# DUAL PAIRS OF QUANTUM MOMENT MAPS AND DOUBLES OF HOPF ALGEBRAS

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Abstract. Given a dual pair of topological Hopf algebras $A, A^*$, under mild conditions there exists a natural associative algebra homomorphism $D(A) \to H(A)$ between the corresponding Drinfeld double $D(A)$ and Heisenberg double $H(A)$. We construct this homomorphism using a pair of commuting quantum moment maps, and then use it to provide a homomorphism of certain reflection equation algebras. We also explain how the quantization of the Grothendieck-Springer resolution arises in this context.

## 1. Introduction

The Grothendieck-Springer simultaneous resolution of a complex simple Lie group $G$ plays a central role in the geometric representation theory. Recall that if $B \subset G$ is a Borel subgroup in $G$, and we write $\mathfrak{g}, \mathfrak{b}$ for the Lie algebras of $G, B$ respectively, then the Grothendieck-Springer resolution is the following map of Poisson varieties:

$$G \times_B \mathfrak{b} \longrightarrow \mathfrak{g}, \qquad (g,x)B \longmapsto gxg^{-1}. \tag{1.1}$$

Indeed, the Poisson map (1.1) admits a quantization, yielding an embedding of the enveloping algebra $U(\mathfrak{g})$ into the ring of global differential operators on the principal affine space $G/N$.

It was shown in [EL07] that both sides of the multiplicative Grothendieck-Springer resolution

$$G \times_B B \longrightarrow G, \qquad (g,b)B \longmapsto gbg^{-1} \tag{1.2}$$

admit natural, nontrivial Poisson structures such that the resolution map is Poisson. In [SS15], we showed that the resolution (1.2) can be also quantized, this time to yield an embedding of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ into a certain ring of quantum differential operators on $G/N$.

One remarkable property of $U_q(\mathfrak{g})$ is that it can be realized as a quotient of the Drinfeld double $D(U_q(\mathfrak{b}))$ of a quantum Borel subalgebra $U_q(\mathfrak{b}) \subset U_q(\mathfrak{g})$. In this note, we observe that an analog of the quantization of the resolution (1.2) exists under mild conditions for the Drinfeld double $D(A)$ of a topological Hopf algebra $A$. The key to the construction of this quantization is the existence of a pair $\mu_L, \mu_R \colon D(A) \to H(D(A)^{*,op})$ of commuting quantum moment maps from $D(A)$ to the Heisenberg double of a certain Hopf algebra $D(A)^{*,op}$ opposite dual to $D(A)$. In this general setting, the role of quantum differential operators on $G/N$ is played by the quantum Hamiltonian reduction of $H(D(A)^{*,op})$ by $\mu_L(A)$, and the resolution map is given by the residual quantum moment map $\mu_R\colon D(A) \to H(D(A)^{*,op})//\mu_L(A)$.

Although a similar construction has appeared before in the context of the quantum Beilinson-Bernstein theorem, we believe that the following results are new. First, we show that the quantum Hamiltonian reduction $H(D(A)^{*,op})//\mu_L(A)$ is isomorphic to the Heisenberg double $H(A)$. Recall [Mon93] that the Heisenberg double $H(A)$ of a finite-dimensional Hopf algebra $A$ is isomorphic to the algebra of its endomorphisms $\mathrm{End}(A)$. Thus, the natural action of $D(A)$ on $A$ yields a homomorphism $D(A) \to H(A)$. We show that it coincides with the map $\mu_R\colon D(A) \to H(A)$ when $A$ is finite dimsensional. Second, we provide an explicit Faddeev-Reshetikhin-Takhtajan type presentation of the map $\mu_R$ in terms of universal $R$-matrices, which leads to a homomorphism between certain reflection equation algebras.

The article is organized as follows. In Section 2, we recall the Poisson geometric constructions of [EL07] in the setting of the double $\mathcal{D}(G)$ of an arbitrary Poisson-Lie group $G$. It serves as a quasi-classical limit of constructions in Sections 3 and 4. In Section 3, we review some of the notions from the theory of Hopf algebras and their doubles that we use in the sequel. Section 4 contains the construction of the quantum resolution, and explains how the general construction specializes to the case of the Grothendieck-Springer resolution. Finally, we conclude in Section 5 by providing a Faddeev-Reshetikhin-Takhtajan presentation of our construction using universal $R$-matrices.

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## 2. Poisson geometry

2.1. **Preliminaries.** Recall that a Poisson-Lie group is a Lie group $G$ with a Poisson structure such that the multiplication map $G\times G \to G$ is a morphism of Poisson varieties. Let $G^*$ be the (connected, simply-connected) Poisson-Lie dual of $G$, and $\mathcal{D}(G)$ be the double of $G$. The Lie algebra $\mathfrak{d} = Lie(\mathcal{D}(G))$ can be written as $\mathfrak{d} = \mathfrak{g}\oplus\mathfrak{g}^*$. We will say that there exist local isomorphisms

$$\mathcal{D}(G) \simeq G \times G^* \simeq G^* \times G.$$

Let us consider a pair of dual bases $(x_i)$ and $(x^i)$ of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Then the element $r \in \mathfrak{d} \wedge \mathfrak{d}$ defined by

$$r = \frac{1}{2} \sum_i (x_i, 0) \wedge (0, x^i)$$

is independent of the choice of bases. Let $X^R$, $X^L$ denote respectively the right- and left-invariant tensor fields on a Lie group, taking the value $X^R(e) = X^L(e) = X$ at the identity element of the group. Then the bivectors

$$\pi_\pm = r^R \pm r^L$$

define a pair of Poisson structures on the Lie group $\mathcal{D}(G)$. We abbreviate the resulting Poisson manifolds by $\mathcal{D}_\pm(G)$. In fact, $\mathcal{D}_-(G)$ is a Poisson-Lie group, while $\mathcal{D}_+(G)$ is only a Poisson manifold.

**Remark 2.1.** As a manifold, the group $\mathcal{D}_-(G)$ is locally isomorphic to $\mathcal{D}_-(G)^*$. In general, however, this is neither an isomorphism of Lie groups, or of Poisson manifolds.

The action of a Poisson-Lie group $G$ on a Poisson variety $P$ is said to be Poisson, if so is the map $G \times P \to P$. Given a Poisson map $P \to G^*$, one can obtain a local Poisson action $G \times P \to P$ using the group-valued moment map. Recall that the group-valued moment map is defined (see [Lu91]) as follows.

**Definition 2.2.** Let $\pi$ be the Poisson bivector field defining the Poisson structure on the manifold $P$. A map $\mu \colon P \to G^*$ is said to be a moment map for the Poisson action $G \times P \to P$, if for every $X \in \mathfrak{g}$ one has

$$\mu_X = \left\langle \pi, \mu^* X^R \otimes - \right\rangle,$$

where $\mu_X$ is the vector field on $P$ generated by the action $\mu_{\exp(tX)}$.

**Remark 2.3.** A moment map is Poisson, if exists.

**Remark 2.4.** Recall that there are open subsets of factorizable elements $G^* \cdot G$ and $G \cdot G^*$ in the double $\mathcal{D}_-(G)$. Hence we may regard $G^*$ as a submanifold in $\mathcal{D}_-(G)/G$, and may regard the moment map $\mu$ in Definition 2.2 as taking values in $\mathcal{D}_-(G)/G$ or $G\backslash\mathcal{D}_-(G)$.

The following theorem is well-known (see e.g. [STS85, Lu91]).

**Proposition 2.5.** *Let $G$ be a Poisson-Lie group, and $\mathcal{D}_\pm(G)$ its double with Poisson bivectors $\pi_\pm$. Then*

1. *the actions of $\mathcal{D}_-(G)$ on $\mathcal{D}_+(G)$ by left and right multiplications are Poisson;*
2. *the moment map for the Poisson action of the subgroup $G \subset \mathcal{D}_-(G)$ on $\mathcal{D}_+(G)$ by left (resp. right) multiplication is the natural projection $\mathcal{D}(G) \to \mathcal{D}(G)/G$ (resp. $\mathcal{D}(G) \to G\backslash\mathcal{D}(G)$).*

**Remark 2.6.** Let $\mathcal{P}$ be the category of Poisson-Lie groups. Consider a Poisson-Lie group $G$ and its connected, simply-connected Poisson-Lie dual $G^*$. Then the assignment $G \to G^*$ defines a functor $\mathcal{P} \to \mathcal{P}^{op}$. Therefore, any Poisson-Lie subgroup $H \subset G$ induces a map $p\colon G^* \to H^*$. Now consider a Poisson action $G \times P \to P$ with the moment map $\mu_G$. It gives rise to the Poisson action $H \times P \to P$ with the moment map $\mu_H = p \circ \mu_G$.

2.2. **Double of the double construction.** Now, let us start with the Poisson-Lie group $D = \mathcal{D}_-(G)$ and consider its double $\mathcal{D}(D) = \mathcal{D}(\mathcal{D}(G))$. The Lie algebra $\mathfrak{D} = Lie(\mathcal{D}(D))$ may be written as $\mathfrak{D} = \mathfrak{d} \oplus \mathfrak{d} = \mathfrak{d}_\Delta \oplus \mathfrak{d}^*$ where

$$\mathfrak{d}_\Delta = \{((x,\alpha),(x,\alpha)) \in \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{g} \oplus \mathfrak{g}^*\}$$

is the diagonal embedding of $\mathfrak{d}$ into $\mathfrak{d} \oplus \mathfrak{d}$ and

$$\mathfrak{d}^* = \{((y,0),(0,\beta)) \in \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{g} \oplus \mathfrak{g}^*\}\,.$$

Using the local isomorphism $\mathcal{D}(D) \simeq D_\Delta \times D^*$ we may coordinatize the moment map $\nu_r$ for the right Poisson action of $D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ as

$$\nu_r\colon \mathcal{D}_+(D) \longrightarrow D^*, \qquad (dg, d\alpha) \longmapsto \alpha^{-1}g$$

for any triple of elements $g \in G$, $\alpha \in G^*$, $d \in \mathcal{D}(\mathcal{D}(G))$. Similarly, using the local isomorphism $\mathcal{D}(D) \simeq D^* \times D_\Delta$ we write the moment map $\nu_l$ for the left Poisson action of $D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ as

$$\nu_l\colon \mathcal{D}_+(D) \longrightarrow D^*, \qquad (gd, \alpha d) \longmapsto g\alpha^{-1}.$$

2.3. **Hamiltonian reduction.** Consider the Poisson action of the subgroup $D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ by left multiplications and the Poisson action of $G \subset D_\Delta \subset \mathcal{D}_-(D)$ on $\mathcal{D}_+(D)$ by right multiplications. Clearly, the two actions commute, because so do the left and right actions of $\mathcal{D}_-(D)$. We illustrate these actions as follows

$$D_\Delta \curvearrowright \mathcal{D}_+(D) \curvearrowleft D_\Delta \supset G.$$

By Remark 2.6, the moment map $\mu_r$ for the right action of $G$ is given as

$$\mu_r\colon \mathcal{D}_+(D) \longrightarrow (G \times D)\backslash\mathcal{D}_+(D).$$

The Hamiltonian reduction of $\mathcal{D}_+(D)$ by the moment map $\mu_r$ becomes

$$\mu_r^{-1}(e)/G_\Delta = \{(dg, d) \,|\, d \in D, g \in G\}\,/G_\Delta.$$

Therefore, we can identify

$$\mu_r^{-1}(e)/G_\Delta \simeq D \times_G G,$$

where $D \times_G G$ denotes the set of $G$-orbits through $D \times G$ under the right action

$$(D \times G) \times G \longrightarrow D \times G, \qquad ((d,g),h) \longmapsto (dh, h^{-1}gh)$$

with $g, h \in G$ and $d \in D$.

On the other hand, since the left and right $D_\Delta$-actions on $\mathcal{D}(D)$ commute, the variety $D\times_G G$ admits the residual $D_\Delta$-action by left multiplication. The corresponding moment map is

$$\mu_l \colon D \times_G G \longrightarrow \mathcal{D}(D)/D_\Delta.$$

As explained in Remark 2.1, we may use local diffeomorphism of $D$ with $D^*$ to write a local expression for the map $\mu_l$ as $\mu_l\left((q,g)G\right) = qgq^{-1} \in D$.

The following Proposition follows easily from considering Poisson bivectors for the Poisson varieties under consideration.

**Proposition 2.7.** *There is a local Poisson isomorphism*

$$\mathcal{D}_+(G) \longrightarrow D \times_G G, \qquad \alpha g \longmapsto (\alpha, g)G,$$

*where $g \in G$, $\alpha \in G^*$ and we identify $\mathcal{D}(G) \simeq G^* \times G$.*

Under this identification, the moment map $\mu_l$ becomes

$$\mathcal{D}_+(G) \longrightarrow \mathcal{D}_-(G), \qquad \alpha g \longmapsto \alpha g \alpha^{-1}. \tag{2.1}$$

## 3. Reminder on Hopf algebras

To fix our notations, we will recall some standard notions from the theory of Hopf algebras. In what follows, we choose to work in the setting of topological Hopf algebras over the ring $k[[\hbar]]$ of formal power series over a ground field $k$. In particular, all tensor products are to be understood as completed in the $\hbar$-adic topology.

3.1. **Basic notations.** Let $A$ be a topological Hopf algebra over $\mathbb{K} := k[[\hbar]]$, with the quadruple $(m, \Delta, \epsilon, S)$ denoting the multiplication, comultiplication, counit, and antipode of $A$ respectively. We say that a pair of topological Hopf algebras $A$ and $A^*$ form a dual pair if there exists a non-degenerate Hopf pairing $\langle -, - \rangle \colon A \otimes A^* \to \mathbb{K}$, that is a non-degenerate pairing satisfying

(1) $\langle ab, x\rangle = \langle a \otimes b, \Delta(x)\rangle$
(2) $\langle a, xy\rangle = \langle \Delta(a), x \otimes y\rangle$
(3) $\langle 1_A, -\rangle = \epsilon_{A^*}$ and $\langle -, 1_{A^*}\rangle = \epsilon_A$
(4) $\langle S(a), x\rangle = \langle a, S(x)\rangle$

for all $a, b \in A$ and $x, y \in A^*$. In fact, condition (4) follows from the other three, see [KS97, Section 1.2.5, Proposition 9]. We will also use the notation $A^{op}$ for the Hopf algebra $(A, m^{op}, \Delta, S^{-1})$, and $A^{cop}$ for the Hopf algebra $(A, m, \Delta^{op}, S^{-1})$.

3.2. **Module algebras.** The category of modules $\mathrm{Mod}_A$ over a Hopf algebra $A$ has a monoidal structure determined by the coproduct $\Delta \colon A \to A \otimes A$. We say that $M$ is an $A$-*module algebra* if it is an algebra object in the monoidal category $\mathrm{Mod}_A$, that is

$$a \cdot 1_M = \epsilon(a) 1_M \qquad \text{and} \qquad a \cdot (mn) = (a_1 \cdot m)(a_2 \cdot n)$$

for any $a \in A$ and $m, n \in M$.

A Hopf algebra $A$ can be naturally regarded as a module algebra over itself using the *adjoint action*

$$\mathrm{ad}\colon A \otimes A \longrightarrow A, \qquad a \otimes b \longmapsto a \triangleright b := a_1 b S a_2.$$

A Hopf algebra $A^*$ dually paired with $A$ can also be regarded as a module algebra over $A$ using the *left coregular action*

$$\mathrm{coreg}\colon A \otimes A^* \longrightarrow A^*, \qquad a \otimes x \longmapsto a \rightharpoonup x := \langle a, x_2 \rangle x_1.$$

There is also a *right coregular action* of $A^{op}$ on $A^*$, defined by

$$a \otimes x \longmapsto x \leftharpoonup a := \langle a, x_1 \rangle x_2.$$

3.3. **The Drinfeld double.** Suppose $A, A^*$ is a dual pair of Hopf algebras. In what follows, we assume that the pairing $\langle \cdot, \cdot \rangle$ is such that a topological basis $\{a_i\}$ for $A$ gives rise to a dual topological basis $\{x^i\}$ in $A^*$ with the property that $\langle a_i, x^j \rangle = \delta_i^j$, and there is a well-defined element $\sum_i a_i \otimes x^i \in A \otimes A^*$, where as usual tensor product is completed in the $\hbar$-adic topology. For instance, this hypothesis will be satisfied whenever $A$ and $A^*$ are a dual pair of QUE-algebras in the sense of Drinfeld [Dri86], and of course whenever $A$ is finitely generated and projective over $k[[\hbar]]$.

Under the above assumption, there exists a Hopf algebra $D(A)$ called the *Drinfeld double* of $A$, with the following properties:

(1) as a coalgebra, $D(A) \simeq (A^*)^{cop} \otimes A$;
(2) the maps $a \mapsto 1 \otimes a$ and $x \mapsto x \otimes 1$ are embeddings of Hopf algebras;
(3) let $(a_i)$ and $(x^i)$ be dual bases for $A$ and $A^*$ respectively. Then the canonical element

$$R = \sum_i (1 \otimes a_i) \otimes (x^i \otimes 1) \in D(A)^{\otimes 2},$$

called the universal $R$-matrix of the Drinfeld double, satisfies

$$R \Delta_D(d) = \Delta_D^{op}(d) R$$

for all $d \in D(A)$.

From the above properties one derives the following explicit formula for the multiplication in $D(A)$:

$$(x \otimes a)(y \otimes b) = \langle a_1, y_3 \rangle \langle a_3, S^{-1} y_1 \rangle x y_2 \otimes a_2 b. \tag{3.1}$$

It also follows from the definition of the double, that the $R$-matrix is invertible, with inverse

$$R^{-1} = (S_D \otimes \mathrm{id})(R)$$

and that the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \in D(A)^{\otimes 3}$$

holds in the triple tensor product $D(A)^{\otimes 3}$.

**Proposition 3.1.** *If $A$ is a Hopf algebra and $D(A)$ its Drinfeld double, the following formula equips $A$ with the structure of a $D(A)$-module algebra:*

$$\begin{aligned}(1 \otimes a) \cdot b &= a_1 b S a_2 \\ (x \otimes 1) \cdot b &= b \leftharpoonup S^{-1} x\end{aligned} \tag{3.2}$$

In the action (3.2), the Hopf subalgebra $A \subset D(A)$ acts adjointly on $A$, while the Hopf subalgebra $(A^*)^{cop} \subset D(A)$ acts by its right coregular action.

3.4. **The dual of the Drinfeld double.** In addition to the Drinfeld double, we will also make use of another Hopf algebra $T(A) = D(A)^*$ dually paired with $D(A)$. As an algebra, we have $T(A) \simeq A^{op} \otimes A^*$, and the pairing $\langle\langle \cdot, \cdot \rangle\rangle : D(A) \otimes T(A) \to k[[\hbar]]$ is defined by

$$\langle\langle x \otimes a, b \otimes y \rangle\rangle = \langle b, x \rangle \langle a, y \rangle. \tag{3.3}$$

The formula for its comultiplication can be found by dualizing (3.1) and reads

$$\Delta_T(a \otimes x) = \left(a_1 \otimes x^r x_1 x^t\right) \otimes \left(S^{-1} a_t a_2 a_r \otimes x_2\right) \in T(A)^{\otimes 2}.$$

Similarly, the antipode in $T(A)$ can be written as

$$S_T(a \otimes x) = a_r S^{-1}(a) S^{-1}(a_t) \otimes x^t S(x) x^r.$$

3.5. **The Heisenberg double.** Given a Hopf algebra $A$ and its module algebra $M$, one defines their *smash-product* $M\#A$ as an associative algebra on the vector space $M \otimes A$ with the multiplication given by

$$(m\#x)(n\#y) = m(x_1 \cdot n)\#y_2 b$$

for any elements $x, y \in A$ and $m, n \in M$. Recall [Lu94], that the *Heisenberg double* $H(A)$ of an associative algebra $A$ is the smash product $H(A) = A\#A^*$ with respect to the coregular action of $A^*$ on $A$. Thus, the multiplication in $H(A)$ is determined by the formula

$$(a\#x)(b\#y) = a(x_1 \rightharpoonup b)\#x_2 y = \langle x_1, b_2 \rangle\, ab_1\#x_2 y$$

for any $a, b \in A$ and $x, y \in A^*$. Note that one has the following inclusions of algebras

$$\begin{aligned} A &\longrightarrow H(A), \qquad a \mapsto a\#1, \\ A^* &\longrightarrow H(A), \qquad x \mapsto 1\#x. \end{aligned}$$

By construction, the Heisenberg double $H(A)$ acts on $A$ via

$$(a\#x) \cdot_L (b) = a(x \rightharpoonup b) = \langle x, b_2 \rangle\, ab_1 \tag{3.4}$$

In fact, $H(A)$ also acts on $A$ via

$$(a\#x) \cdot_R (b) = (b \leftharpoonup S^{-1} x) S a = \langle x, S b_1 \rangle\, b_2 S^{-1} a \tag{3.5}$$

The Heisenberg double $H(A)$ has the following well-known properties:

**Proposition 3.2.** [STS92] *The antipode $S_T$ of $T(A)$, when regarded as an operator $\iota\colon H(A) \to H(A)$ via*

$$\iota\colon H(A) \longrightarrow H(A), \qquad a \otimes x \longmapsto a_r S^{-1}(a) S^{-1}(a_t) \otimes x^t S(x) x^r, \tag{3.6}$$

*defines an algebra automorphism of $H(A)$.*

Note that the automorphism $\iota$ intertwines the two actions 3.4, 3.5 of $H(A)$ on $A$.

**Corollary 3.3.** *One has the following inclusions of algebras*

$$A \longrightarrow H(A), \qquad a \mapsto \iota(a\#1) = a_r S^{-1}(a) S^{-1}(a_t) \otimes x^t x^r,$$
$$A^* \longrightarrow H(A), \qquad x \mapsto \iota(1\#x) = a_r S^{-1}(a_t) \otimes x^t S(x) x^r.$$

Since the actions $(A\#1, \cdot_L)$, $(A\#1, \cdot_R)$ commute, we have

**Proposition 3.4.** [STS92] *The maps*

$$A \otimes A \longrightarrow H(A), \qquad a \otimes b \longmapsto (a\#1)\iota(b\#1),$$
$$A^* \otimes A^* \longrightarrow H(A), \qquad x \otimes y \longmapsto (x\#1)\iota(1\#y)$$

*are homomorphisms of associative algebras.*

3.6. **Quantum Hamiltonian reduction.** Let us briefly recall the notion of quantum Hamiltonian reduction. Suppose that $A$ is a Hopf algebra, $V$ is an associative algebra, $\mu\colon A \to V$ is a homomorphism of associative algebras, and $I$ is a 2-sided ideal in $A$ preserved by the adjoint action of $A$. Then, by the ad-invariance of $I$, the action of $A$ on $V$ defined by the formula

$$a \circ v = \mu(a_1) v \mu(S a_2)$$

descends to an action of $A$ on the $V$-module $V/\mu(I)$, where we abuse notation and write $\mu(I)$ for the left ideal in $V$ generated by $\mu(I)$. The *quantum Hamiltonian reduction* $V//\mu(A)$ of $V$ by the *quantum moment map* $\mu\colon A \to V$ at the ideal $I$ is defined as the set of $A$-invariants

$$\begin{aligned} V//\mu(A) &:= (V/V\mu(I))^A \\ &= \{a \in V/V\mu(I) \mid a \circ v = \epsilon(a) v \quad \text{for all} \quad a \in A\} \end{aligned}$$

One checks that $V//\mu(A)$ inherits a well-defined associative algebra structure from that of $V$, such that $V//\mu(A)$ is an $A$-module algebra.

## 4. Construction of the quantum resolution

4.1. **The double of a double.** Suppose that $A$ is a Hopf algebra, and let $D(A), T(A)$, and $H(A)$ be its Drinfeld double, dual to the Drinfeld double, and the Heisenberg double respectively. Consider the Heisenberg double

$$H(T(A)^{op}) = T(A)^{op} \# D(A)^{cop},$$

of the algebra $T(A)^{op}$. One has an algebra embedding

$$\mu_L\colon D(A) \longrightarrow H(T(A)^{op}), \qquad u \mapsto 1\#u \in H(T(A)^{op})$$

which may be regarded as the quantum moment map for the following $D(A)$-module algebra structure on $H(T(A)^{op})$:

$$u \circ_L (\phi \# v) = (u_3 \rightharpoonup \phi) \# u_2 v S^{-1}_{D(A)} u_1. \tag{4.1}$$

As in Corollary 3.3, there exists another algebra embedding defined by

$$\mu_R \colon D(A) \longrightarrow H(T(A)^{op}), \qquad u \mapsto \iota^{-1}(1 \# u). \tag{4.2}$$

It generates the following $D(A)$-module algebra structure on $H(T(A)^{op})$:

$$u \circ_R (\phi \# v) = (\phi \leftharpoonup S^{-1}_{D(A)} u) \# v \tag{4.3}$$

By Proposition 3.4, the subalgebras $\mu_L(D(A))$ and $\mu_R(D(A))$ commute with each other in $H(T(A)^{op})$. This forces the actions (4.1) and (4.3) to commute as well.

4.2. **Dual pairs of quantum moment maps.** We shall now restrict the action (4.1) to the Hopf subalgebra $A \subset D(A)$, and consider the quantum Hamiltonian reduction of $H(T(A)^{op})$ at the augmentation ideal $I_A = \ker(\epsilon_A)$ of $A$. We denote the algebra obtained as a result of the quantum Hamiltonian reduction by $H(T(A)^{op})//\mu_L(A)$.

We also have the moment map $\mu_R \colon D(A) \to H(T(A)^{op})$ given in (4.2), and the action (4.3) of $D(A)$ on $H(T(A)^{op})$ that it defines.

**Proposition 4.1.** *The action (4.3) of $D(A)$ on $H(T(A)^{op})$ descends to a well-defined action*

$$D(A) \times H(T(A)^{op})//\mu_L(A) \longrightarrow H(T(A)^{op})//\mu_L(A) \tag{4.4}$$

*In turn, the map $\mu_R$ descends to a well-defined homomorphism of $D(A)$-module algebras*

$$\mu_R \colon D(A) \longrightarrow H(T(A)^{op})//\mu_L(A)$$

*which is a moment map for the action (4.4).*

**Proof.** The Proposition is a simple consequence of the fact that the subalgebras $\mu_L(D(A))$ and $\mu_R(D(A))$ commute with one another. Indeed, this commutativity implies that for all $a \in A$, $u \in D(A)$, one has

$$\begin{aligned} a \circ_L (\mu_R(u) + \mu_L(I_A)) &= a_1 \mu_R(u) S a_2 + \mu_L(I_A) \\ &= \mu_R(u) a_1 S a_2 + \mu_L(I_A) \\ &= \epsilon(a) \, (\mu_R(u) + \mu_L(I_A)) \end{aligned}$$

which shows that

$$\mu_R(u) + \mu_L(I_A) \in \big(H(T(A)^{op})/\mu_L(I_A)\big)^A =: H(T(A)^{op})//\mu_L(A).$$

It follows from the definition of the algebra structure of the quantum Hamiltonian reduction $H(T(A)^{op})//\mu_L(A)$ that $\mu_R \colon D(A) \to H(T(A)^{op})//\mu_L(A)$ is a homomorphism of algebras. Regarding this homomorphism as a quantum moment map, we obtain an action of $D(A)$ on $H(T(A)^{op})//\mu_L(A)$ which by construction descends from (4.3), and such that $\mu_R \colon D(A) \to H(T(A)^{op})$ is a morphism of $D(A)$-module algebras. $\square$

4.3. $H(A)$ **from quantum Hamiltonian reduction.** We now examine the algebra structure of the Hamiltonian reduction $H(T(A)^{op})//\mu_L(A)$ in more detail.

**Proposition 4.2.** *There is an isomorphism of algebras*

$$\varphi : H(T(A)^{op})//\mu_L(A) \longrightarrow H(A) \tag{4.5}$$

**Proof.** Let us begin by making explicit the structure of the Hamiltonian reduction $H(T(A)^{op})//\mu_L(A)$. Firstly, note that we can identify the quotient $H(T(A)^{op})/I_A$ with the vector space $T(A)^{op} \otimes A^*$. It is easy to check that induced action of $A$ on $T(A)^{op} \otimes A^*$ is then given by

$$a \circ_L \big((b \otimes y) \otimes x\big) = (b \otimes a_2 \rightharpoonup y) \otimes \mathrm{ad}^*_{a_1}(x)$$

where

$$\mathrm{ad}^*_a(x) = \langle a_1, x_3 \rangle \langle S^{-1} a_2, x_1 \rangle x_2.$$

Hence the algebra $H(T(A)^{op})//\mu_L(A)$ of $A$-invariants in $H(T(A)^{op})/I_A$ may be identified with $H(A) = A \# A^*$, as a vector space, under the map

$$\varphi \colon H(A) \longrightarrow H(T(A)^{op})//\mu_L(A), \qquad a\#x \longmapsto (a \otimes x_1 S x_3) \otimes x_2. \tag{4.6}$$

Finally, we claim that the map (4.6) is in fact an isomorphism of algebras. Indeed, in $H(T(A)^{op})//\mu_L(A)$, one computes

$$\begin{aligned}
\varphi(a\#x)\varphi(b\#y) &= ((a \otimes x_1 S x_3) \otimes x_2)\,((b \otimes y_1 S y_3) \otimes y_2) \\
&= \langle x_2, S^{-1} a_t b_2 a_r \rangle (a b_1 \otimes x^r y_1 S y_3 x^t x_1 S x_3) \otimes x_3 y_2 \\
&= \langle x_3, b_2 \rangle (a b_1 \otimes x_4 y_1 S y_3 S^{-1} x_2 x_1 S x_6) \otimes x_5 y_2 \\
&= \langle x_1, b_2 \rangle (a b_1 \otimes x_2 y_1 S y_3 S x_4) \otimes x_3 y_2 \\
&= \varphi(\langle x_1, b_2 \rangle a b_1 \otimes x_2 y) \\
&= \varphi\left((a\#x)(b\#y)\right)
\end{aligned}$$

which completes the proof. □

**Corollary 4.3.** *Under the isomorphism $\varphi$ defined in* (4.5), *the moment map $\mu_R \colon D(A) \to H(T(A)^{op})//\mu_L(A) \simeq H(A)$ takes the form*

$$\mu_R \colon D(A) \longrightarrow H(A), \qquad by \longmapsto b_1 a_r S b_2 a_t \# S^{-1} x^t S^{-1} y x^r. \tag{4.7}$$

Using the homomorphism $\mu_R$, one can pull back the defining representation (3.4) of $H(A)$ on $A$ to obtain a representation of $D(A)$. A straightforward computation establishes

**Proposition 4.4.** *The pullback under $\mu_R$ of the action* (3.4) *coincides with the representation* (3.2) *of $D(A)$ on $A$.*

**Remark 4.5.** In [Lu96], the formula (4.7) is derived in the finite-dimensional setting from the action (3.2) of $D(A)$, together with the fact, see e.g. [Mon93], that $H(A) \simeq \mathrm{End}(A)$ as algebras.

4.4. **Example: quantized Grothendieck-Springer resolution.** Suppose now that $\mathfrak{g}$ is a complex simple Lie algebra, and denote by $U_\hbar(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$, see [Dri86, CP94]. Recall that $U_\hbar(\mathfrak{g})$ may be regarded as the quantized algebra of functions on a formal neighborhood of the identity element $e \in G^*$, where $G$ is a simple Lie group endowed with its standard Poisson structure. Let us apply our constructions to the case $A = U_\hbar(\mathfrak{b})$, where $U_\hbar(\mathfrak{b})$ is the quantum Borel subalgebra in $U_\hbar(\mathfrak{g})$. Then there is an isomorphism of algebras $D(A) \simeq U_\hbar(\mathfrak{g}) \otimes U_\hbar(\mathfrak{h})$, where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra of $\mathfrak{g}$, see [Dri86]. The restriction of the homomorphism (4.7) to $U_\hbar(\mathfrak{g}) \subset D(A)$ defines a map of algebras $\Phi\colon U_\hbar(\mathfrak{g}) \to H(A)$. In [SS15], it was shown (in the setting of the rational form $U_q(\mathfrak{g})$) that $\Phi$ is injective, and that its image is contained in a certain subalgebra $H(A)^{\mathfrak{h}}$ of $U_\hbar(\mathfrak{h})$-invariants.

In the above setup, the semiclassical limit of the map $\Phi$ is closely related to the well-known Grothendieck-Springer resolution

$$G \times_B B \longrightarrow G, \qquad (g, b)B \longmapsto gbg^{-1}$$

where $G$ is a complex simple Lie group, and $B \subset G$ is a Borel subgroup. More precisely, the algebra $H(A)^{\mathfrak{h}}$ can be regarded as the quantized algebra of functions on a formal neighborhood of $(e, e)B \in G \times_B B$. The Poisson geometric structure is exactly the one described in [EL07].

## 5. $R$-matrix formalism

In this section we rewrite the homomorphism (4.7) in terms of canonical elements of the algebras $D(A)$ and $T(A)$. As before, let

$$R = R_{12} = \sum_i a_i \otimes x^i \in D(A) \otimes D(A)$$

be the universal $R$-matrix of $D(A)$. In what follows we make use of elements

$$R_{21} = \sum_i x^i \otimes a_i$$

and

$$\mathcal{L} = R_{21} R_{12} \in D(A) \otimes D(A).$$

Recall [STS92] that the element $\mathcal{L}$ satisfies the *reflection equation*

$$\mathcal{L}_1 R_{12} \mathcal{L}_2 R_{21} = R_{12} \mathcal{L}_2 R_{21} \mathcal{L}_1 \in D(A)^{\otimes 3} \tag{5.1}$$

where $\mathcal{L}_1 = R_{31} R_{13}$, $\mathcal{L}_2 = R_{32} R_{23}$. Let us also introduce canonical elements $\Theta, \Omega \in D(A) \otimes H(A)$ defined by

$$\Theta = \sum_i a_i \otimes x_i \qquad \text{and} \qquad \Omega = \sum_i x^i \otimes a_i.$$

These elements satisfy the relations

$$\begin{aligned} R_{12}\Theta_1\Theta_2 &= \Theta_2\Theta_1 R_{12} \\ R_{12}\Omega_1\Omega_2 &= \Omega_2\Omega_1 R_{12} \\ R_{12}\Theta_1\Omega_2^{-1} &= \Omega_2^{-1}\Theta_1 \end{aligned} \tag{5.2}$$

If $\iota$ is the automorphism of $H(A)$ defined by (3.6), we write

$$\widetilde{\Theta} = (\mathrm{id} \otimes \iota)(\Theta) \qquad \text{and} \qquad \widetilde{\Omega} = (\mathrm{id} \otimes \iota)(\Omega).$$

The following proposition is straightforward.

**Proposition 5.1.** *Let $\mu_R \colon D(A) \to H(A)$ be the homomorphism defined by* (4.7)*. Then one has*

$$\begin{aligned} (\mathrm{id} \otimes \mu_R)(R_{12}) &= \widetilde{\Theta}, \\ (\mathrm{id} \otimes \mu_R)(R_{21}) &= \Omega\widetilde{\Omega}, \end{aligned}$$

*and hence*

$$(\mathrm{id} \otimes \mu_R)(\mathcal{L}) = \Omega\widetilde{\Omega}\widetilde{\Theta}.$$

Recall [KS97, Section 8.1.3, Proposition 5] that the element $u \in D(A)$ defined by

$$u = Sa_i Sx^i \in D(A)$$

satisfies

$$udu^{-1} = S_D^2(d) \qquad \text{for all} \qquad d \in D(A).$$

**Proposition 5.2.** *The following identity holds in $D(A) \otimes H(A)$*

$$\Theta^{-1}\Omega^{-1} = u_1\widetilde{\Omega}\widetilde{\Theta},$$

*where $u_1 = u \otimes 1 \in D(A) \otimes H(A)$.*

**Proof.** We have

$$\begin{aligned} \Theta^{-1}\Omega^{-1} &= \sum Sa_i Sx^j \otimes x^i a_j \\ &= \sum S(a_k a_t) S(x^r x^k) \otimes (a_r \# x^t) \\ &= \sum a_t u x^r \otimes (Sa_r \# Sx^t) \\ &= u_1 \sum a_t x^r \otimes (Sa_r \# S^{-1}x^t). \end{aligned}$$

Using the formula

$$ax = \langle a_{(1)}, x_{(3)} \rangle \langle S^{-1}a_{(3)}, x_{(1)} \rangle x_{(2)} \otimes a_{(2)}$$

for the multiplication in the Drinfeld double $D(A)$, we arrive at

$$\begin{aligned} \Theta^{-1}\Omega^{-1} &= u_1 \sum a_t x^r \otimes (Sa_r \# S^{-1}x^t) \\ &= u_1 \sum x^p a_q \otimes \left( a_\alpha Sa_q S^{-1}a_\beta \# x^\beta S^{-1}x^p x^\alpha \right) \\ &= u_1\widetilde{\Omega}\widetilde{\Theta} \end{aligned}$$

which completes the proof. □

**Corollary 5.3.** *One has*

$$(\mathrm{id}\otimes\mu_R)(\mathcal{L}) = \Omega u_1^{-1}\Theta\Omega^{-1}. \tag{5.3}$$

**Remark 5.4.** Since the first tensor factor in $\mathcal{L}$ runs over the basis of $D(A)$, the homomorphism $\mu_R$ is completely defined by the formula (5.3). The latter can be thought of as a quantization of the map (2.1), where $u_1$ is a quantum correction, invisible on the level of Poisson geometry.

**Corollary 5.5.** *The element*

$$\widehat{\mathcal{L}} = \Omega u_1^{-1}\Theta\Omega^{-1} \in D(A)\otimes H(A) \tag{5.4}$$

provides a solution to the reflection equation (5.1).

**Remark 5.6.** In fact, one can check using the relations (5.2) that the element

$$\widehat{\mathcal{L}}' = \Omega\Theta\Omega^{-1} \in D(A)\otimes H(A)$$

obtained from (5.4) by omitting $u_1^{-1}$, also satisfies the reflection equation (5.1). In general, however, the linear map $D(A)\to H(A)$ defined by $\mathcal{L}\mapsto\widehat{\mathcal{L}}'$ will fail to be a homomorphism of algebras. On the other hand, suppose that $R\in\mathrm{End}(V\otimes V)$ is a scalar solution of the Yang-Baxter equation. Then, following Faddeev-Reshetikhin-Takhtajan, one can define a *reflection equation algebra* $\mathcal{A}$ as the algebra generated by entries of $\mathcal{L}\in\mathcal{A}\otimes\mathrm{End}(V)$, subject to the defining relations (5.1). Similarly, one can define an algebra $\mathcal{H}$ generated by entries of the elements $\Theta,\Omega\in\mathcal{H}\otimes\mathrm{End}(V)$ subject to the relations (5.2). Then we get a well-defined homomorphism of algebras

$$\mathcal{A}\longrightarrow\mathcal{H}, \qquad \mathcal{L}\longmapsto\Omega\Theta\Omega^{-1}.$$

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