# DYNAMICAL DEGREES OF HURWITZ CORRESPONDENCES

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ABSTRACT. Let $\phi$ be a post-critically finite branched covering of a two-sphere. By work of Koch, the Thurston pullback map induced by $\phi$ on Teichmüller space descends to a multi-valued self-map — a Hurwitz correspondence $\mathcal{H}_\phi$ — of the moduli space $\mathcal{M}_{0,\mathbf{P}}$. We study the dynamics of Hurwitz correspondences via numerical invariants called *dynamical degrees*. We show that the sequence of dynamical degrees of $\mathcal{H}_\phi$ is always non-increasing, and the behavior of this sequence is constrained by the behavior of $\phi$ at and near points of its post-critical set.

## 1. INTRODUCTION

Denote by $S^2$ the oriented two-sphere. Suppose $\phi : S^2 \to S^2$ is an orientation-preserving branched covering whose post-critical set

$$\mathbf{P} := \{\phi^n(x) |\ x \text{ is a critical point of } \phi \text{ and } n > 0\}$$

is finite. Then $\phi$ is called *post-critically finite*. The topological dynamics of $\phi$ induce holomorphic and algebraic dynamical systems:

(I) Thurston [DH93]: a holomorphic, contracting self-map $\mathrm{Th}_\phi : \mathcal{T}_{S^2,\mathbf{P}} \to \mathcal{T}_{S^2,\mathbf{P}}$ of the Teichmüller space of complex structures on $S^2$ punctured at $\mathbf{P}$. This is known as the *Thurston pullback map*, and it descends to

(II) Koch [Koc13]: an algebraic, multivalued self-map $\mathcal{H}_\phi : \mathcal{M}_{0,\mathbf{P}} \rightrightarrows \mathcal{M}_{0,\mathbf{P}}$ of the moduli space of markings of $\mathbb{CP}^1$ by $\mathbf{P}$. Such a multivalued map is called a *Hurwitz correspondence*.

In addition, if

(1) $\mathbf{P}$ contains a periodic and fully ramified point $p_0$ of $\phi$, and
(2) either every other critical point of $\phi$ is also periodic or there is exactly one other critical point of $\phi$,

then we also have

(III) Koch [Koc13]: a meromorphic, single-valued map $\mathcal{H}_\phi^{-1} : \mathcal{M}_{0,\mathbf{P}} \dashrightarrow \mathcal{M}_{0,\mathbf{P}}$.

The branched covering $\phi$ is conjugate, up to homotopy, to a post-critically finite rational map on $\mathbb{CP}^1$ if and only if $\mathrm{Th}_\phi$ has a fixed point. There is a tremendous amount of current research investigating the dynamics of (I). Koch introduced (II) and (III) as algebraic dynamical systems that 'shadow' the holomorphic dynamics of (I).

Dynamical degrees are numerical invariants associated to algebraic dynamical systems; they measure complexity of iteration. Let $X^\circ$ be a smooth quasiprojective variety and $g : X^\circ \dashrightarrow X^\circ$ (resp. $g : X^\circ \dashrightarrow\!\!\!\!\!\rightarrow X^\circ$) a meromorphic map (resp. a meromorphic multi-valued map). Fix a smooth projective birational model $X$ of

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$X^\circ$ and an ample class $\mathfrak{h} \in H^{1,1}(X)$. The $k$th dynamical degree of $g$ is defined to be the non-negative real number

$$\Theta_k(g) = \lim_{n\to\infty} \left(((g^n)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k})\right)^{1/n},$$

The above limit exists and is independent of $X$ and $\mathfrak{h}$; this was proved first by Dinh and Sibony in the complex setting [DS05, DS08] and later by Truong [Tru15, Tru16] in the algebraic setting. The $k$-th dynamical degree of $g$ measures the 'asymptotic growth rate of the degrees of codimension-$k$ subvarieties of $X^\circ$ under iterates of $g$' — amazingly, this is a well-defined notion although the degree of a subvariety of $X^\circ$ is not well-defined.

Now, since $\phi : \mathbf{P} \to \mathbf{P}$ is a self-map of a finite set, every point eventually maps into a periodic cycle. We define the *polynomiality index* of $\phi$ to be the positive real number

$$\mathrm{PI}(\phi) := \max_{\{p\in\mathbf{P}, \ell>0 | \phi^\ell(p)=p\}} \left(\Pi_{i=0}^{\ell-1}(\text{local degree of } \phi \text{ at } \phi^i(p))\right)^{1/\ell}.$$

In fact, $\mathrm{PI}(\phi)$ is the maximum, over *all* periodic cycles of $\phi$ on $S^2$, of the geometric mean of the local degrees of $\phi$ at all the points in the cycle.

**Theorem 3.1.** *For $k = 0, \ldots, |\mathbf{P}| - 4$, we have $\Theta_k(\mathcal{H}_\phi) \geq \mathrm{PI}(\phi) \cdot \Theta_{k+1}(\mathcal{H}_\phi)$.*

Thus the behavior of the sequence of dynamical degrees of $\mathcal{H}_\phi$ is constrained by the behavior of $\phi$ at and near points of $\mathbf{P}$. Note that $1 \leq \mathrm{PI}(\phi) \leq \deg(\phi)$ always holds. $\mathrm{PI}(\phi) = 1$ if and only if no critical point of $\phi$ is periodic, i.e. if every critical point is strictly pre-periodic. $\mathrm{PI}(\phi) = \deg(\phi)$ if and only if $\mathbf{P}$ contains a point $p_0$ that is fixed by and fully ramified under either $\phi$ or $\phi^2$, i.e. if and only if either $\phi$ or $\phi^2$ is a topological polynomial.

**Corollary 1.1.** *$\Theta_k(\mathcal{H}_\phi)$ decreases as $k$ increases, strictly if $\phi$ has a periodic critical point.*

In Section 5, we give an example of a branched covering $\phi$ for which every critical point is strictly pre-periodic so $\mathrm{PI}(\phi) = 1$, such that $\Theta_0(\mathcal{H}_\phi) = \Theta_1(\mathcal{H}_\phi)$. Thus when the polynomiality index equals one, the sequence of dynamical degrees may decrease only weakly.

For any $\phi$, the dynamical degrees of $\mathcal{H}_\phi$ are algebraic integers [Ram18]. As a parallel to Corollary 1.1, the results in [Ram18] show that for $k > 0$, the degree over $\mathbb{Q}$ of $\Theta_k$ 'likely' decreases as $k$ increases. More precisely, there is an upper bound for the degree over $\mathbb{Q}$ of $\Theta_k$ that decreases as $k$ increases. In spite of the parallel, the methods used in this paper are very different from those used in [Ram18].

1.1. **Implications when $\mathcal{H}_\phi^{-1}$ is single-valued.** Dynamical degrees have been studied primarily in the context of single-valued maps. The topological entropy of a holomorphic single-valued map was found to be equal to the logarithm of its largest dynamical degree (Yomdin [Yom87] and Gromov [Gro03]). The topological entropy of a meromorphic single-valued map is bounded from above by the logarithm of its largest dynamical degree (Dinh and Sibony [DS05]); equality is conjectured when there is a unique largest dynamical degree. If $g$ is a single-valued map, either holomorphic or meromorphic, its 0-th dynamical degree is 1 and its top dynamical degree is its topological degree. Guedj [Gue05] found that a map whose top dynamical degree is its largest has especially good ergodic properties. (See Corollary 1.3 for the implications in the context of this paper.)

If $g : \mathbb{CP}^N \to \mathbb{CP}^N$ is a holomorphic map given in coordinates by homogeneous polynomials of degree $d$, its $k$-th dynamical degree is $d^k$. Thus $k \mapsto \log(\Theta_k(g))$ is linear with slope $d$, and the top ($N$-th) dynamical degree of $g$ is its largest. If $g : \mathbb{CP}^N \dashrightarrow \mathbb{CP}^N$ or $g : X \dashrightarrow X$ is a meromorphic map, $k \mapsto \log(\Theta_k(g))$ is known to be concave. Thus the top dynamical degree of $g$ is its largest if and only if $k \mapsto \Theta_k(g)$ is strictly increasing.

Koch and Roeder [KR16] studied the dynamical degrees of $\mathcal{H}_\phi^{-1}$ in the special case that $\phi$ has exactly two critical points, both periodic. They showed in this case that $\Theta_k(\mathcal{H}_\phi^{-1})$ is the absolute value of the largest eigenvalue of the induced pullback action on $H^{k,k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$, where $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is the Deligne-Mumford compactification of $\mathcal{M}_{0,\mathbf{P}}$. Koch [Koc13] studied the maps $\mathcal{H}_\phi^{-1}$ as meromorphic self-maps of $\mathbb{CP}^{|\mathbf{P}|-3}$, another compactification of $\mathcal{M}_{0,\mathbf{P}}$. Koch found that if (1) and (2) hold, and, in addition, the special point $p_0$ is fixed by $\phi$, i.e. $\phi$ is a topological polynomial, then $\mathcal{H}_\phi^{-1} : \mathbb{CP}^{|\mathbf{P}|-3} \dashrightarrow \mathbb{CP}^{|\mathbf{P}|-3}$ is holomorphic, given in coordinates by homogeneous polynomials of degree equal to the topological degree of $\phi$. Thus if $\phi$ is a topological polynomial, $\Theta_k(\mathcal{H}_\phi) = \deg(\phi)^k$. Koch showed that in this case $\mathcal{H}_\phi^{-1} : \mathbb{CP}^{|\mathbf{P}|-3} \to \mathbb{CP}^{|\mathbf{P}|-3}$ is also *critically finite*.

Fix $\phi$ of topological degree $d > 1$ such that (1) and (2) hold. If there are two fully ramified points of $\phi$ in periodic cycles, then pick $p_0$ to be one with minimal cycle length. Set $\ell_0$ to be the length of the cycle containing $p_0$; then $\mathrm{PI}(\phi) \geq d^{1/\ell_0}$. Since $\mathcal{H}_\phi^{-1}$ is single-valued, its 0-th dynamical degree is 1; the results in [Koc13] imply that its top dynamical degree/topological degree is $d^{|\mathbf{P}|-3}$. It follows from definitions that $\Theta_k(\mathcal{H}_\phi^{-1}) = \Theta_{|\mathbf{P}|-3-k}(\mathcal{H}_\phi)$. We obtain from Theorem 3.1:

**Corollary 1.2.** *The dynamical degrees of $\mathcal{H}_\phi^{-1}$ satisfy*

$$\begin{aligned}(d^{1/\ell_0})^{|\mathbf{P}|-3} &= (d^{1/\ell_0})^{|\mathbf{P}|-3} \cdot \Theta_0(\mathcal{H}_\phi^{-1}) \\ &\leq (d^{1/\ell_0})^{|\mathbf{P}|-4} \cdot \Theta_1(\mathcal{H}_\phi^{-1}) \\ &\leq \quad \cdots \\ &\leq \Theta_{|\mathbf{P}|-3}(\mathcal{H}_\phi^{-1}) = \deg(\mathcal{H}_\phi^{-1}) = d^{|\mathbf{P}|-3}\end{aligned}$$

In particular, the topological degree of $\mathcal{H}_\phi^{-1}$ is strictly larger than its other dynamical degrees. Corollary 1.2 provides a theoretical explanation for an aspect of the experimental results in [KR16]: in every example computed, the largest dynamical degree of $\mathcal{H}_\phi^{-1}$ is the topological degree.

A direct application of Guedj's results in [Gue05] yields:

**Corollary 1.3.** *There is a unique $\mathcal{H}_\phi^{-1}$-invariant measure $\mathfrak{m}_\phi$ on $\mathbb{CP}^{|\mathbf{P}|-3}$ of maximal entropy. The measure $\mathfrak{m}_\phi$ is mixing, and all its Lyapunov exponents are bounded from below by*

$$\frac{1}{2}\log(\mathrm{PI}(\phi)) \geq \frac{1}{2\ell_0}\log(d) > 0.$$

*Further, the set of repelling periodic points of $\mathcal{H}_\phi^{-1}$ is equidistributed with respect to $\mathfrak{m}_\phi$.*

If $\ell_0 = 1$, then $p_0$ is fixed and $\phi$ is a topological polynomial. In this case, Corollary 1.2 recovers that $\Theta_k(\mathcal{H}_\phi^{-1}) = d^k$, since by [Koc13], $\mathcal{H}_\phi^{-1}$ is holomorphic on $\mathbb{CP}^{|\mathbf{P}|-3}$. Thus in this case $k \mapsto \log(\Theta_k(\mathcal{H}_\phi^{-1}))$ is linear of slope $\log(d)$. If

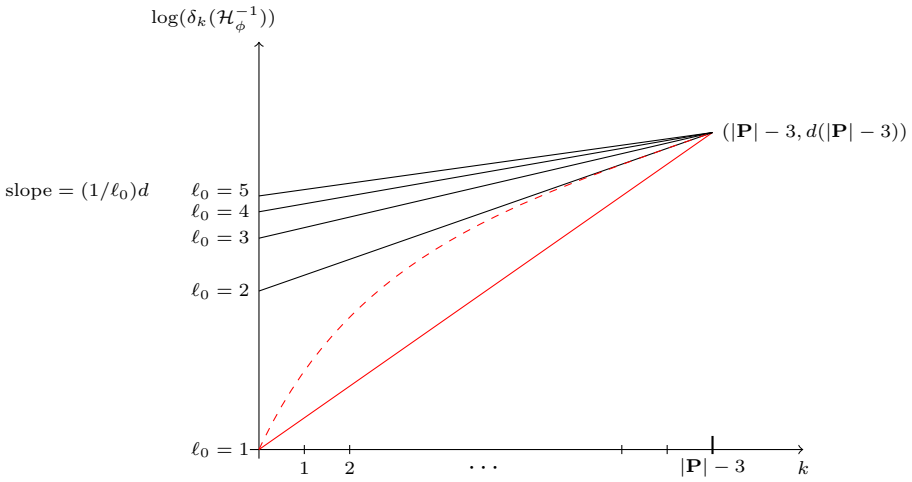


FIGURE 1. Fix $d > 1$ and $|\mathbf{P}| \geq 3$. Given a degree $d$ finite branched covering $\phi$ whose post-critical set has size $|\mathbf{P}|$ such that there exists a fully ramified point in a periodic cycle of length $\ell_0$, the figure shows how the graph of $k \mapsto \log(\Theta_k(\mathcal{H}_\phi^{-1}))$ is constrained by $\ell_0$. If $\ell_0 = 1$, the graph is the line of slope $\log(d)$, pictured in solid red. If $\ell_0 > 1$, the graph is concave of slope at least $(1/\ell_0)\log(d)$, passing through $(0,0)$ and $(|P|-3, d(|P|-3))$, and between the line of slope $(1/\ell_0)\log(d)$ pictured in solid black and the line of slope $\log(d)$ pictured in solid red. The dashed red curve depicts qualitatively what the graph might look like if $\ell_0 = 2$.

$\ell_0 > 1$, then $k \mapsto \log(\Theta_k(\mathcal{H}_\phi^{-1}))$ is a concave function, which by Corollary 1.2 is strictly increasing with slope at least $(1/\ell_0)\log(d)$. This generalizes the result that polynomiality of $\phi$ ensures holomorphicity of $\mathcal{H}_\phi^{-1}$ on $\mathbb{CP}^{|\mathbf{P}|-3}$ as follows:

**Observation 1.4.** The more $\phi$ resembles a topological polynomial, i.e. the smaller the value of $\ell_0$, the more the sequence of dynamical degrees of $\mathcal{H}_\phi^{-1}$ resembles the sequence of dynamical degrees of a holomorphic map on $\mathbb{CP}^{|\mathbf{P}|-3}$ (Figure 1.1).

1.2. **Implications for Hurwitz correspondences as multi-valued maps and an application to enumerative geometry.** Hurwitz correspondences can be defined without reference to the Thurston pullback map. Let $\mathbf{P}$ be a finite set and let $\mathcal{H}$ be a Hurwitz space parametrizing maps $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ together with two injections from $\mathbf{P}$ into the source and target $\mathbb{CP}^1$ respectively, such that $f$ has specified branching behavior at and over the marked points $\mathbf{P}$. The Hurwitz space $\mathcal{H}$ admits two maps to $\mathcal{M}_{0,\mathbf{P}}$: a map $\pi_1$ specifying the configuration of marked points on the "target" $\mathbb{CP}^1$, and a map $\pi_2$ specifying the configuration of marked points on the "source" $\mathbb{CP}^1$. If the marked points on the target $\mathbb{CP}^1$ include all the branch values of $f$, then $\pi_1$ is a covering map, and $\pi_2 \circ \pi_1^{-1}$ defines a multi-valued map from $\mathcal{M}_{0,\mathbf{P}}$ to itself.

The Hurwitz space $\mathcal{H}$ may be disconnected; each connected component of $\mathcal{H}$ parametrizes maps of a single topological type. If $\phi$ is a post-critically finite

branched covering with post-critical set $\mathbf{P}$ and branching type as specified by $\mathcal{H}$, then $\mathcal{H}_\phi$ is the connected component of $\mathcal{H}$ that parametrizes maps $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that there exist homeomorphisms $\chi_1$ and $\chi_2$ from $(\mathbb{CP}^1, \mathbf{P})$ to $(S^2, \mathbf{P})$ with $\chi_2 \circ f = \phi \circ \chi_1$. Every connected component of $\mathcal{H}$ arises as $\mathcal{H}_\phi$ for some post-critically finite $\phi$. Fixing $\mathcal{H}$, all such branched coverings have the same branching, induce the same map $\mathbf{P} \to \mathbf{P}$, and in particular, have the same polynomiality index. Thus polynomiality index is a well-defined invariant of $\mathcal{H}$. We define a Hurwitz correspondence in general to be the multivalued self-map of $\mathcal{M}_{0,\mathbf{P}}$ obtained by restricting $\pi_2 \circ \pi_1^{-1}$ to any non-empty union $\Gamma$ of connected components of a Hurwitz space $\mathcal{H}$. Theorem 3.1 is proved in this more general context, i.e. we have $\Theta_k(\Gamma) \geq \mathrm{PI}(\Gamma) \cdot \Theta_{k+1}(\Gamma)$. This implies that Hurwitz correspondences are a special subclass of multivalued maps: The sequence of dynamical degrees of a multi-valued map may not be log-concave, and appears to be quite unconstrained in general [Tru16].

The topological entropy of a multi-valued map is at most the logarithm of its largest dynamical degree, but can be strictly smaller [DS08]. Thus we have:

**Corollary 1.5.** *The topological entropy of $\mathcal{H}$ (resp. $\mathcal{H}_\phi$ or $\Gamma$) is at most its 0-th dynamical degree $\Theta_0(\mathcal{H})$ (resp. $\Theta_0(\mathcal{H}_\phi)$ or $\Theta_0(\Gamma)$).*

The 0-th dynamical degree of $\mathcal{H}$ is the topological degree of the "target" map $\pi_1 : \mathcal{H} \to \mathcal{M}_{0,\mathbf{P}}$. This degree is called a *Hurwitz number*; it counts covers of $\mathbb{CP}^1$ having specified branch locus on $\mathbb{CP}^1$ and specified ramification profile. It also counts the number of ways to factor the identity in the symmetric group $S_d$ as a product of permutations with specified cycle types that collectively generate a transitive subgroup. Thus the dynamically motivated quantity $\Theta_0(\mathcal{H})$ has a purely combinatorial interpretation.

The top dynamical degree of $\mathcal{H}$ is the topological degree of the "source" map $\pi_2$. In Section 4, we use Theorem 3.1 to prove Proposition 4.1, which has two alternative statements, one relevant to enumerative geometry and another relevant to dynamics. (Thanks to an anonymous referee for pointing out the second statement.)

**Proposition 4.1, Statement 1** *Let $r$ be the maximum local degree of $[f : \mathbb{CP}^1 \to \mathbb{CP}^1] \in \mathcal{H}$ at $p$ where $p$ ranges over $\mathbf{P}$. Then $\deg(\pi_1) \geq r^{|\mathbf{P}|-3} \deg(\pi_2)$. That is, the number of ways a generic configuration of $\mathbf{P}$-marked points on $\mathbb{CP}^1$ arises the configuration of marked points on the target $\mathbb{CP}^1$ of some branched map $[f] \in \mathcal{H}$ is at least $r^{|\mathbf{P}|-3}$ times the number of ways it appears as the configuration of marked points on the source $\mathbb{CP}^1$ of such a map.*

**Proposition 4.1, Statement 2** *Let $r$ be the maximum local degree of a post-critically finite branched covering $\phi$ at $p$ where $p$ ranges over the post-critical set $\mathbf{P}$. Then $\Theta_0(\mathcal{H}_\phi) \geq r^{|\mathbf{P}|-3}\Theta_{|\mathbf{P}|-3}(\mathcal{H}_\phi)$. That is, if there is a critical point that is also post-critical, then there is a strict aggregate decrease between the 0-th and the top dynamical degrees, and an upper bound for this aggregate decrease that is better (in some examples strictly better) than the bound given by the polynomiality index.*

1.3. **Remarks on the polynomiality index.** We are not aware of any previous mention of polynomiality index of a branched covering or rational function as defined in this paper. However, a similar notion is considered in [Sil93], in which Silverman addresses the question of when a rational function on $\mathbb{CP}^1$ has orbits containing infinitely many integers. Silverman defines the *attractive index* of a rational

function $f(z)$ at a point $p \in \mathbb{CP}^1$ of exact period $\ell$ to be

$$\Pi_{i=0}^{\ell-1}(\text{local degree of } \phi \text{ at } f^i(p))^{1/\ell};$$

he shows that the attractive index at $p$ controls the rate at which 'almost all' points in $\mathbb{CP}^1$ approach the orbit of $p$. The polynomiality index of a rational function/branched covering defined in this paper is the maximum attractive index attained by some periodic point in $\mathbb{CP}^1/S^2$. (Thanks to Silverman for mentioning the connection to [Sil93].)

1.4. **Organization.** Section 2 gives background on meromorphic multi-valued maps (henceforth referred to as rational correspondences), the moduli space $\mathcal{M}_{0,\mathbf{P}}$ and its compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$, Hurwitz spaces, and Hurwitz correspondences. Section 3 contains the proof of Theorem 3.1. Section 4 contains an application of the main theorem to enumerative algebraic geometry. Sections 5 and 6 give examples of sepcific Hurwitz correspondences.

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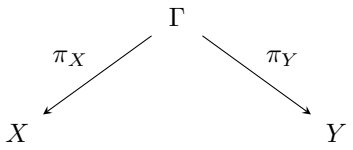
1.6. **Conventions.** All varieties are over $\mathbb{C}$. For $X$ a variety, we denote by $\mathcal{Z}_k(X)$ the group of $k$-cycles on $X$; that is, the free abelian group on the set of $k$-dimensional subvarieties of $X$. We denote by $A_k(X)$ the Chow group of $k$-cycles on $X$ up to rational equivalence. For $X$ a smooth variety, we denote by $A^k(X)$ the Chow group of codimension-$k$ cycles on $X$.

## 2. Background

2.1. **Rational correspondences/meromorphic multi-valued maps.** Rational correspondences generalize the notion of a rational map. A rational correspondence from $X$ to $Y$ is a multi-valued map to $Y$ defined on a dense open set of $X$.

**Definition 2.1.** Let $X$ and $Y$ be irreducible smooth projective varieties. A *rational correspondence* $(\Gamma, \pi_X, \pi_Y) : X \dashrightarrow Y$ is a diagram




where $\Gamma$ is a smooth quasiprojective variety, not necessarily irreducible, and the restriction of $\pi_X$ to every irreducible component of $\Gamma$ is dominant and generically finite.

Over a dense open set in $X$, $\pi_X$ is a covering map, and $\pi_Y \circ \pi_X^{-1}$ defines a multi-valued map to $Y$. However, considered as a multi-valued map from $X$ to $Y$, it is possible that $\pi_Y \circ \pi_X^{-1}$ has indeterminacy, since some fibers of $\pi_X$ may be empty or positive-dimensional.

Like rational maps, rational correspondences induce pushforward and pullback maps of Chow groups, and can be composed with each other.

**Definition 2.2.** Let $\overline{\Gamma}$ be a projective compactification of $\Gamma$ such that $\Gamma$ is dense in $\overline{\Gamma}$ and $\pi_X$ and $\pi_Y$ extend to maps $\overline{\pi_X}$ and $\overline{\pi_Y}$ defined on $\overline{\Gamma}$. The cycle $(\overline{\pi_X} \times \overline{\pi_Y})_*[\overline{\Gamma}] \in \mathcal{Z}_{\dim X}(X \times Y)$ is independent of the choice of compactification $\overline{\Gamma}$, so we denote this cycle by $[\Gamma]$.

*Remark* 2.3. In [DS08], a rational correspondence from $X$ to $Y$ is defined as a cycle $\sum_i m_i[\Gamma_i]$ in $\mathcal{Z}_{\dim X}(X \times Y)$, such that each $\Gamma_i$ maps surjectively onto $X$.

**Definition 2.4.** Let $\overline{\Gamma}$ be a projective compactification of $\Gamma$ as in Definition 2.2. Set

$$[\Gamma]_* := (\overline{\pi_Y})_* \circ (\overline{\pi_X})^* : A_k(X) \to A_k(Y)$$

and

$$[\Gamma]^* := (\overline{\pi_X})_* \circ (\overline{\pi_Y})^* : A^k(Y) \to A^k(X).$$

These pushforward and pullback maps are independent of the choice of compactification $\overline{\Gamma}$; they depend only on the cycle $[\Gamma]$ ([Ful98], Remark 6.2.2).

**Definition 2.5.** Suppose $(\Gamma, \pi_X, \pi_Y) : X \dashrightarrow Y$ and $(\Gamma', \pi'_Y, \pi'_Z) : Y \dashrightarrow Z$ are rational correspondences such that the image under $\pi_Y$ of every irreducible component of $\Gamma$ intersects the domain of definition of the multi-valued map $\pi'_Z \circ (\pi'_Y)^{-1}$. The *composite* $\Gamma' \circ \Gamma$ is a rational correspondence from $X$ to $Z$ defined as follows.

Pick dense open sets $U_X \subseteq X$ and $U_Y \subseteq Y$ such that $\pi_Y(\pi_X^{-1}(U_X)) \subseteq U_Y$, and $\pi_X|_{\pi_X^{-1}(U_X)}$ and $\pi'_Y|_{(\pi'_Y)^{-1}(U_Y)}$ are both covering maps. Set

$$\Gamma' \circ \Gamma := \pi_X^{-1}(U_X) \ {}_{\pi_Y}\times_{\pi'_Y} \ (\pi'_Y)^{-1}(U_Y),$$

to be the fibered product as defined in Theorem 3.3 of [Har77], together with its given maps to $X$ and $Z$.

*Remark* 2.6 (The fibered product). Although the fibered product of two maps of schemes (as cited from [Har77] and used above in Definition 2.5) is complicated, the basic idea is quite simple. If $\pi_1 : U_1 \to Y$ and $\pi_2 : U_2 \to Y$ are two maps of sets, then their fibered product is the subset of $U_1 \times U_2$ where the two maps agree:

$$U_1 \ {}_{\pi_Y}\times_{\pi'_Y} \ U_2 := \{(u_1, u_2) \in U_1 \times U_2 | \pi_1(u_1) = \pi_2(u_2)\}$$

On the other hand, if $\pi_1 : U_1 \to Y$ and $\pi_2 : U_2 \to Y$ are two maps of schemes, then their fibered product is a possibly non-reduced scheme. In the context of this paper, this may be thought of as a variety together with positive integer multiplicities assigned to the irreducible components. However, the underlying set of their fibered product as schemes is their fibered product as sets.

This composite does depend on the choices of open sets $U_X$ and $U_Y$, but the cycle $[\Gamma' \circ \Gamma]$ is well-defined. Note that $[\Gamma' \circ \Gamma]_*$ may not agree with $[\Gamma']_* \circ [\Gamma]_*$ and $[\Gamma' \circ \Gamma]^*$ may not agree with $[\Gamma]^* \circ [\Gamma']^*$. Dynamical degrees, introduced in the next section, are meant to address the discrepancy between $[\Gamma' \circ \Gamma]^*$ and $[\Gamma]^* \circ [\Gamma']^*$ or equivalently between $[\Gamma' \circ \Gamma]_*$ and $[\Gamma']_* \circ [\Gamma]_*$

2.2. **Dynamical degrees.** Dynamical degrees were first introduced as invariants of surjective holomorphic self-maps of a smooth projective variety. The $k$th dynamical degree of $g : X \to X$ is the spectral radius of $g^* : H^{k,k}(X) \to H^{k,k}(X)$. Dynamical degrees were later generalized to rational maps and rational correspondences.

**Definition 2.7.** Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a rational correspondence such that the restriction of $\pi_2$ to every irreducible component of $\Gamma$ is dominant. In this case we say $\Gamma$ is a *dominant* rational self-correspondence.

**Definition 2.8.** Let $\Gamma$ be as in Definition 2.7. Set $\Gamma^n := \Gamma \circ \cdots \circ \Gamma$ ($n$ times), and pick $\mathfrak{h}$ an ample divisor class on $X$. The *$k$th dynamical degree* $\Theta_k$ of $\Gamma$ is defined to be

$$\lim_{n\to\infty} \left(([\Gamma^n]^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X - k})\right)^{1/n}.$$

This limit exists and is independent of choice of ample divisor ([DS05, DS08, Tru15, Tru16]).

The dynamical degrees of $\Gamma$ are determined by the cycle $[\Gamma]$.

**Theorem 2.9** (Birational invariance of dynamical degrees, [DS05, DS08, Tru15, Tru16])**.** *Let $(\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a dominant rational self-correspondence, and let $\beta : X \dashrightarrow X'$ be a birational equivalence. We obtain a dominant rational self-correspondence on $X'$ through conjugation by $\beta$ as follows. Let $U$ be the domain of definition of $\beta$, and set $\Gamma' = \pi_1^{-1}(U) \cap \pi_2^{-1}(U)$. We have a dominant rational self-correspondence*

$$(\Gamma', \beta \circ \pi_1, \beta \circ \pi_2) : X' \dashrightarrow X'.$$

*Then the dynamical degrees of $\Gamma$ and $\Gamma'$ are equal.*

**Definition 2.10.** If $\Gamma = (\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ is a dominant rational correspondence then its inverse $\Gamma^{-1}$ is defined to be $(\Gamma, \pi_2, \pi_1) : X \dashrightarrow X$.

**Lemma 2.11.** *Let $\Gamma = (\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ and $\Gamma' = (\Gamma', \pi_1, \pi_2) : X \dashrightarrow X$ be dominant rational correspondences. Then the correspondence $(\Gamma \circ \Gamma')^{-1}$ is the same as $(\Gamma')^{-1} \circ \Gamma^{-1}$.*

*Proof.* This follows immediately from the definition (2.5) of composition of rational correspondences. □

**Lemma 2.12** (Dynamical degrees of a correspondence and its inverse)**.** *Let $\Gamma = (\Gamma, \pi_1, \pi_2) : X \dashrightarrow X$ be a dominant rational correspondence. Then we have, for $k = 0, \ldots, \dim X$,*

$$\Theta_k(\Gamma) = \Theta_{\dim X - k}(\Gamma^{-1}).$$

*Proof.* Fix an ample divisor class $\mathfrak{h}$ on $X$. Let $(\Gamma^n, \pi_1^n, \pi_2^n) : X \dashrightarrow X$ denote the $n$th iterate of $\Gamma$. By the functoriality of inverse correspondences as in Lemma 2.11, $(\Gamma^n, \pi_2^n, \pi_1^n) : X \dashrightarrow X$ is the $n$th iterate of $\Gamma^{-1}$. By passing to a birational model of $\Gamma^n$ for each $n$ if necessary, we may assume that each $\Gamma^n$ is smooth and projective, and that the maps $\pi_1^n, \pi_2^n : \Gamma^n \to X$ are both regular. We have:

$$\begin{aligned}\Theta_k(\Gamma) &:= \lim_{n\to\infty} \left(((\pi_1^n)_* \circ (\pi_2^n)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X-k})\right)^{1/n} \\ &= \lim_{n\to\infty} \left(((\pi_2^n)^*(\mathfrak{h}^k)) \cdot ((\pi_1^n)^*(\mathfrak{h}^{\dim X-k}))\right)^{1/n} \\ &= \lim_{n\to\infty} \left((\mathfrak{h}^k) \cdot ((\pi_2^n)_* \circ (\pi_1^n)^*\mathfrak{h}^{\dim X-k})\right)^{1/n} \\ &=: \Theta_{\dim X-k}(\Gamma^{-1})\end{aligned}$$

Here, the first equality follows from the definition of pullback by a rational correspondence, and the second and third equalities follow from the projection formula as stated in Proposition 8.3(c) of [Ful98]. □

**The sequence of dynamical degrees of a rational map is log-concave.** Let $g : X \dashrightarrow X$ be a dominant rational map, and let $\mathfrak{h}$ be an ample divisor class on $X$. For $n > 0$ set $\mathrm{Gr}(g^n)$ to be the graph of $g^n$ in $X \times X$, with its two maps $\pi_1^n$ and $\pi_2^n$ to $X$. If $\Theta_k$ denotes the $k$th dynamical degree of $g$, we have

$$\begin{aligned}\Theta_k &= \lim_{n\to\infty} \left(((g^n)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X-k})\right)^{1/n} \\ &= \lim_{n\to\infty} \left(((\pi_1^n)_* \circ (\pi_2^n)^*(\mathfrak{h}^k)) \cdot (\mathfrak{h}^{\dim X-k})\right)^{1/n} \\ &= \lim_{n\to\infty} \left(((\pi_2^n)^*(\mathfrak{h}^k)) \cdot ((\pi_1^n)^*(\mathfrak{h}^{\dim X-k}))\right)^{1/n}.\end{aligned}$$

Here, (as in the proof of Lemma 2.12), the second equality follows from the definition of pullback by a rational map, and the third equality follows from the projection formula as stated in Proposition 8.3(c) of [Ful98]. Since $(\pi_2^n)^*(\mathfrak{h})$ and $(\pi_1^n)^*(\mathfrak{h})$ are nef on $\mathrm{Gr}(g^n)$, and $\mathrm{Gr}(g^n)$ is irreducible, the sequence of intersection numbers

$$\{((\pi_2^n)^*(\mathfrak{h}^k)) \cdot ((\pi_1^n)^*(\mathfrak{h}^{\dim X-k}))\}_k$$

is log-concave ([Laz04], Example 1.6.4). Thus the sequence $\{\Theta_k\}_k$ is log-concave as well.

This statement is false for multi-valued maps/rational correspondences [Tru16]. The argument breaks down since their graphs are not necessarily irreducible. Even if a given rational correspondence $\Gamma$ is irreducible, its iterates $\Gamma^n$ are reducible in general. Our proof of Theorem 3.1 deals separately with every irreducible component of infinitely many iterates of a given Hurwitz correspondence.

2.3. **The moduli spaces $\mathcal{M}_{0,\mathbf{P}}$ and $\overline{\mathcal{M}}_{0,\mathbf{P}}$.** The moduli space $\mathcal{M}_{0,\mathbf{P}}$ is a smooth quasiprojective variety parametrizing ways of marking $\mathbb{CP}^1$ by elements of a finite set, up to change of coordinates on $\mathbb{CP}^1$.

**Definition 2.13.** Let $|\mathbf{P}| \geq 3$. There is a smooth quasiprojective variety $\mathcal{M}_{0,\mathbf{P}}$ of dimension $|\mathbf{P}| - 3$ parametrizing injections $\iota : \mathbf{P} \hookrightarrow \mathbb{CP}^1$ up to post-composition by Möbius transformations of $\mathbb{CP}^1$

There are several compactifications of $\mathcal{M}_{0,\mathbf{P}}$ that extend the interpretation as a moduli space. The most widely-studied of these is the Deligne-Mumford/ stable curves compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$. Projective space $\mathbb{CP}^{|\mathbf{P}|-3}$ is another such compactification.

**Definition 2.14.** A *stable* $\mathbf{P}$*-marked genus zero curve* is a connected projective curve $C$ of arithmetic genus zero whose only singularities are simple nodes, together with an injection $\iota : \mathbf{P} \hookrightarrow$ (smooth locus of $C$), such that the set of automorphisms $C \to C$ that commute with $\iota$ is finite.

**Theorem 2.15** (Deligne, Grothendieck, Knudsen, Mumford)**.** *There is a smooth projective variety* $\overline{\mathcal{M}}_{0,\mathbf{P}}$ *of dimension* $|\mathbf{P}| - 3$ *that parametrizes stable* $\mathbf{P}$*-marked genus zero curves. It contains* $\mathcal{M}_{0,\mathbf{P}}$ *as a dense open subset.*

The complement $\overline{\mathcal{M}}_{0,\mathbf{P}} \setminus \mathcal{M}_{0,\mathbf{P}}$ is a simple normal crossings divisor, referred to as the *boundary* of $\overline{\mathcal{M}}_{0,\mathbf{P}}$. Given a subset $\mathbf{S} \subseteq \mathbf{P}$ such that $|\mathbf{S}|, |\mathbf{S}^C| \geq 2$, define a divisor $\delta_{\mathbf{S}} \subseteq \overline{\mathcal{M}}_{0,\mathbf{P}}$ as follows. Consider the locus of all $[C, \iota]$ in $\overline{\mathcal{M}}_{0,\mathbf{P}}$ such that $C$ has two irreducible components joined at a node, the points $\iota(p)$ with $p \in \mathbf{S}$ are all on one component, and the points $\iota(p)$ with $p \in \mathbf{S}^C$ are all on the other component. Let $\delta_{\mathbf{S}}$ be the closure of this locus; $\delta_{\mathbf{S}}$ is an irreducible divisor contained in the boundary. Every irreducible component of the boundary is obtained in this manner. Note that $\delta_{\mathbf{S}} = \delta_{\mathbf{S}^C}$.

**Definition 2.16.** For an injection $j : \mathbf{P'} \hookrightarrow \mathbf{P}$ with $|\mathbf{P'}| \geq 3$, there is a *forgetful map* $\mu : \mathcal{M}_{0,\mathbf{P}} \to \mathcal{M}_{0,\mathbf{P'}}$ sending $[C, \iota]$ to $[C, \iota \circ j]$. This map extends to $\mu : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \overline{\mathcal{M}}_{0,\mathbf{P'}}$.

**The tautological $\psi$-classes.** $\overline{\mathcal{M}}_{0,\mathbf{P}}$ has a tautological line bundle $\mathcal{L}_p$ corresponding to each marked point $p \in \mathbf{P}$. This line bundle assigns to the point $[C, \iota]$ the 1-dimensional complex vector space $T^\vee_{\iota(p)}C$, namely, the cotangent line to the curve $C$ at the marked point $\iota(p)$. The divisor class associated to $\mathcal{L}_p$ is denoted $\psi_p$.

The space $H^0(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathcal{L}_p)$ is $(|\mathbf{P}| - 2)$-dimensional and basepoint-free. The induced map $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \mathbb{P}(H^0(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathcal{L}_p)^\vee) \cong \mathbb{CP}^{|\mathbf{P}|-3}$ is a birational map onto $\mathbb{CP}^{|\mathbf{P}|-3}$ ([Kap93]).

Consider a forgetful map $\mu : \overline{\mathcal{M}}_{0,\mathbf{P}\cup\{q\}} \to \overline{\mathcal{M}}_{0,\mathbf{P}}$. For $p \in \mathbf{P}$, we have ([AC98])

$$\mu^* \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}}} = \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}\cup\{q\}}} - \delta_{\{p,q\}}.$$

It follows by induction that:

**Lemma 2.17.** *For a forgetful map* $\mu : \overline{\mathcal{M}}_{0,\mathbf{P}\cup\mathbf{Q}} \to \overline{\mathcal{M}}_{0,\mathbf{P}}$*, we have*

$$\mu^* \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}}} = \psi_p^{\overline{\mathcal{M}}_{0,\mathbf{P}\cup\mathbf{Q}}} - \sum_{\substack{\mathbf{S}\subseteq\mathbf{Q} \\ \mathbf{S} \text{ nonempty}}} \delta_{\{p\}\sqcup\mathbf{S}}.$$

2.4. **Hurwitz spaces and Hurwitz correspondences.** Hurwitz spaces are moduli spaces parametrizing finite maps with prescribed ramification between smooth algebraic curves/Riemann surfaces. See [RW06] for a summary.

**Definition 2.18.** A *partition* $\lambda$ of a positive integer $k$ is a multiset of positive integers whose sum with multiplicity is $k$.

**Definition 2.19.** A multiset $\lambda_1$ is a *submultiset* of $\lambda_2$ if for all $r \in \lambda_1$, the multiplicity of occurrence of $r$ in $\lambda_1$ is less than or equal to the multiplicity of occurrence of $r$ in $\lambda_2$.

**Definition 2.20** (*Hurwitz space*, [Ram18], Definition 5.4)**.** Fix discrete data:

- $\mathbf{A}$ and $\mathbf{B}$ finite sets with cardinality at least 3 (marked points on source and target curves, respectively),
- $d$ a positive integer (degree),
- $F : \mathbf{A} \to \mathbf{B}$ a map,
- $\mathrm{br} : \mathbf{B} \to \{\text{partitions of } d\}$ (branching), and
- $\mathrm{rm} : \mathbf{A} \to \mathbb{Z}^{>0}$ (ramification),

such that

- (Condition 1, Riemann-Hurwitz constraint) $\sum_{b\in\mathbf{B}} (d - \text{length of } \mathrm{br}(b)) = 2d-2$, and
- (Condition 2) for all $b \in \mathbf{B}$, the multiset $(\mathrm{rm}(a))_{a\in F^{-1}(b)}$ is a submultiset of $\mathrm{br}(b)$.

There exists a smooth quasiprojective variety $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$, a *Hurwitz space*, parametrizing morphisms $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ up to isomorphism, where

- There are injections from $\mathbf{A}$ and $\mathbf{B}$ into the source and target $\mathbb{CP}^1$ respectively,
- $f$ is degree $d$,
- for all $a \in \mathbf{A}$, $f(a) = F(a)$ via the injections of $\mathbf{A}$ and $\mathbf{B}$ into $\mathbb{CP}^1$,
- for all $b \in \mathbf{B}$, the branching of $f$ over $b$ is given by the partition $\mathrm{br}(b)$, and
- for all $a \in \mathbf{A}$, the local degree of $f$ at $a$ is equal to $\mathrm{rm}(a)$.

The Hurwitz space $\mathcal{H}$ has a "source" map $\pi_{\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{A}}$ sending $[f : (\mathbb{CP}^1, \mathbf{A}) \to (\mathbb{CP}^1, \mathbf{B})]$ to $[\mathbb{CP}^1, \mathbf{A}]$. There is similarly a "target" map $\pi_{\mathbf{B}}$ from $\mathcal{H}$ to $\mathcal{M}_{0,\mathbf{B}}$. Unless $\mathcal{H}$ is empty, $\pi_{\mathbf{B}}$ is a finite covering map. Thus for smooth compactifications $X_{\mathbf{A}}$ of $\mathcal{M}_{0,\mathbf{A}}$ and $X_{\mathbf{B}}$ of $\mathcal{M}_{0,\mathbf{B}}$, $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}}$ is a rational correspondence. We generalize this notion.

**Definition 2.21** (*Hurwitz correspondence*, [Ram18], Definition 5.5)**.** Let $\mathbf{A}'$ be any subset of $\mathbf{A}$ with cardinality at least 3. There is a forgetful morphism $\mu : \mathcal{M}_{0,\mathbf{A}} \to \mathcal{M}_{0,\mathbf{A}'}$. Let $\Gamma$ be a union of connected components of $\mathcal{H}$. If $X_{\mathbf{A}'}$ and $X_{\mathbf{B}}$ are smooth projective compactifications of $\mathcal{M}_{0,\mathbf{A}'}$ and $\mathcal{M}_{0,\mathbf{B}}$ respectively, then

$$(\Gamma, \pi_{\mathbf{B}}, \mu \circ \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}'}$$

is a rational correspondence. We call such a rational correspondence a *Hurwitz correspondence*.

2.5. **Hurwitz self-correspondences and dynamics.** Suppose $\phi : S^2 \to S^2$ is a degree $d$ orientation-preserving branched covering with finite post-critical set $\mathbf{P}$. Define $\mathrm{br} : \mathbf{P} \to \{\text{partitions of } d\}$ sending $p \in \mathbf{P}$ to the branching profile of $\phi$ over $p$. Define $\mathrm{rm} : \mathbf{P} \to \mathbb{Z}^{>0}$ sending $p \in \mathbf{P}$ to the local degree of $\phi$ at $p$. Then

$$\mathcal{H} = \mathcal{H}(\mathbf{P}, \mathbf{P}, d, \phi|_{\mathbf{P}}, \mathrm{br}, \mathrm{rm})$$

parametrizes maps $(\mathbb{CP}^1, \mathbf{P}) \to (\mathbb{CP}^1, \mathbf{P})$ with the same branching as $\phi$. Let $\pi_1$ and $\pi_2$ be the "target" and "source" maps from $\mathcal{H}$ to $\mathcal{M}_{0,\mathbf{P}}$. For $\Gamma$ a non-empty union of connected components of $\mathcal{H}$, and $X_{\mathbf{P}}$ any compactification of $\mathcal{M}_{0,\mathbf{P}}$, $(\Gamma, \pi_1, \pi_2) : X_{\mathbf{P}} \dashrightarrow X_{\mathbf{P}}$ is a rational self-correspondence.

There is a unique connected component $\mathcal{H}_\phi$ of $\mathcal{H}$ parametrizing maps that are topologically isomorphic to $\phi$, i.e. maps $f : (\mathbb{CP}^1, \mathbf{P}) \to (\mathbb{CP}^1, \mathbf{P})$ such that there exist marked-point-preserving homeomorphisms $\chi_1$ and $\chi_2$ from $(\mathbb{CP}^1, \mathbf{P})$ to $(S^2, \mathbf{P})$ with $\chi_2 \circ f = \phi \circ \chi_1$. By [Koc13], the multi-valued map defined by $\mathcal{H}_\phi$ on $\mathcal{M}_{0,\mathbf{P}}$ is descended from the Thurston pullback map $\mathrm{Th}_\phi$.

It is convenient to consider Hurwitz self-correspondences in more generality. Given a Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{P}', \mathbf{P}, d, F, \mathrm{br}, \mathrm{rm})$ together with an injection $\mathbf{P} \hookrightarrow \mathbf{P}'$, if $\mu : \mathcal{M}_{0,\mathbf{P}'} \to \mathcal{M}_{0,\mathbf{P}}$ is the forgetful map, $\Gamma$ is a non-empty union of connected components of $\mathcal{H}$, and $X_{\mathbf{P}}$ is a compactification of $\mathcal{M}_{0,\mathbf{P}}$, then $(\Gamma, \pi_{\mathbf{P}}, \mu \circ \pi_{\mathbf{P}'}) : X_{\mathbf{P}} \dashrightarrow X_{\mathbf{P}}$ is a Hurwitz self-correspondence. Note that by Theorem 2.9, the dynamical degrees of the Hurwitz self-correspondence $\Gamma$ do not depend on the choice of compactification $X_{\mathbf{P}}$.

**Definition 2.22.** As above, let $\mathcal{H} = \mathcal{H}(\mathbf{P}', \mathbf{P}, d, F, \mathrm{br}, \mathrm{rm})$ be a Hurwitz space together with an injection $\mathbf{P} \hookrightarrow \mathbf{P}'$. Since $F : \mathbf{P} \to \mathbf{P}$ is a self-map of a finite set, every point eventually maps into a periodic cycle. We define the *polynomiality index* of $\mathcal{H}$ to be

$$\mathrm{PI}(\mathcal{H}) := \max_{\{p \in \mathbf{P}, \ell > 0 | F^\ell(p) = p\}} \left(\Pi_{i=0}^{\ell-1} \mathrm{rm}(p)\right)^{1/\ell}.$$

If $\Gamma$ is a non-empty union of connected components of $\mathcal{H}$, then we define the polynomiality index of $\Gamma$ to be the polynomiality index of $\mathcal{H}$.

Note that the polynomiality index of $\mathcal{H}_\phi$ as in Definition 2.22 agrees with the polynomiality index of $\phi$ as in Section 1.

2.6. **Fully marked Hurwitz spaces and admissible covers.** Harris and Mumford ([HM82]) constructed compactifications of Hurwitz spaces. These compactifications are called moduli spaces of *admissible covers*. They are projective varieties that parametrize certain ramified maps between nodal curves. They extend the "target curve" and "source curve" maps to the stable curves compactifications of the moduli spaces of target and source curves, respectively.

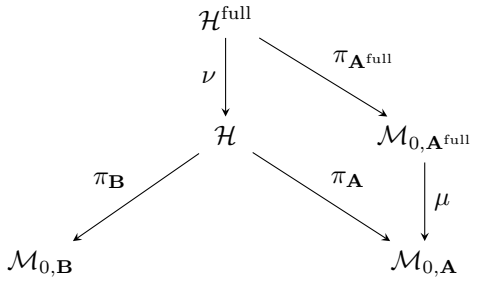
In general, the admissible covers compactifications are only coarse moduli spaces with *orbifold singularities*. For technical ease, we introduce a class of Hurwitz spaces whose admissible covers compactifications are fine moduli spaces. We call these Hurwitz spaces *fully marked*.

**Definition 2.23** ([Ram18], Definition 5.6)**.** Given $(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ as in Definition 2.20 with Condition 2 strengthened to:

- (Condition 2′) For all $b \in \mathbf{B}$, the multiset $(\mathrm{rm}(a))_{a \in F^{-1}(b)}$ is **equal to** $\mathrm{br}(b)$,

we refer to the corresponding Hurwitz space $\mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ as a *fully marked Hurwitz space*.

Given any Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$, there exists a fully marked Hurwitz space $\mathcal{H}^{\mathrm{full}} = \mathcal{H}(\mathbf{A}^{\mathrm{full}}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$, where $\mathbf{A}^{\mathrm{full}}$ is a superset of $\mathbf{A}$ extending the functions $F$ and rm. There is a finite covering map $\nu : \mathcal{H}^{\mathrm{full}} \to \mathcal{H}$, and we have the following commutative diagram (see [Ram18] for details):



For $\Gamma$ a union of connected components of $\mathcal{H}$, and for $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ smooth projective compactifications of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$, respectively, $(\Gamma, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : X_{\mathbf{B}} \dashrightarrow X_{\mathbf{A}}$ is a Hurwitz correspondence. Set $\Gamma^{\text{full}} = \nu^{-1}(\Gamma)$. Then $\Gamma^{\text{full}}$ is a union of connected components of $\mathcal{H}^{\text{full}}$, and in $\mathcal{Z}_{\dim X_{\mathbf{B}}}(X_{\mathbf{B}} \times X_{\mathbf{A}})$,

$$[\Gamma] = \frac{1}{\deg \nu}[\Gamma^{\text{full}}].$$

**Lemma 2.24.** *Let $(\Gamma, \pi_1, \pi_2) : X_{\mathbf{P}} \dashrightarrow X_{\mathbf{P}}$ be a dominant Hurwitz self-correspondence. Then*

$$(k\text{th dynamical degree of } \Gamma) = \frac{1}{\deg \nu}(k\text{th dynamical degree of } \Gamma^{\text{full}}),$$

*where $\Gamma^{\text{full}}$ is a union of connected components of a fully marked Hurwitz space $\mathcal{H}^{\text{full}}$ corresponding to a superset $\mathbf{P}^{\text{full}}$ of $\mathbf{P}$, and $\nu : \Gamma^{\text{full}} \to \Gamma$ is a finite covering map.*

*Proof.* For $\Gamma^{\text{full}}$ as above, we have that for every iterate $\Gamma^n$,

$$[\Gamma^n] = \left(\frac{1}{\deg \nu}\right)^n [(\Gamma^{\text{full}})^n]. \qquad \square$$

This means that arbitrary Hurwitz correspondences may be studied via fully marked Hurwitz spaces. These in turn have convenient compactifications by spaces of admissible covers.

**Theorem 2.25** (Harris and Mumford, [HM82])**.** *Given $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ satisfying Conditions 1 and 2′ as in Definition 2.23, there is a projective variety $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ containing $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ as a dense open subset. This* admissible covers *compactification $\overline{\mathcal{H}}$ extends the maps $\pi_{\mathbf{B}}$ and $\pi_{\mathbf{A}}$ to maps $\overline{\pi_{\mathbf{B}}}$ and $\overline{\pi_{\mathbf{A}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$, respectively, with $\overline{\pi_{\mathbf{B}}} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,\mathbf{B}}$ a finite flat map. $\overline{\mathcal{H}}$ may not be normal, but its normalization is smooth.*

The following comparison of tautological line bundles on moduli spaces of admissible covers is the key ingredient in our proof of Theorem 3.1:

**Proposition 2.26** (Ionel, Lemma 1.17 in [Ion02])**.** *Let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ be a fully marked space of admissible covers with maps $\overline{\pi_{\mathbf{B}}}$ and $\overline{\pi_{\mathbf{A}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$ respectively. Suppose we have $a \in \mathbf{A}$ and $b \in \mathbf{B}$ with $F(a) = b$. Then $(\overline{\pi_{\mathbf{B}}})^*(\mathcal{L}_b) = (\overline{\pi_{\mathbf{A}}})^*(\mathcal{L}_a)^{\otimes \text{rm}(a)}$ as line bundles on $\overline{\mathcal{H}}$.*

## 3. Main Theorem

**Theorem 3.1.** *Let*

$$(\Gamma, \pi_1, \pi_2) : \overline{\mathcal{M}}_{0,\mathbf{P}} \dashrightarrow \overline{\mathcal{M}}_{0,\mathbf{P}}$$

*be a dominant Hurwitz self-correspondence. Let $R$ be the polynomiality index of $\Gamma$, and let $\Theta_k$ be the $k$th dynamical degree of $\Gamma$. Then*

$$\Theta_0 \geq R\Theta_1 \geq \cdots \geq R^{|\mathbf{P}|-3}\Theta_{|\mathbf{P}|-3}.$$

*Proof.* By Lemma 2.24, we may assume $\Gamma$ is a union of connected components of a fully marked Hurwitz space $\mathcal{H} = \mathcal{H}(\mathbf{P}^{\text{full}}, \mathbf{P}, d, F, \text{br}, \text{rm})$ corresponding to a superset $\mathbf{P}^{\text{full}}$ of $\mathbf{P}$. Let $\overline{\mathcal{H}}$ denote the admissible covers compactification of $\mathcal{H}$, and let $\overline{\Gamma}$ be the closure of $\Gamma$ in $\overline{\mathcal{H}}$. For $\ell > 0$ set $\Gamma^\ell$ to be the $\ell$th iterate of $\Gamma$, that is

$$\Gamma \,{}_{\pi_2}\!\times_{\pi_1} \cdots \,{}_{\pi_2}\!\times_{\pi_1} \Gamma \qquad (\ell \text{ times}),$$

Set $\overline{\Gamma^\ell}$ to be its compactification

$$\overline{\Gamma} \,{}_{\overline{\pi_2}}\!\times_{\overline{\pi_1}}\, \cdots \,{}_{\overline{\pi_2}}\!\times_{\overline{\pi_1}}\, \overline{\Gamma} \qquad (\ell \text{ times}),$$

with $\overline{\pi_1^\ell}$ and $\overline{\pi_2^\ell}$ its two maps to $\overline{\mathcal{M}}_{0,\mathbf{P}}$.

Since $\overline{\pi_1^\ell}$ is a flat map, no irreducible component of $\overline{\Gamma^\ell}$ is supported over the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$. This means that $\Gamma^\ell$ is a dense open subset of $\overline{\Gamma^\ell}$. We refer to the complement $\overline{\Gamma^\ell} \smallsetminus \Gamma^\ell$ as the boundary of $\overline{\Gamma^\ell}$. The inverse image under $\overline{\pi_1^\ell}$ of the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is exactly the boundary of $\overline{\Gamma^\ell}$. The inverse image under $\overline{\pi_2^\ell}$ of the boundary of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ is contained in the boundary of $\overline{\Gamma^\ell}$.

The compactification $\overline{\Gamma^\ell}$ is singular. However, for Cartier divisors $D_1, \ldots, D_{\dim \overline{\Gamma^\ell}}$, the intersection product $D_1 \cdot \cdots \cdot D_{\dim \overline{\Gamma^\ell}}$ is a well-defined integer as in Section 1.1.C of [Laz04]. For any subscheme $Y$ of dimension $k$, and Cartier divisors $D_1, \ldots, D_k$, we similarly have the intersection number $D_1 \cdot \cdots \cdot D_k \cdot Y \in \mathbb{Z}$.

**Lemma 3.2.** *For all $p \in \mathbf{P}$ and for all $\ell \geq 0$, there is an equality of Cartier divisors on $\overline{\Gamma^\ell}$ of the form*

$$(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) = \Pi_{i=0}^{\ell-1} \mathrm{rm}(F^i(p)) \cdot (\overline{\pi_2^\ell})^*(\psi_p) + E,$$

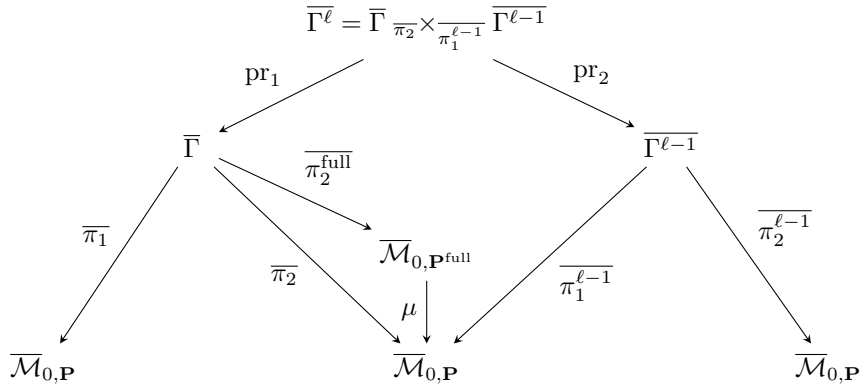
*where $E$ is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^\ell}$.*

*Proof.* We induct on $\ell$. By convention, $\overline{\Gamma^0}$ is the identity rational correspondence

$$(\overline{\mathcal{M}}_{0,\mathbf{P}},\ \overline{\pi_1^0} = \mathrm{Id},\ \overline{\pi_2^0} = \mathrm{Id}) : \overline{\mathcal{M}}_{0,\mathbf{P}} \dashrightarrow \overline{\mathcal{M}}_{0,\mathbf{P}}.$$

For all $p \in \mathbf{P}$, $F^0(p) = p$, so $(\overline{\pi_1^0})^*(\psi_{F^0(p)}) = (\overline{\pi_2^0})^*(\psi_p)$. This gives us the base case $\ell = 0$.

Suppose the Lemma holds for $\ell - 1$. We have




For all $p \in \mathbf{P}$, we have

$$\begin{aligned}(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= \mathrm{pr}_1^*(\overline{\pi_1})^*(\psi_{F^\ell(p)}) \\ &= \mathrm{pr}_1^* \left( \mathrm{rm}(F^{\ell-1}(p)) \cdot (\overline{\pi_2^{\mathrm{full}}})^*(\psi^{\mathbf{P}^{\mathrm{full}}}_{F^{\ell-1}(p)}) \right) \qquad \text{(by Proposition 2.26)}.\end{aligned}$$

By Lemma 2.17,

$$\psi^{\mathbf{P}^{\mathrm{full}}}_{F^{\ell-1}(p)} = \mu^*(\psi_{F^{\ell-1}(p)}) + \sum_{\mathbf{S} \subseteq \mathbf{P}^{\mathrm{full}} \smallsetminus \mathbf{P}} \delta_{\{F^{\ell-1}(p)\} \cup \mathbf{S}}.$$

The inverse image under $\overline{\pi_2^{\text{full}}}$ of the boundary in $\overline{\mathcal{M}}_{0,\mathbf{P}^{\text{full}}}$ is contained in the boundary of $\overline{\Gamma}$ (in fact it is the entire boundary), and the inverse image under $\text{pr}_1$ of the boundary of $\overline{\Gamma}$ is the boundary of $\overline{\Gamma^\ell}$. Thus, the Cartier divisor

$$E_1 := \text{pr}_1^*\left((\overline{\pi_2^{\text{full}}})^*\left(\sum_{\mathbf{S}\subseteq\mathbf{P}^{\text{full}}\smallsetminus\mathbf{P}} \delta_{\{F^{\ell-1}(p)\}\cup\mathbf{S}}\right)\right)$$

is effective and supported on the boundary of $\overline{\Gamma^\ell}$. We continue:

$$\begin{aligned}(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)}) &= \text{rm}(F^{\ell-1}(p))\,\text{pr}_1^*(\overline{\pi_2^{\text{full}}})^*\mu^*(\psi_{F^{\ell-1}(p)}) + \text{rm}(F^{\ell-1}(p))E_1\\ &= \text{rm}(F^{\ell-1}(p))\,\text{pr}_1^*(\overline{\pi_2})^*(\psi_{F^{\ell-1}(p)}) + \text{rm}(F^{\ell-1}(p))E_1\\ &= \text{rm}(F^{\ell-1}(p))\,\text{pr}_2^*(\overline{\pi_1^{\ell-1}})^*(\psi_{F^{\ell-1}(p)}) + \text{rm}(F^{\ell-1}(p))E_1.\end{aligned}$$

By the inductive hypothesis, we can rewrite this as

$$\text{rm}(F^{\ell-1}(p))\,\text{pr}_2^*(\Pi_{i=0}^{\ell-2}\text{rm}(F^i(p))(\overline{\pi_2^{\ell-1}})^*(\psi_p) + E_2) + \text{rm}(F^{\ell-1}(p))E_1,$$

where $E_2$ is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^{\ell-1}}$. Since the inverse image under $\text{pr}_2$ of the boundary of $\overline{\Gamma^{\ell-1}}$ is contained in the boundary of $\overline{\Gamma^\ell}$, $\text{pr}_2^*(E_2)$ is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^\ell}$. Thus we can finally write

$$\begin{aligned}&(\overline{\pi_1^\ell})^*(\psi_{F^\ell(p)})\\ &= \text{rm}(F^{\ell-1}(p))(\Pi_{i=0}^{\ell-2}\text{rm}(F^i(p)))\,\text{pr}_2^*(\overline{\pi_2^{\ell-1}})^*(\psi_p)\\ &\qquad + \text{rm}(F^{\ell-1}(p))\,\text{pr}_2^*(E_2) + \text{rm}(F^{\ell-1}(p))E_1\\ &= \Pi_{i=0}^{\ell-1}\text{rm}(F^i(p))(\overline{\pi_2^\ell})^*(\psi_p) + (\text{rm}(F^{\ell-1}(p))\,\text{pr}_2^*(E_2) + \text{rm}(F^{\ell-1}(p))E_1),\end{aligned}$$

which is as desired. This proves Lemma 3.2. □

Now, since $F : \mathbf{P} \to \mathbf{P}$ is a map of finite sets, every point is eventually periodic. Fix $p \in \mathbf{P}$ that is periodic of period $\ell_0 > 0$ and such that $(\Pi_{i=0}^{\ell_0-1}\text{rm}(F^i(p)))^{\frac{1}{\ell_0}} = R$. Then by Lemma 3.2, for every multiple $m\ell_0$, we have on $\overline{\Gamma^{m\ell_0}}$:

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p) = R^{m\ell_0}(\overline{\pi_2^{m\ell_0}})^*(\psi_p) + E_m, \tag{1}$$

where $E_m$ is an effective Cartier divisor supported on the boundary of $\overline{\Gamma^{m\ell_0}}$.

Let $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \mathbb{CP}^{|\mathbf{P}|-3}$ be the birational morphism to projective space given by the line bundle $\mathcal{L}_p$. Let $\mathfrak{h}$ be the Cartier divisor class of a hyperplane in $\mathbb{CP}^{|\mathbf{P}|-3}$. Then $\rho^*(\mathfrak{h}) = \psi_p$.

The pullback $[\Gamma^n]^*(\mathfrak{h}^k)$ is by definition

$$(\rho\circ\overline{\pi_1^n})_*\circ(\rho\circ\overline{\pi_2^n})^*(\mathfrak{h}^k).$$

So, by the projection formula,

$$([\Gamma^n]^*(\mathfrak{h}^k))\cdot(\mathfrak{h}^{|\mathbf{P}|-3-k}) = ((\rho\circ\overline{\pi_2^n})^*(\mathfrak{h}^k))\cdot((\rho\circ\overline{\pi_1^n})^*(\mathfrak{h}^{|\mathbf{P}|-3-k})).$$

Since dynamical degrees are birational invariants, $\Theta_k$ is also the $k$th dynamical degree of the induced rational correspondence $(\Gamma, \rho\circ\overline{\pi_1}, \rho\circ\overline{\pi_2}) : \mathbb{CP}^{|\mathbf{P}|-3} \dashrightarrow \mathbb{CP}^{|\mathbf{P}|-3}$.

We have

$$\begin{aligned}\Theta_k &= \lim_{n\to\infty}(([\Gamma^n]^*(\mathfrak{h}^k))\cdot(\mathfrak{h}^{|\mathbf{P}|-3-k}))^{1/n}\\ &= \lim_{n\to\infty}(((\rho\circ\overline{\pi_2^n})^*(\mathfrak{h}^k))\cdot((\rho\circ\overline{\pi_1^n})^*(\mathfrak{h}^{|\mathbf{P}|-3-k})))^{1/n}\\ &= \lim_{n\to\infty}(((\overline{\pi_2^n})^*(\psi_p^k))\cdot((\overline{\pi_1^n})^*(\psi_p^{|\mathbf{P}|-3-k})))^{1/n}.\end{aligned}$$

Since this sequence converges, we can find its limit using any subsequence, and

$$\begin{aligned}\Theta_k &= \lim_{m\to\infty}(((\overline{\pi_2^{m\ell_0}})^*(\psi_p^k))\cdot((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{|\mathbf{P}|-3-k})))^{1/m\ell_0}\\ &= \lim_{m\to\infty}(((\overline{\pi_2^{m\ell_0}})^*(\psi_p))^k\cdot((\overline{\pi_1^{m\ell_0}})^*(\psi_p))^{|\mathbf{P}|-3-k})^{1/m\ell_0}.\end{aligned}$$

For $m>0$, set

$$\alpha_{m,k} := ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))^k\cdot((\overline{\pi_1^{m\ell_0}})^*(\psi_p))^{|\mathbf{P}|-3-k},$$

so

$$\Theta_k = \lim_{m\to\infty}(\alpha_{m,k})^{1/m\ell_0}$$

**Lemma 3.3.** *Fix $m>0$. The intersection numbers $\alpha_{m,k}$ on $\overline{\Gamma^{m\ell_0}}$ satisfy*

$$\alpha_{m,0}\geq R^{m\ell_0}\alpha_{m,1}\geq\cdots\geq(R^{m\ell_0})^{|\mathbf{P}|-3}\alpha_{m,|\mathbf{P}|-3}$$

*Proof of Lemma 3.3.* Let $\overline{\mathcal{J}}$ be any irreducible component of $\overline{\Gamma^{m\ell_0}}$, and set

$$\alpha_{\overline{\mathcal{J}},k} := ((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{k}\cdot((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{|\mathbf{P}|-3-k}.$$

Since $(\overline{\pi_1^{m\ell_0}})^*(\psi_p)$ and $(\overline{\pi_2^{m\ell_0}})^*(\psi_p)$ are pullbacks of the ample hyperplane class $\mathfrak{h}$, they are nef on $\overline{\Gamma^{m\ell_0}}$ and $\overline{\mathcal{J}}$. By [Laz04], Example 1.6.4, $\alpha_{\overline{\mathcal{J}},k}$ is a log-concave function of $k$.

Note that $\psi_p^{|\mathbf{P}|-4}=\rho^*(\mathfrak{h}^{|\mathbf{P}|-4})$. The class $\mathfrak{h}^{|\mathbf{P}|-4}$ on $\mathbb{CP}^{|\mathbf{P}|-3}$ may be represented by a line $L$ that does not intersect the codimension-two exceptional locus of $\rho$. Then $\rho^{-1}(L)$ is an irreducible curve in $\overline{\mathcal{M}}_{0,\mathbf{P}}$ not contained in the boundary and $(\overline{\pi_1^{m\ell_0}})^{-1}(\rho^{-1}(L))|_{\overline{\mathcal{J}}}$ is a curve $Y$ none of whose irreducible components lies in the boundary of $\overline{\mathcal{J}}$. Since $\overline{\pi_1^{m\ell_0}}$ is a flat map, and a covering map away from the boundary,

$$((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{|\mathbf{P}|-4}))|_{\overline{\mathcal{J}}}=[Y].$$

By Equation 1, we have

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p)\cdot[Y]=R^{m\ell_0}(\overline{\pi_2^{m\ell_0}})^*(\psi_p)\cdot[Y]+E_m\cdot[Y]$$

Since $(\overline{\pi_1^{m\ell_0}})^*(\psi_p)$ and $(\overline{\pi_2^{m\ell_0}})^*(\psi_p)$ are nef on $\overline{\Gamma^{m\ell_0}}$, the intersection numbers

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p)\cdot[Y]\quad\text{and}\quad(\overline{\pi_2^{m\ell_0}})^*(\psi_p)\cdot[Y]$$

are non-negative. Since $E_m$ is entirely supported on the boundary and no component of $Y$ is supported on the boundary, $E_m\cdot[Y]$ is non-negative as well. Thus we obtain:

$$(\overline{\pi_1^{m\ell_0}})^*(\psi_p)\cdot[Y]\geq R^{m\ell_0}(\overline{\pi_2^{m\ell_0}})^*(\psi_p)\cdot[Y]. \tag{2}$$

Thus,

$$\begin{aligned}
&\alpha_{\overline{\mathcal{J}},0} \\
&\quad = ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{|\mathbf{P}|-3} \\
&\quad = ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{|\mathbf{P}|-4}))|_{\overline{\mathcal{J}}} \\
&\quad = ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot [Y] \\
&\quad \geq R^{m\ell_0}((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot [Y] \qquad \text{by (2)} \\
&\quad = R^{m\ell_0}((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p^{|\mathbf{P}|-4}))|_{\overline{\mathcal{J}}} \\
&\quad = R^{m\ell_0}((\overline{\pi_2^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}} \cdot ((\overline{\pi_1^{m\ell_0}})^*(\psi_p))|_{\overline{\mathcal{J}}}^{|\mathbf{P}|-4} \\
&\quad = R^{m\ell_0}\alpha_{\overline{\mathcal{J}},1}.
\end{aligned}$$

By log-concavity, we conclude that the intersection numbers $\alpha_{\overline{\mathcal{J}},k}$ satisfy

$$\alpha_{\overline{\mathcal{J}},0} \geq R^{m\ell_0}\alpha_{\overline{\mathcal{J}},1} \geq \cdots \geq (R^{m\ell_0})^{|\mathbf{P}|-3}\alpha_{\overline{\mathcal{J}},|\mathbf{P}|-3}$$

Since

$$\alpha_{m,k} = \sum_{\substack{\overline{\mathcal{J}} \text{ irreducible} \\ \text{component of } \overline{\Gamma^{m\ell_0}}}} \alpha_{\overline{\mathcal{J}},k},$$

the lemma follows. □

We now complete the proof of Theorem 3.1. For all $m$,

$$\begin{aligned}
&\alpha_{m,0} \geq R^{m\ell_0}\alpha_{m,1} \geq \cdots \geq (R^{m\ell_0})^{|\mathbf{P}|-3}\alpha_{m,|\mathbf{P}|-3}, \qquad \text{so} \\
&\alpha_{m,0}^{1/m\ell_0} \geq R\alpha_{m,1}^{1/m\ell_0} \geq \cdots \geq R^{|\mathbf{P}|-3}\alpha_{m,|\mathbf{P}|-3}^{1/m\ell_0}
\end{aligned}$$

The theorem follows by taking the limit as $m$ goes to infinity. □

## 4. An application to enumerative algebraic geometry

**Proposition 4.1.** *Let $\mathcal{H} = \mathcal{H}(\mathbf{P}, \mathbf{P}, d, F, \mathrm{br}, \mathrm{rm})$ be a Hurwitz space with "target" and "source" maps $\pi_1$ and $\pi_2$ respectively to $\mathcal{M}_{0,\mathbf{P}}$. Let*

$$r = \max_{p \in \mathbf{P}} \mathrm{rm}(p).$$

*Let $\Gamma$ be any connected component of $\mathcal{H}$. Then*

$$\Theta_0(\Gamma) = \deg(\pi_1|_\Gamma) \geq r^{|\mathbf{P}|-3} \deg(\pi_2|_\Gamma) = \Theta_{|\mathbf{P}|-3}(\Gamma).$$

*By summing over all connected components of $\mathcal{H}$ we obtain*

$$\Theta_0(\mathcal{H}) = \deg(\pi_1) \geq r^{|\mathbf{P}|-3} \deg(\pi_2) = \Theta_{|\mathbf{P}|-3}(\mathcal{H}).$$

*Remark* 4.2. Here, $r$ is the maximum local degree of $[f : \mathbb{CP}^1 \to \mathbb{CP}^1] \in \mathcal{H}$ at $p$ where $p$ ranges over $\mathbf{P}$.

In the introduction, two alternate statements are given for Proposition 4.1. Both follow immediately from the statement as proved here. Note that every connected component $\Gamma$ arises as $\mathcal{H}_\phi$ for some topological branched covering $\phi$, giving us the second statement in the introduction. Note that if there is a $p \in \mathbf{P}$ with local degree $\mathrm{rm}(p)$ strictly bigger than 1, then even if the polynomiality index is 1, by

this proposition we can still guarantee a strict total decrease from the 0-th to the top dynamical degrees.

*Proof.* Fix $p \in \mathbf{P}$ with $\mathrm{rm}(p) = r$. Pick a permutation $\sigma \in \mathrm{Aut}(\mathbf{P})$ such that $\sigma(p) = F(p)$. The permutation $\sigma$ induces an automorphism $\sigma^{\mathrm{moduli}}$ of $\mathcal{M}_{0,\mathbf{P}}$ given by

$$[\iota : \mathbf{P} \hookrightarrow \mathbb{CP}^1] \mapsto [\iota \circ \sigma : \mathbf{P} \hookrightarrow \mathbb{CP}^1].$$

Set

$$\mathcal{H}^{\mathrm{new}} = \mathcal{H}(\mathbf{P}, \mathbf{P}, d, \sigma^{-1} \circ F, \mathrm{br} \circ \sigma, \mathrm{rm}).$$

$\mathcal{H}^{\mathrm{new}}$ is the Hurwitz space obtained by using $\sigma$ to relabel the marked points of $[f : \mathbb{CP}^1 \to \mathbb{CP}^1] \in \mathcal{H}$ on the target $\mathbb{CP}^1$. The point $p$ is a 'fixed point' of maps $[f] \in \mathcal{H}^{\mathrm{new}}$. Note that by construction $\mathrm{PI}(\mathcal{H}^{\mathrm{new}}) \geq r$. (In fact $\mathrm{PI}(\mathcal{H}^{\mathrm{new}}) = r$, although we will not need this stronger fact.) There is an isomorphism $\sigma^{\mathrm{hurwitz}}$ from $\mathcal{H}$ to $\mathcal{H}^{\mathrm{new}}$ as follows. A point in $\mathcal{H}$ is a map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ together with injections $\iota_1$ and $\iota_2$ into the target $\mathbb{CP}^1$ and source $\mathbb{CP}^1$ respectively. The isomorphism $\sigma^{\mathrm{hurwitz}} : \mathcal{H} \to \mathcal{H}^{\mathrm{new}}$ takes $[f, \iota_1, \iota_2] \in \mathcal{H}$ to $[f, \iota_1 \circ \sigma, \iota_2]$. Denote by $\pi_1^{\mathrm{new}}$ and $\pi_2^{\mathrm{new}}$ respectively the "target" and "source" maps from $\mathcal{H}^{\mathrm{new}}$ to $\mathcal{M}_{0,\mathbf{P}}$. Note that $\sigma^{\mathrm{moduli}} \circ \pi_1 = \pi_1^{\mathrm{new}} \circ \sigma^{\mathrm{hurwitz}}$; also that $\pi_2 = \pi_2^{\mathrm{new}} \circ \sigma^{\mathrm{hurwitz}}$.

Now let $\Gamma$ be some connected component of $\mathcal{H}$; denote by $\Gamma^{\mathrm{new}}$ its isomorphic image in $\mathcal{H}^{\mathrm{new}}$. By Theorem 3.1, we have

$$\deg(\pi_1^{\mathrm{new}}|_{\Gamma^{\mathrm{new}}}) = \Theta_0(\Gamma^{\mathrm{new}}) \geq r^{|\mathbf{P}|-3}\Theta_{|\mathbf{P}|-3}(\Gamma^{\mathrm{new}}) = \deg(\pi_2^{\mathrm{new}}|_{\Gamma^{\mathrm{new}}}) \tag{3}$$

Since $\sigma^{\mathrm{moduli}}$ and $\sigma^{\mathrm{hurwitz}}$ are both isomorphisms, we have that

$$\deg(\pi_1|_\Gamma) = \deg(\pi_1^{\mathrm{new}}|_{\Gamma^{\mathrm{new}}})$$

and

$$\deg(\pi_2|_\Gamma) = \deg(\pi_2^{\mathrm{new}}|_{\Gamma^{\mathrm{new}}}).$$

By (3), this proves the proposition.

□

## 5. Equal dynamical degrees when the polynomiality index equals 1

Let $\phi : S^2 \to S^2$ be a branched covering such that every critical point of $\phi$ is strictly pre-periodic; then $\mathrm{PI}(\phi) = 1$. In this case, Theorem 3.1 is at its weakest: it tells us only that $\Theta_k(\mathcal{H}_\phi) \geq \Theta_{k+1}(\mathcal{H}_\phi)$. In fact, equality is possible, as in the following example.

**Example 5.1.** A generic degree 3 rational function on $\mathbb{CP}^1$ has four simple critical points. Let $f$ be such a rational function with simple critical points $x_1, \ldots, x_4$. Set $p_1, \ldots, p_4$ to be the four non-critical points such that $f(p_i) = f(x_i)$ for $i = 1, \ldots, 4$. Now let $\psi$ be any homeomorphism of $\mathbb{CP}^1$ that takes $f(x_i)$ to $p_i$ for $i = 1, \ldots, 4$. Set $\phi = \psi \circ f$; then $\phi$ is a degree 3 branched covering whose simple critical points $x_1, \ldots, x_4$ map respectively to fixed points $p_1, \ldots, p_4$. Thus $\phi$ has finite post-critical set $\mathbf{P} = \{p_1, \ldots, p_4\}$, and $\mathrm{PI}(\phi) = 1$.

Set $\mathbf{P}^{\mathrm{full}} := \mathbf{P} \sqcup \{q_1, \ldots, q_4\}$. Let $\mathcal{H}$ be the Hurwitz space that parametrizes (up to changing coordinates on $\mathbb{CP}^1$) two injections $\iota_1 : \mathbf{P} \hookrightarrow \mathbb{CP}^1$, $\iota_2 : \mathbf{P}^{\mathrm{full}} \hookrightarrow \mathbb{CP}^1$, as well as a degree 3 map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that for $i = 1, \ldots, 4$:

- $f(\iota_2(p_i)) = f(\iota_2(q_i)) = \iota_1(p_i)$,
- $f$ is unramified at $\iota_2(p_i)$, and
- $f$ is simply ramified at $\iota_2(q_i)$.

Each $\iota_2(p_i)$ is called a *co-critical point* of $f$: it is an unramified point mapping to a critical value.

Since $\mathcal{H}$ parametrizes maps with the same markings and branching type as $\phi$, as in Section 2.5 we have that $\mathcal{H}_\phi$ is a connected component of $\mathcal{H}$. On the other hand, since $\mathcal{H}$ parametrizes maps with only simple branching, by [Ful69] $\mathcal{H}$ is connected. Thus $\mathcal{H}_\phi = \mathcal{H}$. The space $\mathcal{H}$ admits two maps to $\mathcal{M}_{0,\mathbf{P}}$: a "target" map $\pi_1$ recording $\iota_1$ and a "source" map $\pi_2$ recording $\iota_2|_{\mathbf{P}}$.

An element of $\mathcal{M}_{0,\mathbf{P}}$ is an injection $\iota : \mathbf{P} \hookrightarrow \mathbb{CP}^1$, considered up to post-composition by Möbius transformations. Given such an equivalence class of injections $[\iota]$, we may post-compose by a Möbius transformation to assume that $\iota(p_1) = 0$, $\iota(p_2) = 1$, and $\iota(p_3) = \infty$. Then $\iota(p_4)$ defines a point of $\mathbb{CP}^1 \smallsetminus \{0, 1, \infty\}$. This gives an identification between $\mathcal{M}_{0,\mathbf{P}}$ and $\mathbb{CP}^1 \smallsetminus \{0, 1, \infty\} = \mathbb{C} \smallsetminus \{0, 1\}$. Thus $\mathcal{H}_\phi$ is one dimensional, and the Hurwitz correspondence $\mathcal{H}_\phi$ has two dynamical degrees: $\Theta_0(\mathcal{H}_\phi) = \deg(\pi_1)$ and $\Theta_1(\mathcal{H}_\phi) = \deg(\pi_2)$.

Now, given an element $[f : (\mathbb{CP}^1, p_1 \dots, p_4, q_1 \dots, q_4) \to (\mathbb{CP}^1, p_1 \dots, p_4)] \in \mathcal{H}_\phi$ we can apply two independent Möbius transformations to assume that $\iota_1(p_1) = \iota_2(q_1) = 0$, $\iota_1(p_2) = \iota_2(q_2) = 1$, and $\iota_1(p_3) = \iota_2(q_3) = \infty$. Thus, in these coordinates, $f$ is a degree 3 rational function such that 0, 1, and $\infty$ are critical fixed points. As a degree 3 rational function, $f$ is of the form

$$f(z) = \frac{z^3 + Az^2 + Cz + D}{Ez^3 + Fz^2 + Gz + H}.$$

The condition that 0 is a critical fixed point forces $C$ and $D$ to vanish; the condition that $\infty$ is a critical fixed point forces $E$ and $F$ to vanish. We re-write $f$ as

$$f(z) = \frac{z^3 + Az^2}{Gz + H}.$$

The condition that 1 be a fixed point forces $H = 1 + A - G$. Re-writing again:

$$f(z) = \frac{z^3 + Az^2}{Gz + 1 + A - G}.$$

Imposing the last condition that 1 be a critical point forces $G = 3 + 2A$. Thus

$$f(z) = \frac{z^3 + Az^2}{(3 + 2A)z - 2 - A}$$

is determined by the parameter $A$. For $A \in \mathbb{C}$, we set

$$f_A(z) = \frac{z^3 + Az^2}{(3 + 2A)z - 2 - A},$$

identifying $\mathcal{H}_\phi$ with a Zariski-open subset of $\mathbb{C}$, parametrized by $A$. A direct computation yields that the fourth and last critical point of $f_A$ is at $(-2A - A^2)/(3+2A)$.

The "target" map $\pi_1$ identifies the positions of the four critical values, $0, 1, \infty$, and

$$f_A\left(\frac{-2A - A^2}{3 + 2A}\right) = -\frac{A^3(2 + A)}{(3 + 2A)^3}.$$

Thus for $A \in \mathcal{H}_\phi$, we have

$$\pi_1(A) = -\frac{A^3(2 + A)}{(3 + 2A)^3} \in \mathbb{C} \smallsetminus \{0, 1\} = \mathcal{M}_{0,\mathbf{P}}.$$

We see from these coordinates that $\pi_1$ has degree 4.

Solving for the inverse images of the critical values, we obtain that the co-critical points mapping respectively to $0$, $1$, $\infty$ and $-(A^3(2+A)/((3+2A)^3)$ are $-A$, $-2-A$, $(2+A)/(3+2A)$, and $A/(3+2A)$. The "source" map $\pi_2$ sends $A \in \mathcal{H}_\phi$ to the cross-ratio of the four co-critical points, which simplifies to

$$\frac{(2+A)^3 A}{-3-2A},$$

which is again degree 4 in $A$. Thus we obtain that

$$\Theta_1(\mathcal{H}_\phi) = \deg(\pi_2) = 4 = \deg(\pi_1) = \Theta_0(\mathcal{H}_\phi).$$

The above example is of a branched covering $\phi$ whose post-critical set has size exactly four, giving us a Hurwitz correspondence on a one-dimensional moduli space. We do not know an example of a branched covering $\phi$ with finite post-critical set with size strictly larger than four (so that $\mathcal{H}_\phi$ is a correspondence on a moduli space of dimension larger than one), such that $\Theta_k(\mathcal{H}_\phi) = \Theta_{k+1}(\mathcal{H}_\phi)$ for some $k$, but we believe that such branched coverings should exist.

We also do not know if there exists a branched covering $\phi$ such that $\mathcal{H}_\phi^{-1}$ is single-valued and $\Theta_k(\mathcal{H}_\phi) = \Theta_{k+1}(\mathcal{H}_\phi)$ for some $k$. Such a map could not satisfy Koch's criteria 1 and 2. On the other hand, every known example of a branched covering $\phi$ with $\mathcal{H}_\phi^{-1}$ single-valued satisfies Koch's criteria.

## 6. A two-dimensional Hurwitz correspondence in coordinates

Example 5.1 describes in coordinates a Hurwitz correspondence on the one-dimensional moduli space $\mathcal{M}_{0,4}$. In this section we describe a simple Hurwitz correspondence on the two-dimensional space $\mathcal{M}_{0,5}$. Let $\mathbf{P} = \{p_1, \ldots, p_5\}$, and let $\mathcal{H}$ be a Hurwitz space that parametrizes

- two injections $\iota_1, \iota_2 : \mathbf{P} \hookrightarrow \mathbb{CP}^1$, and
- a degree 2 map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\iota_1(p_i) = f(\iota_2(p_i))$ for $i = 1, \ldots, 5$, and such that $f$ is ramified at $\iota_2(p_1)$ and $\iota_2(p_3)$,

up to changing coordinates on $\mathbb{CP}^1$. The space $\mathcal{H}$ admits two maps to $\mathcal{M}_{0,\mathbf{P}}$: a "target" map $\pi_1$ recording $\iota_1$ and a "source" map $\pi_2$ recording $\iota_2$. We describe these maps in coordinates below.

An element of $\mathcal{M}_{0,\mathbf{P}}$ is an injection $\iota : \mathbf{P} \hookrightarrow \mathbb{CP}^1$, considered up to post-composition by Möbius transformations. Given such an equivalence class of injections $[\iota]$, we may post-compose by a Möbius transformation to assume that $\iota(p_1) = 0$, $\iota(p_2) = 1$, and $\iota(p_3) = \infty$. Then the tuple $(\iota(p_4), \iota(p_5))$ defines a point of

$$(\mathbb{CP}^1 \smallsetminus \{0, 1, \infty\} \times \mathbb{CP}^1 \smallsetminus \{0, 1, \infty\}) \smallsetminus \{x = y\} \tag{4}$$

This defines an identification between $\mathcal{M}_{0,\mathbf{P}}$ and the space (4). Now, given an element $[\iota_1, \iota_2, f]$ of $\mathcal{H}$, we can apply two independent Möbius transformations to assume that $\iota_1(p_1) = \iota_2(p_1) = 0$, $\iota_1(p_2) = \iota_2(p_2) = 1$, and $\iota_1(p_3) = \iota_2(p_3) = \infty$. Then $f$ must be the map sending $z \in \mathbb{CP}^1$ to $z^2$, and the tuple $(\iota_2(p_4), \iota_2(p_5))$ defines a point of

$$(\mathbb{CP}^1 \smallsetminus \{0, 1, -1, \infty\} \times \mathbb{CP}^1 \smallsetminus \{0, 1, -1, \infty\}) \smallsetminus (\{x = y\} \cup \{x = -y\}) \tag{5}$$

It is straightforward to check that this defines an identification of $\mathcal{H}$ with the space (5). In these coordinates, the map $\pi_1$ sends $(x, y)$ in (5) to $(x^2, y^2)$ in (4), and

$\pi_2$ is the open inclusion from (5) to (4). Thus this Hurwitz correspondence is the 'coordinate-wise square root' multivalued map; it sends $(x, y)$ in

$$\mathcal{M}_{0,\mathbf{P}} \cong (\mathbb{CP}^1 \smallsetminus \{0, 1, \infty\} \times \mathbb{CP}^1 \smallsetminus \{0, 1, \infty\}) \smallsetminus \{x = y\}$$

to the unordered tuple

$$\{(+\sqrt{x}, +\sqrt{y}), (+\sqrt{x}, -\sqrt{y}), (-\sqrt{x}, +\sqrt{y}), (-\sqrt{x}, -\sqrt{y})\}.$$

The correspondence $\mathcal{H}$ arises as $\mathcal{H}_\phi$ for any degree 2 branched cover $\phi : S^2 \to S^2$ with two ramified fixed points $p_1$ and $p_3$, and three labelled unramified (and not post-critical) fixed points $p_2$, $p_4$ and $p_5$. Since we need to label superfluous points that are not post-critical in order to have a correspondence on the two-dimensional $\mathcal{M}_{0,\mathbf{P}}$, this is a somewhat trivial example. In fact, the inverse of $\mathcal{H}$ is the single-valued, holomorphic, 'coordinate-wise squaring' map on $\mathbb{CP}^1 \times \mathbb{CP}^1$ sending $(x, y)$ to $(x^2, y^2)$. Thus we conclude that $\Theta_0(\mathcal{H}) = 4$, $\Theta_1(\mathcal{H}) = 2$, and $\Theta_2(\mathcal{H}) = 1$. Note that the polynomiality index of $\mathcal{H}$ is 2, so the inequalities in Theorem 3.1 are equalities in this example.

However, from this example we obtain certain others by post-composing $\pi_2$ with automorphisms of $\mathcal{M}_{0,\mathbf{P}}$ that are induced by permutations in $S_5$ relabelling the points $p_1, \ldots, p_5$. These new examples are less trivial but also less easily described in coordinates. They correspond to $\mathcal{H}_\phi$ for branched covers $\phi : S^2 \to S^2$ with two ramified period points, either in the same cycle of length 5, or in different cycles of lengths 2 and 3. In [KR16], these correspondences are described in detail in coordinates, and the first dynamical degrees of their single-valued inverse maps are computed.

There is also a computation in Chapter 7 of [Ram17] of the dynamical degrees of a family of two-dimensional Hurwitz correspondences whose inverses are not single-valued. These Hurwitz correspondences are closely related to the one-dimensional correspondence in Example 5.1. For these, computation in coordinates is forbiddingly difficult; instead, a combinatorial algorithm is developed in 7.1 and applied in 7.2.

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