $\mathbb{F}_n$ arising from this construction is precisely the Artin representation (6.13). Now, using the map f_e we identify $\mathbb{D}^2 \backslash \{q_1, \ldots, q_n\}$ with $\mathbb{D}^2 \backslash \{p_1, \ldots, p_n\}$ in (6.15) and apply the loop group functor to this diagram of spaces. As a result, we get the equivalence (6.14), which completes the proof of the proposition. ■

To state our main theorem we introduce some notation. First, observe that, for any algebraic group G, the Artin representation $B_n \hookrightarrow \mathrm{Aut}(\mathbb{F}_n)$ induces naturally a braid group action $B_n \to \mathrm{Aut}[\mathcal{O}(G^n)]$, which we denote by $\beta \mapsto \beta_*$. On the standard generators, this action is defined by

$$(\sigma_i)_* : \mathcal{O}(G^n) \to \mathcal{O}(G^n), \quad f(g_1,\ldots,g_i,g_{i+1},\ldots,g_n) \mapsto f(g_1,\ldots,g_i g_{i+1} g_i^{-1}, g_i,\ldots,g_n).$$

Now, for a braid $\beta \in B_n$, we let $O(G^n)_\beta$ denote the $\mathcal{O}(G^n)$-bimodule whose underlying vector space is $\mathcal{O}(G^n) = \mathcal{O}(G)^{\otimes n}$, the left action of $\mathcal{O}(G^n)$ is given by multiplication, while the right action is twisted by the automorphism β_*.

Theorem 6.1. Let $L = \hat{\beta}$ be a link in $\mathbb{R}^3$ given by the closure of a braid $\beta \in B_n$. Then

$$\mathcal{O}[\mathrm{DRep}_G(\mathbb{R}^3 \backslash L)] \cong \mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta.$$

Consequently,

$$\mathrm{HR}_*(\mathbb{R}^3 \backslash L, G) \cong \mathrm{HH}_*(\mathcal{O}(G^n), \mathcal{O}(G^n)_\beta). \tag{6.16}$$

Proof. By Proposition 6.1 and Theorem 3.2, we have

$$\begin{aligned}
\mathcal{O}[\mathrm{DRep}_G(\mathbb{R}^3 \backslash L)] &\cong \operatorname{hocolim}\,[\,\mathcal{O}(G^n) \xleftarrow{(\beta_*,\mathrm{Id})} \mathcal{O}(G^n) \otimes_k \mathcal{O}(G^n) \xrightarrow{(\mathrm{Id},\mathrm{Id})} \mathcal{O}(G^n)\,] \\
&\cong \operatorname{hocolim}\,[\,\mathcal{O}(G^n) \xleftarrow{(\beta_*,\mathrm{Id})} \mathcal{O}(G^{2n}) \xrightarrow{(\mathrm{Id},\mathrm{Id})} \mathcal{O}(G^n)\,] \\
&\cong \mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta.
\end{aligned}$$

This completes the proof of the theorem. ■

Remark. Theorem 6.1 exhibits an interesting analogy between the representation homology of link complements in $\mathbb{R}^3$ and their Legendrian contact homology in the sense of Ng (see [58]). This analogy is explained in the recent paper [8], where a new algebraic construction of link contact homology is given. Roughly speaking, in terminology of [8], $\mathcal{O}[\mathrm{DRep}_G(\mathbb{R}^3 \backslash L)]$ represents the algebraic "homotopy closure" of the braid $\beta \in B_n$ in the category of simplicial commutative algebras, while the Legendrian contact homology of $\mathbb{R}^3\setminus L$ can be computed from a certain DG category $\mathscr{A}_L$ that represents the homotopy braid closure of $\beta \in B_n$ in the category of (small pointed) DG k-categories.

6.2.2 *Link complements in* $\mathbb{S}^3$

Note that Theorem 6.1 computes the representation homology of the topological *space* $\mathbb{R}^3\setminus L$, not of the *link group* $\pi(L)$, which is the fundamental group of $\mathbb{R}^3\setminus L$. Even when L is a knot in $\mathbb{R}^3$ (i.e., a link with one component), the representation homologies $\mathrm{HR}_*(\mathbb{R}^3\setminus L, G)$ and $\mathrm{HR}_*(\pi(L), G)$ differ, because $\mathbb{R}^3\setminus L$ is not a $K(\pi, 1)$-space (cf. Example 6.1 below). In knot theory, one is usually interested in representation varieties of the knot group $\pi(L)$, so it is important to understand the relation between $\mathrm{HR}_*(\mathbb{R}^3\setminus L, G)$ and $\mathrm{HR}_*(\pi(L), G)$. A natural way to approach this problem is to consider L as a link in $\mathbb{S}^3$ by adding to $\mathbb{R}^3$ one point at infinity. If $L \subset \mathbb{R}^3 \subset \mathbb{S}^3$ is a knot, by Papakyriakopoulos' sphere theorem, the complement $\mathbb{S}^3\setminus L$ is an aspherical space, and $\pi_1(\mathbb{S}^3\setminus L, *) \cong \pi_1(\mathbb{R}^3\setminus L, *) = \pi(L)$. Hence, for any knot L, $\mathrm{HR}_*(\pi(L), G) \cong \mathrm{HR}_*(\mathbb{S}^3\setminus L, G)$, so it suffices to clarify the relation between $\mathrm{HR}_*(\mathbb{R}^3\setminus L, G)$ and $\mathrm{HR}_*(\mathbb{S}^3\setminus L, G)$.

To this end, we observe that the natural inclusion $\mathbb{R}^3\setminus L \hookrightarrow \mathbb{S}^3\setminus L$ fits into the cofibration sequence $\mathbb{S}^2 \overset{i}{\hookrightarrow} \mathbb{R}^3\setminus L \hookrightarrow \mathbb{S}^3\setminus L$, so that

$$\mathbb{S}^3\setminus L \cong \operatorname{hocolim}[\, * \leftarrow \mathbb{S}^2 \xrightarrow{i} \mathbb{R}^3\setminus L\,], \tag{6.17}$$

where $\mathbb{S}^2 \subset \mathbb{R}^3$ is chosen in such a way that it encloses L in $\mathbb{R}^3$. Applying the Kan functor to (6.17), we get

$$\mathbb{G}(\mathbb{S}^3\setminus L) \cong \operatorname{hocolim}[\, 1 \leftarrow \mathbb{G}(\mathbb{S}^2) \xrightarrow{i_*} \mathbb{G}(\mathbb{R}^3\setminus L)\,]. \tag{6.18}$$

To describe the induced map i_*, we note that $\mathbb{S}^2 \cong \Sigma\,\mathbb{S}^1 \cong \operatorname{hocolim}[\, * \leftarrow \mathbb{S}^1 \to *\,]$; hence,

$$\mathbb{G}(\mathbb{S}^2) \cong \operatorname{hocolim}[\, 1 \leftarrow \mathbb{F}_1 \to 1\,]. \tag{6.19}$$

Now, if we identify $\mathbb{G}(\mathbb{R}^3\setminus L)$ as in Proposition 6.1, then i_* is determined by the morphism of diagrams

$$\begin{tikzcd} 1 \arrow[d] & \mathbb{F}_1 \arrow[l] \arrow[r] \arrow[d] & 1 \arrow[d] \\ \mathbb{F}_n & \mathbb{F}_n \amalg \mathbb{F}_n \arrow[l, "(\beta, \mathrm{Id})"'] \arrow[r, "(\mathrm{Id}, \mathrm{Id})"] & \mathbb{F}_n \end{tikzcd}, \tag{6.20}$$

where the map in the middle is given (on free generators) by $x \mapsto (x_1 x_2 \ldots x_n)(y_1 y_2 \ldots y_n)^{-1}$. Note that the left square in (6.20) commutes because the product $x_1 x_2 \ldots x_n \in \mathbb{F}_n$ stays fixed under the Artin representation for any $\beta \in B_n$.

The map $i_* : \mathbb{G}(\mathbb{S}^2) \to \mathbb{G}(\mathbb{R}^3 \backslash L)$ induces a map of simplicial commutative algebras

$$i_* : \mathcal{O}[\mathrm{DRep}_G(\mathbb{S}^2)] \to \mathcal{O}[\mathrm{DRep}_G(\mathbb{R}^3 \backslash L)], \tag{6.21}$$

which (to simplify the notation) we denote by the same symbol. By Theorem 6.1,

$$\mathcal{O}[\mathrm{DRep}_G(\mathbb{R}^3 \backslash L)] \cong \mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta.$$

On the other hand, by (6.19),

$$\mathcal{O}[\mathrm{DRep}_G(\mathbb{S}^2)] \cong k \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k \cong \Lambda^*_k(\mathfrak{m}/\mathfrak{m}^2) \cong \Lambda^*_k(\mathfrak{g}^*),$$

where $\Lambda^*_k(\mathfrak{g}^*)$ denotes the graded exterior algebra of $\mathfrak{g}^*$, with $\mathfrak{g}^*$ being in degree one, equipped with trivial differential. With these identifications, the map (6.21) is induced by the algebra homomorphism

$$\mathcal{O}(G) \to \mathcal{O}(G^{2n}) \quad f(x) \mapsto f((x_1 x_2 \ldots x_n)(y_1 y_2 \ldots y_n)^{-1}). \tag{6.22}$$

Now, we can regard $\mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta$ as a DG module over the DG algebra $k \otimes^{\boldsymbol{L}}_{\mathcal{O}(G)} k \cong \Lambda^*(\mathfrak{g}^*)$. As a consequence of (6.18) and Theorem 6.1, we have then the following.

Theorem 6.2. Let $L = \hat{\beta}$ be a link in $\mathbb{S}^3$ given by the closure of a braid $\beta \in B_n$. Then

$$\mathcal{O}[\mathrm{DRep}_G(\mathbb{S}^3 \backslash L)] \cong k \otimes^{\boldsymbol{L}}_{\Lambda^*(\mathfrak{g}^*)} [\mathcal{O}(G^n) \otimes^{\boldsymbol{L}}_{\mathcal{O}(G^{2n})} \mathcal{O}(G^n)_\beta].$$

Consequently, there is a natural spectral sequence

$$E^2_{*,*} = \mathrm{Tor}^{\Lambda^*(\mathfrak{g}^*)}_*(k, \mathrm{HH}_*(\mathcal{O}(G^n), \mathcal{O}(G^n)_\beta) \implies \mathrm{HR}_*(\mathbb{S}^3 \backslash L, G),$$

converging to the representation homology of $\mathbb{S}^3 \backslash L$.

Example 6.1. Let $L = \bigcirc$ be the unknot in $\mathbb{R}^3$. We can represent L by the trivial braid $\beta = 1 \in B_1$. In this case, Theorem 6.1 combined with the classical Hochschild–Kostant–Rosenberg theorem gives

$$\mathrm{HR}_*(\mathbb{R}^3 \backslash \bigcirc, G) \cong \mathrm{HH}_*(\mathcal{O}(G), \mathcal{O}(G)) \cong \Omega^*(G),$$

where $\Omega^*(G)$ is the de Rham algebra of (algebraic) differential forms on the group G. On the other hand, $\mathrm{HR}_*(\mathbb{S}^3 \setminus \bigcirc, G) \cong \mathrm{HR}_*(\pi(\bigcirc), G) \cong \mathcal{O}(G)$, since $\pi(\bigcirc) \cong \mathbb{Z}$. This simple example illustrates the fact that representation homology does depend on the higher homotopy structure of a space: in particular, it distinguishes the link complements in $\mathbb{R}^3$ and $\mathbb{S}^3$, even though their fundamental groups are the same.

6.2.3 *Lens spaces*

Recall that, for coprime integers p and q, the lens space $L(p, q)$ of type (p, q) is defined as the quotient $\mathbb{S}^3/\mathbb{Z}_p$ of the 3-sphere $\mathbb{S}^3$ viewed as the unit sphere in $\mathbb{C}^2$ modulo the (free) action of the cyclic group $\mathbb{Z}_p$ given by $(z, w) \mapsto (e^{2\pi i/p}\, z,\, e^{2\pi i q/p}\, w)$. This definition shows that $L(p, q)$ is a compact connected 3-manifold, whose universal cover is $\mathbb{S}^3$ and the fundamental group is $\mathbb{Z}_p$. Special cases include $L(1, 0) \cong \mathbb{S}^3$, $L(0, 1) \cong \mathbb{S}^1 \times \mathbb{S}^2$ and $L(2, 1) \cong \mathbb{RP}^3$.

To compute the representation homology of $L(p, q)$ we will use a well-known topological construction of these spaces via Dehn surgery in $\mathbb{S}^3$ (see, e.g., [68, Chap. 3B]). Recall that if $K \subset \mathbb{S}^3$ is a knot in $\mathbb{S}^3$ and p, q are two integer numbers, the p/q *Dehn surgery* on K is a 3-dimensional space obtained by removing from $\mathbb{S}^3$ the interior $\mathring{N}(K)$ of a regular tubular neighborhood $N(K)$, which is a 3-dimensional solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, and then gluing $\mathbb{S}^1 \times \mathbb{D}^2$ back to $\mathbb{S}^3 \setminus \mathring{N}(K)$ in such a way that the meridional curve of $\mathbb{S}^1 \times \mathbb{D}^2$ is identified with a (p, q)-curve on the boundary of $\mathbb{S}^3 \setminus \mathring{N}(K)$. For the trivial knot $K \subset \mathbb{S}^3$, it is easy to see that the p/q Dehn surgery on K gives precisely the lens space $L(p, q)$. In this case, the knot complement $\mathbb{S}^3 \setminus \mathring{N}(K)$ is homeomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, so the space $L(p, q)$ can be obtained by gluing together two solid tori along their boundary.

To describe this in more concrete terms, we consider the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ as a subset in $\mathbb{C}^2$:

$$\mathbb{S}^1 \times \mathbb{D}^2 = \{(z, w) \in \mathbb{C}^2 \,:\, |z| = 1\,,\, |w| \le 1\}.$$

We identify $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ as the boundary of $\mathbb{S}^1 \times \mathbb{D}^2$ in $\mathbb{C}^2$ and denote by $i : \mathbb{T}^2 \hookrightarrow \mathbb{S}^1 \times \mathbb{D}^2$ the natural inclusion.

Now, for the given pair (p, q) of coprime numbers, we choose $m, n \in \mathbb{Z}$, so that $mq - np = 1$, and define the "gluing" map $\gamma : \mathbb{T}^2 \to \mathbb{S}^1 \times \mathbb{D}^2$ by

$$\gamma(z, w) := (z^m w^p,\, z^n w^q). \tag{6.23}$$

Then the p/q Dehn surgery construction of $L(p,q)$ can be described as the pushout in $\mathtt{Top}_{0,*}$:

$$L(p,q) \cong \operatorname{colim}[\,\mathbb{S}^1 \times \mathbb{D}^2 \overset{i}{\hookleftarrow} \mathbb{T}^2 \overset{\gamma}{\hookrightarrow} \mathbb{S}^1 \times \mathbb{D}^2\,]. \tag{6.24}$$

Since i is a cofibration in $\mathtt{Top}_{0,*}$, we can replace the colimit in (6.24) by a homotopy colimit and then replace the diagram of solid tori by a homotopy equivalent diagram of circles:

$$L(p,q) \;\cong\; \operatorname{hocolim}[\,\mathbb{S}^1 \overset{\pi}{\leftarrow} \mathbb{T}^2 \overset{\bar{\gamma}}{\rightarrow} \mathbb{S}^1\,]. \tag{6.25}$$

In this diagram, the map π is given by the canonical projection $(z,w) \mapsto z$ and $\bar{\gamma}$ is the composition $\pi \circ \gamma$ defined by $(z,w) \mapsto z^m w^p$. Now, applying the Kan loop group functor, we get a simplicial group model for $L(p,q)$:

$$\mathbb{G}[L(p,q)] \;\cong\; \operatorname{hocolim}[\,\mathbb{F}_1 \overset{\pi}{\leftarrow} \mathbb{G}(\mathbb{T}^2) \overset{\gamma}{\rightarrow} \mathbb{F}_1\,]. \tag{6.26}$$

Recall (see (6.2)) that $\mathbb{G}(\mathbb{T}^2)$ is given in $\mathtt{sGr}$ by the homotopy cofibre of the commutator map $\alpha : \mathbb{F}_1 \to \mathbb{F}_2,\ c \mapsto [a,b]$, where a and b are the generators of $\mathbb{F}_2$ corresponding to the meridian and longitude in $\mathbb{T}^2$. In terms of these generators, the maps π and γ in (6.26) are induced by

$$\pi\,:\,\mathbb{F}_2 \to \mathbb{F}_1 \;\;(a,b) \mapsto (z,1), \qquad \gamma\,:\,\mathbb{F}_2 \to \mathbb{F}_1 \;\;(a,b) \mapsto (z^m, z^p), \tag{6.27}$$

where z is a generator of $\mathbb{F}_1$.

Now, assume that G admits a global Koszul resolution $K_*(G)$ described in Section 6.1. Then, we have an explicit DG algebra model for $\mathrm{HR}_*(\mathbb{T}^2, G)$ given by $\mathcal{A}_*(\mathbb{T}^2, G) = \mathcal{O}(G \times G) \otimes_{\mathcal{O}(G)} K_*(G)$. Applying to (6.26) the derived representation functor, we get

$$\mathcal{O}[\mathrm{DRep}_G(L(p,q))] \;\cong\; \operatorname{hocolim}[\,\mathcal{O}(G) \overset{\pi_*}{\longleftarrow} \mathcal{A}_*(\mathbb{T}^2, G) \overset{\gamma_*}{\longrightarrow} \mathcal{O}(G)\,]. \tag{6.28}$$

The maps π_* and γ_* in (6.28) are determined by (6.27); on the degree 0 component of the DG algebra $\mathcal{A}_*(\mathbb{T}^2, G)$, they are given by

$$\pi_*\,:\,\mathcal{O}(G \times G) \to \mathcal{O}(G) \quad f(x,y) \mapsto f(z,e), \tag{6.29}$$

$$\gamma_*\,:\,\mathcal{O}(G \times G) \to \mathcal{O}(G) \quad f(x,y) \mapsto f(z^m, z^p). \tag{6.30}$$

Using π_* and γ_*, we can make $\mathcal{O}(G)$ into (left and right) DG modules over the DG algebra $\mathcal{A}_*(\mathbb{T}^2, G)$, which we denote by $\mathcal{O}(G)_\pi$ and $\mathcal{O}(G)_\gamma$ respectively. With this notation, we have the following result that completes our calculation.

Theorem 6.3. The representation homology of a 3-dimensional lens space $L(p,q)$ is given by

$$\mathrm{HR}_*(L(p,q),G) \cong \mathrm{Tor}_*^{\mathcal{A}_*}(\mathcal{O}(G)_\pi, \mathcal{O}(G)_\gamma),$$

where $\mathrm{Tor}_*^{\mathcal{A}_*}$ denotes the differential Tor taken over the DG algebra $\mathcal{A}_* = \mathcal{A}_*(\mathbb{T}^2, G)$. In particular, there is an Eilenberg–Moore homology spectral sequence

$$E^2_{*,*} = \mathrm{Tor}_*^{\mathrm{HR}_*(\mathbb{T}^2,G)}(\mathcal{O}(G)_\pi, \mathcal{O}(G)_\gamma) \implies \mathrm{HR}_*(L(p,q),G)$$

converging to the representation homology of $L(p,q)$.

6.2.4 *Closed* 3-*manifolds*

The above construction of lens spaces generalizes to arbitrary closed 3-manifolds. Specifically, it is well known that every closed connected orientable 3-manifold M admits a Heegaard decomposition $H_g \cup_\gamma H_g$ that can be written as

$$M \cong \operatorname{colim}[H_g \overset{i}{\hookleftarrow} \Sigma_g \overset{\gamma}{\hookrightarrow} H_g], \tag{6.31}$$

where H_g is a handlebody of genus $g \geq 0$, i is the natural inclusion identifying $\Sigma_g = \partial H_g$, and γ is a gluing map defined as the composition $\Sigma_g \xrightarrow{\gamma_A} \Sigma_g \overset{i}{\hookrightarrow} H_g$, where γ_A is an (orientation-preserving) diffeomorphism of Σ_g representing an element in the mapping class group $\mathcal{M}(\Sigma_g) := \pi_0(\mathrm{Diff}^+\,\Sigma_g)$. In particular, for $g = 1$, the Heegaard diagram (6.31) becomes (6.24); in fact, the lens spaces can be characterized as (closed) 3-manifolds that admit Heegaard decompositions of genus 1.

Since H_g is homotopy equivalent as a cell complex to the bouquet of g circles $\vee_{i=1}^g \mathbb{S}^1$, we can represent the homotopy type of M by

$$M \cong \operatorname{hocolim}\left[\bigvee_{i=1}^{g} \mathbb{S}^1 \leftarrow \Sigma_g \rightarrow \bigvee_{i=1}^{g} \mathbb{S}^1\right].$$

This gives the simplicial group model $\mathbb{G}(M) \cong \operatorname{hocolim}[\mathbb{F}_g \xleftarrow{\pi} \mathbb{G}(\Sigma_g) \xrightarrow{\gamma} \mathbb{F}_g]$, and hence

$$\mathcal{O}[\mathrm{DRep}_G(M)] \cong \operatorname{hocolim}[\mathcal{O}(G^g) \xleftarrow{\pi_*} \mathcal{A}_*(\Sigma_g, G) \xrightarrow{\gamma_*} \mathcal{O}(G^g)] \cong \mathcal{O}(G^g) \otimes^{\boldsymbol{L}}_{\mathcal{A}_*} \mathcal{O}(G^g),$$

where $\mathcal{A}_* = \mathcal{A}_*(\Sigma_g, G)$ is an explicit DG algebra model for the representation homology $\mathrm{HR}_*(\Sigma_g, G)$ (see Section 6.1.2). As a result, we have the following generalization of Theorem 6.3 to 3-manifolds of higher genus.

Theorem 6.4. Let M be a closed connected orientable 3-manifold. Assume that M has a Heegaard decomposition (6.31) determined by an element $\gamma \in \mathcal{M}(\Sigma_g)$ in the mapping class group of Σ_g. Then the representation homology of M is given by

$$\mathrm{HR}_*(M,G) \cong \mathrm{Tor}_*^{\mathcal{A}_*}(\mathcal{O}(G^g)_\pi, \mathcal{O}(G^g)_\gamma),$$

where $\mathrm{Tor}_*^{\mathcal{A}_*}$ is the differential Tor taken over the DG algebra $\mathcal{A}_* = \mathcal{A}_*(\Sigma_g, G)$. In particular, there is an Eilenberg–Moore homology spectral sequence

$$E^2_{*,*} = \mathrm{Tor}_*^{\mathrm{HR}_*(\Sigma_g,G)}(\mathcal{O}(G^g)_\pi, \mathcal{O}(G^g)_\gamma) \implies \mathrm{HR}_*(M,G)$$

converging to the representation homology of M.

Remark. If G is a complex *semisimple* group and $g \geq 2$, it is conjectured in [9] (cf. [9, Conjecture 1.3]) that $\mathrm{HR}_i(\Sigma_g, G) = 0$ for all $i > 0$. This conjecture implies, in particular, that the spectral sequence of Theorem 6.4 degenerates for 3-manifolds of Heegaard genus $g \geq 2$, giving an isomorphism

$$\mathrm{HR}_*(M,G) \cong \mathrm{Tor}_*^{\mathcal{A}_G(\Sigma_g)}(\mathcal{O}(G^g)_\pi, \mathcal{O}(G^g)_\gamma),$$

where Tor_* is the ordinary 'Tor' taken over $\mathcal{A}_G(\Sigma_g) := \mathcal{O}[\mathrm{Rep}_G(\Sigma_g)]$, the coordinate ring of the classical representation scheme $\mathrm{Rep}_G(\Sigma_g)$.

7 Representation Cohomology and a Non-abelian Dennis Trace Map

In this section, we define representation homology and cohomology with coefficients in an arbitrary bifunctor on the category of finitely generated free groups $\mathfrak{G}$. Following the analogy with topological Hochschild homology, we construct a natural trace map relating representation homology to the stable homology of automorphism groups $\mathrm{Aut}(\mathbb{F}_n)$ with twisted coefficients.

7.1 Representation cohomology

7.1.1 *(Co)Homology of small categories*

Let $\mathscr{C}$ be a small category. By a $\mathscr{C}$-*bimodule*, we mean a bifunctor $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Vect}_k$, which is contravariant in the 1st argument and covariant in the 2nd. We write $\mathtt{Bimod}(\mathscr{C})$ for the category of $\mathscr{C}$-bimodules. For any $D \in \mathtt{Bimod}(\mathscr{C})$, one can define the (Hochschild–Mitchell) homology $\mathrm{HH}_*(\mathscr{C}, D)$ and cohomology $\mathrm{HH}^*(\mathscr{C}, D)$ of $\mathscr{C}$ with coefficients in D.

For a precise definition and basic properties of these classical (co)homology theories we refer to [3, 34, 57] (a good summary can also be found in [50, Appendix C]). Here, we only recall that $\mathrm{HH}_*(\mathscr{C}, -)$ and $\mathrm{HH}^*(\mathscr{C}, -)$ are functors (covariant and contravariant, respectively) on the category of $\mathscr{C}$-bimodules, such that $\{\mathrm{HH}_n(\mathscr{C}, -)\}_{n\geq 0}$ and $\{\mathrm{HH}^n(\mathscr{C}, -)\}_{n\geq 0}$ are universal δ-sequences, with $\mathrm{HH}_0(\mathscr{C}, D)$ and $\mathrm{HH}^0(\mathscr{C}, D)$ being canonically isomorphic to the coend $\int^{c\in\mathscr{C}} D(c,c)$ and the end $\int_{c\in\mathscr{C}} D(c,c)$ of the bifunctor D. Moreover, the (co)homology theories $\mathrm{HH}_*(\mathscr{C}, D)$ and $\mathrm{HH}^*(\mathscr{C}, D)$ have good functorial properties with respect to the 1st argument: in particular, any functor $F : \mathscr{C}' \to \mathscr{C}$ between small categories induces a natural map on homology $F_* : \mathrm{HH}_*(\mathscr{C}', F^*D) \to \mathrm{HH}_*(\mathscr{C}, D)$, where $F^* : \mathtt{Bimod}(\mathscr{C}) \to \mathtt{Bimod}(\mathscr{C}')$ is the restriction functor on bimodules defined by $F^*D := D \circ (F^{\mathrm{op}} \times F)$.

7.1.2 *Representation cohomology*

To express representation homology in terms of Hochschild–Mitchell homology, we need to slightly extend the above classical setting. Specifically, we will consider chain complexes of $\mathscr{C}$-bimodules, which are simply bifunctors $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$ with values in the category of chain complexes of k-vector spaces, and define $\mathbb{HH}_*(\mathscr{C}, D)$ and $\mathbb{HH}^*(\mathscr{C}, D)$ to be the Hochschild–Mitchell *hyperhomology* and the Hochschild–Mitchell *hypercohomology* of D, respectively. Now, given two chain complexes of right and left $\mathscr{C}$-modules, say $M : \mathscr{C}^{\mathrm{op}} \to \mathtt{Ch}_{\geq 0}(k)$ and $N : \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$, we define the chain complex of $\mathscr{C}$-bimodules $M \boxtimes_k N : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$ by assigning to $(c, c') \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C})$ the tensor product $M(c) \otimes_k N(c')$ of the corresponding chain complexes. With this notation, we have the following.

Lemma 7.1. For any $X \in \mathtt{sSet}_0$ and any commutative Hopf algebra $\mathcal{H}$, there is a natural isomorphism

$$\mathrm{HR}_*(X, \mathcal{H}) \cong \mathbb{HH}_*(\mathfrak{G}, \underline{N}(k[\mathbb{G}X]) \boxtimes_k \underline{\mathcal{H}}). \tag{7.1}$$

Proof. For any small category $\mathscr{C}$ and any right (resp., left) $\mathscr{C}$-modules M and N with values in $\mathtt{Ch}_{\geq 0}(k)$, where k is a commutative ring, there is a natural (Grothendieck) spectral sequence (see, e.g., [50, (C.10.1)]):

$$E^2_{pq} = \mathbb{HH}_p(\mathscr{C}, \mathrm{H}_q[M \boxtimes^{\boldsymbol{L}}_k N]) \implies \mathrm{H}_{p+q}[M \otimes^{\boldsymbol{L}}_{\mathscr{C}} N].$$

When k is a field, this spectral sequence degenerates giving an isomorphism $\mathbb{HH}_*(\mathscr{C}, M \boxtimes_k N) \cong \mathrm{H}_*[M \otimes^{\boldsymbol{L}}_{\mathscr{C}} N]$. In our situation, we have

$$\mathbb{HH}_*(\mathfrak{G}, \underline{N}(k[\mathbb{G}X]) \boxtimes_k \underline{\mathcal{H}}) \;\cong\; \mathrm{H}_*[\underline{N}(k[\mathbb{G}X]) \otimes^{\boldsymbol{L}}_{\mathfrak{G}} \underline{\mathcal{H}}],$$

which in composition with the isomorphism of Theorem 4.1 gives (7.1). ■

Example 7.1. In the case when $X = \mathrm{B}\Gamma$ for a discrete group Γ and $\mathcal{H} = \mathcal{O}(G)$, formula (7.1) reads

$$\mathrm{HR}_*(\Gamma, G) \;\cong\; \mathrm{HH}_*(\mathfrak{G}, k[\Gamma] \boxtimes_k \mathcal{O}(G)).$$

Lemma 7.1 motivates the following definition.

Definition 7.1. The *representation cohomology* of X in $\mathcal{H}$ is defined by

$$\mathrm{HR}^*(X, \mathcal{H}) \;:=\; \mathbb{HH}^*(\mathfrak{G}, \underline{N}(k[\mathbb{G}X]) \boxtimes_k \underline{\mathcal{H}}).$$

More generally, for any $\mathfrak{G}$-bimodule $D : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathtt{Ch}_{\geq 0}(k)$, we define the *representation homology* and the *representation cohomology* of D by

$$\mathrm{HR}_*(D) := \mathbb{HH}_*(\mathfrak{G}, D) \qquad \mathrm{HR}^*(D) := \mathbb{HH}^*(\mathfrak{G}, D).$$

In the case when D is an ordinary $\mathfrak{G}$-bimodule (with values in $\mathtt{Vect}_k$), this definition says that the representation (co)homology of D is just the classical Hochschild–Mitchell (co)homology of D.

Example 7.2. For an affine algebraic group G, consider the $\mathfrak{G}$-bimodule $D := \mathrm{lin}^*_k \boxtimes \mathcal{O}(G)$, where lin^*_k is the dual linearization functor $\mathfrak{G}^{\mathrm{op}} \to \mathtt{Vect}_k$, $\langle n \rangle \mapsto \mathrm{Hom}_{\mathbb{Z}}(\langle n \rangle_{\mathrm{ab}}, k)$. In this case, one can show that there are natural isomorphisms

$$\mathrm{HR}^i(D) \cong \mathrm{H}^{i+1}(G, k) \quad \forall\, i > 0,$$

where $\mathrm{H}^{i+1}(G, k)$ stands for the classical cohomology of the affine algebraic group with coefficients in the trivial (rational) representation.

7.1.3 *Relation to topological Hochschild homology*

For an arbtirary (associative unital) ring R, denote by $F(R)$ the full subcategory of R-$\mathtt{Mod}$ whose objects are the free modules $R^n,\ n \geq 0$. For any R-bimodule N, consider the bifunctor $\mathrm{Hom}(I, N) : F(R)^{\mathrm{op}} \times F(R) \to \mathtt{Mod}(\mathbb{Z})$ defined by $(X, Y) \mapsto \mathrm{Hom}_R(X, N \otimes_R Y))$. Then, a theorem of Pirashvili and Waldhausen [62] asserts that the Hochschild–Mitchell homology $\mathrm{HH}_*(F(R), \mathrm{Hom}(I, N))$ is naturally isomorphic to the *topological Hochschild homology* $\mathrm{THH}_*(R, N)$ of the ring R with coefficients in the bimodule N. It is therefore natural to *define* the topological Hochschild homology of R with coefficients in an arbitrary bifunctor $B : F(R)^{\mathrm{op}} \times F(R) \to \mathtt{Mod}(\mathbb{Z})$ by (cf. [50, Chap. 13])

$$\mathrm{THH}_*(R, B) := \mathrm{HH}_*(F(R), B).$$

For $R = \mathbb{Z}$, the category $F(\mathbb{Z})$ is equivalent to the category $\mathfrak{G}_{\mathrm{ab}}$ of finitely generated free abelian groups, which (as our notation suggests) is the abelianization of the category $\mathfrak{G}$. The abelianization functor $\alpha : \mathfrak{G} \to \mathfrak{G}_{\mathrm{ab}}$ induces a natural map $\mathrm{HR}_*(\alpha^* B) \to \mathrm{THH}_*(\mathbb{Z}, B)$ for any $\mathfrak{G}_{\mathrm{ab}}$-bimodule $B \in \mathtt{Bimod}(\mathfrak{G}_{\mathrm{ab}})$, and conversely, for any $\mathfrak{G}$-bimodule $D \in \mathtt{Bimod}(\mathfrak{G})$, associated to the functor α, there is an André-type spectral sequence (see [34, Theorem 1.20]):

$$E^2_{pq} = \mathrm{THH}_p(\mathbb{Z}, \boldsymbol{L}_q(\alpha^{\mathrm{op}} \times \alpha)_* D) \implies \mathrm{HR}_{p+q}(D),$$

converging to the representation homology of D.

Thus, representation homology may be viewed as a non-abelian analogue of topological Hochschild homology, and it is natural to ask for "non-abelian" analogues of various constructions known for topological Hochschild homology. In the next section, we outline one such construction that may be thought of as a non-abelian version of the Dennis trace map.

7.2 Non-abelian Dennis trace map

Recall (cf. [50, Sect. 13.1.8]) that the classical Dennis trace maps the stable homology of the general linear groups of a ring R to topological Hochschild homology of R:

$$\mathrm{DTr}_\infty(R, B) : \mathrm{H}_*(\mathrm{GL}_\infty(R), B_\infty) \to \mathrm{THH}_*(R, B), \tag{7.2}$$

where B is an arbitrary bimodule over $F(R)$. We generalize this map to the non-abelian setting.

Let $\mathrm{Aut}_n := \mathrm{Aut}(\mathbb{F}_n)$ denote the automorphism group of the free group on generators $x_1, \ldots, x_n$. We will regard Aut_n as the automorphism group $\mathrm{Aut}_{\mathfrak{G}}(\langle n\rangle)$ of the object $\langle n\rangle$ in the category $\mathfrak{G}$. There are obvious inclusions $\mathrm{Aut}_n \hookrightarrow \mathrm{Aut}_{n+1}$ defined by $g \mapsto \tilde{g}$, where $\tilde{g}(x_i) := g(x_i)$ for $i \le n$ and $\tilde{g}(x_{n+1}) = x_{n+1}$. We set $\mathrm{Aut}_\infty := \varinjlim \mathrm{Aut}_n$.

Now, consider an arbitrary bimodule D on the category $\mathfrak{G}$, that is, a bifunctor $D : \mathfrak{G}^{\mathrm{op}} \times \mathfrak{G} \to \mathtt{Vect}_k$. For each $n \ge 1$, let $D_n := D(\langle n\rangle, \langle n\rangle)$ and define the linear maps

$$p^* \circ i_* : D_n \to D(\langle n\rangle, \langle n+1\rangle) \to D_{n+1}, \tag{7.3}$$

where $i_* := D(\mathrm{Id}, i_n)$ and $p^* := D(p_n, \mathrm{Id})$ are induced by the natural inclusion $i : \langle n\rangle \hookrightarrow \langle n+1\rangle$ and the natural projection $p : \langle n+1\rangle \twoheadrightarrow \langle n\rangle$, respectively. Put

$$D_\infty := \varinjlim D_n,$$

where the inductive limit is taken with respect to the linear maps (7.3).

Next, observe that each D_n carries a natural Aut_n-module structure: namely, $\mathrm{Aut}_n \to \mathrm{Aut}(D_n)$, $g \mapsto g^* \circ g_*$, where $g^* := D(g^{-1}, \mathrm{Id})$ and $g_* := D(\mathrm{Id}, g)$. Moreover, for all $g \in \mathrm{Aut}_n$, there is a commutative diagram

$$\begin{tikzcd} D_n \arrow[r, "g^* g_*"] \arrow[d, "p^* i_*"'] & D_n \arrow[d, "p^* i_*"] \\ D_{n+1} \arrow[r, "\tilde{g}^* \tilde{g}_*"] & D_{n+1} \end{tikzcd}$$

where $\tilde{g} \in \mathrm{Aut}_{n+1}$ is the image of g under the natural inclusion $\mathrm{Aut}_n \hookrightarrow \mathrm{Aut}_{n+1}$ defined above. As a consequence, the k-vector space D_∞ carries a natural (inductive) Aut_∞-module structure. Thus, we can consider the homology groups $\mathrm{H}_*(\mathrm{Aut}_n, D_n)$ for all $n \ge 1$ and $\mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)$. Since homology commutes with direct limits, we can identify

$$\mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty) \cong \varinjlim \mathrm{H}_*(\mathrm{Aut}_n, D_n). \tag{7.4}$$

Next, we construct natural maps relating $\mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)$ to the representation homology $\mathrm{HR}_*(D)$. Regarding each group Aut_n as the category $\underline{\mathrm{Aut}}_n$ with a single object, we consider the inclusion functors

$$\gamma_n : \underline{\mathrm{Aut}}_n \to \mathfrak{G},$$

identifying the single object of $\underline{\mathrm{Aut}}_n$ with $\langle n\rangle \in \mathrm{Ob}(\mathfrak{G})$. Since $D_n = \gamma_n^* D$, these functors induce natural maps

$$(\gamma_n)_* : \mathrm{HH}_*(\underline{\mathrm{Aut}}_n, D_n) \to \mathrm{HH}_*(\mathfrak{G}, D) =: \mathrm{HR}_*(D). \tag{7.5}$$

On the other hand, the Hochschild–Mitchell homology of the category $\underline{\mathrm{Aut}}_n$ coincides with the usual group homology of Aut_n:

$$\mathrm{HH}_*(\underline{\mathrm{Aut}}_n, D_n) \cong \mathrm{H}_*(\mathrm{Aut}_n, D_n). \tag{7.6}$$

Indeed, since $\underline{\mathrm{Aut}}_n$ is a category with one object, its Hochschild–Mitchell complex $C_*^{\mathrm{HM}}(\underline{\mathrm{Aut}}_n, D_n)$ is isomorphic to the usual Hochschild complex $C_*(k[\mathrm{Aut}_n], D_n)$ of the group algebra of Aut_n, so that

$$\mathrm{HH}_*(\underline{\mathrm{Aut}}_n, D_n) \cong \mathrm{HH}_*(k[\mathrm{Aut}_n], D_n),$$

while $\mathrm{HH}_*(k[\mathrm{Aut}_n], D_n) \cong \mathrm{H}_*(\mathrm{Aut}_n, D_n)$ via the classical Mac Lane isomorphism (see, e.g., [50, Prop. 7.4.2]). Thus, combining (7.5) and (7.6), for all $n \geq 0$, we get canonical linear maps

$$\mathrm{DTr}_n^{\mathfrak{G}}(D) : \ \mathrm{H}_*(\mathrm{Aut}_n, D_n) \to \mathrm{HR}_*(D). \tag{7.7}$$

As in [50, 13.1.8], it is easy to check that these maps are compatible when passing from n to $n+1$. Hence, we can stabilize (7.7) by passing to the inductive limit as $n \to \infty$. With identification (7.4), the resulting stable map reads

$$\mathrm{DTr}_\infty^{\mathfrak{G}}(D) : \ \mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty) \ \to \ \mathrm{HR}_*(D). \tag{7.8}$$

This is a non-abelian analogue of the Dennis trace map (7.2). As in the classical case, it is natural to ask: When is (7.8) an isomorphism? Motivated by a theorem of Scorichenko (see [28]), we propose a conjectural answer.

Conjecture 2. *The map (7.8) is an isomorphism if D is a polynomial bifunctor (in the sense of [41]).*

We conclude this section a few remarks related to Conjecture 2.

Remark. A famous theorem of Galatius [32] asserts that natural maps from the symmetric group S_n to Aut_n (defined by permuting the generators) induce isomorphisms

$$\mathrm{H}_i(\mathrm{Aut}_n, \mathbb{Z}) \cong \mathrm{H}_i(S_n, \mathbb{Z}) \quad \forall n > 2i + 1.$$

This implies that $\mathrm{H}_i(\mathrm{Aut}_\infty, A) = 0$ for all $i > 0$, where A is any constant k-module provided k has characteristic 0 (which we always assume in this paper). Conjecture 2 implies [22, Theorem 1], which says that $\mathrm{H}_i(\mathrm{Aut}_\infty, D_\infty) = 0$ for all $i > 0$, when D is a polynomial bifunctor, *constant* with respect to its contravariant argument. Indeed, for such bifunctors, we have $\mathrm{HR}_i(D) = \mathrm{HH}_i(\mathfrak{G}, D) = 0$ for $i > 0$ because $\mathfrak{G}$ has a terminal object.

Remark. The direct analogue of Conjecture 2 is false in the abelian case. Indeed, if B is a constant bifunctor on $F(R)$, then $\mathrm{THH}_*(R, B)$ vanishes in positive degrees (since $F(R)$ has terminal object), but $\mathrm{H}_*(\mathrm{GL}_\infty(R), B)$ may be highly nontrivial (see [28]). The correct version of Conjecture 2 replaces the stable group homology with Waldhausen's stable K-theory. In the non-abelian case, one can also state a version of Conjecture 2 for the stable K-theory of automorphism groups Aut_n instead of group homology; however, we expect that the two theories are actually isomorphic. We briefly outline an argument behind this expectation.

Let E_∞ denote the commutator subgroup of Aut_∞. It is known that E_∞ is a perfect normal subgroup; hence, we can form the "plus construction"

$$\Psi : B\mathrm{Aut}_\infty \to B\mathrm{Aut}_\infty^+.$$

Let $F\Psi$ denote the homotopy fiber of the map Ψ. We have a canonical group homomorphism $\pi_1(F\Psi) \to \pi_1(B\mathrm{Aut}_\infty) \cong \mathrm{Aut}_\infty$ that equips any Aut_∞-module with a $\pi_1(F\Psi)$-action. In particular, the Aut_∞-module D_∞ arising from a $\mathfrak{G}$-bimodule D may be viewed as a $\pi_1(F\Psi)$-module, and hence defines a local system on $F\Psi$. The *stable K-theory* $K_*^s(\mathrm{Aut}_\infty, D_\infty)$ is then defined to be $\mathrm{H}_*(F\Psi, D_\infty)$, the homology of $F\Psi$ with coefficients in the local system D_∞. Now, consider the Serre spectral sequence associated to the homotopy fibration $F\Psi \to B\mathrm{Aut}_\infty \to B\mathrm{Aut}_\infty^+$:

$$E^2_{pq} = \mathrm{H}_p(B\mathrm{Aut}_\infty^+, \mathrm{H}_q(F\Psi, D_\infty)) \implies \mathrm{H}_n(B\mathrm{Aut}_\infty, D_\infty).$$

If Aut_∞ acts *trivially* on $K_q^s(\mathrm{Aut}_\infty, D_\infty) = \mathrm{H}_q(F\Psi, D_\infty)$ (as it happens in the classical case, see [50, 13.3.2]), then, since $B\mathrm{Aut}_\infty \to B\mathrm{Aut}_\infty^+$ is a homology equivalence for trivial coefficients, the above spectral sequence becomes

$$E^2_{pq} = \mathrm{H}_p(B\mathrm{Aut}_\infty, K^s_q(\mathrm{Aut}_\infty, D_\infty)) \implies \mathrm{H}_n(B\mathrm{Aut}_\infty, D_\infty).$$

However, by Galatius' theorem [32], we know that $\mathrm{H}_p(\mathrm{Aut}_\infty, A) = 0$ for $p > 0$ for any constant coefficients over k. Hence, the above spectral sequence must collapse on the p-axis, giving the desired isomorphism $K^s_*(\mathrm{Aut}_\infty, D_\infty) \cong \mathrm{H}_*(\mathrm{Aut}_\infty, D_\infty)$.

Remark. As explained in Section 7.1.3, the relation between topological Hochschild homology and functor homology of module categories is based on the Pirashvili–Waldhausen theorem [62]. Schwede [69] generalized this result to arbitrary algebraic theories by associating to an algebraic PROP $\mathfrak{P}$ a ring spectrum $\mathfrak{P}^s$ and identifying the functor homology over $\mathfrak{P}$ with topological Hochschild homology over $\mathfrak{P}^s$ (see [69, Theorem 6.7]). In the case $\mathfrak{P} = \mathfrak{G}$, Schwede's construction provides a topological (spectral) interpretation of representation homology that may be useful for Conjecture 2.

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A Model Approximations and Derived Adjunctions

In this appendix, we collect basic definitions and prove some results in abstract homotopy theory concerning derived functors. We work in the framework of homotopical categories in the sense of Dwyer, Hirschhorn, Kan, and Smith [23]. Apart from the original reference [23], a good introduction to the subject can be found in [66] and a short summary in [72]. The main results of this appendix—Theorem A.2 and Theorem A.3—arise from our attempt to abstract Theorem 3.1 on derived representation adjunctions. We believe that these two theorems as well as Lemma A.1 are of independent interest.

A.1 *Homotopical categories*

A *homotopical category* is a category $\mathcal{C}$ equipped with a class of morphisms $\mathcal{W}$ (called weak equivalences) that contains all identities of $\mathcal{C}$ and satisfies the following *2-of-6 property*: for every composable triple of morphisms $f, g, h \in \mathrm{Mor}(\mathcal{C})$, if gf and hg are in $\mathcal{W}$, then so are f, g, h, and hgf. The 2-of-6 property formally implies, but is stronger than, the usual 2-of-3 property. The class of weak equivalences thus forms a subcategory that contains all objects and all isomorphisms of $\mathcal{C}$. Since the isomorphisms satisfy the 2-of-6 property, any category can be viewed as a homotopical category by taking $\mathcal{W}$ to be the class of all isomorphisms (in [23], such homotopical categories called minimal). Furthermore, by forgetting the fibrations and cofibrations, any model category becomes a homotopical category: that is, the class of weak equivalences in any model category satisfies the 2-of-6 property (see [23, Prop. 9.2]). This is a consequence of the well-known fact that in a model category, the class $\mathcal{W}$ of weak equivalences is *saturated*: that is, it comprises all the arrows of $\mathcal{C}$ that become isomorphisms in the localized category $\mathcal{C}[\mathcal{W}^{-1}]$. Since the isomorphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ satisfy the 2-of-6 property, it follows immediately that the weak equivalences in a saturated category satisfy the 2-of-6 property. Unless stated otherwise, we will assume all our homotopical categories to be saturated. If $\mathcal{C}$ is a homotopical category, the category $\mathtt{Ho}(\mathcal{C}) := \mathcal{C}[\mathcal{W}^{-1}]$ is called the *homotopy category* of $\mathcal{C}$: it comes with the canonical functor $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathtt{Ho}(\mathcal{C})$ called the localization of $\mathcal{C}$. It is often convenient to regard $\mathtt{Ho}(\mathcal{C})$ as a homotopical category itself by taking $\mathcal{W}$ to be the class of isomorphisms; in other words, to think of $\mathtt{Ho}(\mathcal{C})$ as a minimal homotopical category.

A.2 *Derived functors and deformation retracts*

If $\mathcal{C}$ and $\mathcal{D}$ are homotopical categories, a functor $F : \mathcal{C} \to \mathcal{D}$ is called *homotopical* if it preserves weak equivalences. Such a functor induces a unique functor between the homotopy categories of $\mathcal{C}$ and $\mathcal{D}$ that we will denote by $\bar{F} : \mathtt{Ho}(\mathcal{C}) \to \mathtt{Ho}(\mathcal{D})$. In practice, many important functors are not homotopical and hence do not descend to homotopy categories. A standard way to deal with this problem is to replace—or “approximate”—non-homotopical functors with their derived functors that usually come in two kinds: “left” and “right”. We will focus on left derived functors with understanding that all results apply *mutatis mutandis* to the right derived functors as well.

Following [65], we define a *total left derived functor* $\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \to \mathtt{Ho}(\mathcal{D})$ of a functor $F : \mathcal{C} \to \mathcal{D}$ to be the right Kan extension of $\gamma_{\mathcal{D}} \circ F : \mathcal{C} \to \mathcal{D} \to \mathtt{Ho}(\mathcal{D})$ along localization $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathtt{Ho}(\mathcal{C})$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{C}}"'] \arrow[dr, phantom, "\Uparrow"] & \mathcal{D} \arrow[d, "\gamma_{\mathcal{D}}"] \\ \mathtt{Ho}(\mathcal{C}) \arrow[r, dashed, "\boldsymbol{L}F"'] & \mathtt{Ho}(\mathcal{D}). \end{tikzcd}$$

By the universal property of localization, $\boldsymbol{L}F$ is uniquely determined by the homotopical functor $\mathbb{L}F = \boldsymbol{L}F \circ \gamma_{\mathcal{C}}$ defined on the category $\mathcal{C}$. This last functor can be characterized as a universal homotopical functor $\mathbb{L}F : \mathcal{C} \to \mathtt{Ho}(\mathcal{D})$ that comes together with a natural transformation (called the comparison map) $\varepsilon : \mathbb{L}F \to \gamma_{\mathcal{D}} \circ F$ that is terminal among all natural transformations from homotopical functors to $\gamma_{\mathcal{D}} \circ F$. When they exist, both functors $\boldsymbol{L}F$ and $\mathbb{L}F$ are determined by F uniquely up to unique isomorphism. Following [72], we will refer to $\mathbb{L}F$ as a *left derived functor* of F and $\boldsymbol{L}F$ as the corresponding total left derived functor.

It was observed in [53] that a stronger universal property for derived functors—namely, that of an absolute Kan extension—is often very useful (in the additive setting, absolute derived functors between triangulated categories first appeared in the work of Deligne under the name "founcteurs dérivé partout défini" (see [21])). To be precise, a total left derived functor $\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \to \mathtt{Ho}(\mathcal{D})$ is called *absolute* if for any functor H : $\mathtt{Ho}(\mathcal{D}) \to \mathcal{E}$, the right Kan extension of the composition $H \circ \gamma_{\mathcal{D}} \circ F : \mathcal{C} \to \mathcal{D} \to \mathtt{Ho}(\mathcal{D}) \to \mathcal{E}$ along $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathtt{Ho}(\mathcal{C})$ coincides with $H \circ \boldsymbol{L}F$:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{C}}"'] \arrow[dr, phantom, "\Uparrow"] & \mathcal{D} \arrow[d, "\gamma_{\mathcal{D}}"] \\
\mathtt{Ho}(\mathcal{C}) \arrow[r, "\boldsymbol{L}F"'] \arrow[dr, "\mathrm{Ran}_{\gamma_{\mathcal{C}}}(H\gamma_{\mathcal{D}}F)"'] & \mathtt{Ho}(\mathcal{D}) \arrow[d, "H"] \\
& \mathcal{E}
\end{tikzcd}$$

A fundamental theorem of [65] asserts that any left Quillen functor $F : \mathcal{C} \to \mathcal{D}$ between model categories has a total left derived functor $\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \to \mathtt{Ho}(\mathcal{D})$, which can be obtained as the composition $F \circ Q$, where Q is the cofibrant replacement functor on $\mathcal{C}$; moreover, as noticed in [53], such a left derived functor is automatically absolute. This construction of derived functors was axiomatized and extended to homotopical categories in [23]. We briefly recall the main definitions. If $\mathcal{C}$ is a homotopical category, a *left deformation retract* of $\mathcal{C}$ is a full subcategory $i : \mathcal{C}_Q \hookrightarrow \mathcal{C}$ given together with a homotopical functor $Q : \mathcal{C} \to \mathcal{C}_Q$ and natural weak equivalence $q : i \circ Q \to \mathrm{Id}_{\mathcal{C}}$. It is easy to see that, for any left deformation retract of $\mathcal{C}$, the inclusion functor $i : \mathcal{C}_Q \hookrightarrow \mathcal{C}$ induces an equivalence of categories $\mathtt{Ho}(\mathcal{C}_Q) \simeq \mathtt{Ho}(\mathcal{C})$ with inverse induced by Q. Now, we say that a functor $F : \mathcal{C} \to \mathcal{D}$ between two homotopical categories is *left deformable* if there is a left deformation retract $\mathcal{C}_Q$ of the domain category such that the restriction of F to $\mathcal{C}_Q$ is homotopical. For example, if $\mathcal{C}$ and $\mathcal{D}$ are model categories, any left Quillen functor $F : \mathcal{C} \to \mathcal{D}$ is canonically left deformable: for the corresponding deformation retract $\mathcal{C}_Q$, we can always take the subcategory of cofibrant objects in $\mathcal{C}$, with $Q : \mathcal{C} \to \mathcal{C}_Q$ being the cofibrant replacement functor.

Proposition A.1 ([23]). A left deformable functor $F : \mathcal{C} \to \mathcal{D}$ has a left derived functor $\mathbb{L}F : \mathcal{C} \to \mathtt{Ho}(\mathcal{D})$ given by $\mathbb{L}F = \gamma_{\mathcal{D}} \circ F \circ Q$ with comparison map $\varepsilon = \gamma Fq : \mathbb{L}F \to \gamma \circ F$. The corresponding total left derived functor $\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \to \mathtt{Ho}(\mathcal{D})$ is absolute in the sense of [53].

The 1st statement of Proposition A.1 is proved in [23,Sections 41.2–5] (see also [66, Theorem 2.2.8]). The 2nd statement is verified in (the proof of) [66, Proposition 2.2.13].

It is well known that, for any composable pair (F_1, F_2) of left Quillen functors, the derived functor $\boldsymbol{L}(F_1 \circ F_2)$ of their composition coincides with $\boldsymbol{L}F_1 \circ \boldsymbol{L}F_2$. In the more general context of homotopical categories, this is not the case even when both functors F_1 and F_2 are left deformable. To guarantee this property one needs to impose an extra condition on deformation retracts of the functors involved. Following [23], we say that a composable pair (F_1, F_2) of left deformable functors $\mathcal{C} \xrightarrow{F_1} \mathcal{D} \xrightarrow{F_2} \mathcal{E}$ is *left deformable* if F_1 maps the left deformation retract $\mathcal{C}_Q$, on which it is homotopical, into the left deformation retract $\mathcal{D}_Q$, on which F_2 is homotopical: that is, $F_1(\mathcal{C}_Q) \subseteq \mathcal{D}_Q$. With this definition, we have

Proposition A.2 ([23, 42.4]). For any left deformable pair (F_1, F_2), there is a canonical isomorphism of total left derived functors $\boldsymbol{L}(F_1 \circ F_2) \cong \boldsymbol{L}F_1 \circ \boldsymbol{L}F_2$.

A.3 *Derived adjunctions*

We now turn to the important question when an adjunction between two homotopical categories induces a derived adjunction between the corresponding homotopy categories. We begin by stating the main result of [53] (cf. [66, 2.2.15]).

Theorem A.1 ([53]). Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors between homotopical categories. Assume that F has a total left derived functor $\boldsymbol{L}F$, G has a total right derived functor $\boldsymbol{R}G$, and both derived functors are absolute. Then $\boldsymbol{L}F$ and $\boldsymbol{R}G$ are adjoint to each other:

$$\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \rightleftarrows \mathtt{Ho}(\mathcal{D}) : \boldsymbol{R}G. \tag{A.1}$$

Following [23], let us call an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *deformable* if F is left deformable and G is right deformable. As an immediate consequence of Theorem A.1 and Proposition A.1, we get the following.

Corollary A.1 ([23, 44.2]). If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a deformable adjunction, then both total derived functors $\boldsymbol{L}F$ and $\boldsymbol{R}G$ exist and form an adjoint pair (A.1).

This result is one of the key observations of [23], which, in particular, formally implies Quillen's adjunction theorem for model categories [65]. Unfortunately, the assumption that a pair of adjoint functors is deformable is rather restrictive and does not always hold in practice. In what follows we propose a different—somewhat roundabout—way to produce derived adjunctions using model approximations of homotopical categories.

We begin with the following simple lemma that can be viewed as a partial converse of Theorem A.1

Lemma A.1. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors between homotopical categories. Assume the following:

(1) F has an absolute total left derived functor $\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \to \mathtt{Ho}(\mathcal{D})$,

(2) $\boldsymbol{L}F$ has a right adjoint functor $\tilde{G} : \mathtt{Ho}(\mathcal{D}) \to \mathtt{Ho}(\mathcal{C})$.

Then $\tilde{G}$ is an absolute total right derived functor of G: that is, $\boldsymbol{R}G$ exists and $\boldsymbol{R}G \cong \tilde{G}$.

Proof. Let us spell out the universal mapping property of the absolute total right derived functor $\boldsymbol{R}G$: for any functors $E : \mathtt{Ho}(\mathcal{C}) \to \mathcal{E}$ and $H : \mathtt{Ho}(\mathcal{D}) \to \mathcal{E}$, there is a natural (in E and H) bijection:

$$\mathrm{Hom}(E \circ \boldsymbol{R}G,\, H) \;\cong\; \mathrm{Hom}(E \circ \gamma_{\mathcal{C}} \circ G,\, H \circ \gamma_{\mathcal{D}}), \tag{A.2}$$

where by Hom's we denote the sets of natural transformations between the corresponding functors. To prove the lemma it suffices to check that $\tilde{G}$ satisfies this property.

First, $\tilde{G}$ being right adjoint to $\boldsymbol{L}F$ implies that $\tilde{G}^* = (-) \circ \tilde{G}$ is left adjoint to $\boldsymbol{L}F^* = (-) \circ \boldsymbol{L}F$ on the functor category $\mathrm{Fun}(\mathtt{Ho}(\mathcal{C}), \mathcal{E})$, so that there is a natural bijection

$$\mathrm{Hom}(E \circ \tilde{G},\, H) \;\cong\; \mathrm{Hom}(E,\, H \circ \boldsymbol{L}F). \tag{A.3}$$

Second, the universal mapping property of $\boldsymbol{L}F$ being an absolute left derived functor of F gives

$$\mathrm{Hom}(E,\, H \circ \boldsymbol{L}F) \;\cong\; \mathrm{Hom}(E \circ \gamma_{\mathcal{C}},\, H \circ \gamma_{\mathcal{D}} \circ F). \tag{A.4}$$

Third, F being left adjoint to G implies that $F^* = (-) \circ F$ is right adjoint to $G^* = (-) \circ G$; hence,

$$\mathrm{Hom}(E \circ \gamma_{\mathcal{C}},\, H \circ \gamma_{\mathcal{D}} \circ F) \;\cong\; \mathrm{Hom}(E \circ \gamma_{\mathcal{C}} \circ G,\, H \circ \gamma_{\mathcal{D}}). \tag{A.5}$$

Combining now (A.3)–(A.5) and comparing the result with (A.2), we see that $\tilde{G}$ satisfies the same universal mapping property as $\boldsymbol{R}G$. Whence, $\boldsymbol{R}G = \tilde{G}$. ∎

Remark. There is a dual version of Lemma A.1: if the absolute total right derived functor $\boldsymbol{R}G$ for the right adjoint in the pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ exists and has a left adjoint $\tilde{F} : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$, then this left adjoint $\tilde{F}$ is the absolute total left derived functor of F.

A.4 *Model approximations*

Next we recall the notion of a model approximation introduced in [17]. This notion plays an important role in abstract homotopy theory allowing one to define homotopy colimits of arbitrary diagrams in model categories. We will use it, however, for a different purpose: to construct derived adjunctions between homotopical categories.

Definition A.1 ([17]). A *left model approximation* of a homotopical category $\mathcal{C}$ is a model category $\mathcal{M}$ given together with a pair of adjoint functors $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ such that

(1) r is homotopical, that is, $r(\mathcal{W}_{\mathcal{C}}) \subseteq \mathcal{W}_{\mathcal{M}}$;

(2) l is homotopical on cofibrant objects of $\mathcal{M}$;

(3) (l, r) is an 'almost Quillen equivalence' in the sense: for any $A \in \mathrm{Ob}(\mathcal{C})$ and any cofibrant $X \in \mathrm{Ob}(\mathcal{M})$, if $f : X \to r(A)$ is a weak equivalence in $\mathcal{M}$ then the adjoint map $f^{\#} : l(X) \to A$ is a weak equivalence in $\mathcal{C}$.

The intuition behind this definition is that—from the homotopy-theoretical point of view—being a model category or having a model approximation should not make much difference. Our Theorem A.2 below illustrates this principle in the case of derived adjunctions.

We will need one more definition (cf. [17, Def. 5.8]). If $F : \mathcal{C} \to \mathcal{D}$ is a functor between homotopical categories, we say that a left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ is *good* for F if the restriction $F \circ l : \mathcal{M} \to \mathcal{C} \to \mathcal{D}$ is homotopical on cofibrant objects of $\mathcal{M}$. In this case, it follows from property (3) of Definition A.1 that $Q_{\mathcal{C}} := l \circ Q \circ r : \mathcal{C} \to \mathcal{C}$ provides a left deformation for F, where Q is the cofibrant replacement functor on $\mathcal{M}$. Thus, if F admits a good left model approximation, then F is a left deformable functor and hence, by Proposition A.1, has an absolute total left derived functor $\boldsymbol{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$. This applies, in particular, to the functor $l : \mathcal{M} \to \mathcal{C}$ itself (since we can take the identity adjunction on $\mathcal{M}$ as a good model approximation for l). Now, since l is left deformable and r is homotopical, by Corollary A.1, the adjunction $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ induces the adjunction of derived functors

$$\boldsymbol{L}l : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \bar{r}. \tag{A.6}$$

The next lemma clarifies the properties of the derived functors (A.6); it is essentially a reformulation of [17, Proposition 5.5].

Lemma A.2. Let $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ be a left model approximation of a homotopical category $\mathcal{C}$. The functor $\bar{r} : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{M})$ induced by r is fully faithful, and the counit morphism $\boldsymbol{L}l \circ \bar{r} \xrightarrow{\sim} \mathrm{Id}_{\mathrm{Ho}(\mathcal{C})}$ associated with (A.6) is an isomorphism.

Proof. To simplify the notation we write $\bar{X} \in \mathrm{Ho}(\mathcal{C})$ for the image of $X \in \mathrm{Ob}(\mathcal{C})$ under the localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, and similarly for $\mathcal{M}$. We need to prove that, for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, the map

$$\bar{r}_{X,Y} : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(\bar{X},\, \bar{Y}) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(\overline{r(X)},\, \overline{r(Y)})$$

is bijective. For this, we will explicitly construct the inverse map.

Let $Q, R : \mathcal{M} \to \mathcal{M}$ denote the cofibrant and the fibrant replacement functors in $\mathcal{M}$, respectively. Since $\mathcal{M}$ is a model category, any morphism $\bar{f} : \overline{r(X)} \to \overline{r(Y)}$ in $\mathrm{Ho}(\mathcal{M})$ can be represented by a morphism $f : Qr(X) \to RQr(Y)$ in $\mathcal{M}$. Moreover, we have the following natural diagram in $\mathcal{C}$:

$$X \xleftarrow{\sim} lQr(X) \xrightarrow{l(f)} lRQr(Y) \xleftarrow{\sim} lQr(Y) \xrightarrow{\sim} Y. \tag{A.7}$$

The 1st and the last maps in (A.7) are the adjoints of the cofibrant resolutions $Qr(X) \overset{\sim}{\twoheadrightarrow} r(X)$ and $Qr(Y) \overset{\sim}{\twoheadrightarrow} r(Y)$ in $\mathcal{M}$; hence, by property (3) of Definition A.1, they are weak equivalences in $\mathcal{C}$. The 3rd map is obtained by applying the functor l to the fibrant resolution $RQr(Y) \xrightarrow{\sim} Qr(Y)$ of the (cofibrant) object $Qr(Y)$ in $\mathcal{M}$; hence, it is also a weak equivalence, by Definition A.1(2). Now, applying the localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ transforms the weak equivalences in (A.7) into isomorphisms, and by inverting these isomorphisms, we can define a (unique) morphism $\bar{\psi}_{X,Y}(\bar{f}) : \bar{X} \to \bar{Y}$ in $\mathrm{Ho}(\mathcal{C})$, which depends only on $\bar{f}$. It is straightforward to check that the map $\bar{\psi}_{X,Y}$ given by this construction is inverse to $\bar{r}_{X,Y}$. This proves the 1st claim of the lemma. The 2nd claim is equivalent to the 1st by abstract properties of adjunctions (see, e.g., [31, Prop. I.1.3]). ■

We are now in position to state the main result of this appendix.

Theorem A.2. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors between homotopical categories. Assume that $\mathcal{C}$ admits a left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ together with adjoint functors $\hat{F} : \mathcal{M} \rightleftarrows \mathcal{D} : \hat{G}$, such that

(*i*) $(\hat{F}, \hat{G})$ is a deformable adjunction,

(*ii*) $(\hat{F}, r)$ is a left deformable pair, and there is a natural weak equivalence,

$$\hat{F} \circ r \xrightarrow{\sim} F$$

(*iii*) $\mathrm{Im}(\boldsymbol{R}\hat{G}) \subseteq \mathrm{Im}(\bar{r})$.

Then F and G have total (left and right) derived functors given by

$$\boldsymbol{L}F = \boldsymbol{L}\hat{F} \circ \bar{r}, \quad \boldsymbol{R}G = \boldsymbol{L}l \circ \boldsymbol{R}\hat{G}. \tag{A.8}$$

The derived functors $\boldsymbol{L}F$ and $\boldsymbol{R}G$ are both absolute and adjoint to each other:

$$\boldsymbol{L}F : \mathtt{Ho}(\mathcal{C}) \rightleftarrows \mathtt{Ho}(\mathcal{D}) : \boldsymbol{R}G.$$

Proof. First, note that, by (*i*) and Proposition A.1, the derived functors $\mathbb{L}\hat{F}$ and $\mathbb{R}\hat{G}$ exist, and by Corollary A.1, the corresponding total derived functors $\boldsymbol{L}\hat{F}$ and $\boldsymbol{R}\hat{G}$ are adjoint to each other. By (*ii*), the functor $\mathbb{L}F := \mathbb{L}(\hat{F} \circ r)$ satisfies the universal property of a left derived functor of F provided we define the comparison map $\varepsilon : \mathbb{L}F \to \gamma \circ F$ to be the composition $\varepsilon := \gamma(\varphi) \circ \hat{\varepsilon}$, where $\varphi : \hat{F} \circ r \to F$ is the natural weak equivalence of (*ii*) and $\hat{\varepsilon}$ is the comparison map for the derived functor $\mathbb{L}(\hat{F} \circ r)$. By Proposition A.2, the total left derived functor $\boldsymbol{L}(\hat{F} \circ r)$ is absolute and isomorphic to $\boldsymbol{L}\hat{F} \circ \bar{r}$. Hence, $\boldsymbol{L}F = \boldsymbol{L}\hat{F} \circ \bar{r}$ is an absolute total left derived functor of F.

Now, by (*iii*), we can factor $\boldsymbol{R}\hat{G}$ as a composition: $\mathtt{Ho}(\mathcal{D}) \xrightarrow{\bar{G}_0} \bar{\mathcal{C}} \xhookrightarrow{\bar{i}} \mathtt{Ho}(\mathcal{M})$, where $\bar{\mathcal{C}} := \mathrm{Im}(\bar{r})$ denotes the essential image of $\bar{r}$ in $\mathtt{Ho}(\mathcal{M})$ and $\bar{i}$ is the inclusion functor. By Lemma A.2, we can also factor $\bar{r} = \bar{i} \circ \bar{r}_0$, where $\bar{r}_0 : \mathtt{Ho}(\mathcal{C}) \xrightarrow{\sim} \bar{\mathcal{C}}$ is an equivalence, with quasi-inverse $\bar{l}_0 := \boldsymbol{L}l \circ \bar{i} : \bar{\mathcal{C}} \to \mathtt{Ho}(\mathcal{C})$. Combining these two factorizations, we can write $\boldsymbol{L}l \circ \boldsymbol{R}\hat{G} = \boldsymbol{L}l \circ \bar{i} \circ \bar{G}_0 = \bar{l}_0 \circ \bar{G}_0$. Then, for any objects $X \in \mathtt{Ho}(\mathcal{C})$ and $A \in \mathtt{Ho}(\mathcal{D})$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathtt{Ho}(\mathcal{C})}(X, (\boldsymbol{L}l \circ \boldsymbol{R}\hat{G})(A)) &= \mathrm{Hom}_{\mathtt{Ho}(\mathcal{C})}(X, \bar{l}_0(\bar{G}_0(A))) \\
&\cong \mathrm{Hom}_{\bar{\mathcal{C}}}(\bar{r}_0(X), \bar{G}_0(A)) \\
&\cong \mathrm{Hom}_{\mathtt{Ho}(\mathcal{M})}(\bar{i}(\bar{r}_0(X)), \bar{i}(\bar{G}_0(A))) \\
&\cong \mathrm{Hom}_{\mathtt{Ho}(\mathcal{M})}(\bar{r}(X), \boldsymbol{R}\hat{G}(A)) \\
&\cong \mathrm{Hom}_{\mathtt{Ho}(\mathcal{D})}(\boldsymbol{L}\hat{F}(\bar{r}(X)), A) \\
&\cong \mathrm{Hom}_{\mathtt{Ho}(\mathcal{D})}(\boldsymbol{L}F(X), A).
\end{aligned}$$

This shows that $\boldsymbol{L}l \circ \boldsymbol{R}\hat{G}$ is right adjoint to $\boldsymbol{L}F$, which is an absolute left derived functor. Hence, by Lemma A.1, we conclude that $\boldsymbol{R}G$ exists and $\boldsymbol{R}G \cong \boldsymbol{L}l \circ \boldsymbol{R}\hat{G}$. ■

Remark. **1.** Under the assumptions of Theorem A.2, there is a natural isomorphism of functors

$$\boldsymbol{L}\hat{F} \cong \boldsymbol{L}F \circ \boldsymbol{L}l, \tag{A.9}$$

which is *a priori* a stronger condition than $\boldsymbol{L}F \cong \boldsymbol{L}\hat{F} \circ \bar{r}$. Indeed, by Theorem A.2, the functor $\boldsymbol{L}F$ has a right adjoint $\boldsymbol{R}G$ that can be written, using the notation introduced in the proof, as $\boldsymbol{R}G = \bar{l}_0 \circ \bar{G}_0$. Since $\bar{l}_0$ is an equivalence with quasi-inverse $\bar{r}_0$, this implies

$$\bar{r} \circ \boldsymbol{R}G = \bar{r} \circ \bar{l}_0 \circ \bar{G}_0 = \bar{i} \circ \bar{r}_0 \circ \bar{l}_0 \circ \bar{G}_0 \cong \bar{i} \circ \bar{G}_0 = \boldsymbol{R}\hat{G}.$$

Thus, we have an isomorphism of functors $\boldsymbol{R}\hat{G} \cong \bar{r} \circ \boldsymbol{R}G$, where each functor has a left adjoint. By adjunction, this gives (A.9).

2. The main assumption of Theorem A.2—namely, the condition that the adjunction $\hat{F} : \mathcal{M} \rightleftarrows \mathcal{D} : \hat{G}$ is defined on the whole model category $\mathcal{M}$—can be weakened. The proof shows that it suffices to assume that $\hat{F}$ exists on a full subcategory $\mathcal{M}'$ of $\mathcal{M}$, which is closed under the weak equivalences in $\mathcal{M}$ and whose image in $\mathtt{Ho}(\mathcal{M})$ contains $\mathrm{Im}(\bar{r})$.

A.5 *Homotopy colimits*

Recall that any adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ extends formally to an adjunction $F^I : \mathcal{C}^I \rightleftarrows \mathcal{D}^I : G^I$ of the diagram categories $\mathcal{C}^I := \mathrm{Fun}(I, \mathcal{C})$ and $\mathcal{D}^I := \mathrm{Fun}(I, \mathcal{D})$ for any small category I. The corresponding functors F^I and G^I are given by compositions $F^I(X) = F \circ X$ and $G^I(Y) = G \circ Y$, where $X \in \mathrm{Ob}(\mathcal{C}^I)$ and $Y \in \mathrm{Ob}(\mathcal{D}^I)$. If C is a homotopical category, the diagram category $\mathcal{C}^I$ has a natural homotopical structure in which a morphism of I-diagrams $\varphi : X \to X'$ is a weak equivalence if $\varphi_i : X(i) \xrightarrow{\sim} X'(i)$ is a weak equivalence in $\mathcal{C}$ for every object $i \in \mathrm{Ob}(I)$. Moreover, as observed in [23], if the functor $F : \mathcal{C} \to \mathcal{D}$ is left deformable, then so is $F^I : \mathcal{C}^I \to \mathcal{D}^I$: in fact, if $Q : \mathcal{C} \to \mathcal{C}_Q$ is a left deformation retract for F, then $Q^I : \mathcal{C}^I \to \mathcal{C}_Q^I$ is a left deformation retract for F^I. By Proposition A.1, this implies that for any left deformable functor $F : \mathcal{C} \to \mathcal{D}$, the functor $F^I : \mathcal{C}^I \to \mathcal{D}^I$ has an absolute total left derived functor $\boldsymbol{L}F^I : \mathtt{Ho}(\mathcal{C}^I) \to \mathtt{Ho}(\mathcal{D}^I)$ induced by $\mathbb{L}F^I = \gamma_{\mathcal{D}^I} F^I Q^I$. Informally speaking, the left derived functor of F^I is just the left derived functor of F applied objectwise.

Now, for a small category I, let $\mathrm{diag}_I^{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^I$ denote the diagonal functor that assigns to an object $A \in \mathrm{Ob}(\mathcal{C})$ the constant diagram $\mathrm{diag}_I^{\mathcal{C}}(A) : I \to \mathcal{C},\ i \mapsto A$. Recall that the colimit $\mathrm{colim}_I^{\mathcal{C}} : \mathcal{C}^I \to \mathcal{C}$ is the left adjoint functor of $\mathrm{diag}_I^{\mathcal{C}}$. If $\mathcal{C}$ is a homotopical category, we define the *homotopy colimit* $\mathbb{L}\mathrm{colim}_I^{\mathcal{C}} : \mathcal{C}^I \to \mathtt{Ho}(\mathcal{C})$ to be the left derived functor of $\mathrm{colim}_I^{\mathcal{C}}$, and following our convention, write $\boldsymbol{L}\mathrm{colim}_I^{\mathcal{C}} : \mathtt{Ho}(\mathcal{C}^I) \to \mathtt{Ho}(\mathcal{C})$ for the corresponding total left derived functor. By Proposition A.1, $\mathbb{L}\mathrm{colim}_I^{\mathcal{C}}$ exists if $\mathrm{colim}_I^{\mathcal{C}}$ exists and is left deformable; in that case, since the diagonal functor is homotopical, we have a deformable adjunction $\mathrm{colim}_I^{\mathcal{C}} : \mathcal{C}^I \rightleftarrows \mathcal{C} : \mathrm{diag}_I^{\mathcal{C}}$, and hence, by Corollary A.1, the derived adjunction

$$\boldsymbol{L}\mathrm{colim}_I^{\mathcal{C}} : \mathtt{Ho}(\mathcal{C}^I) \rightleftarrows \mathtt{Ho}(\mathcal{C}) : \overline{\mathrm{diag}}_I^{\mathcal{C}}.$$

After these preliminary remarks, we can state our 2nd main theorem.

Theorem A.3. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors satisfying the conditions of Theorem A.2. Assume, in addition, that for a small category I, the functors $\mathrm{colim}_I^{\mathcal{C}}$ and $\mathrm{colim}_I^{\mathcal{D}}$ exist and are left deformable. Then there is a natural isomorphism of functors

$$\boldsymbol{L}F \circ \boldsymbol{L}\mathrm{colim}_I^{\mathcal{C}} \cong \boldsymbol{L}\mathrm{colim}_I^{\mathcal{D}} \circ \boldsymbol{L}(F^I). \tag{A.10}$$

In other words, the functor $\boldsymbol{L}F$ preserves homotopy colimits.

Theorem A.3 follows readily from Theorem A.2 and the main results of [17] concerning homotopy colimits. For reader's convenience, we will summarize these results below, before proving Theorem A.3. We start with a simple lemma, which is probably well known to experts, but since we could not find a reference, we provide a quick proof.

Lemma A.3. Let $\hat{F} : \mathcal{M} \rightleftarrows \mathcal{D} : \hat{G}$ be a deformable adjunction between homotopical categories. Assume that, for a small category I, the functors $\mathrm{colim}_I^{\mathcal{M}}$ and $\mathrm{colim}_I^{\mathcal{D}}$ exist and are left deformable. Then there is a natural isomorphism

$$\boldsymbol{L}\hat{F} \circ \boldsymbol{L}\mathrm{colim}_I^{\mathcal{M}} \cong \boldsymbol{L}\mathrm{colim}_I^{\mathcal{D}} \circ \boldsymbol{L}(\hat{F}^I). \tag{A.11}$$

Proof. Since $\hat{G}$ is right deformable, so is $\hat{G}^I$, and there is a right deformation functor on $\mathcal{D}$, say $R : \mathcal{D} \to \mathcal{D}$, such that $\mathbb{R}\hat{G} = \gamma_{\mathcal{M}} \circ \hat{G} \circ R$ and $\mathbb{R}(\hat{G}^I) = \gamma_{\mathcal{M}^I} \circ \hat{G}^I \circ R^I$ are the right derived functors of $\hat{G}$ and $\hat{G}^I$, respectively. Now, since diag_I is homotopical, we have obvious isomorphisms:

$$\begin{aligned}
\overline{\mathrm{diag}}_I^{\mathcal{M}} \circ \mathbb{R}\hat{G} &= \overline{\mathrm{diag}}_I^{\mathcal{M}} \circ \gamma_{\mathcal{M}} \circ \hat{G} \circ R \\
&\cong \gamma_{\mathcal{M}^I} \circ \mathrm{diag}_I^{\mathcal{M}} \circ \hat{G} \circ R \\
&\cong \gamma_{\mathcal{M}^I} \circ \hat{G}^I \circ \mathrm{diag}_I^{\mathcal{D}} \circ R \\
&\cong \gamma_{\mathcal{M}^I} \circ \hat{G}^I \circ R^I \circ \mathrm{diag}_I^{\mathcal{D}} \\
&\cong \mathbb{R}(\hat{G}^I) \circ \mathrm{diag}_I^{\mathcal{D}},
\end{aligned}$$

which induce an isomorphism of the total right derived functors

$$\overline{\mathrm{diag}}_I^{\mathcal{M}} \circ \boldsymbol{R}\hat{G} \cong \boldsymbol{R}(\hat{G}^I) \circ \overline{\mathrm{diag}}_I^{\mathcal{D}}. \tag{A.12}$$

By lemma's assumptions, each functor in (A.12) has a left adjoint; hence, (A.12) implies (A.11). ■

Now, we briefly review the results of [17] needed for the proof of our Theorem A.3. We warn the reader that our notation differs from that of [17] but this should not cause confusion. For a small category I, we denote by ΔI the simplex category of I, that is, the category of simplices $\Delta \downarrow N(I)$ of the nerve of I, and write $\mathcal{M}_b^{\Delta I}$ for the full subcategory (In [17], the category $\mathcal{M}_b^{\Delta I}$ is denoted $Fun^b(\mathbf{N}(I), \mathcal{M})$.) of $\mathcal{M}^{\Delta I}$ consisting of *bounded* ΔI-diagrams in a category $\mathcal{M}$. Recall that a functor $X : \Delta I \to \mathcal{M}$ is *bounded* if it maps every degeneracy map $s^i : s^i\sigma \to \sigma$ in ΔI to an isomorphism in $\mathcal{M}$; thus, modulo isomorphisms, a bounded functor is determined by its values on nondegenerate simplices in ΔI. The simplex category comes together with a forgetful functor $\tau : \Delta I \to I$ that takes an n-simplex σ in ΔI, that is a chain $\sigma = (i_0 \leftarrow i_1 \leftarrow \ldots \leftarrow i_n)$ of n composable maps in I, to its target $\tau(\sigma) = i_0$. This forgetful functor yields the restriction functor $\tau^* : \mathcal{M}^I \to \mathcal{M}^{\Delta I}$ whose image is in $\mathcal{M}_b^{\Delta I}$ (in fact, it is easy to check that $\mathrm{Im}(\tau^*)$ consists of bounded functors $X : \Delta I \to M$ which, in addition to inverting all the degeneracy maps in ΔI, also invert all the boundary maps $d^i : d^i\sigma \to \sigma$ with $i > 0$). Now, if the category $\mathcal{M}$ is closed under colimits, the functor $\tau^* : \mathcal{M}^I \to \mathcal{M}_b^{\Delta I}$ has a left adjoint $\tau_* : \mathcal{M}_b^{\Delta I} \to \mathcal{M}^I$, which is given by restricting to $\mathcal{M}_b^{\Delta I}$ the left Kan extension $\mathrm{Lan}_\tau : \mathcal{M}^{\Delta I} \to \mathcal{M}^I$ taken along $\tau : \Delta I \to I$. In this way, for any cocomplete category $\mathcal{M}$, we get the adjunction

$$\tau_* : \mathcal{M}_b^{\Delta I} \rightleftarrows \mathcal{M}^I : \tau^*. \tag{A.13}$$

In $\mathcal{M}$ is a model category, (A.13) is called the *Bousfield-Kan approximation* of $\mathcal{M}^I$. More generally, in $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ is a left model approximation of a homotopical category $\mathcal{C}$, the composition of adjunctions

$$l^I \circ \tau_* : \mathcal{M}_b^{\Delta I} \rightleftarrows \mathcal{M}^I \rightleftarrows \mathcal{C}^I : \tau^* \circ r^I \tag{A.14}$$

is called the *Bousfield–Kan approximation* of $\mathcal{C}^I$. Now, the main results of [17] can be encapsulated into the following theorem.

Theorem A.4 ([17, Theorem 11.2 and Theorem 11.3]). Let I be a small category.

(1) For any model category $\mathcal{M}$, the category $\mathcal{M}_b^{\Delta I}$ has a model structure, where the weak equivalences (resp., fibrations) are the objectwise weak equivalences (resp., fibrations) of bounded ΔI-diagrams in $\mathcal{M}$.

(2) For any left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$, the Bousfield–Kan approximation (A.14) is a left model approximation of $\mathcal{C}^I$. In particular, (A.13) is a left model approximation of $\mathcal{M}^I$.

(3) If $\mathcal{C}$ is closed under colimits and admits a left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$, the corresponding Bousfield–Kan approximation (A.14) is good for $\mathrm{colim}_I^{\mathcal{C}} : \mathcal{C}^I \to \mathcal{C}$.

In particular, the functor $\mathrm{colim}_I^{\mathcal{C}}$ is left deformable and its left derived functor (the homotopy colimit) $\mathbb{L}\mathrm{colim}_I^{\mathcal{C}}$ exists.

We now explain how to construct homotopy colimits using the Bousfield–Kan model structure on $\mathcal{M}_b^{\Delta I}$. To this end, we need another important observation of [17] that, for any model category $\mathcal{M}$, the functor $\mathrm{colim}_{\Delta I}^{\mathcal{M}} : \mathcal{M}_b^{\Delta I} \to \mathcal{M}$, obtained by restricting the usual colimit to bounded diagrams, is homotopical on cofibrant objects in $\mathcal{M}_b^{\Delta I}$, and hence has a left derived functor (see [17, Cor. 13.4 and Prop. 14.2]). Following [17], we denote this derived functor by

$$\mathrm{ocolim}_{\Delta I}^{\mathcal{M}} : \mathcal{M}_b^{\Delta I} \to \mathrm{Ho}(\mathcal{M}). \tag{A.15}$$

(It is important to note that the functor $\mathrm{ocolim}_{\Delta I}^{\mathcal{M}}$ is *not* equivalent, in general, to the usual homotopy colimit $\mathbb{L}\mathrm{colim}_{\Delta I}^{\mathcal{M}}$ restricted to $\mathcal{M}_b^{\Delta I}$ (see [17,Remark 14.3]).) In terms of (A.15), the homotopy colimit functor on arbitrary I-diagrams $\mathbb{L}\mathrm{colim}_I^{\mathcal{M}} : \mathcal{M}^I \to \mathrm{Ho}(\mathcal{M})$ is given by

$$\mathbb{L}\mathrm{colim}_I^{\mathcal{M}} \cong \mathrm{ocolim}_{\Delta I}^{\mathcal{M}} \circ \tau^*, \tag{A.16}$$

where $\tau^* : \mathcal{M}^I \to \mathcal{M}_b^{\Delta I}$ is the restriction functor in the Bousfield–Kan approximation (A.13). More generally, for a left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$, the homotopy colimit $\mathbb{L}\mathrm{colim}_I^{\mathcal{C}} : \mathcal{C}^I \to \mathrm{Ho}(\mathcal{C})$ is given by the composition

$$\mathcal{C}^I \xrightarrow{r^I} \mathcal{M}^I \xrightarrow{\tau^*} \mathcal{M}_b^{\Delta I} \xrightarrow{\mathrm{ocolim}_{\Delta I}^{\mathcal{M}}} \mathrm{Ho}(\mathcal{M}) \xrightarrow{\boldsymbol{L}l} \mathrm{Ho}(\mathcal{C}),$$

that is

$$\mathbb{L}\mathrm{colim}_I^{\mathcal{C}} \cong \boldsymbol{L}l \circ \mathrm{ocolim}_{\Delta I}^{\mathcal{M}} \circ \tau^* \circ r^I. \tag{A.17}$$

Combining the isomorphisms (A.16) and (A.17) and passing to total derived functors, we arrive at the following result that we will use in the proof of Theorem A.3.

Corollary A.2. Assume that a homotopical category $\mathcal{C}$ admits a left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ and is closed under colimits. Then, for any small category I, $\boldsymbol{L}\mathrm{colim}_I^{\mathcal{C}}$ exists and

$$\boldsymbol{L}\mathrm{colim}_I^{\mathcal{C}} \cong \boldsymbol{L}l \circ \boldsymbol{L}\mathrm{colim}_I^{\mathcal{M}} \circ \bar{r}^I. \tag{A.18}$$

Finally, we turn to

Proof of Theorem A.3. By Lemma A.3, we have a natural isomorphism (A.11) that yields by restriction:

$$\boldsymbol{L}\hat{F} \circ \boldsymbol{L}\mathrm{colim}_I^{\mathcal{M}} \circ \bar{r}^I \cong \boldsymbol{L}\mathrm{colim}_I^{\mathcal{D}} \circ \boldsymbol{L}(\hat{F}^I) \circ \bar{r}^I. \tag{A.19}$$

Now, by Theorem A.2 (see (A.9)) and Corollary A.2, the composition of functors in the left-hand side of (A.19) is isomorphic to

$$\boldsymbol{L}F \circ \boldsymbol{L}l \circ \boldsymbol{L}\mathrm{colim}_I^{\mathcal{M}} \circ \bar{r}^I \cong \boldsymbol{L}F \circ \boldsymbol{L}\mathrm{colim}_I^{\mathcal{C}}.$$

On the other hand, by condition (*ii*) of Theorem A.2, the pair of functors $(\hat{F}^I, r^I)$ is left deformable, and there is a natural weak equivalence $\hat{F}^I \circ r^I = (\hat{F} \circ r)^I \xrightarrow{\sim} F^I$, inducing an isomorphism

$$\boldsymbol{L}(\hat{F}^I) \circ \bar{r}^I \cong \boldsymbol{L}(\hat{F}^I \circ r^I) \cong \boldsymbol{L}(F^I).$$

Hence, the right-hand side of (A.19) is isomorphic to $\boldsymbol{L}\mathrm{colim}_I^{\mathcal{D}} \circ \boldsymbol{L}(F^I)$. Combining (A.19) with these two isomorphisms gives (A.10). ■

Remark. The assumption of Theorem A.3 that $\mathrm{colim}_I^{\mathcal{C}}$ is a left deformable functor is superfluous. Indeed, thanks to Theorem A.4(3), it suffices only to assume the existence of $\mathrm{colim}_I^{\mathcal{C}}$.