ALGEBRAIC STABILITY OF MEROMORPHIC MAPS DESCENDED FROM THURSTON'S PULLBACK MAPS

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ABSTRACT. Let $\phi : S^2 \to S^2$ be an orientation-preserving branched covering whose post-critical set has finite cardinality n. If ϕ has a fully ramified periodic point p_∞ and satisfies certain additional conditions, then, by work of Koch, ϕ induces a meromorphic self-map R_ϕ on the moduli space $\mathcal{M}_{0,n}$; R_ϕ descends from Thurston's pullback map on Teichmüller space. Here, we relate the dynamics of R_ϕ on $\mathcal{M}_{0,n}$ to the dynamics of ϕ on S^2. Let ℓ be the length of the periodic cycle in which the fully ramified point p_∞ lies; we show that R_ϕ is algebraically stable on the heavy-light Hassett space corresponding to ℓ heavy marked points and $(n - \ell)$ light points.

1. INTRODUCTION

Suppose that $\phi : S^2 \to S^2$ is an orientation preserving branched covering from a topological 2-sphere to itself, of topological degree $d > 1$. A *critical point* of ϕ is a point at which ϕ is not a local homeomorphism. If x is a critical point of ϕ then x has a punctured neighborhood on which ϕ is an r-to1 covering map, with $2 \leq r \leq d$. In this case the *multiplicity* of x is $(r - 1)$; the map ϕ has $(2d - 2)$ critical points counted with multiplicity. Suppose further that the *post-critical set* of ϕ:

$$\mathbf{P} := \{\phi^n(x) | \ x \text{ is a critical point of } \phi \text{ and } n > 0\}$$

is finite. Then ϕ is called *post-critically finite/ PCF*. Thurston [DH93] introduced a holomorphic *pullback* map Th_ϕ induced by ϕ on the Teichmüller space $\mathcal{T}(S^2, \mathbf{P})$ of complex structures on $(S^2, \mathbf{P})$; the branched covering ϕ is homotopic to a PCF rational function on $\mathbb{CP}^1$ if and only if Th_ϕ has a fixed point.

Teichmüller space $\mathcal{T}(S^2, \mathbf{P})$ is a non-algebraic complex manifold but is the universal cover of the algebraic moduli space $\mathcal{M}_{0,\mathbf{P}}$ of markings of $\mathbb{CP}^1$ by the set $\mathbf{P}$. Koch has introduced algebraic dynamical systems on $\mathcal{M}_{0,\mathbf{P}}$ that descend from the transcendental Thurston pullback map. We say that a critical point x of ϕ is *fully ramified* if it has the maximum possible multiplicity of $(d - 1)$, i.e. if the local degree of ϕ at x equals the global degree of ϕ. We say that $y \in S^2$ is a *periodic* point of ϕ if $\exists \ell > 0$ such that $\phi^\ell(y) = y$; if ℓ is chosen to be minimal we say y is periodic of period ℓ. If $\ell = 1$, i.e. if $\phi(y) = y$, then y is a fixed point of ϕ. We say ϕ is a *topological polynomial* if there is a point on S^2 that is fully ramified and fixed. Now, suppose $\phi : (S^2, \mathbf{P}) \to (S^2, \mathbf{P})$ is PCF and satisfies:

Criteria 1.1. *(1) $\mathbf{P}$ contains a periodic and fully ramified point p_∞ of ϕ, and*
(2) either every other critical point of ϕ is also periodic or there is exactly one other critical point of ϕ,

then [Koc13] the "inverse" of Th_ϕ descends to $\mathcal{M}_{0,\mathbf{P}}$. More precisely, there is a meromorphic map $R_\phi : \mathcal{M}_{0,\mathbf{P}} \dashrightarrow \mathcal{M}_{0,\mathbf{P}}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{T}(S^2, \mathbf{P}) & \xrightarrow{\ \mathrm{Th}_\phi\ } & \mathcal{T}(S^2, \mathbf{P}) \\
\text{universal cover} \downarrow & & \downarrow \text{universal cover} \\
\mathcal{M}_{0,\mathbf{P}} & \xleftarrow{\quad\quad R_\phi \quad\quad} & \mathcal{M}_{0,\mathbf{P}}
\end{array}
$$

The moduli space $\mathcal{M}_{0,\mathbf{P}}$ is not compact. It is natural to ask whether R_ϕ extends to a holomorphic self-map of some compactification. Projective space $\mathbb{CP}^{|\mathbf{P}|-3}$ is a compactification of $\mathcal{M}_{0,\mathbf{P}}$. Koch also showed that if the fully ramified point p_∞ in criterion (1.1, 1) is a fixed point of ϕ, i.e. if ϕ is a topological polynomial,

then $R_\phi : \mathbb{CP}^{|\mathbf{P}|-3} \to \mathbb{CP}^{|\mathbf{P}|-3}$ is holomorphic. Moreover, in this case, the union of the forward orbits of the critical loci of R_ϕ is an algebraic set in $\mathbb{CP}^{|\mathbf{P}|-3}$, i.e. R_ϕ is a higher-dimensional analog of a post-critically finite map.

In general, if ϕ is not a topological polynomial, we ask whether R_ϕ extends "nicely" to some compactification of $\mathcal{M}_{0,\mathbf{P}}$. It might be too much to expect that R_ϕ extends to a holomorphic self-map of some compactification. Instead, we study a weaker property called *algebraic stability*. Let X be some smooth projective compactification of $\mathcal{M}_{0,\mathbf{P}}$, so R_ϕ can be considered to be a meromorphic self-map of X. Although $R_\phi : X \dashrightarrow X$ may not extend to a holomorphic or even continuous map from X to itself, it induces pullback actions R_ϕ^* on the singular cohomology groups of X (see Section 3.2 for details of how this action is defined). This action preserves the Hodge decomposition and therefore induces a pullback action on the groups $H^{k,k}(X)$. However, crucially, this action does not respect iteration, i.e. in general we do not have $(R_\phi^n)^* = (R_\phi^*)^n$. Suppose we do have, for some fixed k and all $n > 0$ that $(R_\phi^n)^* = (R_\phi^*)^n$ on $H^{k,k}(X)$; in this case we say that R_ϕ is *k-stable* on X. We say R_ϕ is *algebraically stable* on X if it is k-stable on X for all k. If R_ϕ extends to a holomorphic self-map of X then it is automatically algebraically stable on X, so being algebraically stable may be thought of as 'acting on cohomology like a holomorphic map does'.

Koch and Roeder [KR16] showed that if ϕ has exactly two critical points, both periodic, then R_ϕ is algebraically stable on the Deligne-Mumford compactification of $\mathcal{M}_{0,\mathbf{P}}$. This was generalized by Koch, Speyer and the author [Ram18]: If ϕ is PCF and R_ϕ exists, then R_ϕ is algebraically stable on the Deligne-Mumford compactification. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$ of $\mathcal{M}_{0,\mathbf{P}}$ is "large" as measured by the ranks of its cohomology groups and the number of irreducible components of $\overline{\mathcal{M}}_{0,\mathbf{P}} \smallsetminus \mathcal{M}_{0,\mathbf{P}}$. On the other hand, by [Koc13], if ϕ satisfies criteria (1.1, 1) and (1.1, 2) and is a topological polynomial, then R_ϕ is holomorphic, thus algebraically stable, on the much smaller compactification $\mathbb{CP}^{|\mathbf{P}|-3}$.

In this paper, we interpolate between [Koc13] and [Ram18] by identifying a relationship between the topological dynamics of ϕ and the algebraic dynamics of R_ϕ. We find a sequence $\{X_\ell\}_{\ell=1,\dots,|\mathbf{P}|}$ of of smooth projective compactifications of $\mathcal{M}_{0,\mathbf{P}}$, with $X_1 = \mathbb{CP}^{|\mathbf{P}|-3}$ and $X_{|\mathbf{P}|} = \overline{\mathcal{M}}_{0,\mathbf{P}}$, such that for all ℓ, there is a birational holomorphic map $\rho_{\ell+1,\ell} : X_{\ell+1} \to X_\ell$. We show:

Theorem 1.2. *If ϕ is a branched covering with post-critical set $\mathbf{P}$ satisfying criteria (1.1, 1) and (1.1, 2) and such that the fully ramified point p_∞ of (1.1, 1) is in a cycle of period ℓ, then the meromorphic map $R_\phi : \mathcal{M}_{0,\mathbf{P}} \dashrightarrow \mathcal{M}_{0,\mathbf{P}}$ is algebraically stable on X_ℓ.*

The ℓ-th compactification X_ℓ is the *heavy/light Hassett space* corresponding to ℓ heavy weights and $(|\mathbf{P}|-\ell)$ light weights, constructed by Hassett and parametrizing *weighted stable curves* ([Has03], see Sections 1.3 and 2.3 for details). The space X_ℓ can be obtained as an iterated blow-up of $\mathbb{CP}^{|P|-3}$. The last three compactifications, $X_{|\mathbf{P}|-2}$, $X_{|\mathbf{P}|-1}$ and $X_{|\mathbf{P}|}$, are isomorphic to each other, but for $\ell \leq (|P|-3)$, the birational map $\rho_{\ell+1,\ell} : X_{\ell+1} \to X_\ell$ contracts in dimension certain subvarieties in the boundary $X_{\ell+1} \smallsetminus \mathcal{M}_{0,\mathbf{P}}$. Under the pushforward map $(\rho_{\ell+1,\ell})_*$ on homology, the classes of the contracted subvarieties go to zero. Thus for $\ell = 1, \dots, (|P| - 2)$, the spaces X_ℓ are all distinct. For small ℓ, the compactification X_ℓ is "small", as measured by the number of components in its boundary $X_\ell \smallsetminus \mathcal{M}_{0,\mathbf{P}}$ and the ranks of its cohomology groups. If ϕ is a branched covering with a fully ramified point p_∞ in a periodic cycle (i.e. satisfying criterion (1.1, 1)), then the length ℓ of that cycle measures how much ϕ "resembles" a topological polynomial: If $\ell = 1$ then ϕ is a topological polynomial; if $\ell > 1$ is small then ϕ resembles a topological polynomial. We give a non-rigorous interpretation for Theorem 1.2:

If ϕ resembles a topological polynomial, then R_ϕ is algebraically stable on a small compactification of $\mathcal{M}_{0,\mathbf{P}}$.

1.1. **Dynamical degrees and the significance of algebraic stability.** Let $g : U \dashrightarrow U$ be a meromorphic self-map of a smooth quasiprojective complex variety, and let X be some smooth projective compactification of U. As discussed above and described in Section 3.2, g induces a pullback action on $H^{k,k}(X)$, but we may not have $(g^n)^* = (g^*)^n$. However, we obtain an important numerical invariant of g by considering the asymptotics of the operators $(g^n)^*$. Pick any norm on $H^{k,k}(X)$. The *k-th dynamical degree* of g is the non-negative real number

$$\lim_{n\to\infty} \left\| (g^n)^* : H^{k,k}(X) \to H^{k,k}(X) \right\|^{1/n}.$$

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This limit exists, is independent of the choice of norm, and also of the choice of compactification X (Dinh and Sibony [DS05] in the complex setting and Truong [Tru15] in the algebraic setting). Thus the k-th dynamical degree is intrinsic to the action of g on the possibly non-compact space U. The dynamical degrees of a map measure its complexity: The topological entropy of a holomorphic map is equal to the logarithm of its largest dynamical degree (Gromov [Gro03] and Yomdin [Yom87]) and the topological entropy of a meromorphic map is at most the logarithm of its largest dynamical degree (Dinh and Sibony [DS05]).

Given $g : U \dashrightarrow U$, if there exists a compactification X of U on which g is k-stable, then the k-th dynamical degree of g is the absolute value of the largest eigenvalue of g^* acting on $H^{k,k}(X)$, thus an algebraic integer whose degree over $\mathbb{Q}$ is at most $\mathrm{Rank}(H^{k,k}(X))$. The degree over $\mathbb{Q}$ of an algebraic integer is a measure of its complexity. Thus if g is k-stable on X, then $\mathrm{Rank}(H^{k,k}(X))$ gives an upper bound on a certain type of complexity of the map g.

A common strategy to compute the dynamical degrees of a given map is to look for birational models on which the map is k-stable/algebraically stable. However, Favre [F+03] has given examples of monomial maps $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ for which no such birational models exist. Computing the k-th dynamical degree of a map which is either provably not k-stable or not known to be k-stable on any model involves dealing with the pullbacks along infinitely many iterates, and is difficult to impossible. Also, given a meromorphic map, there is no known strategy to find a birational model on which it is k-stable. Thus there are only a few examples of meromorphic maps whose dynamical degrees have been computed.

In this regard, monomial maps are perhaps the best understood. A monomial map $g : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ is determined by an n-by-n integer matrix M_g of exponents. Work of Jonsson and Wulcan [JW11] for $k = 1$ and Lin [Lin12a] in general gives criteria on M_g for the existence of a compactification on which g is k-stable. When those criteria are satisfied, they give explicit descriptions of toric compactifications on which the maps are 1-stable (Jonsson-Wulcan)/algebraically stable (Lin). These works also lead to formulas for the dynamical degrees of monomial maps: the k-th dynamical degree of g is the absolute value of the product of the k largest eigenvalues of the integer matrix M_g, thus an algebraic integer of degree at most $\binom{n}{k}$ [Lin12b]. In addition to monomial maps, birational surface transformations are also well-studied: Diller and Favre ([DF01]) showed that every birational transformation g of a projective surface X is 1-stable on some birational model of X. They use this result to show that the first dynamical degree of g is either 1, a Salem number, or a Pisot number. Blanc and Cantat ([BC16]) describe the set of Salem and Pisot numbers that arise as dynamical degrees of birational surface transformations. Given the difficulty in computing dynamical degrees, there are several open questions about them. Until recently, it was not known whether every dynamical degree is an algebraic integer: Bell-Diller-Jonsson [BDJ19] have recently found a map with a transcendental dynamical degree.

It had already been established in [Ram18] that every R_ϕ is algebraically stable on the Deligne-Mumford compactifcation, and thus has all dynamical degrees are algebraic integers. Theorem 1.2 offers a more refined view, by relating a type of complexity of R_ϕ (the length of the periodic cycle of p_∞) to a type of complexity of R_ϕ (the degree over $\mathbb{Q}$ of its k-th dynamical degree). As a corollary to Theorem 1.2, we obtain:

Corollary 1.3. *If ϕ is a branched covering with finite post-critical set $\mathbf{P}$ satisfying criteria (1.1, 1) and (1.1, 2) and such that the fully ramified point p_∞ of (1.1, 1) is in a cycle of period ℓ, then the kth dynamical degree of R_ϕ is an algebraic integer whose degree over $\mathbb{Q}$ is at most $\mathrm{Rank}(H^{k,k}(X_\ell))$.*

The isomorphism class of X_ℓ depends on ℓ and $|\mathbf{P}|$. For fixed $|\mathbf{P}|$, fixed $k \in \{1, \ldots, (\dim_{\mathbb{C}}(\mathcal{M}_{0,\mathbf{P}}) - 1)\}$ and for $\ell_1, \ell_2 \in \{1, \ldots, (|\mathbf{P}| - 2)\}$, if $\ell_1 < \ell_2$ then $\mathrm{Rank}(H^{k,k}(X_{\ell_1})) < \mathrm{Rank}(H^{k,k}(X_{\ell_2}))$. Thus, if we fix $N > 0$ the cardinality of post-critical set, and consider

$$\phi \in \{\text{Branched coverings with } |\text{post-critical set}| = N \text{ satisfying } (1.1, 1) \text{ and } (1.1, 2)\},$$

then the shorter the length ℓ of the periodic cycle of the fully ramified point of ϕ, the better an upper bound one can obtain on the degree over $\mathbb{Q}$ of the k-th dynamical degree of R_ϕ. More informally:

If ϕ resembles a topological polynomial, then the k-th dynamical degree of R_ϕ is an algebraic integer of low degree over $\mathbb{Q}$.

The sequence of dynamical degrees of any meromorphic map is log-concave [DS05]; the sequence of dynamical degrees of a holomorphic map on $\mathbb{CP}^r$ is log-linear. There is an analysis, in [Ram19], of how the k-th dynamical degree of R_ϕ depends on k. It is shown that the sequence $\{k\text{th dynamical degree of } R_\phi\}_k$

increases strictly with k, and that the sequence of logarithms dynamical degrees of R_ϕ is less concave if ϕ resembles a topological polynomial. The precise statement of the result in [Ram19] is very different from the statement of Theorem 1.2, and the proofs are unrelated as well. However, the two statements share the following informal interpretation (generalizing [Koc13]):

If ϕ resembles a topological polynomial, then the dynamics of R_ϕ resemble those of a holomorphic map on $\mathbb{CP}^{|P|-3}$.

It would be interesting to have a conceptual explanation for the relationship between Theorem 1.2 and the results in [Ram19].

1.2. **Hurwitz correspondences.**
Koch's results in [Koc13] are more general than described above. Let $\phi : S^2 \to S^2$ be a degree d post-critically finite branched covering with post-critical set $\mathbf{P}$. If ϕ does not satisfy criteria (1.1, 1) and (1.1, 2), then the meromorphic map R_ϕ need not exist. However, it is still true that the transcendental pullback map Th_ϕ induced by ϕ on $\mathcal{T}(S^2, \mathbf{P})$ descends to an algebraic dynamical system on $\mathcal{M}_{0,\mathbf{P}}$. However, in general, this is a multivalued map. More precisely, there is an algebraic variety $\mathcal{H}_\phi$ admitting a covering map from $\mathcal{T}(S^2, \mathbf{P})$ as well as two maps π_1 and π_2 to $\mathcal{M}_{0,\mathbf{P}}$ such that π_1 is a covering map and the following diagram commutes [Koc13]:

The variety $\mathcal{H}_\phi$ is a *Hurwitz space*, a moduli space parametrizing degree d regular maps $f : (\mathbb{CP}^1, \mathbf{P}) \to (\mathbb{CP}^1, \mathbf{P})$, with analogous branching to ϕ. The Hurwitz space $\mathcal{H}_\phi$ is *non-dynamical* in the sense that it parametrizes maps up to separate changes of coordinates on source and target $\mathbb{CP}^1$; this means that the behavior under iteration of $[f] \in \mathcal{H}_\phi$ is not well-defined. The multivalued map $\pi_2 \circ \pi_1^{-1}$ is called a *Hurwitz correspondence*, and considered to be an algebraic "shadow" of Th_ϕ. Hurwitz correspondences can be defined purely algebro-geometrically, with no reference to branched coverings of the sphere and to Thurston's pullback map. (Section 3; see [Ram17] or [Ram18] for more details). In the case that ϕ satisfies criteria (1.1, 1) and (1.1, 2), Koch showed that π_2 is generically one-to-one; the meromorphic map R_ϕ is $\pi_1 \circ \pi_2^{-1}$, i.e. a single-valued but meromorphic "inverse" of the multi-valued but holomorphic Hurwitz correspondence. Thus the graph of R_ϕ is (up to birational equivalence) the Hurwitz space $\mathcal{H}_\phi$.

1.3. **Combinatorial compactifications of moduli spaces and the proof of Theorem 1.2.**
The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$ of $\mathcal{M}_{0,\mathbf{P}}$ is a moduli space of stable nodal genus zero curves with smooth distinct points marked by elements of $\mathbf{P}$ (see Section 2 for definitions and details). The *boundary* $\overline{\mathcal{M}}_{0,\mathbf{P}} \smallsetminus \mathcal{M}_{0,\mathbf{P}}$ has a combinatorial stratification that is very useful: for example, this stratification is used to give explicit descriptions of the cohomology groups of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ [Kee92].

Hassett's [Has03] alternate weighted stable curves compactifications of $\mathcal{M}_{0,\mathbf{P}}$ parametrize nodal genus zero curves with smooth points — not necessarily distinct — marked by elements of $\mathbf{P}$. Let $\boldsymbol{\epsilon} : \mathbf{P} \to \mathbb{Q} \cap (0, 1]$ be an assignment, to every $p \in \mathbf{P}$ of some rational 'weight', such that the sum of the weights of all $p \in \mathbf{P}$ is greater than 2. Then there is a smooth projective compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon})$ parametrizing nodal genus zero curves with smooth points marked by $\mathbf{P}$; the marked points corresponding to a subset of $\mathbf{P}$ may coincide as long as the sum of the weights of the points in that set is no greater than 1. The *boundary* $\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon}) \smallsetminus \mathcal{M}_{0,\mathbf{P}}$ has a combinatorial stratification that is related to the stratification of the boundary of the Deligne-Mumford compactification. Also, $\overline{\mathcal{M}}_{0,\mathbf{P}}$ admits a regular birational map $\rho_{\boldsymbol{\epsilon}}$ to $\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon})$, with fibers that may be positive dimensional over the boundary.

The Hurwitz space $\mathcal{H}_\phi$ admits an *admissible covers* compactification $\overline{\mathcal{H}}_\phi$ constructed by Harris and Mumford [HM82]; this compactification extends π_1 and π_2 to regular maps to $\overline{\mathcal{M}}_{0,\mathbf{P}}$. The map $\pi_1 : \overline{\mathcal{H}}_\phi \to \overline{\mathcal{M}}_{0,\mathbf{P}}$ is finite and flat; this fact was used in [Ram18] to conclude algebraic stability of all Hurwitz correspondences

on $\overline{\mathcal{M}}_{0,\mathbf{P}}$. The boundary of $\overline{\mathcal{H}}_\phi$ has a stratification analogous to and compatible with the stratification of $\overline{\mathcal{M}}_{0,\mathbf{P}}$.

Now, suppose ϕ satisfies criteria (1.1, 1) and (1.1, 2), so as per the previous section, $\mathcal{H}_\phi$ is the graph of the meromorphic map R_ϕ. We set $\mathbf{P}_\infty \subseteq \mathbf{P}$ to be the subset of points in the periodic cycle containing the fully ramified point p_∞. We assign weight $\boldsymbol{\epsilon}(p)$ to $p \in \mathbf{P}$ by the rule $\boldsymbol{\epsilon}(p) = 1$ for $p \in \mathbf{P}_\infty$ (these are heavy points), and $\boldsymbol{\epsilon}(p) \ll 1$ for $p \in (\mathbf{P} \smallsetminus \mathbf{P}_\infty)$ (these points are vanishingly light). We formulate and apply a combinatorial analysis of the stratification of $\overline{\mathcal{H}}_\phi$ and of the fibers of $\rho_{\boldsymbol{\epsilon}}$ to show, roughly speaking, that positive-dimensional fibers of $\rho_{\boldsymbol{\epsilon}} \circ \pi_1$ are also positive-dimensional fibers of $\rho_{\boldsymbol{\epsilon}} \circ \pi_2$, and so the meromorphic map R_ϕ has finite fibers on $\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon})$. When this analysis is applied to the induced map on cohomology, we obtain the algebraic stability result of Theorem 1.2.

1.4. **Organization.** We begin by introducing $\mathcal{M}_{0,\mathbf{P}}$ and its compactifications in Section 2 and Hurwitz spaces in Section 3. In these background sections we frequently refer back to [Ram18]. Section 4 contains the key technical contribution of this paper: here, we relate the combinatorics of compactifications of Hurwitz spaces with the combinatorics of certain Hassett spaces. In Section 5, we bring all the ingredients together to prove Theorem 1.2.

1.5. **Conventions.** We work over $\mathbb{C}$.

2. THE MODULI SPACE $\mathcal{M}_{0,\mathbf{P}}$ AND ITS COMPACTIFICATIONS

Let $\mathbf{P}$ be a finite set of cardinality at least 3. There is a smooth quasiprojective variety $\mathcal{M}_{0,\mathbf{P}}$ of complex dimension $(|\mathbf{P}| - 3)$ that parametrizes injections $\iota : \mathbf{P} \hookrightarrow \mathbb{CP}^1$ up to the equivalence $\iota \sim \psi \circ \iota$ for any Möbius transformation ψ. The variety $\mathcal{M}_{0,\mathbf{P}}$ is not compact — the limit of a one-parameter family of injections $\mathbf{P} \hookrightarrow \mathbb{CP}^1$ may irreparably fail to be an injection into $\mathbb{CP}^1$. There are a number of smooth projective compactifications of $\mathcal{M}_{0,\mathbf{P}}$. The *boundary* of a compactification X is the complement $X \smallsetminus \mathcal{M}_{0,\mathbf{P}}$. If X is a modular compactification of $\mathcal{M}_{0,\mathbf{P}}$, i.e. one that extends its interpretation as a moduli space of maps from $\mathbf{P}$ to an algebraic curve, then points on the boundary of X must correspond to degenerate cases in which either the map is not injective, or the curve has singularities, or both.

2.1. **Stable curves and the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,\mathbf{P}}$.** The Deligne-Mumford compactification is in some sense the universal and largest modular compactification of $\mathcal{M}_{0,\mathbf{P}}$: It admits a holomorphic birational map to every other modular compactification (Smyth, [Smy09]).

Definition 2.1. Let $\mathbf{P}$ be a finite set. A $\mathbf{P}$-*marked nodal genus zero algebraic curve* is a connected, proper, possibly nodal algebraic curve C of arithmetic genus zero, together with an injection ι from $\mathbf{P}$ into the smooth locus of C. We say that (C, ι) is *stable* if C has no nontrivial automorphisms that commute with ι.

Concretely, since C has arithmetic genus zero, it is isomorphic to a tree of $\mathbb{CP}^1$s attached at nodes. A *special point* on C is a point of C that is either a node, or in the image of ι. The condition that (C, ι) have no non-trivial automorphism is equivalent to the condition that every irreducible component of C have at least three special points. For the remainder of this section we suppose that $\mathbf{P}$ is a finite set of cardinality at least 3; by works of Deligne, Grothendieck, Knudsen, and Mumford, there is a smooth projective variety $\overline{\mathcal{M}}_{0,\mathbf{P}}$ that parametrizes stable $\mathbf{P}$-marked genus zero algebraic curves, and that contains $\mathcal{M}_{0,\mathbf{P}}$ as a dense open subset. The boundary $\overline{\mathcal{M}}_{0,\mathbf{P}} \smallsetminus \mathcal{M}_{0,\mathbf{P}}$ has codimension one. Points in the boundary correspond to injective maps from $\mathbf{P}$ to a nodal algebraic curve; for a general point on the boundary this curve has two irreducible components. The homeomorphism class of a marked nodal curve is encoded combinatorially by a marked tree. For this reason, we introduce below some notation and terminology for describing marked trees and nodal curves. Note that every node on a genus zero curve is disconnecting, in fact, the complement of any node has exactly two connected components.

Definition 2.2. Let (C, ι) be a $\mathbf{P}$-marked nodal genus zero curve. If C_α is an irreducible component of C, $x \in C \smallsetminus C_\alpha$ and $\eta \in C \smallsetminus \{x\}$ is a node, we say η *connects* C_α *to* x if $C_\alpha \smallsetminus \{\eta\}$ and x are in distinct connected components of $C \smallsetminus \{\eta\}$. If η connects C_α to $\iota(p)$ for $p \in \mathbf{P}$, we say that η connects C_α to p. Similarly, if C_α and C_β are two irreducible components of C and $\eta \in C$ is a node, we say η *connects* C_α *to* C_β if $C_\alpha \smallsetminus \{\eta\}$ and $C_\beta \smallsetminus \{\eta\}$ are in distinct connected components of $C \smallsetminus \{\eta\}$.

If $x \notin C_\alpha$ for some irreducible component C_α of C, then there is a unique node $\eta \in C_\alpha$ that connects C_α to x. Similarly, if C_α and C_β are distinct irreducible components, then there is a unique node η on C_α that connects C_α to C_β.

Definition 2.3. A $\mathbf{P}$-*marked tree* is a 'graph with legs' σ defined as follows: σ has vertices, edges joining pairs of vertices, and *legs* marked by elements of $\mathbf{P}$ that are attached to vertices, such that the resulting graph is connected and has no cycles. More formally, σ carries the data of: a finite set $\mathrm{Verts}(\sigma)$ of vertices, a finite set $\mathrm{Edges}(\sigma)$ of edges, a map $\mathrm{Edges}(\sigma) \to \mathrm{Sym}^2(\mathrm{Verts}(\sigma))$ encoding the adjacency, a set of legs of σ that is canonically identified with $\mathbf{P}$, and a map $\mathrm{Mark} : \mathbf{P} \to \mathrm{Verts}(\sigma)$ encoding how the legs are attached. For a vertex v on σ, set of *flags* on v is defined as: $\mathrm{Flags}_v := \{\text{Legs attached to } v\} \cup \{\text{edges incident to } v\}$. The *valence* of v, denoted $|v|$ is defined to be the cardinality of Flags_v. We define the *moduli dimension* $\mathrm{md}(v)$ of $v \in \mathrm{Verts}(\sigma)$ to be $|v| - 3$. We say that σ is stable if every vertex on σ has valence at least 3, or, equivalently, if very vertex has non-negative moduli dimension. Suppose σ is a $\mathbf{P}$-marked tree, and v is a vertex of σ. For $p \in \mathbf{P}$, we define $\delta(v \to p)$ to be the unique flag in Flags_v that connects the leg p to v, i.e. is part of the unique (non-repeating) path in σ from v to p. If $\mathrm{Mark}(p) = v$ then $\delta(v \to p) = p$; otherwise $\delta(v \to p)$ is an edge. Similarly, for v_1 and v_2 two distinct vertices of σ, we define $\delta(v_1 \to v_2)$ to be the unique flag in Flags_{v_1} that is part of the path in σ from v_1 to v_2.

Definition 2.4. Let (C, ι) be a $\mathbf{P}$-marked nodal genus zero curve. Its *dual tree* is the $\mathbf{P}$-marked tree σ defined as follows. The vertices v of σ correspond to irreducible components C_v of C. Two vertices v_1 and v_2 are joined by an edge if and only if the components C_{v_1} and C_{v_2} intersect at a node. Thus nodes of C correspond to edges of σ. For each marked point $\iota(p)$ on C_v, we attach a leg marked by p to the vertex v, i.e. $\mathrm{Mark}(p) = v$. The graph σ has no loops because C has arithmetic genus zero. Note that σ is stable if and only if (C, ι) is.

For fixed $\mathbf{P}$, there are finitely many isomorphism classes of stable $\mathbf{P}$-marked trees, and each of these arises as the dual tree of some $\mathbf{P}$-marked stable genus zero curve. The classification of stable curves by topological type gives a stratification of $\overline{\mathcal{M}}_{0,\mathbf{P}}$.

Definition 2.5. Given σ a stable $\mathbf{P}$-marked tree, the closure S_σ of the locus S_σ° of curves with dual graph σ is an irreducible subvariety of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ isomorphic to

$$\tag{1} \prod_{v \in \mathrm{Verts}(\sigma)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$$

We refer to S_σ as a *boundary stratum* of $\overline{\mathcal{M}}_{0,\mathbf{P}}$; Boundary strata on $\overline{\mathcal{M}}_{0,\mathbf{P}}$ are in bijection with isomorphism classes of stable $\mathbf{P}$-marked trees.

From the above decomposition 1 of S_σ into a product we obtain that the dimension of a boundary stratum S_σ is

$$\tag{2} \sum_{v \in \mathrm{Verts}(\sigma)} \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{0,\mathrm{Flags}_v}) = \sum_{v \in \mathrm{Verts}(\sigma)} \mathrm{md}(v)$$

2.2. Stabilization and forgetful maps. Suppose $|\mathbf{P}| \geq 3$ and (C, ι) is a $\mathbf{P}$-marked nodal genus zero curve. Then there is a unique curve C', together with a surjective map $\mathrm{st} : C \to C'$, such that $(C', \mathrm{st} \circ \iota)$ is stable. The curve C' is called the *stabilization* of C, and is obtained from C as follows. Let σ be the dual tree of C. Given an irreducible component C_v of C corresponding to vertex v of σ, we say that C_v (resp. v) is $\mathbf{P}$-*stable* if there are at least three special points on C_v of the form either a marked point $\iota(p)$ or a node η that connects C_v to some marked point p. This is equivalent to the condition that there are at least three flags on v of the form $\delta(v \to p)$ for some $p \in \mathbf{P}$. We obtain C' from C by contracting to a point each connected component of the closure of $C \smallsetminus \bigcup_{v \ \mathbf{P}\text{-stable}} C_v$. The map $\mathrm{st} : C \to C'$ is the resulting map: a component C_v maps isomorphically onto its image in C' if and only if it is stable; otherwise $\mathrm{st}(C_v)$ is a point.

Now, let $j : \mathbf{P}' \hookrightarrow \mathbf{P}$ be an injection of finite sets, where $|\mathbf{P}|, |\mathbf{P}'| \geq 3$. There is a *forgetful map* $\mu : \mathcal{M}_{0,\mathbf{P}} \to \mathcal{M}_{0,\mathbf{P}'}$ sending $[(\mathbb{CP}^1, \iota)]$ to $[(\mathbb{CP}^1, \iota \circ j)]$. If (C, ι) is a $\mathbf{P}$-marked stable curve, then $(C, \iota \circ j)$ is not necessarily stable. However, we can obtain from $(C, \iota \circ j)$ a stable curve by stabilizing as described above. In this way, μ extends to a regular map from $\overline{\mathcal{M}}_{0,\mathbf{P}}$ to $\overline{\mathcal{M}}_{0,\mathbf{P}'}$. If σ is a $\mathbf{P}$-marked stable tree, forgetting the points in $\mathbf{P} \smallsetminus \mathbf{P}'$ yields a $\mathbf{P}'$-marked tree, in general not stable. We have:

Lemma 2.6. *If S_σ is a boundary stratum of $\overline{\mathcal{M}}_{0,\mathbf{P}}$, then $\mu(S_\sigma)$ is a boundary stratum of $\overline{\mathcal{M}}_{0,\mathbf{P}'}$, and the restriction of μ to S_σ factors through the projection:*

$$(3) \qquad S_\sigma = \prod_{v \in \mathrm{Verts}(\sigma)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v} \to \prod_{\substack{v \in \mathrm{Verts}(\sigma) \\ \mathbf{P}'\text{-stable}}} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v},$$

2.3. **Hassett spaces/Moduli spaces of weighted stable curves.** These are alternate compactifications of $\mathcal{M}_{0,\mathbf{P}}$ constructed by Hassett in [Has03]. Points in the boundary of these compactifications parametrize possibly nodal curves C that are marked by elements of $\mathbf{P}$; but here the marked points are assigned rational weights that prescribe the extent to which they are allowed to coincide.

Definition 2.7. A *weight datum* on $\mathcal{M}_{0,\mathbf{P}}$ is a map $\epsilon : \mathbf{P} \to \mathbb{Q} \cap (0,1]$ such that $\sum_{p \in \mathbf{P}} \epsilon(p) > 2$.

Definition 2.8. Let ϵ be a weight datum on $\mathcal{M}_{0,\mathbf{P}}$. A $\mathbf{P}$-*marked ϵ-stable genus zero curve* is a possibly nodal curve C of arithmetic genus zero, together with a (not necessarily injective) map $\mathrm{mp} : \mathbf{P} \to$ (smooth locus of C), such that

(1) If $\mathrm{mp}(p_1) = \cdots = \mathrm{mp}(p_s)$ then $\epsilon(p_1) + \cdots + \epsilon(p_s) \leq 1$, and
(2) For every irreducible component C_v,

$$\#\{\text{nodes on } C_v\} + \sum_{\{p \mid \mathrm{mp}(p) \in C_v\}} \epsilon(p) > 2.$$

Like a stable curve, C is isomorphic to a tree of $\mathbb{CP}^1$s attached at nodes, and is marked by elements of $\mathbf{P}$. Condition (1) specifies that marked points may coincide if their combined weights don't exceed one. Condition (2) ensures that any node on C partitions the set $\mathbf{P}$ into two sets, both of which have total weight greater than one.

Definition-Theorem 2.9 (Hassett, [Has03]). (1) Given a weight datum ϵ on $\mathcal{M}_{0,\mathbf{P}}$, there is a smooth projective variety $\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ that parametrizes $\mathbf{P}$-marked ϵ-stable genus zero curves and contains $\mathcal{M}_{0,\mathbf{P}}$ as a dense open set.
(2) There is a *reduction map* $\rho_\epsilon : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ that respects the open inclusion of $\mathcal{M}_{0,\mathbf{P}}$ into both spaces.
(3) If ϵ_1 and ϵ_2 are two weight data on $\mathcal{M}_{0,\mathbf{P}}$ such that for every $p \in \mathbf{P}$, $\epsilon_1(p) \geq \epsilon_2(p)$, then there is a *generalized reduction map* $\rho_{\epsilon_1,\epsilon_2} : \overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon_1) \to \overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon_2)$ such that $\rho_{\epsilon_2} = \rho_{\epsilon_1,\epsilon_2} \circ \rho_{\epsilon_1}$.

Example 2.10. (1) Set $\epsilon(p) = 1$ for all $p \in \mathbf{P}$; this is a weight datum as long as $|\mathbf{P}| \geq 3$. Then the notions of stability and ϵ-stability coincide, so $\overline{\mathcal{M}}_{0,\mathbf{P}} \cong \overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$. Thus the Deligne-Mumford compactification is a special case of a Hassett space.
(2) Fix $p_\infty \in \mathbf{P}$, and $\epsilon \in \mathbb{Q}$ such that $(|\mathbf{P}| - 2) < (1/\epsilon)$ but $(|\mathbf{P}| - 1) > (1/\epsilon)$. Set $\epsilon(p_\infty) = 1$, and $\epsilon(p) = \epsilon$ for all $p \neq p_\infty$. Then $\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon) \cong \mathbb{CP}^{|\mathbf{P}|-3}$.

The reduction map ρ_ϵ can be described explicitly: Suppose (C, ι) is a $\mathbf{P}$-marked nodal genus zero curve. Then there is a unique curve C', together with a surjective map $\mathrm{st}_\epsilon : C \to C'$, such that $(C', \mathrm{st}_\epsilon \circ \iota)$ is ϵ-stable. The curve C' is called the ϵ-*stabilization* of C, and is obtained from C as follows. Let σ be the dual tree of C.

Definition 2.11. Given an irreducible component C_v of C corresponding to vertex v of σ, we say that v (resp. C_v) is ϵ-*stable* if:

$$(4) \qquad \sum_{\mathrm{fl} \in \mathrm{Flags}_v} \min\left\{ 1, \sum_{\{p \mid \mathrm{fl} = \delta(v \to p)\}} \epsilon(p) \right\} > 2.$$

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We obtain C' by contracting to a point each connected component of the closure of $(C \smallsetminus \bigcup_{(v\ \epsilon\text{-stable})} C_v)$. The induced stabilization map $\mathrm{st}_\epsilon : C \to C'$ maps a component C_v isomorphically onto its image in C' if and only if C_v is ϵ-stable; otherwise $\mathrm{st}_\epsilon(C_v)$ is a point. The reduction map $\rho_\epsilon : \overline{\mathcal{M}}_{0,\mathbf{P}} \to \overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ sends $[(C,\iota)] \in \overline{\mathcal{M}}_{0,\mathbf{P}}$ to its ϵ-stabilization. Given a boundary stratum S_σ of $\overline{\mathcal{M}}_{0,\mathbf{P}}$, we consider the natural projection map

$$(5) \qquad S_\sigma = \prod_{v \in \mathrm{Verts}(\sigma)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v} \to \prod_{\substack{v \in \mathrm{Verts}(\sigma) \\ \epsilon\text{-stable}}} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v} =: \mathrm{PS}_\sigma,$$

where PS_σ is defined to be the product on the right. Set PS_σ° to be the product $\prod_{\substack{v \in \mathrm{Verts}(\sigma) \\ \epsilon\text{-stable}}} \mathcal{M}_{0,\mathrm{Flags}_v}$; this is a dense open subset of PS_σ. We have the following lemma.

Lemma 2.12. *Let $S_\sigma \subseteq \overline{\mathcal{M}}_{0,\mathbf{P}}$ be a boundary stratum. Then:*
  (1) the restriction $\rho_\epsilon|_{S_\sigma}$ factors through the projection in equation (5),
  (2) the induced map from PS_σ to $\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ is birational onto its image, and an isomorphism onto its image when restricted to PS_σ°, and
  (3) $\dim_{\mathbb{C}}(\rho_\epsilon(S_\sigma)) = \sum_{\substack{v \in \mathrm{Verts}(\sigma) \\ \epsilon\text{-stable}}} \mathrm{md}(v)$

2.4. **Heavy/light Hassett spaces.** In this paper we will be primarily concerned with a certain subclass of spaces of weighted stable curves. These spaces are called *heavy/light* Hassett spaces and have appeared in studies of tropical moduli spaces of curves [CHMR16, KKL19].

Definition 2.13. Suppose $|\mathbf{P}| \geq 3$, and there is a decomposition $\mathbf{P} = \mathbf{P}_{hvy} \sqcup \mathbf{P}_{lt}$ with $|\mathbf{P}_{hvy}| \geq 2$. Let $\epsilon > 0$ be any rational number such that $|\mathbf{P}_{lt}| < (1/\epsilon)$. Then the weight datum ϵ sending $p \in \mathbf{P}_{hvy}$ to 1 (these are the heavy points) and $p \in \mathbf{P}_{lt}$ to ϵ (these are the light points) is called a heavy/light weight datum, and the resulting moduli space $\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ is called a heavy/light Hassett space.

Heavy/light weight data ϵ can be characterized in the following manner: on a ϵ-stable curve, heavy marked points may not coincide with each other or with light marked points, but light marked points may coincide with each other to an arbitrary extent. Thus the isomorphism class of the moduli space $\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ does not depend on the value of the rational number ϵ; it depends only on the numbers of heavy and light points. If the number of light points is one or two, then the resulting heavy/light space is isomorphic to the Deligne-Mumford compactification. The number of heavy points must be at least 2; if that number is exactly 2, the resulting heavy/light space is a toric variety called a *Losev-Manin* space and has been studied independently.

Since we will be interested in understanding the reduction maps from $\overline{\mathcal{M}}_{0,\mathbf{P}}$ to various heavy/light Hassett spaces, the following characterization of ϵ-stableness for heavy/light data will be useful.

Lemma 2.14. *Suppose ϵ is a heavy/light weight datum, with $\mathbf{P} = \mathbf{P}_{hvy} \sqcup \mathbf{P}_{lt}$. Then*
  (1) (Statement about curves.)
      (a) An irreducible component C_1 of a nodal $\mathbf{P}$-marked curve (C,ι) is not ϵ-stable if

$$(6) \qquad |\{p \in \mathbf{P}_{hvy}\ s.t.\ \iota(p) \in C_1\} \cup \{\eta \in C_1\ node\ connecting\ C_1\ to\ some\ p \in \mathbf{P}_{hvy}\}| = 1$$

      (b) If (C,ι) is a stable $\mathbf{P}$-marked curve then the converse also holds.
  (2) (Equivalent statement about trees.)
      (a) A vertex v of a $\mathbf{P}$-marked tree σ is not ϵ-stable if

$$(7) \qquad |\{\mathrm{fl} \in \mathrm{Flags}_v \mid \mathrm{fl} = \delta(v \to p)\ for\ some\ p \in \mathbf{P}_{hvy}\}| = 1$$

      (b) If σ is a stable $\mathbf{P}$-marked tree then the converse also holds.

Proof. Since the equivalence of items (1) and (2) is clear, we prove only (2). First, we claim that for any vertex v on a $\mathbf{P}$-marked tree σ, the cardinality of set in (7) must be at least one: Since there exists some $p_0 \in \mathbf{P}_{hvy}$, and σ is connected, $\exists \mathrm{fl}_0 \in \mathrm{Flags}_v$ connecting v to p_0.

Now suppose v is a vertex of σ such that the set in (7) has cardinality one. Then

$$(8) \quad \sum_{\text{fl} \in \text{Flags}_v} \min \left\{ 1, \sum_{\{p \mid \text{fl} = \delta(v \to p)\}} \boldsymbol{\epsilon}(p) \right\} = \sum_{\substack{\text{fl} \in \text{Flags}_v \\ \text{fl} = \delta(v \to p) \\ \text{for some} \\ p \in \mathbf{P}_{hvy}}} 1 + \sum_{\substack{\text{fl} \in \text{Flags}_v \\ \text{fl} \neq \delta(v \to p) \\ \text{for any} \\ p \in \mathbf{P}_{hvy}}} \sum_{\substack{p \text{ s.t.} \\ \text{fl} = \delta(v \to p)}} \epsilon \leq 1 + |\mathbf{P}_{lt}| \, \epsilon < 2.$$

So, according to Definition 2.11, v is not $\boldsymbol{\epsilon}$-stable, proving part (2a).

Finally, we suppose that σ is a stable $\mathbf{P}$-marked tree, and v on σ. Since σ is stable, v is $\mathbf{P}$-stable, so there are at least three flags on v of the form $\delta(v \to p)$ for some $p \in \mathbf{P}$. If there are two or more flags on v for the form $\delta(v \to p)$ for some $p \in \mathbf{P}_{hvy}$, then

$$(9) \quad \sum_{\text{fl} \in \text{Flags}_v} \min \left\{ 1, \sum_{\{p \mid \text{fl} = \delta(v \to p)\}} \boldsymbol{\epsilon}(p) \right\} \geq 2 + \epsilon > 2,$$

so v is $\boldsymbol{\epsilon}$-stable. So if v is not $\boldsymbol{\epsilon}$-stable, then the set (7) has cardinality exactly one, proving the lemma. $\quad\square$

2.5. **A tower of compactifications.** Let $|\mathbf{P}| \geq 3$, and suppose $\epsilon > 0$ is such that $(|\mathbf{P}| - 2) < (1/\epsilon)$ and $(|\mathbf{P}| - 1) > 1/\epsilon$. Fix $p_\infty \in \mathbf{P}$ and subsets of $\mathbf{P}$

$$\{p_\infty\} = \mathbf{P}_1 \subset \mathbf{P}_2 \subset \mathbf{P}_3, \cdots, \subset \mathbf{P}_{|\mathbf{P}|} = \mathbf{P}$$

such that $|\mathbf{P}_\ell| = \ell$. For $\ell = 1, \ldots, |\mathbf{P}|$, let X_ℓ be the Hassett space corresponding to the weight datum assigning the points in $\mathbf{P}_\ell$ weight 1 and all other points weight ϵ. For $\ell = 2, \ldots, |\mathbf{P}|$, X_ℓ is a heavy/light Hassett space with ℓ heavy points and $(|\mathbf{P}| - \ell)$ light points. As stated in the previous section, $\overline{\mathcal{M}}_{0,\mathbf{P}} \cong X_{|\mathbf{P}|} \cong X_{|\mathbf{P}|-1} \cong X_{|\mathbf{P}|-2}$, and X_2 is a Losev-Manin space. On the other hand, X_1 is not a heavy/light space: $X_1 \cong \mathbb{CP}^{|\mathbf{P}|-3}$, as described in Example 2.10, (2). By Theorem 2.9, we have reduction maps $\rho_{\ell+1,\ell} : X_{\ell+1} \to X_\ell$ for $\ell = 1, \ldots, |\mathbf{P}| - 1$. These are the spaces and maps referred to in the statement of Theorem 1.2.

2.6. **(Co)homology groups of compactifications of $\mathcal{M}_{0,\mathbf{P}}$.** In this work, we only consider the Deligne-Mumford compactifications and the Hassett weighted stable curves compactification of $\mathcal{M}_{0,\mathbf{P}}$. For any such compactification $X_\mathbf{P}$ we have [Kee92, Cey09] that $H_{2k}(X_\mathbf{P}, \mathbb{Z})$ is a finitely generated free abelian group generated by fundamental classes of boundary strata. We also have identifications $H^{2k}(X_\mathbf{P}, \mathbb{Z}) = H_{2(\dim(X_\mathbf{P})-k)}(X_\mathbf{P}, \mathbb{Z})$ and $H^{2k}(X_\mathbf{P}, \mathbb{R}) = H^{k,k}(X_\mathbf{P})$. A boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon)$ is the image, under ρ_ϵ, of a boundary stratum in $\overline{\mathcal{M}}_{0,\mathbf{P}}$. This tells us that $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon), \mathbb{Z})$ is the quotient of $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathbb{Z})$ by the kernel of the pushforward $(\rho_\epsilon)_*$. By [Ram18] (Lemma 10.9), the kernel of $(\rho_\epsilon)_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}(\epsilon), \mathbb{Q})$ is spanned by fundamental classes of boundary strata. Lemma 2.12 allows us to describe $\ker((\rho_\epsilon)_*)$ as follows:

Lemma 2.15. *Suppose $\boldsymbol{\epsilon}$ is a weight datum on $\mathcal{M}_{0,\mathbf{P}}$, with ρ_ϵ the reduction morphism from $\overline{\mathcal{M}}_{0,\mathbf{P}}$ to $\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon})$. Then*

  (1) *For S_σ any k-dimensional boundary stratum of $\overline{\mathcal{M}}_{0,\mathbf{P}}$, the pushforward $(\rho_\epsilon)_*([S_\sigma])$ is nonzero in $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon}), \mathbb{Q})$ if and only if every vertex $v \in \text{Verts}(\sigma)$ with $\text{md}(v) > 0$ is $\boldsymbol{\epsilon}$-stable.*
  (2) *The kernel of $(\rho_\epsilon)_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon}), \mathbb{Q})$ equals the span of $\{[S_\sigma] \text{ } k\text{-dim} \mid \exists v \in \text{Verts}(\sigma) \text{ not } \boldsymbol{\epsilon}\text{-stable with } \text{md}(v) > 0\}$.*

Change of notation. In the subsequent sections, for a $\mathbf{P}$-marked curve (C, ι) or (C, mp), we will suppress the notation ι/mp for the marking map, and just write $(C, \mathbf{P})$.

3. Hurwitz correspondences

Hurwitz spaces are moduli spaces parametrizing finite maps with prescribed ramification between smooth curves. We refer the reader to [RW06] for a general summary and to [Ram18] for the definitions as used in this paper. In particular, we use Definition 5.4 of [Ram18] for our definition of the *Hurwitz space*: Fix discrete data: $\mathbf{A}$ and $\mathbf{B}$ finite sets, $d \in \mathbb{Z}^{>0}$, $F : \mathbf{A} \to \mathbf{B}$, $\text{br} : \mathbf{B} \to \{\text{partitions of } d\}$, and $\text{rm} : \mathbf{A} \to \mathbb{Z}^{>0}$. Then $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$ is a smooth quasiprojective variety parametrizing morphisms $f : (C, \mathbf{A}) \to (D, \mathbf{B})$,

where C and D are, respectively, $\mathbf{A}$-marked and $\mathbf{B}$-marked smooth connected genus zero curves, f is degree d, and maps the points in $\mathbf{A}$ to those in $\mathbf{B}$ as specified by F, with ramification at points in $\mathbf{A}$ and branching over points in $\mathbf{B}$ as specified by rm and br respectively. The Hurwitz space $\mathcal{H}$ has a "source curve" map $\pi_{\mathbf{A}}$ to $\mathcal{M}_{0,\mathbf{A}}$ sending $[f : (C, \mathbf{A}) \to (D, \mathbf{B})]$ to the marked curve $[(C, \mathbf{A})]$. There is similarly a "target curve" map $\pi_{\mathbf{B}}$ from $\mathcal{H}$ to $\mathcal{M}_{0,\mathbf{B}}$. Unless $\mathcal{H}$ is empty, $\pi_{\mathbf{B}}$ is a finite covering map. Thus the triple $(\mathcal{H}, \pi_{\mathbf{B}}, \pi_{\mathbf{A}}) : \mathcal{M}_{\mathbf{B}} \rightrightarrows \mathcal{M}_{\mathbf{A}}$ is a multi-valued map. We generalize this notion.

Definition 3.1 (*Hurwitz correspondence*, [Ram18], Definition 5.5)**.** With notation as above, let $\mathbf{A}'$ be any subset of $\mathbf{A}$ with cardinality at least 3. There is a forgetful map $\mu : \mathcal{M}_{0,\mathbf{A}} \to \mathcal{M}_{0,\mathbf{A}'}$. Let Γ be a union of connected components of $\mathcal{H}$. We call the triple $(\Gamma, \pi_{\mathbf{B}}, \mu \circ \pi_{\mathbf{A}}) : \mathcal{M}_{0,\mathbf{B}} \rightrightarrows \mathcal{M}_{0,\mathbf{A}'}$ a *Hurwitz correspondence*.

3.1. Hurwitz correspondences and meromorphic maps from PCF maps.

Suppose $\phi : S^2 \to S^2$ is a degree d orientation-preserving branched covering with finite post-critical set $\mathbf{P}$. Define rm $: \mathbf{P} \to \mathbb{Z}^{>0}$ sending $p \in \mathbf{P}$ to the local degree of ϕ at p. Define br $: \mathbf{P} \to \{$partitions of $d\}$ sending $p \in \mathbf{P}$ to the branching profile of ϕ over p. Then $\mathcal{H} = \mathcal{H}(\mathbf{P}, \mathbf{P}, d, \phi|_{\mathbf{P}}, \text{br}, \text{rm})$ parametrizes regular maps $(\mathbb{CP}^1, \mathbf{P}) \to (\mathbb{CP}^1, \mathbf{P})$ with the same branching as ϕ. Let π_1 and π_2 be the "target" and "source" maps from $\mathcal{H}$ to $\mathcal{M}_{0,\mathbf{P}}$. There is a unique connected component $\mathcal{H}_\phi$ of $\mathcal{H}$ parametrizing maps that are topologically isomorphic to ϕ, i.e. maps $f : (\mathbb{CP}^1, \mathbf{P}) \to (\mathbb{CP}^1, \mathbf{P})$ such that there exist marked-point-preserving homeomorphisms χ_1 and χ_2 from $(\mathbb{CP}^1, \mathbf{P})$ to $(S^2, \mathbf{P})$ with $\chi_2 \circ f = \phi \circ \chi_1$. By [Koc13], the Hurwitz correspondence $(\mathcal{H}_\phi, \pi_1, \pi_2)$ on $\mathcal{M}_{0,\mathbf{P}}$ is descended from the Thurston pullback map Th_ϕ. When, in addition, ϕ satisfies criteria 1.1, 1 and 1.1, 2, Koch showed that $\pi_2 : \mathcal{H}_\phi \to \mathcal{M}_{0,\mathbf{P}}$ is generically one-to-one; the meromorphic map R_ϕ is $\pi_1 \circ \pi_2^{-1}$. Thus the graph of R_ϕ is (up to birational equivalence) the Hurwitz space $\mathcal{H}_\phi$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
 & \mathcal{H}_\phi & \\
{}^{\pi_1}\swarrow & & \nwarrow {}^{\pi_2} \\
\mathcal{M}_{0,\mathbf{P}} & \xleftarrow{\quad\quad} & \mathcal{M}_{0,\mathbf{P}} \\
 & {}_{\pi_2^{-1}} \qquad R_\phi &
\end{array}
$$

As described in [DS08, TT16, Ram18], correspondences can be composed and dynamical correspondences such as $(\mathcal{H}_\phi, \pi_1, \pi_2)$ can be iterated. When the meromorphic map R_ϕ exists, then the multivalued map $(\mathcal{H}_\phi^n, \pi_{1,n}, \pi_{2,n})$ given by the n-th iterate of $(\mathcal{H}_\phi, \pi_1, \pi_2)$ is the inverse of R_ϕ^n.

3.2. The maps on (co)homology induced by Hurwitz correspondences.

Suppose $(\Gamma, \pi_{\mathbf{B}}, \mu \circ \pi_{\mathbf{A}}) : \mathcal{M}_{0,\mathbf{B}} \rightrightarrows \mathcal{M}_{0,\mathbf{A}'}$ is a Hurwitz correspondence from $\mathcal{M}_{0,\mathbf{B}}$ to $\mathcal{M}_{0,\mathbf{A}'}$, $X_{\mathbf{B}}$ is some smooth projective compactification of $\mathcal{M}_{0,\mathbf{B}}$ and $X_{\mathbf{A}'}$ is some smooth projective compactification of $\mathcal{M}_{0,\mathbf{A}'}$. Then there are induced pushforward maps on homology groups, (and, dually, pullback maps on cohomology groups) as follows. Let $\overline{\Gamma}$ be any smooth projective compactification of Γ such that the maps $\pi_{\mathbf{B}} : \overline{\Gamma} \to X_{\mathbf{B}}$ and $(\mu \circ \pi_{\mathbf{A}}) : \overline{\Gamma} \to X_{\mathbf{A}'}$ are both regular. Then $[\Gamma]_* := (\mu \circ \pi_{\mathbf{A}})_* \circ \pi_{\mathbf{B}}^* : H_{2k}(X_{\mathbf{B}}, \mathbb{Z}) \to H_{2k}(X_{\mathbf{A}'}, \mathbb{Z})$. The pushforward and pullback maps are well-defined, i.e. they do not depend on the choice of compactification $\overline{\Gamma}$, but they are not in general functorial with respect to composition of correspondences. (See [DS08, Ram18] for details). However, the maps induced by Hurwitz correspondences on the (co)homology groups of the Deligne-Mumford compactifications in particular are functorial with respect to composition [Ram18]. Now suppose $\phi : S^2 \to S^2$ is PCF and satisfies criteria (1.1, 1) and (1.1, 2) so R_ϕ exists. By the definition of pullback by a meromorphic map given in [Roe13], and the fact that $\mathcal{H}_\phi$ is the graph of R_ϕ, we have, for any compactification $X_{\mathbf{P}}$ of $\mathcal{M}_{0,\mathbf{P}}$, and $\forall n > 0$, that $[\mathcal{H}_\phi^n]_* = (R_\phi^n)^*$ and $[\mathcal{H}_\phi^n]^* = (R_\phi^n)$ on $H_{2k}(X_{\mathbf{P}}, \mathbb{Z})$. This implies that by [Ram18], R_ϕ is algebraically stable on $\overline{\mathcal{M}}_{0,\mathbf{P}}$.

3.3. Compactifications of Hurwitz spaces.

An *admissible cover* is a ramified map between nodal curves that satisfies a certain balancing condition at nodes. Harris and Mumford [HM82] defined admissible covers and constructed their moduli spaces, which compactify Hurwitz spaces. We refer the reader to Definition 7.3 of [Ram18] for the definition of a $(\mathbf{A}, \mathbf{B}, d, F, \text{br}, \text{rm})$-*admissible cover*, as it is used here. In general, the admissible covers compactifications are only coarse moduli spaces with orbifold singularities. For technical ease, [Ram18], Definition 7.1 introduces *fully marked Hurwitz spaces*, a class of Hurwitz spaces parametrizing maps of curves $f : (C, \mathbf{A}) \to (D, \mathbf{B})$ with the property that $\forall a \in C$ with $f(a) \in \mathbf{B}$, we must have $a \in \mathbf{A}$. In

other words, a point on the source curve is marked if (and only if) its image on the target curve is marked. The admissible covers compactifications of fully marked Hurwitz spaces are fine moduli spaces.

Theorem 3.2 (Harris and Mumford, [HM82]). *Given $\mathcal{H} = (\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ a fully marked Hurwitz space as in [Ram18], Definition 7.1, there is a projective variety $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ parametrizing admissible covers, and containing $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ as a dense open subset. The compactification $\overline{\mathcal{H}}$ extends the maps $\pi_{\mathbf{B}}$ and $\pi_{\mathbf{A}}$ to maps $\overline{\pi_{\mathbf{B}}}$ and $\overline{\pi_{\mathbf{A}}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$, respectively, with $\overline{\pi_{\mathbf{B}}} : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{0,\mathbf{B}}$ a finite flat map. $\overline{\mathcal{H}}$ may not be normal, but its normalization is smooth.*

Remark 3.3. The irreducible components of $\overline{\mathcal{H}}$ are the Zariski closures of the connected components of $\mathcal{H}$.

3.4. **Boundary strata in $\overline{\mathcal{H}}$.** Moduli spaces of admissible covers have a stratification analogous to and compatible with that of $\overline{\mathcal{M}}_{0,n}$. This stratification has been studied in detail in [CMR16]. In this section we fix $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ a fully marked Hurwitz space, and let $\overline{\mathcal{H}} = \overline{\mathcal{H}}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$. If $[f : (C, \mathbf{A}) \to (D, \mathbf{B})] \in \overline{\mathcal{H}}$ is an admissible cover, then there is an induced map of graphs from the dual tree of C to that of D, as described in [Cap]. The *combinatorial type* of an admissible cover records this map of graphs together with other discrete data that describe how the irreducible components of C map to those of D. We refer the reader to [CMR16] for the general definition of combinatorial type of admissible cover, and to Definition 7.8 of [Ram18] for the specific definition, used here, of the combinatorial type γ of $[f : (C, \mathbf{A}) \to (D, \mathbf{B})] \in \overline{\mathcal{H}}$, where:

$$\gamma = (\sigma, \tau, d_{\mathrm{Verts}}, f_{\mathrm{Verts}}, F_{\mathrm{Edges}}, (\mathrm{br}_v)_{v \in \mathrm{Verts}(\sigma)}, \mathrm{rm}_{\mathrm{Edges}}). \tag{10}$$

Here, σ and τ denote the dual trees of C and D respectively, f_{Verts} and F_{Edges} record, respectively, how the irreducible components and the nodes of C map to those of D, d_{Verts} records the degrees of the restrictions of f to the irreducible components of C, and $(\mathrm{br}_v)_{v \in \mathrm{Verts}(\sigma)}$ and $\mathrm{rm}_{\mathrm{Edges}}$ record, respectively, the branching of f over nodes of D and at nodes of C.

Definition 3.4. Set G_γ° to be the locally closed subvariety $\{[f'] \in \overline{\mathcal{H}} | f'$ has combinatorial type $\gamma\}$ and set G_γ to be its closure in $\overline{\mathcal{H}}$. We refer to G_γ as a *boundary stratum* of $\overline{\mathcal{H}}$.

The boundary stratum G_γ can be decomposed into a product of lower-dimensional spaces of admissible covers. For $v \in \mathrm{Verts}(\sigma)$, the Hurwitz space $\mathcal{H}_v := \mathcal{H}(\mathrm{Flags}_v, \mathrm{Flags}_{f_{\mathrm{Verts}}(v)}, d_{\mathrm{Verts}}, F_{\mathrm{Edges}}, \mathrm{br}_v, \mathrm{rm}_{\mathrm{Edges}})$ is fully marked. Denote by $\overline{\mathcal{H}}_v$ the corresponding space of admissible covers; this is reducible in general. The space $\overline{\mathcal{H}}_v$ admits maps to the moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$ of source curves and the moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_{f_{\mathrm{Verts}}(v)}}$ of target curves. For $w \in \mathrm{Verts}(\tau)$, set $\overline{\mathcal{H}}_w := \prod_{v \in (f_{\mathrm{Verts}})^{-1}(w)} \overline{\mathcal{H}}_v$, where the product is fibered over the common moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_w}$ of target curves. The fibered product $\overline{\mathcal{H}}_w$ is itself a moduli space of possibly disconnected admissible covers, admitting a map $\overline{\pi}_w^{\mathrm{source}}$ to the moduli space $\prod_{v \in (f_{\mathrm{Verts}})^{-1}(w)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_v}$ of source curves and a finite flat map $\overline{\pi}_w^{\mathrm{target}}$ to the moduli space $\overline{\mathcal{M}}_{0,\mathrm{Flags}_w}$ of target curves. The stratum G_γ is isomorphic to $\prod_{w \in \mathrm{Verts}(\tau)} \overline{\mathcal{H}}_w$. The boundary stratum G_γ is not necessarily irreducible. Its irreducible components are of the form

$$J = \prod_{w \in \mathrm{Verts}(\tau)} \overline{\mathcal{J}_w} \tag{11}$$

where $\overline{\mathcal{J}_w}$ is an irreducible component of $\overline{\mathcal{H}}_w$.

4. Moduli spaces of static polynomials

A polynomial $f(z)$ in one variable defines a regular map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ for which $\infty \in \mathbb{CP}^1$ is a fully ramified fixed point. More generally, a regular map $f' : \mathbb{CP}^1 \to \mathbb{CP}^1$ is called a *polynomial* if there is a fully ramified fixed point $a_\infty \in \mathbb{CP}^1$; such a map f' is conjugate to a regular map defined by a polynomial in one variable. We recall from Section 1 that a topological polynomial is a branched covering $\phi : S^2 \to S^2$ that has a fully ramified fixed point. The condition of having a fully ramified point is invariant under separate changes of coordinates on source and target, i.e. it is a *non-dynamical/static* feature. On the other hand, the condition of being a fixed point is invariant under the same change of coordinates on source and

target, but not invariant under separate changes of coordinates on source and target. In other words, the condition of being a fixed point is a *dynamical* feature. Now, suppose ϕ satisfies criteria (1.1, 1) and (1.1, 2). Then, although ϕ may not be a topological polynomial, it shares a non-dynamical feature with topological polynomials, i.e. there is a point p_∞ that is a fully ramified point of ϕ, although it may not be fixed. This motivates the following definition:

Definition 4.1. We say that a regular map $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ of degree d is a *static polynomial* if it has a fully ramified point, i.e. if there exists $a_\infty \in \mathbb{CP}^1$ such that the local degree of f at a_∞ is d. Similarly, we say that a degree d admissible cover $f : C \to D$ is a static polynomial if there exists a smooth point $a_\infty \in C$ such that the local degree of f at a_∞ is d.

Note that if ϕ is PCF and satisfies criteria (1.1, 1) and (1.1, 2), then $\mathcal{H}_\phi$ and $\overline{\mathcal{H}}_\phi$ are moduli spaces that parametrize static polynomials.

4.1. **Degenerations of static polynomials.** Here, we describe a few basic features of the combinatorics of static polynomial admissible covers.

Lemma 4.2. *Suppose $f : C \to D$ is a degree d admissible cover with an irreducible component C_∞ of C such that the restriction $f|_{C_\infty}$ has degree equal to d. If $\theta \in D$ is any node, and D_0 is the connected component of $D \setminus \{\theta\}$ that contains $f(C_\infty) \setminus \{\theta\}$, then $f^{-1}(D_0)$ is connected.*

Proof. Set $\overline{D_0}$ be the closure of D_0 in D, and set $\overline{C_0}$ to be the closure of $f^{-1}(D_0)$ in C. Then the restriction $f|_{\overline{C_0}} : \overline{C_0} \to \overline{D_0}$ is also an admissible cover of degree d, and it has degree equal to d on the irreducible component C_∞ of $\overline{C_0}$. Thus the source curve $\overline{C_0}$ must be connected. Since $\overline{C_0} \setminus f^{-1}(D_0)$ is a set of isolated smooth points (these are nodes of C but smooth points of $\overline{C_0}$), connectedness of $\overline{C_0}$ is equivalent of to connectedness of $f^{-1}(D_0)$; we conclude that the latter is connected, as desired. $\square$

Corollary 4.3. *Suppose $f : C \to D$ is a degree d static polynomial admissible cover, fully ramified over a smooth point $b_\infty \in D$. If $\theta \in D$ is any node, and D_0 is the connected component of $D \setminus \{\theta\}$ that contains b_∞, then $f^{-1}(D_0)$ is connected.*

Proof. This follows from Lemma 4.2 and the fact that if C_∞ is the irreducible component containing the fully ramified smooth point $a_\infty = f^{-1}(b_\infty)$, then the restriction $f|_{C_\infty}$ has degree equal to d. $\square$

We further conclude that if $f : C \to D$ is a degenerate static polynomial as in Corollary 4.3, then the restriction of f to any irreducible component of C is a static polynomial of possibly smaller degree. More precisely, let C_1 be an irreducible component of C. If $a_\infty \in C_1$, then it's clear that $f|_{C_1}$ is a static polynomial. Otherwise, let η be the node on C_1 connecting it to a_∞; we claim that η is a fully ramified point of $f|_{C_1}$. To see this, set $\theta = f(\eta)$, and D_0 to be the connected component of $D \setminus \{\theta\}$ that contains b_∞. Since by Corollary 4.3, $f^{-1}(D_0)$ is connected, and since C has genus zero, there is a unique node connecting C_1 to $f^{-1}(D_0)$; this node must be η. Thus η is the only point of C_1 mapping to θ; this forces $f|_{C_1}$ to be fully ramified at η.

4.2. **Static polynomials and weighted stable curves.** In this section we study Hurwitz spaces $\mathcal{H}$ parametrizing static polynomials. We will relate the combinatorics of static polynomials to the combinatorics of stable curves to find compactifications $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ on which the Hurwitz correspondence induced by $\mathcal{H}$ behaves well.

Definition 4.4. Let $\mathcal{H} = \mathcal{H}(\mathbf{A}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be a Hurwitz space parametrizing static polynomials, and let $b_\infty \in \mathbf{B}$ be the image of the fully ramified point, i.e. we have $\mathrm{br}(b_\infty) = (d)$. We define a *compatible pair of heavy/light Hassett spaces* with respect to $\mathcal{H}$ to be a pair $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ of compactifications of $\mathcal{M}_{0,\mathbf{B}}$ and $\mathcal{M}_{0,\mathbf{A}}$ respectively, obtained as follows. Let $\mathbf{B} = \mathbf{B}_{hvy} \sqcup \mathbf{B}_{lt}$ be a set partition such that: $b_\infty \in \mathbf{B}_{hvy}$, $|\mathbf{B}_{hvy}| \geq 2$, and $\left|F^{-1}(\mathbf{B}_{hvy})\right| \geq 2$. Set $\mathbf{A}_{hvy} = F^{-1}(\mathbf{B}_{hvy})$, and $\mathbf{A}_{lt} = F^{-1}(\mathbf{B}_{lt})$. Let $\epsilon > 0$ be such that $|\mathbf{B}_{lt}| < (1/\epsilon)$ and $|\mathbf{A}_{lt}| < (1/\epsilon)$. Let $\epsilon_{\mathbf{B}}$ be the weight datum on $\mathcal{M}_{0,\mathbf{B}}$ that assigns points in $\mathbf{B}_{hvy}$ weight 1 and points in $\mathbf{B}_{lt}$ weight ϵ, and let $\epsilon_{\mathbf{A}}$ be the weight datum on $\mathcal{M}_{0,\mathbf{A}}$ that assigns points in $\mathbf{A}_{hvy}$ weight 1 and points in $\mathbf{A}_{lt}$ weight ϵ. Set $X_{\mathbf{B}}$ and $X_{\mathbf{A}}$ to be the corresponding spaces of $\mathbf{B}$- and $\mathbf{A}$-marked weighted stable curves respectively.

In other words, we require the special point b_∞ (the marked image of the fully ramified point) to be heavy, we require all of the points in $\mathbf{A}$ that map, under F, to heavy points in $\mathbf{B}$ to be heavy themselves, and we require points in $\mathbf{A}$ that map to light points in $\mathbf{B}$ to be light.

Now, we fix $\mathcal{H}$, together with a pair of compatible pair $X_\mathbf{B}$ and $X_\mathbf{A}$ of heavy/light Hassett spaces, along with all the notation in Definition 4.4. Let $\rho_\mathbf{B}$ and $\rho_\mathbf{A}$ be the reduction morphisms from $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}}$ to $X_\mathbf{B}$ and $X_\mathbf{A}$ respectively. We are interested in studying the correspondence induced by $\mathcal{H}$ from $X_\mathbf{B}$ to $X_\mathbf{A}$. In order to be able to use an admissible covers compactification, we pass to the fully marked Hurwitz space: Let $\mathcal{H}^{\mathrm{full}} = \mathcal{H}(\mathbf{A}^{\mathrm{full}}, \mathbf{B}, d, F, \mathrm{br}, \mathrm{rm})$ be the fully marked Hurwitz space of $\mathcal{H}$ as in Section 3.3, with $\mathbf{A}^{\mathrm{full}} \supseteq \mathbf{A}$ the full marked preimage of $\mathbf{B}$. Let $\overline{\mathcal{H}}^{\mathrm{full}}$ be the admissible covers compactification of $\mathcal{H}^{\mathrm{full}}$; it admits a covering map ν to $\mathcal{H}$. Set $\overline{\pi_\mathbf{B}}$ and $\overline{\pi_{\mathbf{A}^{\mathrm{full}}}}$ to be the maps from $\overline{\mathcal{H}}^{\mathrm{full}}$ to $\overline{\mathcal{M}}_{0,\mathbf{B}}$ and $\overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{full}}}$ respectively, and $\mu : \overline{\mathcal{M}}_{0,\mathbf{A}^{\mathrm{full}}} \to \overline{\mathcal{M}}_{0,\mathbf{A}}$ to be the forgetful map. Throughout the section we will refer back to the notation defined above:

$$
\text{(12)} \qquad
\begin{array}{lllllllllll}
\mathbf{A}, & \mathbf{A}_{hvy}, & \mathbf{A}_{lt}, & \mathbf{B}, & \mathbf{B}_{hvy}, & \mathbf{B}_{lt}, & \mathbf{A}^{\mathrm{full}}, & a_\infty, & b_\infty, & d, & F, & \mathrm{br}, & \mathrm{rm}, \\
\epsilon, & \epsilon_\mathbf{A}, & \epsilon_\mathbf{B}, & \mathcal{H}, & \overline{\mathcal{H}}^{\mathrm{full}}, & X_\mathbf{B}, & X_\mathbf{A}, & \overline{\pi_\mathbf{B}}, & \overline{\pi_{\mathbf{A}^{\mathrm{full}}}}, & \rho_\mathbf{B}, & \rho_\mathbf{A}, & \mu & \text{and } \nu
\end{array}
$$

Lemma 4.5. *With notation as in (12), suppose we have $[f : (C, \mathbf{A}^{\mathrm{full}}) \to (D, \mathbf{B})] \in \overline{\mathcal{H}}^{\mathrm{full}}$. Then, considering C as a (not necessarily stable) $\mathbf{A}$-marked curve, we have:*

> (1) *(Statement about the map of curves.) If C_1 is an irreducible component of C such that $f(C_1)$ is not $\epsilon_\mathbf{B}$-stable, then C_1 is not $\epsilon_\mathbf{A}$-stable.*
> (2) *(Equivalent statement about the induced map of dual trees.) If v is a vertex on the dual tree of C such that $f_{\mathrm{Verts}}(v)$ on the dual tree of D is not $\epsilon_\mathbf{B}$-stable, then v is not $\epsilon_\mathbf{A}$-stable.*

Proof. Since the equivalence of items (1) and (2) is clear, we prove only (1). Since $(D, \mathbf{B})$ is a stable curve and $f(C_1)$ is not $\epsilon_\mathbf{B}$-stable, we conclude from part (1b) of Lemma 2.14 that there is a unique node θ on $f(C_1)$ connecting it to every heavy point, i.e. to every point in $\mathbf{B}_{hvy}$. Since b_∞ is heavy, θ connects $f(C_1)$ to b_∞. Now, let D_0 be the connected component of $D \smallsetminus \{\theta\}$ that contains b_∞ (and contains every other point in $\mathbf{B}_{hvy}$, and does not contain $f(C_1) \smallsetminus \{\theta\}$). By Corollary 4.3, $C_0 := f^{-1}(D_0)$ is connected. Since the pair $\epsilon_\mathbf{A}$ and $\epsilon_\mathbf{B}$ is a compatible pair of weights as in Definition 4.4, every point in $\mathbf{A}_{hvy}$ maps, via, f, to some point in $\mathbf{B}_{hvy}$, and thus every point in $\mathbf{A}_{hvy}$ lies on C_0. Now, since C is genus zero, there is a unique note η on C_1 connecting it to C_0, i.e. η is the node on C_1 that connects it to every point in $\mathbf{A}_{hvy}$. By the criterion in part (1a) of Lemma 2.14, we conclude that C_1 is not $\epsilon_\mathbf{A}$-stable. $\qquad\square$

Lemma 4.6. *With notation as in (12), suppose G_γ is any boundary stratum of $\overline{\mathcal{H}}^{\mathrm{full}}$ and that J is any irreducible component of G_γ. Then the two maps $(\rho_\mathbf{B} \circ \overline{\pi_\mathbf{B}})$ and $(\rho_\mathbf{A} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\mathrm{full}}}})$ from J to $X_\mathbf{B}$ and to $X_\mathbf{A}$ respectively both factor through the projection*

$$
\text{(13)} \qquad J = \prod_{w \in \mathrm{Verts}(\tau)} \overline{\mathcal{J}_w} \to \prod_{\substack{w \in \mathrm{Verts}(\tau) \\ \epsilon_\mathbf{B}\text{-}stable}} \overline{\mathcal{J}_w},
$$

where the decomposition of J as a product is as per Section 3.4, equation (11).

Proof. Recall that τ is the dual tree of the target curve $(D, \mathbf{B})$ of a generic admissible cover $[f : (C, \mathbf{A}^{\mathrm{full}}) \to (D, \mathbf{B})] \in J \subset \overline{\mathcal{H}}^{\mathrm{full}}$. As described in Section 3.4, the above decomposition of J into a product is induced by the decomposition $G_\gamma = \prod_{w \in \mathrm{Verts}(\tau)} \overline{\mathcal{H}}_w$, where $\overline{\mathcal{H}}_w$ is an admissible covers space of (pure) dimension $\mathrm{md}(w)$. The factor $\overline{\mathcal{J}_w}$ in the decomposition of J is an irreducible component of $\overline{\mathcal{H}}_w$, and thus also has dimension $\mathrm{md}(w)$. Under $\overline{\pi_\mathbf{B}}$, J maps to the boundary stratum T_τ in $\overline{\mathcal{M}}_{0,\mathbf{B}}$, and the restriction $\overline{\pi_\mathbf{B}} : J \to T_\tau$ decomposes into a product as follows:

$$
J = \prod_{w \in \mathrm{Verts}(\tau)} \overline{\mathcal{J}_w} \xrightarrow{\;\overline{\pi_\mathbf{B}}|_J = \prod_{w \in \mathrm{Verts}(\tau)} \overline{\pi}_w^{\mathrm{target}}\;} \prod_{w \in \mathrm{Verts}(\tau)} \overline{\mathcal{M}}_{0,\mathrm{Flags}_w} = T_\tau
$$

13

Each factor map $\overline{\pi}_w^{\text{target}}$ is a finite flat map from the admissible covers space $\overline{\mathcal{J}}_w$ to an appropriate moduli space of target curves. Thus $\overline{\pi_{\mathbf{B}}} : J \to T_\tau$ is finite and flat, so T_τ is the full image of J. Now, by Lemma 2.12, the restriction $\rho_{\mathbf{B}} : T_\tau \to X_{\mathbf{B}}$ factors through the projection

$$(14) \qquad T_\tau = \prod_{w \in \text{Verts}(\tau)} \overline{\mathcal{M}}_{0,\text{Flags}_w} \to \prod_{\substack{w \in \text{Verts}(\tau) \\ \boldsymbol{\epsilon_{\mathbf{B}}}\text{-stable}}} \overline{\mathcal{M}}_{0,\text{Flags}_w}$$

We conclude that the restriction $\overline{\pi_{\mathbf{B}}} : J \to X_{\mathbf{B}}$ factors through the projection in (13).

Now, the boundary stratum S_σ in $\overline{\mathcal{M}}_{0,\mathbf{A}^{\text{full}}}$ is isomorphic to $\prod_{v \in \text{Verts}(\sigma)} \overline{\mathcal{M}}_{0,\text{Flags}_v}$, and the restriction $\overline{\pi_{\mathbf{A}^{\text{full}}}} : J \to S_\sigma$ also factors as a product:

$$J = \prod_{w \in \text{Verts}(\tau)} \overline{\mathcal{J}}_w \xrightarrow{\overline{\pi_{\mathbf{A}}^{\text{full}}}|_J = \prod_{w \in \text{Verts}(\tau)} \overline{\pi}_w^{\text{source}}} \prod_{w \in \text{Verts}(\tau)} \prod_{v \in f_{\text{Verts}}^{-1}(w)} \overline{\mathcal{M}}_{0,\text{Flags}_v} = S_\sigma$$

Each factor map $\overline{\pi}_w^{\text{source}}$ is a map from the space $\overline{\mathcal{J}}_w$ of admissible covers to a moduli space of possibly disconnected source curves. Note that every vertex on σ that is $\boldsymbol{\epsilon_{\mathbf{A}}}$-stable is also $\mathbf{A}$-stable, thus by Lemmas 2.6 and 2.12, the restriction $\rho_{\mathbf{A}} \circ \mu : S_\sigma \to X_{\mathbf{A}}$ factors through the composition of projections

$$(15) \qquad S_\sigma = \prod_{v \in \text{Verts}(\sigma)} \overline{\mathcal{M}}_{0,\text{Flags}_v} \to \prod_{\substack{v \in \text{Verts}(\sigma) \\ \mathbf{A}\text{-stable}}} \overline{\mathcal{M}}_{0,\text{Flags}_v} \to \prod_{\substack{v \in \text{Verts}(\sigma) \\ \boldsymbol{\epsilon_{\mathbf{A}}}\text{-stable}}} \overline{\mathcal{M}}_{0,\text{Flags}_v}$$

From item (2) of Lemma 4.5, we conclude that if v is a $\boldsymbol{\epsilon_{\mathbf{A}}}$-stable vertex on σ, then $f_{\text{Verts}}(v)$ is $\boldsymbol{\epsilon_{\mathbf{B}}}$-stable as a vertex on τ. Thus the restriction $\rho_{\mathbf{A}} \circ \mu : S_\sigma \to X_{\mathbf{A}}$ factors through the projection

$$(16) \qquad S_\sigma = \prod_{v \in \text{Verts}(\sigma)} \overline{\mathcal{M}}_{0,\text{Flags}_v} \to \prod_{\substack{v \in \text{Verts}(\sigma) \\ f_{\text{Verts}}(v)\, \boldsymbol{\epsilon_{\mathbf{B}}}\text{-stable}}} \overline{\mathcal{M}}_{0,\text{Flags}_v} = \prod_{\substack{w \in \text{Verts}(\tau) \\ \boldsymbol{\epsilon_{\mathbf{B}}}\text{-stable}}} \prod_{v \in f_{\text{Verts}}^{-1}(w)} \overline{\mathcal{M}}_{0,\text{Flags}_v}$$

Thus the composite $\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}}^{\text{full}}} : J \to X_{\mathbf{A}}$ factors though the projection in (13), proving the lemma. $\qquad \square$

Using Lemma 4.6, we conclude that any irreducible component J of a boundary stratum of $\overline{\mathcal{H}}^{\text{full}}$ that is contracted in dimension by the map to $X_{\mathbf{B}}$ must also be contracted in dimension by the map to $X_{\mathbf{A}}$.

Lemma 4.7. *With notation as in (12), suppose G_γ is a boundary stratum of $\overline{\mathcal{H}}^{\text{full}}$ and J is some irreducible component of G_γ such that $\dim_{\mathbb{C}}(\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}}(J)) < \dim_{\mathbb{C}}(J)$. Then $\dim_{\mathbb{C}}(\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\text{full}}}}(J)) < \dim_{\mathbb{C}}(J)$.*

Proof. The map $\overline{\pi_{\mathbf{B}}}$ is finite, so $\dim_{\mathbb{C}}(\overline{\pi_{\mathbf{B}}}(J)) = \dim_{\mathbb{C}}(J)$. In fact, $\overline{\pi_{\mathbf{B}}}(J)$ is the boundary stratum T_τ of $\overline{\mathcal{M}}_{0,\mathbf{B}}$. We conclude that $\dim_{\mathbb{C}}(\rho_{\mathbf{B}}(T_\tau)) < \dim_{\mathbb{C}}(T_\tau)$. By Lemma 4.6, $\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}}^{\text{full}}} : J \to X_{\mathbf{A}}$ factors though the projection in 13. We have that $\dim_{\mathbb{C}}(J) = \sum_{w \in \text{Verts}(\tau)} \text{md}(w)$. By the above, $\dim_{\mathbb{C}}(\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\text{full}}}}(J)) \leq \sum_{\substack{w \in \text{Verts}(\tau) \\ \boldsymbol{\epsilon_{\mathbf{B}}}\text{-stable}}} \text{md}(w)$. By Lemma 2.15, τ has at least one vertex with positive moduli dimension that is not $\boldsymbol{\epsilon_{\mathbf{B}}}$-stable. Thus $\sum_{\substack{w \in \text{Verts}(\tau) \\ \boldsymbol{\epsilon_{\mathbf{B}}}\text{-stable}}} \text{md}(w) < \sum_{w \in \text{Verts}(\tau)} \text{md}(w)$, proving the lemma. $\qquad \square$

Proposition 4.8. *With notation as in (12), let Γ be any non-empty union of connected components of $\mathcal{H}$, and let $k \in \{0, \dots, \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{0,\mathbf{B}})\}$. Then the pushforward maps*

$$[\Gamma]_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Q}) \to H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Q})$$

$$(\rho_{\mathbf{B}})_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{B}}, \mathbb{Q}) \to H_{2k}(X_{\mathbf{B}}, \mathbb{Q})$$

$$(\rho_{\mathbf{A}})_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{A}}, \mathbb{Q}) \to H_{2k}(X_{\mathbf{A}}, \mathbb{Q})$$

satisfy: $[\Gamma]_*(\ker((\rho_{\mathbf{B}})_*)) \subseteq \ker((\rho_{\mathbf{A}})_*)$.

Remark 4.9. The very beginning of the proof of Proposition 4.8 follows the beginning the proof of Theorem 9.7 of [Ram18]: In order to understand the pushforward by a Hurwitz correspondence on the homology groups of the Deligne-Mumford compactifications we reduce to the case of a fully marked Hurwitz space, then frame the action of the pushforward on boundary strata in terms of the stratification of the space of

admissible covers. After this point, the two proofs diverge. Here, the key content lies in Lemma 4.6, via Lemma 4.7.

Proof. First, we reformulate the problem in order to allow ourselves to work solely with fully marked Hurwitz spaces. Set $\Gamma^{\mathrm{full}} = \nu^{-1}(\Gamma)$ and set $\overline{\Gamma^{\mathrm{full}}}$ to be its closure in $\overline{\mathcal{H}}^{\mathrm{full}}$. We have that $[\Gamma^{\mathrm{full}}]_* = \mu_* \circ ((\overline{\pi_{\mathbf{A}}^{\mathrm{full}}})|_{\overline{\Gamma^{\mathrm{full}}}})_* \circ (\overline{\pi_{\mathbf{B}}})|_{\overline{\Gamma^{\mathrm{full}}}}^*$. By Lemma 7.2 of [Ram18], we have that $[\Gamma]_* = (1/\deg\nu)[\Gamma^{\mathrm{full}}]_*$, so it suffices to show that $[\Gamma^{\mathrm{full}}]_*(\ker((\rho_{\mathbf{B}})_*)) \subseteq \ker((\rho_{\mathbf{A}})_*)$. Suppose T_τ is a k-dimensional boundary stratum of $\overline{\mathcal{M}}_{0,\mathbf{B}}$ such that $[T_\tau] \in \ker((\rho_{\mathbf{B}})_*)$; in this case $\dim_{\mathbb{C}}(\rho_{\mathbf{B}}(T_\tau)) < \dim_{\mathbb{C}}(T_\tau)$. Since the map $\overline{\pi_{\mathbf{B}}}$ is flat, by Lemma 1.7.1 of [Ful98] we have that

$$(\overline{\pi_{\mathbf{B}}})|_{\overline{\Gamma^{\mathrm{full}}}}^*([T_\tau]) = \sum_{\substack{J \text{ irred. comp} \\ \text{of } (\overline{\pi_{\mathbf{B}}})|_{\overline{\Gamma^{\mathrm{full}}}}^{-1}(T_\tau)}} m_J[J],$$

where the (m_J)s are positive integer multiplicities. Let J be an arbitrary term appearing in the above sum. Since the inverse image under $\overline{\pi_{\mathbf{B}}}$ of a boundary stratum of $\overline{\mathcal{M}}_{0,\mathbf{B}}$ is a union of boundary strata of $\overline{\mathcal{H}}^{\mathrm{full}}$, and since $\overline{\pi_{\mathbf{B}}}$ is finite, J is an irreducible component of some k-dimensional boundary stratum of $\overline{\mathcal{H}}^{\mathrm{full}}$, and $\overline{\pi_{\mathbf{B}}}(J) = T_\tau$. We have that

$$\dim_{\mathbb{C}}(\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}}(J)) = \dim_{\mathbb{C}}(\rho_{\mathbf{B}}(T_\tau)) < \dim_{\mathbb{C}}(T_\tau) = \dim_{\mathbb{C}}(J).$$

By Lemma 4.7, $\dim(\rho_{\mathbf{A}} \circ \mu \circ (\overline{\pi_{\mathbf{A}}^{\mathrm{full}}})|_{\overline{\Gamma^{\mathrm{full}}}}(J) < \dim(J)$, so $(\rho_{\mathbf{A}} \circ \mu \circ (\overline{\pi_{\mathbf{A}}^{\mathrm{full}}})|_{\overline{\Gamma^{\mathrm{full}}}})_*([J]) = 0 \in H_{2k}(X_{\mathbf{A}}, \mathbb{Q})$. Since J was arbitrary, we conclude:

$$(\rho_{\mathbf{A}})_*([\Gamma^{\mathrm{full}}]_*([T_\tau])) = (\rho_{\mathbf{A}})_* \circ \mu_* \circ ((\overline{\pi_{\mathbf{A}}^{\mathrm{full}}})|_{\overline{\Gamma^{\mathrm{full}}}})_* \circ \overline{\pi_{\mathbf{B}}}|_{\overline{\Gamma^{\mathrm{full}}}}^*([T_\tau])$$

$$= \sum_{\substack{J \text{ irred. comp} \\ \text{of } (\overline{\pi_{\mathbf{B}}})|_{\overline{\Gamma^{\mathrm{full}}}}^{-1}(T_\tau)}} m_J (\rho_{\mathbf{A}} \circ \mu \circ (\overline{\pi_{\mathbf{A}}^{\mathrm{full}}})|_{\overline{\Gamma^{\mathrm{full}}}})_*[J] = 0 \in H_{2k}(X_{\mathbf{A}}, \mathbb{Q}).$$

Thus $[\Gamma^{\mathrm{full}}]_*([T_\tau]) \in \ker((\rho_{\mathbf{A}})_*)$. We conclude that for an arbitrary boundary stratum T_τ of $\overline{\mathcal{M}}_{0,\mathbf{B}}$ such that $[T_\tau] \in \ker((\rho_{\mathbf{B}})_*)$, we have that $[\Gamma^{\mathrm{full}}]_*([T_\tau]) \in \ker((\rho_{\mathbf{A}})_*)$. By [Ram18], Lemma 10.9, $\ker((\rho_{\mathbf{B}})_*)$ is spanned by fundamental classes of boundary strata. We conclude that $[\Gamma^{\mathrm{full}}]_*(\ker((\rho_{\mathbf{B}})_*)) \subseteq \ker((\rho_{\mathbf{A}})_*)$, proving the proposition. $\square$

Remark 4.10. Lemma 4.6, which is the key technical lemma of this paper, is used in Lemma 4.7 and Proposition 4.8 to study the pushforward maps induced by Hurwitz spaces of static polynomials on the homology groups of compatible pairs of heavy/light Hassett spaces. Proposition 4.8 is then used in Theorem 5.1 to conclude algebraic stability on certain heavy/light Hassett spaces. Lemma 4.6 can be independently used to provide a geometric perspective on Theorem 5.1. While there isn't an agreed-upon definition of a regular/holomorphic (as opposed to a rational/meromorphic) correspondence from X to Y, it should be something like "a variety Γ with a finite map (proper and having finite fibers) $\pi_X : \Gamma \to X$ and a map $\pi_Y : \Gamma \to Y$". This candidate definition captures the informal notion of a multivalued map from X to Y taking only finitely many values for any given $x \in X$. However, it might not be strong enough to ensure good behavior, for example to ensure algebraic stability of a "regular" self-correspondence on a smooth projective variety. For that, either Γ should be required to be smooth, or π_X should be required to be flat [Roe13, Ram18]. Below, we use Lemma 4.6 to show (Proposition 4.12) that a Hurwitz space of static polynomials induces a "regular" correspondence (according to the weaker definition given above) on a compatible pair of heavy/light Hassett spaces. Proposition 4.12 is too weak to be used to prove Theorem 5.1.

Lemma 4.11. *With notation as in (12), suppose that $V \subseteq \overline{\mathcal{H}}^{\mathrm{full}}$ is an irreducible subvariety with the property that $\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}}(V)$ is a single point in $X_{\mathbf{B}}$. Then $\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\mathrm{full}}}}(V)$ is a single point $X_{\mathbf{A}}$.*

Proof. Since V is irreducible, there is a unique smallest boundary stratum G_γ of $\overline{\mathcal{H}}^{\mathrm{full}}$ that contains V. Set $V^\circ = V \cap G_\gamma^\circ$, where G_γ° is as in Definition 3.4. Then V° is open and dense in V. Since V is irreducible, there is a unique irreducible component J of G_γ such that $V \subseteq J$. By Lemma 4.6, the restrictions of both $\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}}$ and $\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\mathrm{full}}}}$ to J factor through the projection to $\mathrm{PJ} := \prod_{\substack{w \in \mathrm{Verts}(\tau) \\ \boldsymbol{\epsilon}_{\mathbf{B}}\text{-stable}}} \overline{\mathcal{J}}_w$ described in (13).

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(Here, τ is the dual tree of the target curve of every $[f] \in G_\gamma^\circ$.) When restricted to V°, $\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}}$ factors through the projection to the product of (not compactified) Hurwitz spaces $\mathrm{PJ}^\circ := \prod_{w \in \mathrm{Verts}(\tau)} \mathcal{J}_w$, followed by the map $\pi_{target} : \mathrm{PJ}^\circ \to \mathrm{PT}_\tau^\circ := \prod_{w \in \mathrm{Verts}(\tau)}^{\boldsymbol{\epsilon_{\mathbf{B}}}\text{-stable}} \mathcal{M}_{0,\mathrm{Flags}_w}$. By Lemma 2.12, the map induced by $\rho_{\mathbf{B}}$ from PT_τ° is an isomorphism onto its image in $X_{\mathbf{B}}$. Since by assumption $\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}}(V)$ is a single point in $X_{\mathbf{B}}$, the image of V° in PT_τ° is a single point y. Now, π_{target} is a product of covering maps thus itself a covering map, so $\pi_{target}^{-1}(y)$ is a finite set $\{x_1, \ldots x_r\}$. The image of V° in PJ° is contained in $\pi_{target}^{-1}(y)$, and since V° is irreducible, its image must be a single point $x_i \in \mathrm{PJ}^\circ$. Since V° is dense in V, the image of V in PJ is the single point x_i. On the other hand, the restriction of $\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\mathrm{full}}}}$ to J and thus also to V factors through the projection to PJ. Since the image of V in PJ is a single point, this forces the image of V under $\rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\mathrm{full}}}}$ to be a single point of $X_{\mathbf{A}}$. $\qquad\square$

From Lemma 4.11 we obtain:

Proposition 4.12. *With notation as in (12): The induced map to $X_{\mathbf{B}}$ from $(\rho_{\mathbf{B}} \circ \overline{\pi_{\mathbf{B}}} \times \rho_{\mathbf{A}} \circ \mu \circ \overline{\pi_{\mathbf{A}^{\mathrm{full}}}})(\overline{\mathcal{H}}^{\mathrm{full}}) \subset X_{\mathbf{B}} \times X_{\mathbf{A}}$ is finite.*

5. Algebraic stability of $\mathcal{H}_\phi$ on heavy/light Hassett spaces

In this section, we fix a degree d branched covering ϕ with finite post-critical set $\mathbf{P}$ that satisfies criteria (1.1, 1) and (1.1, 2). Let $p_\infty \in \mathbf{P}$ be the fully ramified cyclic point, and let $\mathbf{P}_\infty \subset \mathbf{P}$ be the forward orbit of p_∞ i.e. the set of all points in its periodic cycle. Fix a positive rational number ϵ such that $\frac{1}{|\mathbf{P}|-1} < \epsilon < \frac{1}{|\mathbf{P}|-2}$ and let $\boldsymbol{\epsilon}$ be the weight datum that assigns weight 1 to the elements of $\mathbf{P}_\infty$ and assigns weight ϵ to all elements not in $\mathbf{P}_\infty$. Let $X = \overline{\mathcal{M}}_{0,\mathbf{P}}(\boldsymbol{\epsilon})$ be the corresponding Hassett space. If $|\mathbf{P}_\infty| = 1$ then p_∞ is fixed so ϕ is a topological polynomial. Also, as described in Section 2.5, the Hassett space X is isomorphic to $\mathbb{CP}^{|\mathbf{P}|-3}$. By [Koc13], R_ϕ is holomorphic thus algebraically stable on X. If $|\mathbf{P}_\infty| \geq 2$ then X is a heavy/light Hassett space. Here, we show:

Theorem 5.1. *Let $\phi, \mathbf{P}, p_\infty, \mathbf{P}_\infty, \epsilon, \boldsymbol{\epsilon}$, and X be as above, and suppose $|\mathbf{P}_\infty| \geq 2$. Then $\mathcal{H}_\phi$ and R_ϕ are algebraically stable on X.*

Proof. By [Ram18], The Hurwitz correspondence $\mathcal{H}_\phi$ is algebraically stable on $\overline{\mathcal{M}}_{0,\mathbf{P}}$. Now, let $\rho : \overline{\mathcal{M}}_{0,\mathbf{P}} \to X$ be the reduction map. Note that (X, X) is a compatible pair of heavy/light Hassett spaces with respect to the Hurwitz space $\mathcal{H}_\phi$, as in Definition 4.4. Thus $\mathcal{H}_\phi$, X and ρ together satisfy the assumptions of Proposition 4.8. We conclude that for all k, the kernel of $\rho_* : H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) \to H_{2k}(X)$ is an invariant subspace of $[\mathcal{H}_\phi]_*$. On the other hand, it is shown in [Ram18] Lemma 4.16 that if a correspondence Γ is algebraically stable on X_1, and if $r : X_1 \to X_2$ is a regular birational map such that for all m, the kernel of the pushforward $r_* : H_m(X_1) \to H_m(X_2)$ is invariant under the action of $[\Gamma]_*$ on $H_m(X_1)$, then Γ is also algebraically stable on X_2. Applying this result here tells us that $\mathcal{H}_\phi$ is algebraically stable on X, i.e. for every iterate n, $[\mathcal{H}_\phi^n]_* = [\mathcal{H}_\phi]_*^n$ on $H_{2k}(X)$. On the other hand, for all $k = 0, \ldots, \dim_{\mathbb{C}}(X)$, and for iterates $n > 0$, the action of $(R_\phi^n)^*$ on $H^{2k}(X) = H^{k,k}(X)$ is identified with the action of $[\mathcal{H}_\phi^n]_*$ on $H_{2(\dim_{\mathbb{C}}(X)-k)}(X)$. We conclude that R_ϕ is algebraically stable on X. $\qquad\square$

Theorem 1.2 follows as an immediate consequence: it is a restatement of Theorem 5.1 above.

Remark 5.2. There is a variant of Theorem 5.1 obtained by applying Proposition 4.8 repeatedly. Suppose ϕ is a degree d branched covering with finite post-critical set $\mathbf{P}$ that satisfies criterion (1.1, 1) with p_∞ the fully ramified periodic point, and such that every critical point of ϕ is periodic. Then ϕ satifies (1.1, 2) as well. Let $\mathbf{P}_\infty$ and $\boldsymbol{\epsilon}$ be as in the statement of Theorem 5.1. Let $\mathbf{P}_\infty = \mathbf{P}_1 \subset \mathbf{P}_2 \ldots, \mathbf{P}_r = \mathbf{P}$ be any filtration of $\mathbf{P}$ such that each $\mathbf{P}_i$ is a union of periodic cycles of ϕ. For $i = 1, \ldots, r$, let $\boldsymbol{\epsilon}_i$ be the weight datum on $\mathcal{M}_{0,\mathbf{P}}$ assigning weight 1 to points in $\mathbf{P}_i$ and weight ϵ to points in the complement of $\mathbf{P}_i$, and let X_i be the corresponding space of weighted stable curves. Note that there is a generalized reduction map $\rho_{i,i-1} : X_i \to X_{i-1}$ commuting with the reduction maps ρ_i and ρ_{i-1} from $X_r = \overline{\mathcal{M}}_{0,\mathbf{P}}$ to X_i and X_{i-1} respectively. For $i = 1, \ldots, r-1$, set $V_{i,k} = \ker((\rho_i)_*) \subset H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}})$. By Proposition 4.8:

(1) The subspace $V_{i,k}$ is invariant under the action $[\mathcal{H}_\phi]_* = R_\phi^*$ on $H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) = H^{2(\dim(\overline{\mathcal{M}}_{0,\mathbf{P}})-k)}(\overline{\mathcal{M}}_{0,\mathbf{P}})$. Thus by Lemma 4.16 of [Ram18]:

(2) The Hurwitz correspondence $\mathcal{H}_\phi$ and the rational map R_ϕ are algebraically stable on each X_i.

Thus $V_{r-1,k} \subset V_{r-2,k} \subset \ldots \subset H_{2k}(\overline{\mathcal{M}}_{0,\mathbf{P}}) = H^{2(\dim(\overline{\mathcal{M}}_{0,\mathbf{P}})-k)}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ is a filtration of $H^{2(\dim(\overline{\mathcal{M}}_{0,\mathbf{P}})-k)}(\overline{\mathcal{M}}_{0,\mathbf{P}})$ by R_ϕ^*-invariant subspaces. This filtration allows us to write R_ϕ^* as a block-lower-triangular matrix. In contrast, the main result of [Ram18] is a completely different filtration of the (co)homology groups of $\overline{\mathcal{M}}_{0,\mathbf{P}}$ by subspaces invariant for every Hurwitz correspondence, giving us in this specific context another — different and utterly independent — expression of R_ϕ^* as a block-lower-triangular matrix. The latter expression makes no use of the specifics of ϕ, in particular of criteria (1.1, 1) and (1.1, 2).

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