Letter

Bounding temperature dissipation in time-modulated Rayleigh-Bénard convection

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(Received 17 December 2020; accepted 17 May 2021; published 28 May 2021)

We use the background method to find an upper bound on the nondimensional temperature dissipation, $\langle \|\nabla T\|_2^2 \rangle$, for Rayleigh-Bénard convection with the temperature of one boundary modulated in time. The resulting bound depends on characteristics of the temperature modulation profile, $f(t)$, including the nondimensional parameter ω, which is defined as the supremum of $|f'(t)|$. It is found that the resulting bound on $\langle \|\nabla T\|_2^2 \rangle$ grows like $\sqrt{\omega}$ for $\omega \gg \mathrm{Ra}$, with Ra fixed, and like $\sqrt{\mathrm{Ra}}$ for $\mathrm{Ra} \gg \omega$, with ω fixed. Asymptotically, the bound for large Ra has the same leading order behavior as the nonmodulated case, with modulation effects appearing only at $O(\mathrm{Ra}^{-1/2})$.

DOI: 10.1103/PhysRevFluids.6.L051501

I. INTRODUCTION

Investigations of the linear and nonlinear stability of modulated convection configurations go back at least to Gershuni and Zhukhovitskii [1] for linear stability and to Homsy [2] for nonlinear stability. The review in Davis [3] describes the work to that time, and work has continued steadily to the present day, including recent linear stability calculations such as Hazra *et al.* [4] and numerical simulations such as Yang *et al.* [5].

Work on bounding flow quantities, where an upper bound is sought on a flow quantity of interest, has its roots in Malkus [6] and Howard [7]. The method of bounding known as the background flow technique was pioneered by Doering and Constantin [8]. We are not aware of any work on bounds for modulated convection. To our knowledge, the only bounding analysis dealing with modulation is Marchioro [9], who found a bound on energy dissipation in shear flow between horizontal parallel plates with one plate modulated horizontally.

In this paper, we use the background flow technique to find an upper bound on the temperature dissipation for the standard Rayleigh-Bénard configuration but with one boundary temperature that can vary in time. We follow Marchioro [9] closely, but we frame the analysis in the notation of Doering and Gibbon [10].

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II. SETUP

We work with two parallel plates extending infinitely far in the x and y directions containing fluid satisfying the Boussinesq equations,

$$\partial_{t_*}\boldsymbol{u}_* + \boldsymbol{u}_* \cdot \nabla_* \boldsymbol{u}_* = -\frac{1}{\rho_0}\nabla_* p_* + \alpha g T_* \hat{\boldsymbol{z}} + \nu \nabla_*^2 \boldsymbol{u}_*, \tag{1}$$

$$\nabla_* \cdot \boldsymbol{u}_* = 0, \tag{2}$$

$$\partial_{t_*} T_* + \boldsymbol{u}_* \cdot \nabla_* T_* = \kappa \nabla_*^2 T_*, \tag{3}$$

where asterisks represent dimensional quantities, $\boldsymbol{u}_*$ is the velocity, T_* is the temperature measured with respect to the reference temperature at the upper boundary, ρ_0 is the density at the reference temperature, g is the acceleration due to gravity, ν is the kinematic viscosity, κ is the thermal diffusivity, α is the thermal expansion coefficient, and p_* is the pressure. The boundary conditions are

$$\boldsymbol{u}_* = 0 \quad \text{at} \quad z_* = 0, d, \tag{4}$$

$$T_* = f_*(t_*) \quad \text{at} \quad z_* = 0, \tag{5}$$

$$T_* = 0 \quad \text{at} \quad z_* = d, \tag{6}$$

where d is the domain size in the z direction. On the horizontal boundaries, we take $\boldsymbol{u}_*$ and T_* to be periodic (this can be relaxed to, e.g., the no-slip adiabatic conditions $\boldsymbol{u}_* = \boldsymbol{0}$ and $\partial_n T_* = 0$). We nondimensionalize using

$$t \equiv \frac{\kappa t_*}{d^2}, \quad \boldsymbol{x} \equiv \frac{\boldsymbol{x}_*}{d}, \quad \boldsymbol{u} \equiv \frac{d\boldsymbol{u}_*}{\kappa}, \quad T \equiv \frac{T_*}{\Delta T}, \quad f(t) \equiv \frac{f_*(t_*)}{\Delta T} \tag{7}$$

and an appropriate scaling for pressure. The temperature scale ΔT is left unspecified for now. The most convenient choice for ΔT will in general depend on the boundary temperature profile $f_*(t_*)$. This is discussed further below.

Using the nondimensional variables in (7), the nondimensional governing equations become

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \mathrm{Ra}\,\mathrm{Pr} T \hat{\boldsymbol{z}} + \mathrm{Pr} \nabla^2 \boldsymbol{u}, \tag{8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{9}$$

$$\partial_t T + \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \nabla^2 T. \tag{10}$$

The boundary conditions become

$$\boldsymbol{u} = 0 \quad \text{at} \quad z = 0, 1, \tag{11}$$

$$T = f(t) \quad \text{at} \quad z = 0, \tag{12}$$

$$T = 0 \quad \text{at} \quad z = 1. \tag{13}$$

The Rayleigh number, Ra, and Prandtl number, Pr, are defined as

$$\mathrm{Ra} \equiv \frac{\alpha g \Delta T d^3}{\nu\kappa}, \quad \mathrm{Pr} \equiv \frac{\nu}{\kappa}. \tag{14}$$

For later use in the bounding argument, we introduce the following two dimensionless parameters, both assumed to be finite:

$$M = \sup|f(t)|, \quad \omega = \sup|f'(t)|. \tag{15}$$

III. FINDING THE BOUND

We decompose using the background flow technique by writing the temperature as

$$T(\boldsymbol{x}, t) = \tau(z, t) + \theta(\boldsymbol{x}, t), \tag{16}$$

where τ satisfies the boundary conditions on T so that

$$\tau(0, t) = f(t), \quad \tau(1, t) = 0, \quad \theta(0, t) = 0 = \theta(1, t). \tag{17}$$

We now find appropriate power integrals. We begin by taking the inner product of $\boldsymbol{u}$ with the momentum equation (8) and integrating over the volume. The power integral for velocity is

$$\frac{d}{dt}\left(\frac{\|\boldsymbol{u}\|_2^2}{2}\right) = \mathrm{Ra}\,\mathrm{Pr}\int_V w\theta\, dV - \mathrm{Pr}\|\nabla\boldsymbol{u}\|_2^2, \tag{18}$$

where we have integrated by parts. The usual norms will be employed for scalars, vectors, and tensors:

$$\|f\|_2^2 = \int_V f^2 dV, \quad \|\boldsymbol{a}\|_2^2 = \int_V |\boldsymbol{a}|^2 dV, \quad \|\nabla\boldsymbol{u}\|_2^2 = \int_V \left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right) dV. \tag{19}$$

For the temperature, we use the background decomposition (16) in (10), multiply the result by θ, and integrate to find

$$\frac{d}{dt}\left(\frac{\|\theta\|_2^2}{2}\right) = -\int_V \theta\partial_t\tau\, dV - \int_V \theta w\,\partial_z\tau\, dV - \frac{1}{2}\|\nabla T\|_2^2 + \frac{1}{2}\|\partial_z\tau\|_2^2 - \frac{1}{2}\|\nabla\theta\|_2^2. \tag{20}$$

Noticing that each of the power integrals has an integral involving the product $w\theta$, we divide (18) by Ra Pr and combine the two integrals to get

$$\frac{1}{2}\frac{d}{dt}\left(\|\theta\|_2^2 + \frac{\|\boldsymbol{u}\|_2^2}{\mathrm{Ra}\,\mathrm{Pr}}\right) = -\int_V \theta\partial_t\tau\, dV - \frac{\|\nabla T\|_2^2}{2} + \frac{\|\partial_z\tau\|_2^2}{2} - \mathcal{H}[\theta, \tau, \boldsymbol{u}], \tag{21}$$

where we have defined the quadratic-form functional

$$\mathcal{H}[\theta, \tau, \boldsymbol{u}] = \int_V \left(\frac{|\nabla\theta|^2}{2} + \frac{|\nabla\boldsymbol{u}|^2}{\mathrm{Ra}} + \theta w(\partial_z\tau - 1)\right) dV. \tag{22}$$

We now take a time average, denoted by the operator $\langle\cdot\rangle = T^{-1}\int_0^T(\cdot)dt$, to get

$$\begin{aligned}\left\langle\frac{\|\nabla T\|_2^2}{2}\right\rangle = \left\langle\frac{\|\partial_z\tau\|_2^2}{2}\right\rangle - \left\langle\int_V \theta\partial_t\tau\, dV\right\rangle - \langle\mathcal{H}[\theta, \tau, \boldsymbol{u}]\rangle \\ - \frac{1}{2T}\left(\|\nabla\theta\|_2^2 + \frac{\|\nabla\boldsymbol{u}\|_2^2}{\mathrm{Ra}\,\mathrm{Pr}}\right)\Bigg|_{t=T} + \frac{1}{2T}\left(\|\nabla\theta\|_2^2 + \frac{\|\nabla\boldsymbol{u}\|_2^2}{\mathrm{Ra}\,\mathrm{Pr}}\right)\Bigg|_{t=0},\end{aligned} \tag{23}$$

which allows us to write

$$\left\langle\frac{\|\nabla T\|_2^2}{2}\right\rangle \leqslant \left\langle\frac{\|\partial_z\tau\|_2^2}{2}\right\rangle - \left\langle\int_V \theta\partial_t\tau\, dV\right\rangle - \langle\mathcal{H}[\theta, \tau, \boldsymbol{u}]\rangle + \frac{1}{2T}\left(\|\nabla\theta\|_2^2 + \frac{\|\nabla\boldsymbol{u}\|_2^2}{\mathrm{Ra}\,\mathrm{Pr}}\right)\Bigg|_{t=0}. \tag{24}$$

If the initial state is finite, we take $T \to \infty$ from now on, so that the last term goes to zero, leading to

$$\left\langle\frac{\|\nabla T\|_2^2}{2}\right\rangle \leqslant \left\langle\frac{\|\partial_z\tau\|_2^2}{2}\right\rangle - \left\langle\int_V \theta\partial_t\tau\, dV\right\rangle - \langle\mathcal{H}[\theta, \tau, \boldsymbol{u}]\rangle. \tag{25}$$

Now if we impose $\mathcal{H} \geqslant 0$, which we shall follow Doering and Gibbon [10] in calling the "spectral constraint," then we are left with a bound on the temperature dissipation, albeit with a dependence on the fluctuation θ that we will have to work to eliminate.

With the spectral constraint enforced, we have

$$\langle \|\nabla T\|_2^2 \rangle \leqslant \langle \|\partial_z \tau\|_2^2 \rangle - 2\left\langle \int_V \theta \partial_t \tau \, dV \right\rangle, \tag{26}$$

which corresponds to the solution of the variational problem

$$\langle \|\nabla T\|_2^2 \rangle \leqslant \inf \left\{ \langle \|\partial_z \tau\|_2^2 \rangle - 2\left\langle \int_V \theta \partial_t \tau \, dV \right\rangle \middle| \begin{array}{l} \mathcal{H}[\theta, \tau, \boldsymbol{u}] = \displaystyle\int_V \left(\frac{|\nabla \theta|^2}{2} + \frac{|\nabla \boldsymbol{u}|^2}{\mathrm{Ra}} + \theta w(\partial_z \tau - 1) \right) dV \geqslant 0, \\ \tau(0,t) = f(t), \ \ \tau(1,t) = 0, \\ \theta(x,y,0,t) = 0 = \theta(x,y,1,t), \\ \nabla \cdot \boldsymbol{u} = 0, \\ \boldsymbol{u}(x,y,0,t) = 0 = \boldsymbol{u}(x,y,1,t) \end{array} \right\}. \tag{27}$$

In theory, we could use Lagrange multipliers to enforce the constraints, find the Euler-Lagrange equations, and thereby solve the problem. In practice, because an exact solution is difficult, we simply try different background profiles. Any background profile satisfying the constraints will provide an upper bound on the temperature dissipation.

The guiding principle in choosing the background profile will be to satisfy the spectral constraint, and to do that we need the term $\theta w(\partial_z \tau - 1)$, to be as small as we can make it because it is the only term in the spectral constraint that can be negative. From the boundary conditions, we know that the velocity, $\boldsymbol{u}$, and the temperature fluctuations, θ, are small near the boundaries but possibly large away from the boundaries. To neutralize this, we try to make $(\partial_z \tau - 1)$ small away from the boundaries.

The simplest profile to try is a piecewise linear profile with two sections. From the above discussion, we make $\partial_z \tau = 1$ away from the bottom boundary while enforcing $\tau(0,t) = f(t)$ at the bottom boundary, which gives us

$$\tau(z,t) = \begin{cases} f(t) - z\delta^{-1}(f(t) - \delta + 1), & 0 \leqslant z \leqslant \delta \\ z - 1, & \delta \leqslant z \leqslant 1. \end{cases} \tag{28}$$

In order to make progress on the bound in (26), we use the inequalities listed in Appendix. First we ensure that we meet the spectral constraint, $\mathcal{H}[\theta, \tau, \boldsymbol{u}] \geqslant 0$, or

$$\left(\int_V \frac{|\nabla \theta|^2}{2} + \frac{|\nabla \boldsymbol{u}|^2}{\mathrm{Ra}} + \theta w(\partial_z \tau - 1) \right) dV \geqslant 0, \tag{29}$$

which clearly requires us to work on the last term. Using our definition of the linear piecewise profile leads to

$$\int_V \theta w(\partial_z \tau - 1) dV = -\frac{f(t)+1}{\delta} \int_0^\delta \left(\int_A w\theta \, dA \right) dz, \tag{30}$$

where A is the nondimensional horizontal area of the system. This allows us to find

$$\left| \int_V \theta w(\partial_z \tau - 1) dV \right| \leqslant \frac{(M+1)}{\delta} \int_0^\delta \left| \int_A w\theta \, dA \right| dz \tag{31}$$

$$\leqslant \frac{(M+1)\delta}{2} \|\partial_z w\|_2 \|\partial_z \theta\|_2 \tag{32}$$

$$\leqslant \frac{(M+1)\delta}{4}\left(\eta\|\partial_z w\|_2^2+\frac{1}{\eta}\|\partial_z\theta\|_2^2\right) \tag{33}$$

$$\leqslant \frac{(M+1)\delta}{4}\left(\frac{\eta\|\nabla \boldsymbol{u}\|_2^2}{4}+\frac{\|\nabla\theta\|_2^2}{\eta}\right), \tag{34}$$

for any positive η. Here we used (A1) to go from (31) to (32); Young's inequality, $2ab \leqslant \eta a^2 + \eta^{-1}b^2$ $(\eta > 0)$, to go from (32) to (33); and the incompressibility inequality (A5) and the definition of the two-norm to go from (33) to (34).

We use this result to find the necessary δ and η to enforce the spectral constraint,

$$\mathcal{H}[\theta,\tau,\boldsymbol{u}] = \left(\int_V \frac{|\nabla\theta|^2}{2}+\frac{|\nabla\boldsymbol{u}|^2}{\mathrm{Ra}}+\theta w(\partial_z\tau-1)\right)dV$$

$$\geqslant \frac{\|\nabla\theta\|_2^2}{2}+\frac{\|\nabla\boldsymbol{u}\|_2^2}{\mathrm{Ra}}-\frac{(M+1)\delta}{4}\left(\frac{\eta\|\nabla\boldsymbol{u}\|_2^2}{4}+\frac{\|\nabla\theta\|_2^2}{\eta}\right) \tag{35}$$

$$= \|\nabla\boldsymbol{u}\|_2^2\left(\frac{1}{\mathrm{Ra}}-\frac{(M+1)\eta\delta}{16}\right)+\|\nabla\theta\|_2^2\left(\frac{1}{2}-\frac{(M+1)\delta}{4\eta}\right), \tag{36}$$

which means that we require

$$\frac{1}{\mathrm{Ra}} \geqslant \frac{(M+1)\eta\delta}{16}, \quad \frac{1}{2} \geqslant \frac{(M+1)\delta}{4\eta} \tag{37}$$

to satisfy $\mathcal{H} \geqslant 0$. Provided that

$$\delta \leqslant \frac{4\sqrt{2}}{(M+1)\sqrt{\mathrm{Ra}}}, \tag{38}$$

any value of η satisfying $(M+1)\delta/2 \leqslant \eta \leqslant 16/[\mathrm{Ra}(M+1)\delta]$ can be used to satisfy (37).

From (26) we work towards the desired bound. We use (A2) to get

$$\langle\|\nabla T\|_2^2\rangle \leqslant \langle\|\partial_z\tau\|_2^2\rangle + \tfrac{8}{15}\omega\delta^{3/2}\sqrt{A}\langle\|\nabla\theta\|_2\rangle. \tag{39}$$

Next we use the convexity inequality (A3) to obtain

$$\langle\|\nabla T\|_2^2\rangle \leqslant \langle\|\partial_z\tau\|_2^2\rangle + \tfrac{8}{15}\omega\delta^{3/2}\sqrt{A}\langle\|\nabla\theta\|_2^2\rangle^{1/2}, \tag{40}$$

and (A4) to eliminate the temperature fluctuation, leaving

$$\langle\|\nabla T\|_2^2\rangle \leqslant \langle\|\partial_z\tau\|_2^2\rangle + \frac{8\sqrt{2}}{15}\omega\delta^{3/2}\sqrt{A}\left(\langle\|\nabla T\|_2^2\rangle+\langle\|\partial_z\tau\|_2^2\rangle\right)^{1/2}, \tag{41}$$

which we can write as

$$\varepsilon_T \leqslant a + b\sqrt{\varepsilon_T + a}. \tag{42}$$

Solving inequality (42) results in

$$\varepsilon_T \leqslant a + \frac{b^2}{2} + \frac{b}{2}\sqrt{8a+b^2}. \tag{43}$$

We have

$$\varepsilon_T \equiv \langle\|\nabla T\|_2^2\rangle, \tag{44}$$

$$a \equiv \langle\|\partial_z\tau\|_2^2\rangle = A\left(\frac{\langle f^2\rangle + 2\langle f\rangle + 1}{\delta} - 2\langle f\rangle - 1\right), \tag{45}$$

$$b \equiv \frac{8\sqrt{2}}{15}\omega\delta^{3/2}\sqrt{A}. \tag{46}$$

Written out, the bound becomes

$$\frac{\|\nabla T\|_2^2}{A} \leqslant \frac{F}{\delta} - 2\langle f\rangle - 1 + \left(\frac{8}{15}\right)^2\omega^2\delta^3 + \frac{4\sqrt{2}\omega\delta^{3/2}}{15}\left[2\left(\frac{8}{15}\right)^2\omega^2\delta^3 + 8\left(\frac{F}{\delta} - 2\langle f\rangle - 1\right)\right]^{1/2}, \tag{47}$$

where $F = \langle f^2\rangle + 2\langle f\rangle + 1$, and the bound is valid for all $\delta \leqslant \delta_{\max}$, with

$$\delta_{\max} \equiv \frac{4\sqrt{2}}{(M+1)\sqrt{\mathrm{Ra}}} \leqslant 1 \Rightarrow \mathrm{Ra} \geqslant \frac{32}{(M+1)^2}. \tag{48}$$

Let us denote the right-hand side of the bound in (47) as $h(\delta)$. It now remains to choose an optimal $\delta_{\rm opt} \leqslant \delta_{\max}$ in order to obtain the smallest bound in (47). This leads to an optimization problem for $\delta_{\rm opt}$ that in general must be solved numerically. We note that Ra does not appear in h explicitly, instead serving only to restrict the maximum value of δ. For large ω and Ra, we can find asymptotic representations for $\delta_{\rm opt}$, and therefore for the bound.

IV. LIMITS AND EXAMPLES

A. Large-ω limit

Examining the order of the terms in (47) shows that $\delta_{\rm opt} \sim \omega^{-1/2}$ provides the optimal bound for large ω. Calculating $h'(\delta)$ with $\omega = \Omega\delta_{\rm opt}^{-2}$ and expanding for small δ leads to a quadratic for Ω^2. The appropriate solution leads to

$$\delta_{\rm opt} = C_0 F^{1/4}\omega^{-1/2}, \tag{49}$$

where $C_0 = (\sqrt{15}/4)(13 - 5\sqrt{19/3})^{1/4} \approx 0.78$. To leading order we find that the minimum bound corresponds to

$$\frac{\|\nabla T\|_2^2}{A} \leqslant D_0 F^{3/4}\sqrt{\omega} - E_0(2\langle f\rangle + 1) + O(\omega^{-1/2}), \tag{50}$$

where $D_0 \approx 2.26$ and $E_0 \approx 1.32$. The large ω bound depends on Ra only through the restriction that $\omega \gg \mathrm{Ra}$, as Ra does not appear explicitly in the bound. The bound breaks down when $\delta_{\rm opt} \sim \delta_{\max}$, which, from equating (49) with the expression for $\delta_{\max}$ in (48), corresponds to

$$\omega = \frac{C_0^2}{32}(M+1)^2F^{1/2}\mathrm{Ra} \approx 0.02(M+1)^2F^{1/2}\mathrm{Ra}. \tag{51}$$

B. Large-Ra limit

With no time dependence, $\omega = 0$ and the bound in (47) looks like δ^{-1}, so that choosing the largest value for δ gives the best bound. Furthermore, from consideration of the large ω limit, we know that $\delta_{\rm opt} = \delta_{\max}$ when $\mathrm{Ra} \sim \omega$ as in (51) and for larger Ra. We therefore take $\delta_{\rm opt} = \delta_{\max}$ for large Ra. The bound on the temperature dissipation in (47) is then

$$\frac{\|\nabla T\|_2^2}{A} \leqslant \frac{(M+1)F}{4\sqrt{2}}\sqrt{\mathrm{Ra}} - 2\langle f\rangle - 1 + \frac{c^2\omega^2\mathrm{Ra}^{-3/2}}{2(M+1)^3} + \frac{c\omega F^{1/2}\mathrm{Ra}^{-1/2}}{2^{3/4}(M+1)}\left(1 + \frac{c^2\omega^2\mathrm{Ra}^{-2}}{(M+1)^4\sqrt{2}} - \frac{8(2\langle f\rangle + 1)\mathrm{Ra}^{-1/2}}{\sqrt{2}(M+1)F}\right)^{1/2}, \tag{52}$$

where $c = 2^{(29/4)}/15$ and Ra must satisfy the restriction in (48). Expanding this for $\omega \ll \mathrm{Ra}$ and $\mathrm{Ra} \gg 1$, we can write the bound as

$$\frac{\|\nabla T\|_2^2}{A} \leqslant \frac{(M+1)F}{4\sqrt{2}}\sqrt{\mathrm{Ra}} - 2\langle f\rangle - 1 + \frac{c\omega\sqrt{F}\,\mathrm{Ra}^{-1/2}}{2^{3/4}(M+1)} - \frac{2^{3/4}c\omega(2\langle f\rangle+1)}{(M+1)^2\sqrt{F}}\mathrm{Ra}^{-1} + O(\omega\,\mathrm{Ra}^{-3/2}, \omega^2\mathrm{Ra}^{-3/2}). \tag{53}$$

Note that ω does not appear at leading order in this expansion. We again see the crossover to the high-ω limit when $\sqrt{\mathrm{Ra}} \sim \omega\,\mathrm{Ra}^{-1/2}$, so that the first and fourth terms are of the same order.

C. Bound in terms of M

The use of $\langle f^2\rangle$ and $\langle f\rangle$ in (43)is more general than the approach of Marchioro [9], but the resulting bound is more complicated. We can obtain a simpler result by bounding a using $\langle f^2\rangle \leqslant M^2$ and $|\langle f\rangle| \leqslant M$, and using the resulting bound on a in (43), since the form of (43) means that it is satisfied when a is replaced by an upper bound. The result is simpler but less precise, and takes the form

$$\frac{\|\nabla T\|_2^2}{A} \leqslant \frac{(M+1)^2}{\delta} + 2M - 1 + \left(\frac{8}{15}\right)^2\omega^2\delta^3 + \frac{4\sqrt{2}\omega\delta^{3/2}}{15}\left[2\left(\frac{8}{15}\right)^2\omega^2\delta^3 + 8\left(\frac{(M+1)^2}{\delta} + 2M - 1\right)\right]^{1/2}. \tag{54}$$

D. Examples

We now consider specific example profiles. When the average temperature at the top and bottom is different, one can define ΔT to be the difference in average temperature, which is the usual approach in steady Rayleigh-Bénard convection. For example, we can consider the sinusoidal profile $f_*(t_*) = \Delta T_0 + \Delta T_1 \cos\omega_* t_*$. For this case we choose to define ΔT as ΔT_0, so that

$$f(t) = 1 + \alpha\cos(\omega_1 t), \tag{55}$$

with $\alpha = \Delta T_1/\Delta T_0$ and $\omega_1 = \omega_* d^2/\kappa$. For this profile, we have $M = 1+\alpha$, $\omega = \alpha\omega_1$, $\langle f\rangle = 1$, and $\langle f^2\rangle = 1 + \alpha^2/2$. For $\omega \gg \mathrm{Ra}$, the bound on temperature dissipation is (50) with the appropriate values of $\langle f\rangle$ and $\langle f^2\rangle$. For any fixed ω_1 and small enough α so that $\omega \ll \mathrm{Ra}$, the bound for large Ra in (53) becomes

$$\frac{\|\nabla T\|_2^2}{A} \leqslant (1+\alpha/2)(1+\alpha^2/8)\sqrt{2}\sqrt{\mathrm{Ra}} - 3 + O(\omega\,\mathrm{Ra}^{-1/2}). \tag{56}$$

As $\alpha \to 0$ we recover a bound for the standard setup with no modulation, namely, $\|\nabla T\|_2^2/A \leqslant \sqrt{2}\sqrt{\mathrm{Ra}} - 3$. If instead we take the bottom boundary temperature to have the same mean as the top boundary temperature, then $\Delta T_0 = 0$. Hence we take $\Delta T = \Delta T_1$, the maximum of the difference between the top and bottom temperatures. This leads to $f(t) = \cos(\omega_1 t)$, with $M = 1$, $\omega = \omega_1$, $\langle f\rangle = 0$, and $\langle f^2\rangle = 1/2$. As in the previous case, the bound on temperature dissipation for $\omega \gg \mathrm{Ra}$ is (50) with the appropriate values of $\langle f\rangle$ and $\langle f^2\rangle$. For large Rayleigh numbers with $\omega \ll \mathrm{Ra}$, the bound (53) becomes

$$\frac{\|\nabla T\|_2^2}{A} \leqslant \frac{3}{4\sqrt{2}}\sqrt{\mathrm{Ra}} - 1 + O(\omega\,\mathrm{Ra}^{-1/2}). \tag{57}$$

V. DISCUSSION

Using the background method, we have found a bound on the temperature dissipation rate for a Rayleigh-Bénard-like setup with a modulated bottom boundary. We have investigated the bound for large Ra and ω, and we have considered specific simple examples of modulated temperature profiles. The resulting bound is expressed in terms of the maximum of the temperature on the boundary, M, as well as its maximum rate of change, ω, mean, $\langle f\rangle$, and mean square, $\langle f^2\rangle$. The spectral constraint must be satisfied at all times, so that it involves M, which hence is present in the bound. However, the inequalities used to obtain the final bound itself use the mean and mean square, resulting in a sharper bound than using M alone. The calculation of the bound may also be carried out with an additional boundary layer of thickness δ at the top. If the interior profile for $\tau(z,t)$ is chosen simply as z to make $(\partial_z\tau - 1)$ equal zero, then it turns out that $\delta_{\max}$ is the same as in the single boundary layer case. The only difference in (47) is that F now becomes $F = \langle f^2\rangle + 1$. When $\langle f\rangle = 0$, there is therefore no difference between using one or two boundary layers in the background profile. For the example with different average temperatures, this leads to an improvement in the small-α limit of the prefactor with $\|\nabla T\|_2^2/A \leqslant \sqrt{\mathrm{Ra}/2} - 3$.

The Nusselt number, which is the ratio of the total to conductive heat transfer, is commonly used as the quantity to bound in convection problems in Rayleigh-Bénard convection, because with no time dependence the Nusselt number is proportional to the temperature dissipation, meaning that the bound on temperature dissipation yields a bound on heat transport. The time dependence in more complicated problems, such as the one considered here, makes the relationship between the Nusselt number and the temperature dissipation more complicated, and therefore the Nusselt number is not as useful in these cases as the temperature dissipation. The latter appears naturally in the bounding process. Furthermore, the temperature dissipation can be shown to be directly proportional to entropy production from heat conduction under the Boussinesq approximation, as discussed in Howard [7] and also more recently in the context of horizontal convection in Rocha *et al.* [11].

Besides the background flow technique, more recent bounding methods include polynomial sum of squares, as reviewed in Chernyshenko *et al.* [12], and the auxiliary functional method, as applied in Fantuzzi [13] to fixed-flux convection with free-slip conditions. These other methods have the potential for finding better bounds, while generally reducing to the background flow technique if certain parameter choices are made.

ACKNOWLEDGMENTS

The authors acknowledge helpful conversations with Charles R. Doering. Suggestions from the anonymous referees led to improvements in the material in Sec. IV. This research was partly supported by NSF Award No. OCE-1829919.

APPENDIX: INEQUALITIES

We make use of the following inequalities:

$$\left|\int_A w\theta\, dA\right| \leqslant z\|\partial_z w\|_2\|\partial_z\theta\|_2, \tag{A1}$$

$$\left|\int_V \theta\partial_t\tau\, dV\right| \leqslant \frac{4}{15}\omega\delta^{3/2}\sqrt{A}\|\nabla\theta\|_2, \tag{A2}$$

$$\langle\|\nabla\theta\|_2\rangle \leqslant \left\langle\|\nabla\theta\|_2^2\right\rangle^{1/2}, \tag{A3}$$

$$\|\nabla\theta\|_2^2 \leqslant 2\|\nabla T\|_2^2 + 2\|\partial_z\tau\|_2^2, \tag{A4}$$

$$\|\partial_z w\|_2^2 \leqslant \frac{1}{4}\|\nabla \boldsymbol{u}\|_2^2. \tag{A5}$$

(A1) is a consequence of the fundamental theorem of calculus, the Cauchy-Schwarz inequality, extending the domain of integration, and the definition of the two-norm, in that order. It also holds with z replaced by $1 - z$. Inequality (A2) is equivalent to (19) in Marchioro [9], but with the prefactor improved from $1/2$ to $4/15$. Jensen's inequality leads to (A3), while (A4) is obtained using Young's inequality or the arithmetic mean–geometric mean inequality. A derivation of (A5) can be found in Doering and Constantin [14], Eqs. (5.13)–(5.16).

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