PULLBACKS OF κ CLASSES ON $\overline{\mathcal{M}}_{0,n}$

ROHINI RAMADAS

Abstract. The moduli space $\overline{\mathcal{M}}_{0,n}$ carries a codimension-d Chow class κ_d. We consider the subspace $\mathcal{K}_n^d$ of $A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ spanned by pullbacks of κ_d via forgetful maps. We find a permutation basis for $\mathcal{K}_n^d$, and describe its annihilator under the intersection pairing in terms of d-dimensional boundary strata. As an application, we give a new permutation basis of the divisor class group of $\overline{\mathcal{M}}_{0,n}$.

1. Introduction

Mumford [D.83] introduced the tautological, codimension-d class κ_d in the cohomology/Chow group of the moduli space $\mathcal{M}_{g,n}$. This class extends to the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves as well as to various partial compactifications of $\mathcal{M}_{g,n}$. Ring-theoretic relations involving κ classes have been studied by Faber, Ionel, Pandharipande, Pixton, Zagier, Zvonkine, and several others, and play a role in the study of Gromov-Witten theory and mirror symmetry ([Fab99, Ion05, Pan12, PP, PPZ16]; see [Pan11, Pan18] for overviews).

Here, we investigate κ classes on $\overline{\mathcal{M}}_{0,n}$ from a linear-algebraic and representation-theoretic perspective. The symmetric group S_n acts on $\overline{\mathcal{M}}_{0,n}$, and thus acts on its cohomology and Chow groups. Given any $T \subseteq \{1,\ldots,n\}$ with $|T| \geq 3$, there is a forgetful map $\pi_T : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,T}$. We set $\kappa_d^T := \pi_T^*(\kappa_d)$, and consider the subspace $\mathcal{K}_n^d \subseteq A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ spanned by $\{\kappa_d^T\}_{T\subseteq\{1,\ldots,n\}}$; this subspace is clearly S_n-invariant. Recall that a *permutation basis* of a G-representation is one whose elements are permuted by the action of G. We show:

Theorem A. (Theorem 3.14(iii).) *If $n \geq 4$ and $1 \leq d \leq n-3$, then $\mathcal{K}_n^d$ has a permutation basis given by $\{\kappa_d^T \mid |T| \geq (d+3), |T| \equiv (d+3) \mod 2\}$.*

1.1. Does $A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ have a permutation basis?

Getzler [Get95] and Bergström-Minabe [BM13] have given algorithms to compute the character of $A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ as an S_n-representation. It is not clear from these algorithms whether $A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ has a permutation basis. Farkas and Gibney [FG03] have given a permutation basis for $A^1(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$. Theorem 3.14 implies that $\mathcal{K}_n^1 = A^1(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$, so:

Theorem B. *The set $\{\kappa_1^T \mid |T| \geq 4, |T|$ even$\}$ is a permutation basis of $A^1(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$.*

The basis given by Theorem B is different from the one given in [FG03], which consists of certain boundary divisors and ψ classes. For odd n, the two bases are isomorphic as S_n-sets, but for even n they are not.

Silversmith and the author [RS20] have produced a permutation basis for $A_2(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$, using Theorem 3.14 as an ingredient. Very recent work of Castravet and Tevelev [CT20] on the derived category of $\overline{\mathcal{M}}_{0,n}$ gives a permutation basis of $A^*(\overline{\mathcal{M}}_{0,n},\mathbb{Q}) = \bigoplus_{d=0}^{n-3} A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$; its elements, however, are not of pure degree. The question of whether or not $A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ has a permutation basis for all d and n remains open.

1.2. The dual story in $A_d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ and the proof of Theorem 3.14.

There is an S_n-equivariant intersection pairing $A_d(\overline{\mathcal{M}}_{0,n},\mathbb{Q}) \times A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q}) \to \mathbb{Q}$. To prove Theorem A, we show:

Theorem C. (Theorem 3.14(i),(ii).) If $n \geq 4$ and $1 \leq d \leq n-3$, we have

(1) *The annihilator of $\mathcal{K}_n^d \subseteq A^d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ is the subspace $\mathcal{V}_{d,n} \subseteq A_d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})$ spanned by boundary strata whose dual trees have two or more vertices with valence at least four.*

(2) *$\mathcal{Q}_{d,n} := \frac{A_d(\overline{\mathcal{M}}_{0,n},\mathbb{Q})}{\mathcal{V}_{d,n}}$ is the dual of $\mathcal{K}_n^d$.*

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It is straightforward to show that $\mathcal{V}_{d,n}$ is contained in the annihilator of $\mathcal{K}_n^d$, but to show equality involves a complicated induction on n. We use the fact that if π denotes the forgetful morphism from $\overline{\mathcal{M}}_{0,n+1}$ to $\overline{\mathcal{M}}_{0,n}$, then:

Theorem D. (Theorem 3.14(v)) *If $n \geq 4$ and $1 \leq d \leq n-3$, then we have the following (dual) exact sequences:*

$$0 \to \mathcal{Q}_{d,n} \xrightarrow{\pi^*} \mathcal{Q}_{d+1,n+1} \xrightarrow{\pi_*} \mathcal{Q}_{d+1,n} \to 0$$
$$0 \to \mathcal{K}_n^{d+1} \xrightarrow{\pi^*} \mathcal{K}_{n+1}^{d+1} \xrightarrow{\pi_*} \mathcal{K}_n^d \to 0$$

In general, it is difficult to use induction to study $A^d(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, partly due to the fact that

$$A^{d+1}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \xrightarrow{\pi^*} A^{d+1}(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q}) \xrightarrow{\pi_*} A^d(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$$

is not exact. This failure of exactness is also responsible for the fact that the dimensions of $A_d(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) = A^{n-3-d}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$ grow exponentially with n, whereas $\dim(\mathcal{K}_n^{n-3-d})$ grows as a degree-d polynomial in n.

1.3. **Significance for dynamics on $\mathcal{M}_{0,n}$.** *Hurwitz correspondences* are a class of multivalued dynamical systems on $\mathcal{M}_{0,n}$. They were introduced by Koch [Koc13] in the context of Teichmüller theory and complex dynamics on $\mathbb{P}^1$, and their dynamics were studied by the author [Ram18, Ram19b, Ram19a]. A Hurwitz correspondence $\mathcal{H}$ on $\mathcal{M}_{0,n}$ induces a linear pushforward action on $\mathcal{Q}_{d,n}$, and the d-th dynamical degree of $\mathcal{H}$ (a numerical invariant of algebraic dynamical systems) is the largest eigenvalue of this action [Ram18]. Theorem 3.14 can be used to re-interpret Theorem 10.6 of [Ram18] to conclude that $\mathcal{H}$ acts on pullbacks of κ classes, and that this action encodes important information about the dynamics of $\mathcal{H}$:

Theorem E. *Suppose $\mathcal{H}$ is a Hurwitz correspondence on $\mathcal{M}_{0,n}$. If $1 \leq d \leq n-3$, then $\mathcal{K}_n^d$ is invariant under the pullback $\mathcal{H}^* : A^d(\overline{\mathcal{M}}_{0,n}, \mathbb{Q}) \to A^d(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, and the d-th dynamical degree of $\mathcal{H}$ is the largest eigenvalue of the action of $\mathcal{H}^*$ on $\mathcal{K}_n^d$.*

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Notation and conventions. For n a positive integer, we denote by $[n]$ the set $\{1, \ldots, n\}$. For $\mathbf{A}$ a finite set, we denote by $\mathbb{Q}\mathbf{A}$ the free $\mathbb{Q}$-vector space on $\mathbf{A}$. For $\mathcal{V}$ a vector space, we denote by $\mathcal{V}^\vee$ its dual. For a linear map $\mu : \mathcal{V} \to \mathcal{X}$, we denote by $\mu^\vee$ its dual map. If $\mathcal{W}$ is a subspace of $\mathcal{V}$, we denote by $\mathcal{W}^\perp$ its annihilator in $\mathcal{V}^\vee$. For Y a variety, we denote by $A_d(Y)$ its Chow group, and by $A_d(Y, \mathbb{Q})$ the tensor product of $A_d(Y)$ with $\mathbb{Q}$. For X a d-dimensional subvariety of Y, we denote by $[X]$ its class in $A_d(Y)$ (resp. $A_d(Y, \mathbb{Q})$).

2. Chow classes on $\overline{\mathcal{M}}_{0,n}$

The moduli space $\mathcal{M}_{0,n}$ is an $(n-3)$-dimensional smooth variety parametrizing configurations of n distinct labelled points on $\mathbb{P}^1$, up to changes of coordinates. Its *stable curves* compactification $\overline{\mathcal{M}}_{0,n}$ parametrizes stable nodal rational curves with n distinct smooth marked points [Knu83]. We set $\mathcal{A}_{d,n} := A_d(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$, and $\mathcal{A}_n^d := \mathcal{A}_{n-3-d,n}$. There is an S_n-equivariant non-degenerate intersection pairing $\mathcal{A}_{d,n} \times \mathcal{A}_n^d \to \mathbb{Q}$; this identifies $\mathcal{A}_n^d$ with $\mathcal{A}_{d,n}^\vee$. In this section, we introduce certain classes in $\mathcal{A}_{d,n}$ and $\mathcal{A}_n^d$, and reformulate the the intersection pairing between these classes in purely combinatorial terms.

2.1. **Boundary strata, dual trees, and set partitions.** Given a point in $\overline{\mathcal{M}}_{0,n}$, i.e. stable n-marked rational curve C, its *dual tree* is a graph σ decorated with n marked *legs*, defined as follows: Vertices of σ correspond to irreducible components of C, two vertices are connected by an edge if the corresponding irreducible components of C meet at a node, and a vertex is decorated by the i-th leg if the corresponding irreducible component of C contains the i-th marked point. Note that σ is a *stable n-marked tree*, i.e. every vertex has valence at least 3 (counting legs). Given a stable n-marked tree σ, we set X_σ to be the closure, in $\overline{\mathcal{M}}_{0,n}$, of the locus of stable curves whose dual tree is σ. The locus X_σ is a subvariety, and is called a *boundary stratum*. By [Kee92], $A_d(\overline{\mathcal{M}}_{0,n})$ is a finitely generated free abelian group generated, though not freely, by the fundamental classes of d dimensional boundary strata; additive relations among boundary strata are described in [KM94]. Boundary strata are isomorphic to products of smaller moduli spaces:

$$X_\sigma \cong \prod_{v \text{ vertex of } \sigma} \overline{\mathcal{M}}_{0,\text{Valence}(v)}. \tag{1}$$

We conclude that X_σ is positive-dimensional if and only if its dual tree has at least one vertex with valence at least four. If σ has exactly one vertex v with valence at least four, then X_σ is isomorphic to $\overline{\mathcal{M}}_{0,\text{Valence}(v)}$, since the factors in the above product decomposition of X_σ corresponding to vertices other than v are all isomorphic to single-point spaces.

Definition 2.1. We say that a positive-dimensional boundary stratum X_σ is *Type I* if its dual tree σ has exactly one vertex with valence at least four; in this case we also say that σ is a Type I stable tree. We say that a positive-dimensional boundary stratum X_σ is *Type II* if its dual tree σ has two or more vertices with valence at least four; in this case we also say that σ is a type II stable tree. For $n \geq 4$ and $d = 1, \ldots, n-4$, we set $\mathcal{V}_{d,n} \subset \mathcal{A}_{d,n}$ to be the subspace generated by the fundamental classes of Type II boundary strata. We set $\mathcal{Q}_{d,n}$ to be the quotient $\mathcal{A}_{d,n}/\mathcal{V}_{d,n}$. Note that since $\mathcal{V}_{d,n}$ is S_n-invariant, $\mathcal{Q}_{d,n}$ inherits an action of S_n. Also note that $\mathcal{Q}_{d,n}$ is generated by the fundamental classes of Type I boundary strata.

Definition 2.2. Suppose σ is a stable n-marked tree and v a vertex on σ. We obtain from the pair (σ, v) a set partition $\Pi_*(\sigma, v)$ of $[n]$ as follows: i and j are in the same part of $\Pi_*(\sigma, v)$ if and only if the i- and j-marked legs on σ are on the same connected component of $\sigma \smallsetminus \{v\}$. Note that there is a canonical bijection $\delta : \Pi_*(\sigma, v) \to \{\text{edges adjacent to } v\} \cup \{\text{legs attached to } v\}$: If $\{i\} \in \Pi_*(\sigma, v)$, this implies that the $i-$th leg is attached to v, and we set $\delta(\{i\}) := (i\text{-th leg})$; and if $P \in \Pi_*(\sigma, v)$ is such that $|P| \geq 2$, then there is a unique edge e adjacent to v such that the legs corresponding to the elements of $[n] \smallsetminus P$ are all in the same connected component of $\sigma \smallsetminus e$; we set $\delta(P) := e$. Thus $|\Pi_*(\sigma, v)| = \text{Valence}(v)$. If σ is Type I and v is its unique vertex with valence at least four, then the partition $\Pi_*(\sigma, v)$ is intrinsically associated to σ, so we denote it by $\Pi_*(\sigma)$. In this case we have $\dim(X_\sigma) = |\Pi_*(\sigma)| - 3 (= \text{Valence}(v) - 3)$.

Definition 2.3. We recall certain special relations (introduced in Section 7.2 of [KM94]) that hold, in $\mathcal{A}_{d,n}$, among d-dimensional boundary strata. Suppose τ is the dual tree of a $(d+1)-$dimensional boundary stratum, v is a vertex of τ with valence four or more, and $i, j, k, l \in [n]$ are such that the parts P_1, P_2, P_3, P_4 of $\Pi = \Pi_*(\tau, v)$ containing i, j, k, l respectively are distinct. If we have $\Pi = \Pi_1 \sqcup \Pi_2$ with $|\Pi_1|, |\Pi_2| \geq 2$, then there is a stable n-marked tree $\tau(\Pi_1, \Pi_2)$ obtained from τ by splitting v into two vertices v_1 and v_2 joined by a new edge, and attaching to v_1 all the edges/legs of τ corresponding to elements of Π_1 (under the bijection δ introduced in Definition 2.2 above), and attaching to v_2 all the edges/legs of τ corresponding to elements of $P\Pi_2$. By [KM94], we obtain from the data $(\tau, \{i,j,k,l\}, v)$ a relation, denoted $R(\tau, \{i,j,k,l\}, v)$, in $\mathcal{A}_{d,n}$:

$$\sum_{\Pi = \Pi_1 \sqcup \Pi_2; P_1, P_2 \in \Pi_1; P_3, P_4 \in \Pi_2} [X_{\tau(\Pi_1,\Pi_2)}] - \sum_{\Pi = \Pi_1 \sqcup \Pi_2; P_1, P_3 \in \Pi_1; P_2, P_4 \in \Pi_2} [X_{\tau(\Pi_1,\Pi_2)}] = 0. \tag{2}$$

Lemma 2.4. *Suppose that σ_1 and σ_2 are two Type I stable n-marked trees, and suppose $\Pi_*(\sigma_1) = \Pi_*(\sigma_2)$. Then $[X_{\sigma_1}] = [X_{\sigma_2}] \in \mathcal{A}_{d,n}$, where $d = \dim(X_{\sigma_1}) = \dim(X_{\sigma_2})$.*

Proof. For $i = 1, 2$, denote by v_i the unique vertex on σ_i with valence at least four. Observe that the tree obtained by collapsing to a point every edge of σ_1 except those adjacent to v_1 is the same as the tree obtained by collapsing every edge of σ_2 except those adjacent to v_2; we denote this common tree by σ and we denote by v_0 the vertex on σ obtained from v_1 as well as from v_2. We use the product decomposition of X_σ given in (1), and, for v a vertex of σ, set pr_v to be the projection from X_σ to $\overline{\mathcal{M}}_{0,\text{Valence}(v)}$. Observe that in $A_d(X_\sigma)$, we have, for $i=1,2$, that $[X_{\sigma_i}] = (\mathrm{pr}_{v_0})^*([\overline{\mathcal{M}}_{0,\mathrm{Valence}(v_0)}]) \cdot \prod_{v \neq v_0} \mathrm{pr}_v^*([\mathrm{point}])$. Pushing this relation forward to $\overline{\mathcal{M}}_{0,n}$, we obtain the desired equality.

□

For $n \geq 1$ and $d \geq -3$, we set $\mathbf{SP}_{d,n}$ to be the set of all set partitions of $[n]$ having exactly $d+3$ parts. By Lemma 2.4, for $n \geq 4$ and $d \geq 1$, there is a well-defined map $\mathbf{SP}_{d,n} \to \mathcal{A}_{d,n}$ sending Π to $[X_\sigma]$, where σ is any Type I stable n-marked tree such that $\Pi = \Pi_*(\sigma)$ (it is clear that such a σ exists). Extending by linearity and composing with the quotient map from $\mathcal{A}_{d,n}$ to $\mathcal{Q}_{d,n}$, we obtain a surjective, S_n-equivariant, linear map $\mathbb{Q}\mathbf{SP}_{d,n} \to \mathcal{Q}_{d,n}$.

Lemma 2.5. *The kernel of the surjective linear map $\mathbb{Q}\mathbf{SP}_{d,n} \to \mathcal{Q}_{d,n}$ is the subspace $\mathcal{R}_{d,n}$ of $\mathbb{Q}\mathbf{SP}_{d,n}$ generated by elements of the form:*

$$\begin{aligned} &\{P_1 \cup P_2, P_3, P_4, \ldots, P_{d+4}\} + \{P_1, P_2, P_3 \cup P_4, \ldots, P_{d+4}\} \\ &- \{P_1 \cup P_3, P_2, P_4, \ldots, P_{d+4}\} - \{P_1, P_3, P_2 \cup P_4, \ldots, P_{d+4}\}, \end{aligned} \tag{3}$$

where $\{P_1, P_2, P_3, P_4, \ldots, P_{d+4}\}$ is a set partition of $[n]$ with $d+4$ parts.

Proof. An element $\sum_{s=1}^t a_s \Pi_s$ of $\mathbb{Q}\mathbf{SP}_{d,n}$ is the kernel of the map to $\mathcal{Q}_{d,n}$ if and only if there is a linear relation in $\mathcal{A}_{d,n}$ of the form: $\sum_{s=1}^t a_s [X_{\sigma_s}] =$ (linear combination of classes of Type II strata), where each X_{σ_s} is a Type I stratum such that $\Pi_*(\sigma_s) = \Pi_s$. By Theorem 7.3 of [KM94], the relations, in $\mathcal{A}_{d,n}$, among d-dimensional strata are generated by the relations $R(\tau, \{i,j,k,l\}, v)$ as in Definition 2.3. Given $R(\tau, \{i,j,k,l\}, v)$, there are three cases: **Case 1** is that τ is Type II, v has valence exactly four, and τ has exactly one vertex v' other than v with valence four or more. In this case, the resulting relation is of the form: $[X_{\tau_1}] + [X_{\tau_2}] - [X_{\tau_3}] - [X_{\tau_4}] = 0$, where for $x = 1, \ldots, 4$, the tree τ_x is Type I and $\Pi_*(\tau_x) = \Pi_*(\tau, v')$; in particular, applying Π_* to the left side of the above relation, we obtain $0 \in \mathbb{Q}\mathbf{SP}_{d,n}$. **Case 2** is that τ is Type II and either v has valence five or more, or τ has at least three vertices each with valence four or more. In this case, all the terms in the resulting relation are Type II. **Case 3** is that τ is Type I. In this case, we may write $\Pi_*(\tau) = \{P_1, \ldots, P_{d+4}\}$ in such a way that $i \in P_1$, $j \in P_2$, $k \in P_3$, and $l \in P_4$, and we observe that the resulting relation has the form (after rearranging terms):

$$[X_{\tau_{1,2}}] + [X_{\tau_{3,4}}] - [X_{\tau_{1,3}}] - [X_{\tau_{2,4}}] = (\text{linear combination of classes of Type II strata}), \tag{4}$$

where each $\tau_{x,y}$ is a type I tree and $\Pi_*(\tau_{x,y}) = \{P_x \cup P_y\} \cup (\Pi_*(\tau) \smallsetminus \{P_x, P_y\})$. In particular, applying Π_* to the left side of the relation (4), we obtain an expression of the form (3). We conclude if we have a relation in $\mathcal{A}_{d,n}$ of the form:

$$(\text{linear combination of classes of Type I strata}) = (\text{linear combination of classes of Type II strata}) \tag{5}$$

then applying Π_* to the left side yields a linear combinations of set partitions in the $\mathbb{Q}$-span of expressions of the form (3). This implies that the kernel of the map from $\mathbb{Q}\mathbf{SP}_{d,n}$ to $\mathcal{Q}_{d,n}$ is contained in the $\mathbb{Q}$-span of expressions of the form (3). Conversely, given any element $\theta \in \mathbb{Q}\mathbf{SP}_{d,n}$ of the form (3), i.e. any set partition $\{P_1, \ldots, P_{d+4}\}$ of $[n]$ with distinguished parts $P_1, \ldots, P_4$, one can find a Type I tree τ (with v its unique vertex of valence at least four) such that $\Pi_*(\tau) = \{P_1, \ldots, P_{d+4}\}$. If, further, we pick $i \in P_1$, $j \in P_2$, $k \in P_3$, and $l \in P_4$, then applying Π_* to the Type I terms of the induced relation $R(\tau, \{i,j,k,l\}, v)$ yields θ, showing that θ is in the kernel of the map to $\mathcal{Q}_{d,n}$.

□

Corollary 2.6. *There is an S_n-equivariant isomorphism $\mathcal{Q}_{d,n} \cong \mathbb{Q}\mathbf{SP}_{d,n}/\mathcal{R}_{d,n}$ identifying the class $[X_\sigma]$ of a Type I stratum with the set partition $\Pi_*(\sigma)$.*

2.2. **Kappa classes and the intersection pairing.** For $d = 0, \ldots, n-3$, $\overline{\mathcal{M}}_{0,n}$ carries a codimension d kappa class $\kappa_d \in \mathcal{A}_n^d$. It can be defined as follows: Set $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ to the map forgetting the $(n+1)-$st marked point, and ψ_{n+1} to be the divisor class on $\overline{\mathcal{M}}_{0,n+1}$ corresponding to the line bundle whose fiber at $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,n+1}$ is the cotangent space of C at p_{n+1}. Then $\kappa_d := \pi_*(\psi_{n+1}^{d+1})$. (See [AC96, AC98] for details and properties.)

Lemma 2.7. *Let $n \geq 4$ and $1 \leq d \leq n-3$. Suppose X_σ is a dimension d boundary stratum on $\overline{\mathcal{M}}_{0,n}$. Then*

$$[X_\sigma]\cdot\kappa_d = \begin{cases} 1 & X_\sigma \text{ is Type I} \\ 0 & X_\sigma \text{ is Type II} \end{cases} \tag{6}$$

Proof. We use the product decomposition of X_σ given in (1), and, for v a vertex of σ, set pr_v to be the projection from X_σ to $\overline{\mathcal{M}}_{0,\mathrm{Valence}(v)}$. By Equation 1.8 of [AC96], we have that, as a class supported on X_σ:

$$[X_\sigma]\cdot\kappa_d = \sum_{v \text{ vertex of } \sigma} \mathrm{pr}_v^*(\kappa_d^{\overline{\mathcal{M}}_{0,\mathrm{Valence}(v)}}), \tag{7}$$

where $\kappa_d^{\overline{\mathcal{M}}_{0,\mathrm{Valence}(v)}}$ denotes the codimension-d κ class on the factor $\overline{\mathcal{M}}_{0,\mathrm{Valence}(v)}$. If X_σ is Type II, then for each vertex v, we have $(\mathrm{Valence}(v))-3) < d$. This implies that $\kappa_d^{\overline{\mathcal{M}}_{0,\mathrm{Valence}(v)}}$ has negative dimension so must be zero, and so $[X_\sigma]\cdot\kappa_d = 0$. On the other hand, if X_σ is Type I and v_0 is the unique vertex of σ with valence at least four, then $\mathrm{Valence}(v_0) - 3 = d$, and for any vertex $v \neq v_0$, $\mathrm{Valence}(v) - 3 = 0$. This implies that, for such X_σ, the term of the sum in (7) corresponding to v_0 equals $\mathrm{pr}_{v_0}^*(\kappa_d^{\overline{\mathcal{M}}_{0,d+3}})$, which by Lemma 1.1 (12) of [CY11] equals 1, while dimension-counting tells us that each of the other terms of the sum in (7) equals 0, showing that $[X_\sigma]\cdot\kappa_d = 1$.

□

Definition 2.8. For $T \subseteq [n]$, we set κ_d^T to be the pullback, to $\overline{\mathcal{M}}_{0,n}$, of the codimension d kappa class on $\overline{\mathcal{M}}_{0,T}$, via the natural forgetful map $\pi_T : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,T}$. We also set $\mathcal{K}_n^d$ to be the subspace of $\mathcal{A}_n^d$ spanned by the classes $\{\kappa_d^T \mid T \subseteq [n]\}$. Note that $\mathcal{K}_n^d$ is S_n-invariant.

Definition 2.9. Define a pairing $\langle\cdot,\cdot\rangle : \{\text{set partitions of } [n]\} \times \{\text{subsets of } [n]\} \to \mathbb{Z}$. For a set partition Π and subset T, set

$$\langle \Pi, T\rangle = \begin{cases} 1 & \forall P \in \Pi, \quad P\cap T \neq \emptyset \\ 0 & \exists P \in \Pi \text{ s.t. } P \cap T = \emptyset. \end{cases}$$

Lemma 2.10. *Let $n \geq 4$ and $1 \leq d \leq n-3$. Suppose X_σ is a dimension d boundary stratum on $\overline{\mathcal{M}}_{0,n}$, and $T \subseteq [n]$. Then*

$$[X_\sigma]\cdot\kappa_d^T = \begin{cases} \langle \Pi_*(\sigma), T\rangle & X_\sigma \text{ is Type I} \\ 0 & X_\sigma \text{ is Type II.} \end{cases} \tag{8}$$

Proof. By the projection formula, $[X_\sigma]\cdot\kappa_d^T = (\pi_T)_*([X_\sigma])\cdot\kappa_d$. By Lemma 9.5 of [Ram18], if X_σ is Type II, then $(\pi_T)_*([X_\sigma])$ is either zero, or the fundamental class of a Type II boundary stratum of $\overline{\mathcal{M}}_{0,T}$. If X_σ is Type I, then if $\exists P \in \Pi_*(\sigma)$ s.t. $P \cap T = \emptyset$, then $(\pi_T)_*([X_\sigma]) = 0$, while if $\forall P \in \Pi_*(\sigma)$, $P \cap T \neq \emptyset$, then $(\pi_T)_*([X_\sigma])$ is the fundamental class of a Type I boundary stratum of $\overline{\mathcal{M}}_{0,T}$. Applying Lemma 2.7, we obtain the desired result. □

Corollary 2.11. *(1) The subspace $\mathcal{K}_n^d \subseteq \mathcal{A}_n^d$ is orthogonal, with respect to the intersection pairing, to $\mathcal{V}_{d,n} \subseteq \mathcal{A}_{d,n}$, i.e. we have $\mathcal{K}_n^d \subseteq \mathcal{V}_{d,n}^\perp$.*
(2) The intersection pairing on $\overline{\mathcal{M}}_{0,n}$ descends to a pairing $\mathcal{Q}_{d,n} \times \mathcal{K}_n^d \to \mathbb{Q}$.

We will eventually show that $\mathcal{K}_n^d = \mathcal{V}_{d,n}^\perp$, which implies that $\mathcal{K}_n^d = \mathcal{Q}_{d,n}^\vee$. Note that by Lemma 2.10, the pairing between $\mathcal{Q}_{d,n}$ and $\mathcal{K}_n^d$ obtained in Corollary 2.11 can be expressed in purely combinatorial terms.

2.3. **Pushing forward and pulling back via forgetful maps.** The forgetful map $\pi : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ induces pushforward maps $\pi_* : \mathcal{A}_{d,n+1} \to \mathcal{A}_{d,n}$ and pullback maps $\pi^* : \mathcal{A}_{d,n} \to \mathcal{A}_{d+1,n+1}$. By [Ram18], $\pi_*(\mathcal{V}_{d,n+1}) \subseteq \mathcal{V}_{d,n}$. Also, if X_σ is a Type II boundary stratum, then $\pi^*([X_\sigma])$ is a sum of fundamental classes of Type II boundary strata of $\overline{\mathcal{M}}_{0,n+1}$. This implies that $\pi^*(\mathcal{V}_{d,n}) \subseteq \mathcal{V}_{d+1,n+1}$. Thus there are induced pushforward maps $\pi_* : \mathcal{Q}_{d,n+1} \to \mathcal{Q}_{d,n}$ and pullback maps $\pi^* : \mathcal{Q}_{d,n} \to \mathcal{Q}_{d+1,n+1}$.

Lemma 2.12. *(1) The pushforward $\pi_* : \mathcal{Q}_{d,n+1} \to \mathcal{Q}_{d,n}$ lifts to $\tilde{\pi}_* : \mathbb{Q}\mathbf{SP}_{d,n+1} \to \mathbb{Q}\mathbf{SP}_{d,n}$ where*

$$\tilde{\pi}_*(\Pi) := \begin{cases} 0 & \{n+1\} \in \Pi \\ \{P \smallsetminus \{n+1\} \mid P \in \Pi\} & \textit{otherwise} \end{cases}$$

(2) The pullback $\pi^* : \mathcal{Q}_{d,n} \to \mathcal{Q}_{d+1,n+1}$ lifts to $\tilde{\pi}^* : \mathbb{Q}\mathbf{SP}_{d,n} \to \mathbb{Q}\mathbf{SP}_{d+1,n+1}$ where, for $\Pi \in \mathbf{SP}_{d,n}$, $\tilde{\pi}^*(\Pi) := \Pi \cup \{\{n+1\}\}$.

Proof. For (1), suppose that X_σ is a Type I d-dimensional boundary stratum on $\overline{\mathcal{M}}_{0,n+1}$. Observe that if, $\forall P \in \Pi_*(\sigma)$, we have that $P \smallsetminus \{n+1\} \neq \emptyset$, then $\pi_*([X_\sigma]) = [X_{\sigma'}]$, where $\Pi_*(\sigma') = \{P \smallsetminus \{n+1\} \mid P \in \Pi_*(\sigma)\}$. On the other hand, if $\{n+1\} \in \Pi_*(\sigma)$, then $\dim(\pi(X_\sigma)) < \dim(X_\sigma)$, so $\pi_*([X_\sigma]) = 0$. For (2), suppose that X_τ is a Type I d-dimensional boundary stratum on $\overline{\mathcal{M}}_{0,n}$, and observe that

$$\pi^*([X_\tau]) = [X_{\tau'}] + (\text{sum of classes of Type II boundary strata}), \tag{9}$$

where τ' is a Type I stable $(n+1)$-marked tree and $\Pi_*(\tau') = \Pi_*(\tau) \cup \{\{n+1\}\}$. $\square$

The pushforward maps $\pi_* : \mathcal{A}_{d,n+1} \to \mathcal{A}_{d,n}$ and $\pi_* : \mathcal{Q}_{d,n+1} \to \mathcal{Q}_{d,n}$ are easily seen to be surjective. Since π has positive relative dimension (equal to one), $\pi_* \circ \pi^* = 0$ on $\mathcal{A}_{*,*}$, thus also on $\mathcal{Q}_{*,*}$.

Lemma 2.13. *For $n \geq 4$ and $k \geq 1$, the complex $\mathcal{Q}_{d,n} \xrightarrow{\pi^*} \mathcal{Q}_{d+1,n+1} \xrightarrow{\pi_*} \mathcal{Q}_{d+1,n}$ is exact.*

Proof. We use the lifts of π^* and π_* to $\tilde{\pi}^* : \mathbb{Q}\mathbf{SP}_{d,n} \to \mathbb{Q}\mathbf{SP}_{d+1,n+1}$ and $\tilde{\pi}_* : \mathbb{Q}\mathbf{SP}_{d+1,n+1} \to \mathbb{Q}\mathbf{SP}_{d+1,n}$ respectively. Note that $\mathrm{Im}(\tilde{\pi}^*) = \mathrm{Span}(\{\Pi \mid \{n+1\} \in \Pi\})$. On the other hand, $\mathrm{Ker}(\tilde{\pi}_*)$ is spanned by the set partitions in which $n+1$ appears as a singleton set (i.e. set partitions in $\mathrm{Im}(\tilde{\pi}^*)$), together with differences of two set partitions that differ only in the placement of $n+1$, i.e.:

$$\mathrm{Ker}(\tilde{\pi}_*) = \mathrm{Im}(\tilde{\pi}^*) + \mathrm{Span}(\{\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots\} - \{P_1, P_2 \cup \{n+1\}, P_3, P_4, \ldots\}\}).$$

Note that:

$$\begin{aligned} &\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots\} - \{P_1, P_2 \cup \{n+1\}, P_3, P_4, \ldots\} \\ &= \Big(\{P_1 \cup \{n+1\}, P_2, P_3, P_4, \ldots\} + \{P_1, \{n+1\}, P_2 \cup P_3, P_4, \ldots\} \\ &\qquad -\{P_1, P_3, P_2 \cup \{n+1\}, P_4, \ldots\} - \{P_1 \cup P_3, P_2, \{n+1\}, P_4, \ldots\}\Big) \\ &\qquad - \Big(\{P_1, \{n+1\}, P_2 \cup P_3, P_4, \ldots\} - \{P_1 \cup P_3, P_2, \{n+1\}, P_4, \ldots\}\Big) \in \mathcal{R}_{d+1,n+1} + \mathrm{Im}(\tilde{\pi}^*) \end{aligned}$$

This implies that $\mathrm{Ker}(\tilde{\pi}_*) = \mathrm{Im}(\tilde{\pi}^*) + \mathcal{R}_{d+1,n+1}$, which in turn implies that $\mathrm{Ker}(\pi_*) = \mathrm{Im}(\pi^*)$. $\square$

The pullback $\pi^* : \mathcal{A}^d_n \to \mathcal{A}^d_{n+1}$, which by the projection formula is dual to $\pi_* : \mathcal{A}_{d,n} \to \mathcal{A}_{d,n+1}$, restricts to $\pi^* : \mathcal{K}^d_n \to \mathcal{K}^d_{n+1}$, and sends κ^T_d on $\overline{\mathcal{M}}_{0,n}$ to κ^T_d on $\overline{\mathcal{M}}_{0,n+1}$. The pushforward $\pi_* : \mathcal{A}^{d+1}_{n+1} \to \mathcal{A}^d_n$ is dual to $\pi^* : \mathcal{A}_{d+1,n+1} \to \mathcal{A}_{d,n}$.

Lemma 2.14. *The pushforward $\pi_* : \mathcal{A}^{d+1}_{n+1} \to \mathcal{A}^d_n$ restricts to $\pi_* : \mathcal{K}^{d+1}_{n+1} \to \mathcal{K}^d_n$, with*

$$\pi_*(\kappa^T_{d+1}) = \begin{cases} \kappa_d^{T \smallsetminus \{n+1\}} & n+1 \in T \\ 0 & n+1 \notin T \end{cases}$$

Proof. By Lemma 2.11, (1), for all n and d, we have that $\mathcal{K}^d_n \subseteq \mathcal{V}^\perp_{d,n}$. Since $\pi^*(\mathcal{V}_{d,n}) \subseteq \mathcal{V}_{d+1,n+1}$, and since, by the projection formula, π_* and π^* are dual maps, we have that $\pi_*(\mathcal{K}^{d+1}_{n+1}) \subseteq \mathcal{V}^\perp_{d,n}$. This means that for $T \subseteq [n+1]$, the class $\pi_*(\kappa^T_{d+1})$ is determined by the functional that it defines on $\mathcal{Q}_{d,n}$, i.e. by the values of $[X_\sigma] \cdot \pi_*(\kappa^T_{d+1})$, where X_σ ranges over all Type I d-dimensional boundary strata on $\overline{\mathcal{M}}_{0,n}$. Given such an X_σ, we have, by the projection formula, by the expression for $\pi^*([X_\sigma])$ given in Equation 9, and by applying

Lemma 2.10 twice, that

$$\begin{aligned}[X_\sigma]\cdot\pi_*(\kappa_{d+1}^T) = \pi^*([X_\sigma])\cdot\kappa_{d+1}^T &= \langle \Pi_*(\sigma)\cup\{\{n+1\}\}, T\rangle \\ &= \begin{cases} 1 & \text{if } \forall P\in\Pi_*(\sigma)\cup\{\{n+1\}\}, P\cap T\neq\emptyset \\ 0 & \text{if } \exists P\in\Pi_*(\sigma)\cup\{\{n+1\}\} \text{ s.t. } P\cap T=\emptyset\end{cases} \\ &= \begin{cases} 1 & \text{if } \forall P\in\Pi_*(\sigma), P\cap(T\smallsetminus\{n+1\})\neq\emptyset, \text{ and } n+1\in T \\ 0 & \text{if } \exists P\in\Pi_*(\sigma) \text{ s.t. } P\cap(T\smallsetminus\{n+1\})=\emptyset, \text{ or if } n+1\notin T\end{cases} \\ &= \begin{cases} \langle\Pi_*(\sigma), T\smallsetminus\{n+1\}\rangle & \text{if } n+1\in T \\ 0 & \text{if } n+1\notin T\end{cases} \\ &= \begin{cases} [X_\sigma]\cdot\kappa_d^{T\smallsetminus\{n+1\}} & \text{if } n+1\in T \\ 0 & \text{if } n+1\notin T.\end{cases}\end{aligned}$$

□

3. Main results and proofs

3.1. **The set-up.** Throughout Section 3, we use the identification $\mathcal{Q}_{d,n}\cong\mathbb{Q}\mathbf{SP}_{d,n}/\mathcal{R}_{d,n}$ introduced in Corollary 2.6: we write an element of $\mathcal{Q}_{d,n}$ as a $\mathbb{Q}$-linear combination of set partitions of $[n]$ with $d+3$ parts, rather than as a $\mathbb{Q}$-linear combination of fundamental classes of Type I boundary strata.

Definition 3.1. For $n\geq 1$ and $d\geq -3$, we set $\mathbf{K}_n^d := \{T\subseteq[n] \mid |T|\geq(d+3), |T|\equiv(d+3) \mod 2\}$, and set $\zeta_{d,n}:\mathbb{Q}\mathbf{K}_n^d\to\mathcal{K}_n^d$ to be the natural linear map $\zeta_{d,n}:\mathbb{Q}\mathbf{K}_n^d\to\mathcal{K}_n^d$ sending T to κ_d^T.

The map $\zeta_{d,n}$, together with the intersection pairing $\mathcal{Q}_{d,n}\times\mathcal{K}_n^d\to\mathbb{Q}$ induces the pairing $\mathcal{Q}_{d,n}\times\mathbb{Q}\mathbf{K}_n^d\to\mathbb{Q}$, where, for $\Pi\in\mathbf{SP}_{d,n}$ and $T\in\mathbf{K}_n^d$, $\Pi\cdot T=\langle\Pi,T\rangle$ as in Definition 2.9. Note that if $\mathbf{S}$ is an S_n-set, then $\mathbb{Q}\mathbf{S}$ has a natural basis $\mathbf{S}$: identifying each basis element of $\mathbb{Q}\mathbf{S}$ with the corresponding dual basis element in $\mathbb{Q}\mathbf{S}^\vee$ induces a natural S_n-equivariant identification between $\mathbb{Q}\mathbf{S}$ and its dual. Thus we write $\mathbb{Q}\mathbf{K}_n^d=(\mathbb{Q}\mathbf{K}_n^d)^\vee$.

Definition 3.2. For $d\geq -1$, the pairing $\langle .,.\rangle$ (Definition 2.9) between set partitions and subsets of $[n]$ induces a map $\tilde\phi_{d,n}:\mathbb{Q}\mathbf{SP}_{d,n}\to(\mathbb{Q}\mathbf{K}_n^d)^\vee=\mathbb{Q}\mathbf{K}_n^d$. For $d\geq 1$, $\tilde\phi_{d,n}$ descends to a map $\phi_{d,n}:\mathcal{Q}_{d,n}\to\mathbb{Q}\mathbf{K}_n^d$. The map $\tilde\phi_{d,n}$ on generators $\Pi\in\mathbf{SP}_{d,n}$ is given explicitly by: $\tilde\phi_{d,n}(\Pi)=\sum_{T\in\mathbf{K}_n^d}\langle\Pi,T\rangle\cdot T$.

Definition 3.3. Define maps $\alpha:\mathbb{Q}\mathbf{K}_n^d\to\mathbb{Q}\mathbf{K}_{n+1}^{d+1}$ and $\beta:\mathbb{Q}\mathbf{K}_{n+1}^d\to\mathbb{Q}\mathbf{K}_n^d$, where:

$$\alpha(T)=T\cup\{n+1\} \qquad\qquad \beta(T)=\begin{cases} T & n+1\notin T\\ 0 & n+1\in T.\end{cases}$$

Lemma 3.4. *The following is an exact sequence:*

$$0\to\mathbb{Q}\mathbf{K}_n^d\xrightarrow{\alpha}\mathbb{Q}\mathbf{K}_{n+1}^{d+1}\xrightarrow{\beta}\mathbb{Q}\mathbf{K}_n^{d+1}\to 0. \tag{10}$$

Proof. First, we observe that α is injective, since it maps the natural basis of $\mathbb{Q}\mathbf{K}_n^d$ injectively to a subset of the natural basis of $\mathbb{Q}\mathbf{K}_{n+1}^{d+1}$. Also, β is surjective, since every basis element of $\mathbb{Q}\mathbf{K}_n^{d+1}$ is in the image. Finally, for exactness in the middle, we observe that $\mathrm{Im}(\alpha)=\mathrm{Ker}(\beta)=\mathbb{Q}\{T\in\mathbf{K}_{n+1}^{d+1}\mid n+1\in T\}$. □

Lemma 3.5. *The following diagram commutes:*

$$\begin{array}{ccccc}\mathbb{Q}\mathbf{SP}_{d,n} & \xrightarrow{\tilde\pi^*} & \mathbb{Q}\mathbf{SP}_{d+1,n+1} & \xrightarrow{\tilde\pi_*} & \mathbb{Q}\mathbf{SP}_{d+1,n}\\ \downarrow{\scriptstyle\tilde\phi_{d,n}} & & \downarrow{\scriptstyle\tilde\phi_{d+1,n+1}} & & \downarrow{\scriptstyle\tilde\phi_{d+1,n}}\\ \mathbb{Q}\mathbf{K}_n^d & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{K}_{n+1}^{d+1} & \xrightarrow{\beta} & \mathbb{Q}\mathbf{K}_n^{d+1}\end{array}$$

Proof. **Commutativity of the left square:** Given $\Pi \in \mathbf{SP}_{d,n}$, we have

$$\begin{aligned}\tilde{\phi}_{d+1,n+1}(\tilde{\pi}^*(\Pi)) = \sum_{T \in \mathbf{K}^{d+1}_{n+1}} \langle \Pi \cup \{\{n+1\}\}, T\rangle \cdot T &= \sum_{\substack{T \in \mathbf{K}^{d+1}_{n+1} \\ n+1 \in T}} \langle \Pi \cup \{\{n+1\}\}, T\rangle \cdot T \\ &= \sum_{T' \in \mathbf{K}^{d}_{n}} \langle \Pi, T'\rangle \cdot T' \cup \{n+1\} \\ &= \alpha(\sum_{T' \in \mathbf{K}^{d}_{n}} \langle \Pi, T'\rangle \cdot T') = \alpha(\tilde{\phi}_{d,n}(\Pi)).\end{aligned}$$

Commutativity of the right square: Given $\Pi' = \{P_1, \ldots, P_{d+4}\} \in \mathbf{SP}_{d+1,n+1}$, we may assume without loss of generality that $n+1 \in P_1$. There are two cases:
Case 1: $P_1 = \{n+1\}$. Then $\tilde{\pi}_*(\Pi') = 0$ so $\tilde{\phi}_{d+1,n}(\tilde{\pi}_*(\Pi')) = 0$. Note that for $T \subset [n+1]$, we have that if $n+1 \notin T$, then $\langle \Pi', T\rangle = 0$, while if $n+1 \in T$, then $\beta(T) = 0$. This implies that

$$\beta(\tilde{\phi}_{d+1,n+1}(\Pi')) = \beta(\sum_{T \in \mathbf{K}^{d+1}_{n+1}} \langle \Pi', T\rangle \cdot T) = \beta(\sum_{\substack{T \in \mathbf{K}^{d+1}_{n+1} \\ n+1 \in T}} \langle \Pi', T\rangle \cdot T) = 0.$$

Case 2: $P_1 \neq \{n+1\}$. Then

$$\begin{aligned}\tilde{\phi}_{d+1,n}(\tilde{\pi}_*(\Pi')) &= \sum_{T' \in \mathbf{K}^{d+1}_{n}} \langle \{P_1 \smallsetminus \{n+1\}, P_2, \ldots, P_{d+4}\}, T'\rangle \cdot T' \\ &= \sum_{\substack{T' \in \mathbf{K}^{d+1}_{n} \\ T' \cap P_1 \smallsetminus \{n+1\} \neq \emptyset \\ T' \cap P_2, \ldots, T' \cap P_{d+1} \neq \emptyset}} T' = \sum_{\substack{T' \in \mathbf{K}^{d+1}_{n+1} \\ n+1 \notin T \\ \langle \Pi', T\rangle = 1}} T' = \beta(\tilde{\phi}_{d+1,n+1}(\Pi')).\end{aligned}$$

□

We use the following lemma several times; its proof follows from a standard diagram chase.

Lemma 3.6. *[Variant of the Four Lemma] Suppose we have a commutative diagram of vector spaces as follows*

$$\begin{array}{ccccccc} \mathcal{W}_1 & \xrightarrow{f_1} & \mathcal{W}_2 & \xrightarrow{f_2} & \mathcal{W}_3 & \longrightarrow & 0 \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\ \mathcal{X}_1 & \xrightarrow{g_1} & \mathcal{X}_2 & \xrightarrow{g_2} & \mathcal{X}_3 & & \end{array}$$

Suppose further that the bottom row is exact at $\mathcal{X}_2$, that the top row is exact at $\mathcal{W}_3$, and that h_1 and h_3 are surjective. Then h_2 is surjective.

3.2. **A preliminary lemma.** In this section, we prove some technical results — Lemmas 3.8, 3.9 and 3.10. These are not of independent interest, but are necessary to prove Theorem 3.13 in Section 3.3. The proofs (and statements) of these three lemmas are conceptually similar to each other, as well as to those of Theorem 3.13; all four proofs use the Four Lemma or its variant Lemma 3.6 to induct on n. Lemma 3.8 is required in the inductive step of Lemma 3.9, which is required in the inductive step of Lemma 3.10, which in turn is required in the inductive step of Theorem 3.13. The proofs of Lemmas 3.8 and 3.9 also involve some intricate combinatorics of set partitions and subsets.

Definition 3.7. For $n > 0$, we set:

$$\mathbf{E}_n := \{T \subseteq [n] \,|\, |T| \text{ even}\}; \quad \mathbf{O}_n := \{T \subseteq [n] \,|\, |T| \text{ odd}\}; \quad \mathbf{F}_n := \{(P_1, P_2) | P_1 \cup P_2 = [n], P_1 \cap P_2 = \emptyset, 1 \in P_1\}$$

Note that $\mathbf{F}_n \smallsetminus \{([n], \emptyset)\}$ is in canonical bijection with $\mathbf{SP}_{-1,n}$, so $\mathbb{Q}\mathbf{F}_n$ is canonically isomorphic to $\mathbb{Q}\{([n], \emptyset)\} \oplus \mathbb{Q}\mathbf{SP}_{-1,n}$. There are maps $\alpha : \mathbb{Q}\mathbf{E}_n \to \mathbb{Q}\mathbf{O}_{n+1}$, $\alpha : \mathbb{Q}\mathbf{O}_n \to \mathbb{Q}\mathbf{E}_{n+1}$, $\beta : \mathbb{Q}\mathbf{E}_{n+1} \to \mathbb{Q}\mathbf{E}_n$ and

$\beta : \mathbb{Q}\mathbf{O}_{n+1} \to \mathbb{Q}\mathbf{O}_n$, analogous to the maps α and β as in Definition 3.3. Define maps $\mathrm{odd}_n : \mathbb{Q}\mathbf{F}_n \to \mathbb{Q}\mathbf{O}_n$ and $\mathrm{even}_n : \mathbb{Q}\mathbf{F}_n \to \mathbb{Q}\mathbf{E}_n$, where

$$\mathrm{odd}_n((P_1, P_2)) = \sum_{\substack{T \subseteq P_1 \\ |T| \text{ odd}}} (-T) + \sum_{\substack{T \subseteq P_2 \\ |T| \text{ odd}}} T; \qquad \mathrm{even}_n((P_1, P_2)) = \sum_{\substack{T \subseteq P_1 \\ |T| \text{ even}}} (-T) + \sum_{\substack{T \subseteq P_2 \\ |T| \text{ even}}} (-T)$$

Lemma 3.8. *For $n \geq 1$, the maps odd_n and even_n are surjective.*

Proof. We induct on n. **Base case:** $n = 1$. We have:

$$\mathbf{F}_1 = \{(\{1\}, \emptyset)\}; \quad \mathbf{E}_1 = \{\emptyset\}; \quad \mathbf{O}_1 = \{\{1\}\}; \quad \mathrm{odd}_1((\{1\}, \emptyset)) = -\{1\}; \quad \mathrm{even}_1((\{1\}, \emptyset)) = -2\emptyset.$$

This establishes the base case.

Inductive hypothesis: The proposition holds up to some $n \geq 1$.

Inductive step: Define maps $\gamma : \mathbb{Q}\mathbf{F}_n \to \mathbb{Q}\mathbf{F}_{n+1}$ and $\delta : \mathbb{Q}\mathbf{F}_{n+1} \to \mathbb{Q}\mathbf{F}_n$, where:

$$\gamma((P_1', P_2')) = (P_1' \cup \{n+1\}, P_2') - (P_1', P_2' \cup \{n+1\})$$
$$\delta((P_1, P_2)) = (P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\})$$

The diagram below has exact rows; we claim it commutes.

$$\begin{array}{ccccccccc}
 & & \mathbb{Q}\mathbf{F}_n & \xrightarrow{\gamma} & \mathbb{Q}\mathbf{F}_{n+1} & \xrightarrow{\delta} & \mathbb{Q}\mathbf{F}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{even}_n} & & \downarrow{\scriptstyle \mathrm{odd}_{n+1}} & & \downarrow{\scriptstyle \mathrm{odd}_n} & & \\
0 & \longrightarrow & \mathbb{Q}\mathbf{E}_n & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{O}_{n+1} & \xrightarrow{\beta} & \mathbb{Q}\mathbf{O}_n & \longrightarrow & 0
\end{array}$$

Commutativity of the left square: For $(P_1', P_2') \in \mathbb{Q}\mathbf{F}_n$,

$$\begin{aligned}
\mathrm{odd}_{n+1}(\gamma(P_1', P_2')) &= \mathrm{odd}_{n+1}(P_1' \cup \{n+1\}, P_2') - \mathrm{odd}_{n+1}(P_1', P_2' \cup \{n+1\}) \\
&= \sum_{\substack{T \subseteq P_1' \cup \{n+1\} \\ |T| \text{ odd}}} (-T) + \sum_{\substack{T \subseteq P_2' \\ |T| \text{ odd}}} (T) - \left(\sum_{\substack{T \subseteq P_1' \\ |T| \text{ odd}}} (-T) + \sum_{\substack{T \subseteq P_2' \cup \{n+1\} \\ |T| \text{ odd}}} (T) \right) \\
&= \sum_{\substack{T \subseteq P_1' \cup \{n+1\} \\ |T| \text{ odd} \\ n+1 \in T}} (-T) + \sum_{\substack{T \subseteq P_2' \cup \{n+1\} \\ |T| \text{ odd} \\ n+1 \in T}} (-T) \\
&= \sum_{\substack{T' \subseteq P_1' \\ |T'| \text{ even}}} (-(T' \cup \{n+1\})) + \sum_{\substack{T' \subseteq P_2' \\ |T'| \text{ even}}} (-(T' \cup \{n+1\})) \\
&= \alpha(\mathrm{even}_n(P_1', P_2')).
\end{aligned}$$

Commutativity of the right square: For $(P_1, P_2) \in \mathbb{Q}\mathbf{F}_{n+1}$,

$$\begin{aligned}
\mathrm{odd}_n(\delta(P_1, P_2)) &= \mathrm{odd}_n(P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\}) \\
&= \sum_{\substack{T \subseteq P_1 \setminus \{n+1\} \\ |T| \text{ odd}}} (-T) + \sum_{\substack{T \subseteq P_2 \setminus \{n+1\} \\ |T| \text{ odd}}} (T) \\
&= \beta(\mathrm{odd}_{n+1}(P_1, P_2)).
\end{aligned}$$

This proves the claim. By the inductive hypothesis, even_n and odd_n are surjective, so by the Four Lemma, odd_{n+1} is surjective.

The digram below has exact rows; we claim it commutes.

$$\begin{array}{ccccccccc}
 & & \mathbb{Q}\mathbf{F}_n & \xrightarrow{\gamma} & \mathbb{Q}\mathbf{F}_{n+1} & \xrightarrow{\delta} & \mathbb{Q}\mathbf{F}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{odd}_n} & & \downarrow{\scriptstyle \mathrm{even}_{n+1}} & & \downarrow{\scriptstyle \mathrm{even}_n} & & \\
0 & \longrightarrow & \mathbb{Q}\mathbf{O}_n & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{E}_{n+1} & \xrightarrow{\beta} & \mathbb{Q}\mathbf{E}_n & \longrightarrow & 0
\end{array}$$

Commutativity of the left square: For $(P_1', P_2') \in \mathbb{Q}\mathbf{F}_n$,

$$\begin{aligned}
\text{even}_{n+1}(\gamma(P_1', P_2')) &= \text{even}_{n+1}(P_1' \cup \{n+1\}, P_2') - \text{even}_{n+1}(P_1', P_2' \cup \{n+1\}) \\
&= \sum_{\substack{T \subseteq P_1' \cup \{n+1\} \\ |T| \text{ even}}} (-T) + \sum_{\substack{T \subseteq P_2' \\ |T| \text{ even}}} (-T) - \left(\sum_{\substack{T \subseteq P_1' \\ |T| \text{ even}}} (-T) + \sum_{\substack{T \subseteq P_2' \cup \{n+1\} \\ |T| \text{ even}}} (-T) \right) \\
&= \sum_{\substack{T \subseteq P_1' \cup \{n+1\} \\ |T| \text{ even} \\ n+1 \in T}} (-T) + \sum_{\substack{T \subseteq P_2' \cup \{n+1\} \\ |T| \text{ even} \\ n+1 \in T}} (T) \\
&= \sum_{\substack{T' \subseteq P_1' \\ |T'| \text{ odd}}} -(T' \cup \{n+1\}) + \sum_{\substack{T' \subseteq P_2' \\ |T'| \text{ odd}}} (T' \cup \{n+1\}) \\
&= \alpha(\text{odd}_n(P_1', P_2')).
\end{aligned}$$

Commutativity of the right square: For $(P_1, P_2) \in \mathbb{Q}\mathbf{F}_{n+1}$,

$$\begin{aligned}
\text{even}_n(\delta(P_1, P_2)) &= \text{even}_n(P_1 \setminus \{n+1\}, P_2 \setminus \{n+1\}) \\
&= \sum_{\substack{T \subseteq P_1 \setminus \{n+1\} \\ |T| \text{ even}}} (-T) + \sum_{\substack{T \subseteq P_2 \setminus \{n+1\} \\ |T| \text{ even}}} (-T) \\
&= \beta(\text{even}_{n+1}(P_1, P_2)).
\end{aligned}$$

This proves the claim. Again, by the inductive hypothesis, odd_n and even_n are surjective, so by Lemma 3.6, even_{n+1} is surjective. □

We only use the fact that odd_n is surjective to proceed; we use it to prove Lemma 3.9

Lemma 3.9. *For all $n \geq 2$, the map $\tilde{\phi}_{-1,n} : \mathbb{Q}\mathbf{SP}_{-1,n} \to \mathbb{Q}\mathbf{K}_n^{-1}$ is surjective.*

Proof. We induct on n.
Base case: $n = 2$. We have $\mathbf{SP}_{-1,2} = \{\{\{1\}, \{2\}\}\}$ and $\mathbf{K}_2^{-1} = \{\{1, 2\}\}$. We have

$$\tilde{\phi}_{2,2}(\{\{1\}, \{2\}\}) = \langle \{\{1\}, \{2\}\}, \{1, 2\} \rangle \cdot \{1, 2\} = 1 \cdot \{1, 2\},$$

which shows that $\tilde{\phi}_{2,2}$ is surjective.
Inductive hypothesis: The lemma holds up to some $n \geq 2$.
Inductive step: Define a map $\gamma : \mathbb{Q}\mathbf{F}_n \to \mathbb{Q}\mathbf{SP}_{-1,n+1}$, where

$$\gamma((P_1', P_2')) = \{P_1' \cup \{n+1\}, P_2'\} - \{P_1', P_2' \cup \{n+1\}\}.$$

Consider the digram

$$\begin{array}{ccccccccc}
 & & \mathbb{Q}\mathbf{F}_n & \xrightarrow{\gamma} & \mathbb{Q}\mathbf{SP}_{-1,n+1} & \xrightarrow{\tilde{\pi}_*} & \mathbb{Q}\mathbf{SP}_{-1,n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \text{odd}_n} & & \downarrow{\scriptstyle \tilde{\phi}_{-1,n+1}} & & \downarrow{\scriptstyle \tilde{\phi}_{-1,n}} & & \\
0 & \longrightarrow & \mathbb{Q}\mathbf{O}_n & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{K}_{n+1}^{-1} & \xrightarrow{\beta} & \mathbb{Q}\mathbf{K}_n^{-1} & \longrightarrow & 0
\end{array}$$

Here, $\tilde{\pi}_* \circ \gamma = 0$, the bottom row is exact, and $\tilde{\pi}_*$ is surjective. Note that the right square commutes by Lemma 3.5; we claim the left square commutes as well.

Commutativity of the left square: For $(P'_1, P'_2) \in \mathbb{Q}\mathbf{F}_n$,

$$\begin{aligned}
\tilde{\phi}_{-1,n+1}(\gamma(P'_1,P'_2)) &= \tilde{\phi}_{-1,n+1}(\{P'_1 \cup \{n+1\}, P'_2\}) - \tilde{\phi}_{-1,n+1}(\{P'_1\}, P'_2 \cup \{n+1\}\}) \\
&= \sum_{\substack{T \subseteq [n+1] \\ |T| \geq 2 \\ |T| \text{ even} \\ T \cap (P'_1 \cup \{n+1\}) \neq \emptyset \\ T \cap P'_2 \neq \emptyset}} (T) - \sum_{\substack{T \subseteq [n+1] \\ |T| \geq 2 \\ |T| \text{ even} \\ T \cap (P'_1) \neq \emptyset \\ T \cap P'_2 \cup \{n+1\} \neq \emptyset}} (T) \\
&= \sum_{\substack{T \subseteq [n+1] \\ |T| \geq 2 \\ |T| \text{ even} \\ n+1 \in T \\ T \smallsetminus \{n+1\} \subseteq P'_2}} (T) - \sum_{\substack{T \subseteq [n+1] \\ |T| \geq 2 \\ |T| \text{ even} \\ n+1 \in T \\ T \smallsetminus \{n+1\} \subseteq P'_1}} (T) \\
&= \sum_{\substack{T' \subseteq [n] \\ |T| \text{ odd} \\ T' \subseteq P'_2}} (T' \cup \{n+1\}) - \sum_{\substack{T' \subseteq [n] \\ |T| \text{ odd} \\ T' \subseteq P'_1}} (T' \cup \{n+1\}) \\
&= \alpha(\mathrm{odd}_n(P'_1, P'_2)).
\end{aligned}$$

This proves the claim. Since odd_n is surjective, and by the inductive hypothesis so is $\tilde{\phi}_{-1,n}$, By Lemma 3.6, $\tilde{\phi}_{-1,n+1}$ is surjective. □

Lemma 3.10. *For all $n \geq 3$, the map $\tilde{\phi}_{0,n} : \mathbb{Q}\mathbf{SP}_{0,n} \to \mathbb{Q}\mathbf{K}_n^0$ is surjective.*

Proof. We induct on n.
Base case: We have $\mathbf{SP}_{0,3} = \{\{\{1\},\{2\},\{3\}\}\}$, $\mathbf{K}_3^0 = \{\{1,2,3\}\}$, and $\tilde{\phi}_{0,3}(\{\{1\},\{2\},\{3\}\}) = \{1,2,3\}$, so $\tilde{\phi}_{0,3}$ is surjective.
Inductive hypothesis: The proposition holds up to some $n \geq 4$.
Inductive step: Consider the following diagram, which commutes by Lemma 3.5

$$\begin{array}{ccccccccc}
 & & \mathbb{Q}\mathbf{SP}_{-1,n} & \xrightarrow{\tilde{\pi}^*} & \mathbb{Q}\mathbf{SP}_{0,n+1} & \xrightarrow{\tilde{\pi}_*} & \mathbb{Q}\mathbf{SP}_{0,n} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\phi}_{-1,n}} & & \downarrow{\scriptstyle \tilde{\phi}_{0,n+1}} & & \downarrow{\scriptstyle \tilde{\phi}_{0,n}} & & \\
0 & \longrightarrow & \mathbb{Q}\mathbf{K}_n^{-1} & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{K}_{n+1}^0 & \xrightarrow{\beta} & \mathbb{Q}\mathbf{K}_n^0 & \longrightarrow & 0
\end{array}$$

By the inductive hypothesis, $\tilde{\phi}_{0,n}$ is surjective. By Lemma 3.9, $\tilde{\phi}_{-1,n}$ is surjective, so by the Four Lemma, $\tilde{\phi}_{0,n+1}$ is surjective, as desired. □

3.3. An inductive proof of Theorem 3.13.

Lemma 3.11. *For all $n \geq 4$, we have $\dim \mathcal{Q}_{1,n} = \dim \mathbb{Q}\mathbf{K}_n^1$.*

Proof. There are no Type II 1-dimensional boundary strata, so $\forall n \geq 4$, $\mathcal{V}_{1,n} = \{0\}$ and $\mathcal{Q}_{1,n} \cong \mathcal{A}_{1,n}$. By [FG03], $\dim \mathcal{A}_{1,n} = 2^{n-1} - \binom{n}{2} - 1$. On the other hand,

$$\dim \mathbb{Q}\mathbf{K}_n^1 = \#\{T \subseteq [n] \mid |T| \text{ even}, |T| \geq 4\} = 2^{n-1} - \binom{n}{2} - 1.$$

□

Lemma 3.12. *For all $n \geq 4$, the map $\phi_{n-3,n}$ is an isomorphism.*

Proof. For all $n \geq 4$, we have that $\mathcal{V}_{n-3,n} = \{0\}$, $\mathcal{A}_{n-3,n} = \mathcal{Q}_{n-3,n} = \mathbb{Q}\{\{\{1\},\dots,\{n\}\}\}$, $\mathbf{K}_n^{n-3} = \{[n]\}$, and $\phi_{n-3,n} : \mathcal{Q}_{n-3,n} \to \mathbb{Q}\mathbf{K}_n^{n-3}$ sends $\{\{1\},\dots,\{n\}\}$ to $[n]$. Thus $\phi_{n-3,n}$ is an isomorphism. □

Theorem 3.13. *For $n \geq 4$ and d such that $1 \leq d \leq n-3$, $\phi_{d,n} : \mathcal{Q}_{d,n} \to \mathbb{Q}\mathbf{K}_n^d$ is an isomorphism.*

Inductive proof of Theorem 3.13. **Base case:** $n = 4$; then $1 \le d \le n-3$ implies that $d = 1 = n-3$. By Lemma 3.12, $\phi_{1,4}$ is an isomorphism.
Inductive hypothesis: For some $n \ge 4$, and for all d with $1 \le d \le n-3$, we have that $\phi_{d,n} : \mathcal{Q}_{d,n} \to \mathbb{Q}\mathbf{K}_n^d$ is an isomorphism.
Inductive step: For $1 \le d \le (n-4)$, we have the following diagram, which commutes by Lemma 3.5:

$$\begin{array}{ccccccccc}
\mathrm{Ker}(\pi^*) & \longrightarrow & \mathcal{Q}_{d,n} & \xrightarrow{\pi^*} & \mathcal{Q}_{d+1,n+1} & \xrightarrow{\pi_*} & \mathcal{Q}_{d+1,n} & \longrightarrow & 0 \\
\downarrow & & \cong\downarrow \phi_{d,n} & & \downarrow \phi_{d+1,n+1} & & \cong\downarrow \phi_{d+1,n} & & \\
0 & \longrightarrow & \mathbb{Q}\mathbf{K}_n^d & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{K}_{n+1}^{d+1} & \xrightarrow{\beta} & \mathbb{Q}\mathbf{K}_n^{d+1} & \longrightarrow & 0
\end{array}$$

The top row is exact by Lemma 2.13, and the bottom row is exact by Lemma 3.4. By the inductive hypothesis, $\phi_{d,n}$ and $\phi_{d+1,n}$ are isomorphisms. So, by the Five Lemma, $\phi_{d+1,n+1}$ is an isomorphism. Combining the above argument with Lemma 3.12, we conclude that for $2 \le d \le (n+1)-3$, the map $\phi_{d,n+1}$ is an isomorphism. We also have the following diagram, which commutes by Lemma 3.5:

$$\begin{array}{ccccccccc}
 & & \mathbb{Q}\mathbf{SP}_{0,n} & \xrightarrow{\tilde{\pi}^*} & \mathcal{Q}_{1,n+1} & \xrightarrow{\pi_*} & \mathcal{Q}_{1,n} & \longrightarrow & 0 \\
 & & \downarrow \tilde{\phi}_{0,n} & & \downarrow \phi_{1,n+1} & & \cong\downarrow \phi_{1,n} & & \\
0 & \longrightarrow & \mathbb{Q}\mathbf{K}_n^0 & \xrightarrow{\alpha} & \mathbb{Q}\mathbf{K}_{n+1}^1 & \xrightarrow{\beta} & \mathbb{Q}\mathbf{K}_n^1 & \longrightarrow & 0
\end{array}$$

where the bottom row is exact and the top row is a complex, exact at $\mathcal{Q}_{1,n}$. By Lemma 3.10, $\tilde{\phi}_{0,n}$ is surjective, and by the inductive hypothesis, $\phi_{1,n}$ is an isomorphism. By Lemma 3.6), $\phi_{1,n+1}$ is surjective. But by Lemma 3.11, $\dim \mathcal{Q}_{1,n+1} = \dim \mathbb{Q}\mathbf{K}_{n+1}^1$, so $\phi_{1,n+1}$ is an isomorphism.

□

3.4. Theorem 3.14 and its proof.

Theorem 3.14. *For $n \ge 4$ and d such that $1 \le d \le n-3$:*

(i) We have $\mathcal{K}_n^d = \mathcal{V}_{d,n}^\perp$.
(ii) The pairing $\mathcal{Q}_{d,n} \times \mathcal{K}_n^d \to \mathbb{Q}$ is perfect.
(iii) The set $\{\kappa_d^T \mid |T| \ge (d+3), |T| \equiv (d+3) \mod 2\}$ is an S_n-equivariant basis for $\mathcal{K}_n^d$.
(iv) The S_n actions on $\mathcal{Q}_{d,n}$ and $\mathcal{K}_n^d$ are isomorphic to the permutation representation induced by the natural action of S_n on the set $\{T \subseteq [n] \mid |T| \ge (d+3), |T| \equiv (d+3) \mod 2\}$.
(v) The following (dual) sequences are exact:

$$0 \to \mathcal{Q}_{d,n} \xrightarrow{\pi^*} \mathcal{Q}_{d+1,n+1} \xrightarrow{\pi_*} \mathcal{Q}_{d+1,n} \to 0 \tag{11}$$

$$0 \to \mathcal{K}_n^{d+1} \xrightarrow{\pi^*} \mathcal{K}_{n+1}^{d+1} \xrightarrow{\pi_*} \mathcal{K}_n^d \to 0 \tag{12}$$

Proof. Recall the map $\zeta_{d,n} : \mathbb{Q}\mathbf{K}_n^d \to \mathcal{K}_n^d$ given in Definition 3.1. We have compatible pairings $\mathcal{Q}_{d,n} \times \mathcal{K}_n^d \to \mathbb{Q}$ and $\mathcal{Q}_{d,n} \times \mathbb{Q}\mathbf{K}_n^d \to \mathbb{Q}$, inducing maps $\eta_{d,n} : \mathcal{Q}_{d,n} \to (\mathcal{K}_n^d)^\vee$ and $\phi_{d,n} : \mathcal{Q}_{d,n} \to (\mathbb{Q}\mathbf{K}_n^d)^\vee = \mathbb{Q}\mathbf{K}_n^d$, where, $\phi_{d,n}$ is as in Definition 3.2. These maps satisfy: $\phi_{d,n} = (\zeta_{d,n})^\vee \circ \eta_{d,n}$. By Theorem 3.13, $\phi_{d,n}$ is an isomorphism, which implies that $\eta_{d,n}$ is injective. On the other hand, we have by Corollary 2.11 that $\mathcal{K}_n^d \subset \mathcal{V}_{d,n}^\perp = (\mathcal{Q}_{d,n})^\vee$, so $(\eta_{d,n})^\vee$ is injective as well. This implies that $\eta_{d,n}$ is an isomorphism, proving items (ii) and (i). Since $\eta_{d,n}$ and $\phi_{d,n}$ are both isomorphisms, we conclude that so is $\zeta_{d,n}$, proving item (iii), and thus also item (iv). Finally, by Theorem 3.13 and Lemma 3.5, the sequence in Equation 11 is dual to the sequence in Equation 10, which is exact by Lemma 3.4. We conclude that the sequence in Equation 11 is exact. The sequence in Equation 11 is dual to the sequence in Equation 12, so the latter sequence is exact, proving item (v). □

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Email address: rohini_ramadas@brown.edu

Department of Mathematics, Brown University, Providence, RI