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Translating hollow vortex pairs

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ABSTRACT

A new derivation, and representation, of the classical solution for a translating hollow vortex pair originally discovered by Pocklington in 1895 [H.C. Pocklington, The configuration of a pair of equal and opposite hollow straight vortices of finite cross-section, moving steadily through fluid, Proc. Cambridge Philos. Soc. 8 (1895) 178–187] is presented. The derivation makes use of a special function known as the Schottky–Klein prime function. The new representation of the hollow vortex pair facilitates an investigation of the linear stability properties of this configuration, something apparently not previously studied in the literature. We show here that the hollow vortex pair is linearly stable to infinitesimal irrotational perturbations provided the area of the two vortices is below a critical value.

1. Introduction

The theoretical study of vortex pairs, or vortex dipoles, comprising two counter-rotating vortices of equal and opposite sign is an important one in fluid mechanics. Such vortex configurations generally travel at constant speed in a rectilinear motion: for example, two point vortices [1] of circulations $\pm\Gamma$ and separated by distance b will travel at speed

$$U_0 = \frac{\Gamma}{2\pi b}. \tag{1}$$

The Lamb dipole [2], for which the vorticity is linearly related to the stream function, is the best-known vortex pair with distributed vorticity [1]. Deem & Zabusky [3] used numerical methods to compute the equilibrium configurations of two counter-rotating vortex patches. A vortex patch is a two-dimensional region of constant vorticity. It is a popular vortex model that has found abundant application in the literature [1]. Pierrehumbert [4] has also performed a numerical study of the counter-rotating pair of vortex patches. Another travelling vortex pair solution with distributed vorticity is due to Pocklington [5], who used elliptic function theory to derive analytical solutions for two counter-rotating hollow vortices travelling at uniform speed. Here we understand a hollow vortex, following earlier authors [5,6,1], to be a finite-area, constant pressure region having a non-zero circulation around it. It is a much less commonly used model of vorticity, although it has been the subject of a revival of interest in recent work [7–9]. Tanveer [10] used a numerical scheme based on conformal mappings to generalize Pocklington's solution to the case where the interior of the vortex has non-zero uniform vorticity. Tanveer's solutions are also natural extensions of the solution due to Sadovskii [11], who considered the case of two touching vortices and allowed discontinuities in the tangential velocity across the boundaries of the uniform vorticity regions.

When it comes to stability, the point vortex pair is easily shown to be linearly stable to infinitesimal perturbations. Cavazza et al. [12] have numerically studied the stability of the Lamb dipole by adding initial perturbations. They find that if the dipole is subjected to elliptic deformation, it sheds some vorticity in its wake and then returns to a circular shape with a smaller radius. The linear stability properties of the uniform vortex patch solutions, as calculated numerically by Deem & Zabusky [3], do not appear to have been previously studied in the literature but, according to Saffman [1], "it is believed that they are stable". To the best of the authors' knowledge, the linear stability properties of Pocklington's hollow vortex pair have not been studied either. The purpose of the present paper is to revisit Pocklington's hollow vortex pair and to present, apparently for the first time, a full analysis of its linear stability. Interestingly, we find that the hollow vortex pair is only linearly stable provided the size of the vortices is sufficiently small.

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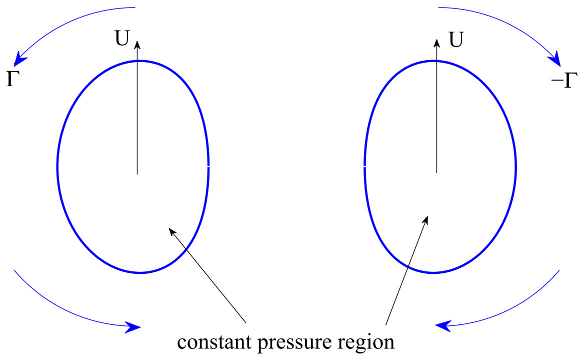

Fig. 1. Schematic of a hollow vortex pair with circulations $\pm\Gamma$ propagating in the positive y direction with speed U.

This paper also presents a new mathematical derivation of Pocklington's solution. This not only provides a convenient basis for the linear stability analysis, but it is also valuable in that it avoids complications associated with the use of standard elliptic functions exploited by Pocklington in his original derivation [5]. Indeed, Pocklington's original paper [5] contains some mathematical errors that have been pointed out, and corrected, by Tanveer [13]. Here we find a family of solutions for steadily translating hollow vortex pairs with a single mathematical parameter ρ governing the size of the vortices. The solutions are described by a conformal mapping from the annulus $\rho < |\zeta| < 1$ to the fluid region exterior to the vortex pair given by

$$z(\zeta) = C\int_{-\sqrt{\rho}}^{\zeta}\left\{\frac{P(\zeta'\sqrt{\rho}\mathrm{e}^{\mathrm{i}\Theta},\rho)P(\zeta'\sqrt{\rho}\mathrm{e}^{-\mathrm{i}\Theta},\rho)}{P(\zeta'/\sqrt{\rho},\rho)P(\zeta'\sqrt{\rho},\rho)}\right\}^2 \mathrm{d}\zeta', \qquad (2)$$

where C is a normalization constant and $\Theta = \Theta(\rho)$ is a real constant dependent on the choice of ρ. The special function $P(\zeta,\rho)$ is a version of the *Schottky–Klein prime function* [14] on the annulus $\rho < |\zeta| < 1$. The two circles $|\zeta| = \rho, 1$ map to the two boundaries of the hollow vortices. The solution for the point vortex pair is retrieved in the limit $\rho \to 0$. It is our view that formula (2) is a more concise and convenient, representation of the translating hollow vortex pair than that given by Pocklington [5].

2. Mathematical formulation

We will seek solutions, in a complex $z = x + \mathrm{i}y$-plane, for which the pair is steadily translating parallel to the imaginary axis with speed U. One vortex is assumed to have its centroid on the positive real axis and to have circulation $-\Gamma$ with $\Gamma > 0$; the other is on the negative real axis with circulation $+\Gamma$. This configuration is expected to translate steadily upwards parallel to the positive imaginary axis with some speed $U > 0$. Fig. 1 shows a schematic of this configuration.

It is appropriate to move to a co-travelling frame in which the configuration is steady. The complex potential associated with the flow in the co-travelling frame is $w(z)$ where, as $z \to \infty$,

$$w(z) \sim U\exp(-\mathrm{i}\chi)z + \text{locally analytic function}, \qquad (3)$$

where $\chi = -\pi/2$ so that the uniform flow is in the direction of the negative imaginary axis in the co-travelling frame. Apart from this singularity at infinity, $w(z)$ is analytic everywhere outside the hollow vortices.

Consider a conformal mapping $z(\zeta)$ from some annulus $\rho < |\zeta| < 1$ to the fluid region exterior to two hollow vortices. Suppose that $|\zeta| = 1$ maps to the right-most vortex and $|\zeta| = \rho$ maps to the left-most vortex. We will seek symmetric solutions where each hollow vortex is the reflection of the other about the imaginary axis. In this case, it can be argued that the circle $|\zeta| = \sqrt{\rho}$ maps to the vertical axis of symmetry between the two vortices. Let β be the point on this circle mapping to infinity and suppose that

$$z(\zeta) = \frac{a}{\zeta - \beta} + \text{locally analytic function}, \qquad (4)$$

where, using a rotational degree of freedom of the Riemann mapping theorem, we can suppose that a is real and positive. This means that we must have $\beta = \sqrt{\rho}$ so that the portion of real ζ-axis $\sqrt{\rho} < \zeta < 1$ maps to the positive real z-axis between the vortex centred on the positive real axis and $x = +\infty$. Similarly, the portion $\rho < \zeta < \sqrt{\rho}$ maps to the real z-axis between $x = -\infty$ and the vortex centred on the negative real axis.

3. Function theory

The basis of our method is to make use of properties of the special function defined by

$$P(\zeta,\rho) = (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}), \qquad (5)$$

together with its derivatives. This function is, to within a normalization, the *Schottky–Klein prime function* [14] of the annulus $\rho < |\zeta| < 1$. Clearly $P(\zeta,\rho)$ has a simple zero at $\zeta = 1$. It is easy to show, directly from the infinite product definition, that

$$P(\rho^2\zeta,\rho) = -\zeta^{-1}P(\zeta,\rho), \qquad P(\zeta^{-1},\rho) = -\zeta^{-1}P(\zeta,\rho). \qquad (6)$$

We also introduce

$$K(\zeta,\rho) = \frac{\zeta P'(\zeta,\rho)}{P(\zeta,\rho)} \qquad (7)$$

where $P'(\zeta,\rho)$ denotes the derivative with respect to ζ. This function can be represented by the infinite sum

$$K(\zeta,\rho) = -\frac{\zeta}{1-\zeta} - \sum_{k=1}^{\infty}\frac{\rho^{2k}\zeta}{1-\rho^{2k}\zeta} + \sum_{k=1}^{\infty}\frac{\rho^{2k}\zeta^{-1}}{1-\rho^{2k}\zeta^{-1}}. \qquad (8)$$

It is easy to show, from (6), that

$$K(\rho^2\zeta,\rho) = K(\zeta,\rho) - 1, \qquad K(\zeta^{-1},\rho) = 1 - K(\zeta,\rho). \qquad (9)$$

It is also worth noting that, near $\zeta = 1$,

$$K(\zeta,\rho) \sim \frac{1}{\zeta - 1} + \text{locally analytic function}, \qquad (10)$$

so it has a simple pole there.

We will also need the function

$$L(\zeta,\rho) = \zeta K'(\zeta,\rho), \qquad (11)$$

which can be shown, from (9), to satisfy the functional relations

$$L(\rho^2\zeta,\rho) = L(\zeta,\rho), \qquad L(1/\zeta,\rho) = L(\zeta,\rho). \qquad (12)$$

It is easy to show that $L(\zeta,\rho)$ can be represented by

$$L(\zeta,\rho) = -\sum_{k=0}^{\infty}\frac{\rho^{2k}\zeta}{(1-\rho^{2k}\zeta)^2} - \sum_{k=1}^{\infty}\frac{\rho^{2k}\zeta^{-1}}{(1-\rho^{2k}\zeta^{-1})^2}. \qquad (13)$$

From (10), $L(\zeta,\rho)$ has a second-order pole at $\zeta = 1$.

4. The function $W(\zeta)$

The conditions on $W(\zeta) = w(z(\zeta))$ are that it has a simple pole, of appropriate residue, at $\zeta = \beta = \sqrt{\rho}$ and constant imaginary part on $|\zeta| = \rho, 1$ so that the boundaries of both vortices are streamlines. It is also necessary to arrange for the vortices to have the correct circulations. The mathematical problem is equivalent to finding the complex potential for uniform flow past two solid obstacles with equal and opposite circulations around them. Such a result follows immediately from the work of Crowdy [15,16,14] and is given by

$$W(\zeta) = \frac{Ua}{\sqrt{\rho}}[e^{-i\chi}K(\zeta/\sqrt{\rho}, \rho) - e^{i\chi}K(\zeta\sqrt{\rho}, \rho)] - \frac{i\Gamma}{2\pi}\log\zeta. \qquad (14)$$

Putting aside the derivation, it is easy to verify directly that $W(\zeta)$ as given in (14) has all the required properties. From (4) and (10), it follows that $W(\zeta) \sim Ue^{-i\chi}z$ as $\zeta \to \sqrt{\rho}$. By using the properties in (9) it can be verified that $W(\zeta)$ has constant imaginary part on $|\zeta| = 1, \rho$. Since $|\zeta| = \rho$ maps to the left-hand vortex with positive circulation Γ then (14) gives the required circulation since it changes by Γ on traversing the circle $|\zeta| = \rho$ in an anticlockwise direction and that corresponds, under the mapping, to encircling the left-most hollow vortex in an anticlockwise direction. Formula (14) also changes by Γ on traversing the circle $|\zeta| = 1$ in an anticlockwise fashion but that corresponds, under the mapping, to going clockwise around the vortex on the right.

5. The function $\mathrm{d}w/\mathrm{d}z$

Now consider the complex velocity function

$$R(\zeta) = \frac{\mathrm{d}w}{\mathrm{d}z} = u - iv. \qquad (15)$$

It must be analytic and single-valued in the flow region and must tend to $Ue^{-i\chi} = iU$ as $z \to \infty$. Based on an analysis of the problem of a co-travelling point vortex pair – which has two stagnation points, one ahead of and another behind the vortex pair – we similarly expect this function to have two zeros in the annulus $\rho < |\zeta| < 1$. By Bernoulli's theorem, and the fact that the hollow vortices contain regions at constant pressure, $R(\zeta)$ must also have constant modulus on the circles $|\zeta| = \rho$ and 1.

Consider the candidate function

$$R(\zeta) = \frac{A}{\zeta}\left\{\frac{P(\zeta\alpha^{-1}, \rho)P(\zeta\overline{\alpha}^{-1}, \rho)}{P(\zeta\overline{\alpha}, \rho)P(\zeta\alpha, \rho)}\right\}, \qquad (16)$$

where α is some point in the annulus $\rho < |\zeta| < 1$ and A is a constant. It will be shown that this function has all the desired properties. First, it is analytic and single-valued in $\rho < |\zeta| < 1$ and tends to a constant as $\zeta \to \sqrt{\rho}$. It also has two simple zeros in the annulus $\rho < |\zeta| < 1$ at points α and $\overline{\alpha}$ (and no others in this annulus). Furthermore, on use of the functional relations in (6), it follows that

$$|R| = q_0 = \frac{|A|}{|\alpha|^2} \quad \text{on } |\zeta| = 1, \qquad |R| = q_1 = \frac{|A|}{\rho} \quad \text{on } |\zeta| = \rho. \qquad (17)$$

Hence, $R(\zeta)$ has constant modulus on the two boundary circles $|\zeta| = \rho, 1$ and, from the symmetry of the problem, we expect the constant fluid speed on each vortex to be the same which implies

$$|\alpha| = \sqrt{\rho}. \qquad (18)$$

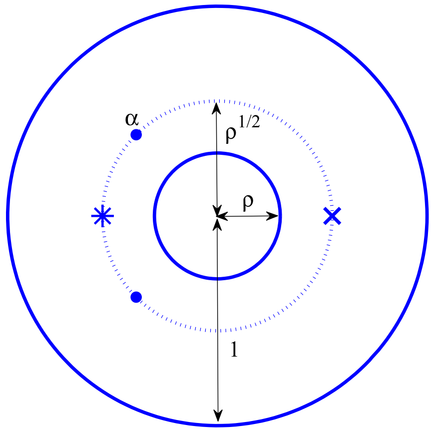

Fig. 2. Schematic of the preimage ζ domain. The two circles $|\zeta| = \rho, 1$ map to the boundaries of the hollow vortices. The cross at $\sqrt{\rho}$ is the preimage of the point at infinity, the asterisk at $-\sqrt{\rho}$ marks the preimage of $z = 0$ and α and its conjugate (the stagnation point preimages) are depicted by dots.

We therefore write

$$\alpha = \sqrt{\rho}\exp(i\Theta) \qquad (19)$$

for some real parameter Θ. As $\zeta \to \sqrt{\rho}$,

$$R(\zeta) \to R(\sqrt{\rho}) = \frac{A}{\sqrt{\rho}}\left(\frac{P(e^{i\Theta}, \rho)P(e^{-i\Theta}, \rho)}{P(\rho e^{i\Theta}, \rho)P(\rho e^{-i\Theta}, \rho)}\right) = iU, \qquad (20)$$

implying that

$$A = iU\sqrt{\rho}\left(\frac{P(\rho e^{i\Theta}, \rho)P(\rho e^{-i\Theta}, \rho)}{P(e^{i\Theta}, \rho)P(e^{-i\Theta}, \rho)}\right) = iU\sqrt{\rho}\left|\frac{P(\rho e^{i\Theta}, \rho)}{P(e^{i\Theta}, \rho)}\right|^2. \qquad (21)$$

By the chain rule,

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}W/\mathrm{d}\zeta}{\mathrm{d}z/\mathrm{d}\zeta}, \qquad (22)$$

and so, since $\mathrm{d}z/\mathrm{d}\zeta$ cannot vanish inside the annulus owing to the requirement that the conformal mapping be one-to-one, α and $\overline{\alpha}$ are also zeros of the function $\mathrm{d}W/\mathrm{d}\zeta$. On taking a derivative of (14) with respect to ζ and multiplying by ζ, we find

$$\zeta\frac{\mathrm{d}W}{\mathrm{d}\zeta} = \frac{iUa}{\sqrt{\rho}}\left(L(\zeta/\sqrt{\rho}, \rho) + L(\zeta\sqrt{\rho}, \rho)\right) - \frac{i\Gamma}{2\pi}. \qquad (23)$$

Therefore, if the solution exists, Θ must be a real solution of

$$L(\exp(i\Theta), \rho) + L(\rho\exp(i\Theta), \rho) = \mu, \qquad (24)$$

where

$$\mu \equiv \frac{\Gamma\sqrt{\rho}}{2\pi Ua}. \qquad (25)$$

It can be shown by using (12) that if Θ is a real solution of (24) then so is $-\Theta$, which is consistent with choosing the zeros of (16) to be at α and $\overline{\alpha}$. Fig. 2 shows the ζ domain.

6. Derivation of the conformal map

From the properties of $L(\zeta, \rho)$ and the discussion above, the function $\zeta\,\mathrm{d}W/\mathrm{d}\zeta$ has second-order poles at β and $1/\beta$ and simple zeros at $\alpha, \overline{\alpha}, \alpha^{-1}$ and $\overline{\alpha}^{-1}$. It follows that

$$\zeta\frac{\mathrm{d}W}{\mathrm{d}\zeta} = B\left\{\frac{P(\zeta/\alpha, \rho)P(\zeta\alpha, \rho)P(\zeta/\overline{\alpha}, \rho)P(\zeta\overline{\alpha}, \rho)}{P^2(\zeta/\sqrt{\rho}, \rho)P^2(\zeta\sqrt{\rho}, \rho)}\right\}, \qquad (26)$$

where B is a constant, is an alternative representation of $\zeta dW/d\zeta$. The validity of this representation can be directly verified using the functional relations (6). A rearrangement of (22) implies that

$$\frac{dz}{d\zeta} = \frac{\zeta dW/d\zeta}{\zeta dw/dz}. \quad (27)$$

On substitution of (26) and (16) into (27) we deduce that

$$\frac{dz}{d\zeta} = C\left\{\frac{P(\zeta\sqrt{\rho}e^{i\Theta},\rho)P(\zeta\sqrt{\rho}e^{-i\Theta},\rho)}{P(\zeta/\sqrt{\rho},\rho)P(\zeta\sqrt{\rho},\rho)}\right\}^2, \quad (28)$$

where C is some real constant. On integration, the required conformal mapping is found to be

$$z(\zeta) = C\int_{-\sqrt{\rho}}^{\zeta}\left\{\frac{P(\zeta'\sqrt{\rho}e^{i\Theta},\rho)P(\zeta'\sqrt{\rho}e^{-i\Theta},\rho)}{P(\zeta'/\sqrt{\rho},\rho)P(\zeta'\sqrt{\rho},\rho)}\right\}^2 d\zeta'. \quad (29)$$

This mapping depends on three real parameters: ρ, Θ and C.

To fix a time scale for the vortex motion we set $\Gamma = 1$. We expect to be able to specify the size of the hollow vortices in the steadily translating configuration. A natural choice for controlling the vortex size is to specify ρ. For fixed Γ, we now calculate a one-parameter family of hollow vortex pairs parametrized by ρ.

For a given ρ, the angle Θ must be chosen so that the conformal mapping (29) is a single-valued function in the annulus $\rho < |\zeta| < 1$. This is necessary in order that the images of the circles $|\zeta| = \rho, 1$ are closed curves. Hence we must solve

$$\oint_{|\zeta|=1}\left\{\frac{P(\zeta'\sqrt{\rho}e^{i\Theta},\rho)P(\zeta'\sqrt{\rho}e^{-i\Theta},\rho)}{P(\zeta'/\sqrt{\rho},\rho)P(\zeta'\sqrt{\rho},\rho)}\right\}^2 d\zeta' = 0 \quad (30)$$

to find Θ. Eq. (30) is the only nonlinear equation to be solved because, once Θ is determined from (30), μ is given explicitly by (24), while the normalization constant C can be chosen such that the vortex centroids are at ± 1. A local expansion of (28) about the point $\zeta = \sqrt{\rho}$ gives

$$a = -\frac{\rho C P^2(\rho e^{i\Theta},\rho)P^2(\rho e^{-i\Theta},\rho)}{\hat{P}^2(1)P^2(\rho,\rho)}, \quad (31)$$

where

$$\hat{P}(\zeta,\rho) = \frac{P(\zeta,\rho)}{1-\zeta} = \prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \quad (32)$$

From (25), the propagation speed U is

$$U = \frac{\Gamma\sqrt{\rho}}{2\pi\mu a}. \quad (33)$$

The vortex area is found to be a monotonic increasing function of ρ, vindicating our choice of ρ as the parameter governing the solution family. Fig. 3 shows Θ as a function of ρ. For $\rho = 0$, we find $\Theta = \pi/3$.

Fig. 4 shows a range of hollow vortex equilibria for small ρ increasing up to 0.4. As $\rho \to 0$ the vortices have small area and are nearly circular in shape. As ρ increases, their area increases monotonically and the vortices progressively assume a kidney shape. As ρ approaches unity the area of the vortices appears to increase without bound and there is no limiting state with finite area. Fig. 5 shows the equilibria for $\rho = 0.5, 0.6, 0.7$ and 0.8: as the vortex area increases the vortex shapes each develop a well-defined flat edge separated by a thin layer of irrotational fluid. The thickness of this intermediate layer appears to reach a limiting value as $\rho \to 1$.

Fig. 6 shows that, as a function of vortex area, the speed of translation U/U_0 is monotonic decreasing. For vanishing vortex area U tends, as expected, to the value U_0 given in (1) and relevant for two point vortices with circulations ± 1 separated by distance $b = 2$. It is worth pointing out that an existence proof due to [17] and based on a variational approach shows that desingularizing a point vortex as a vortex patch with the same circulation, and the same centroid also generally decreases the propagation speed of the pair. The result here confirms that the same is true if the point vortex pair is desingularized using hollow vortices.

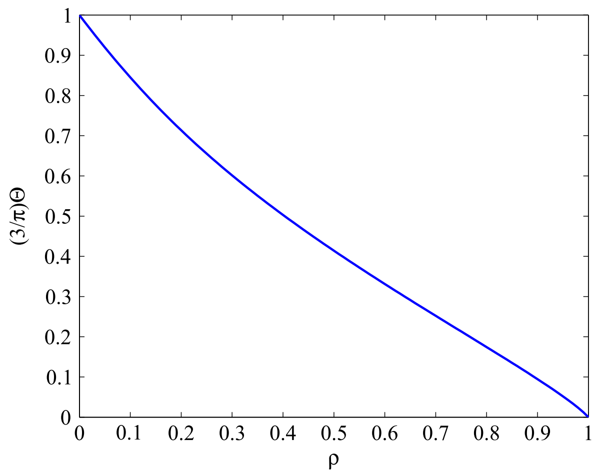

Fig. 3. Graph of Θ as a function of ρ obtained by solving (30).

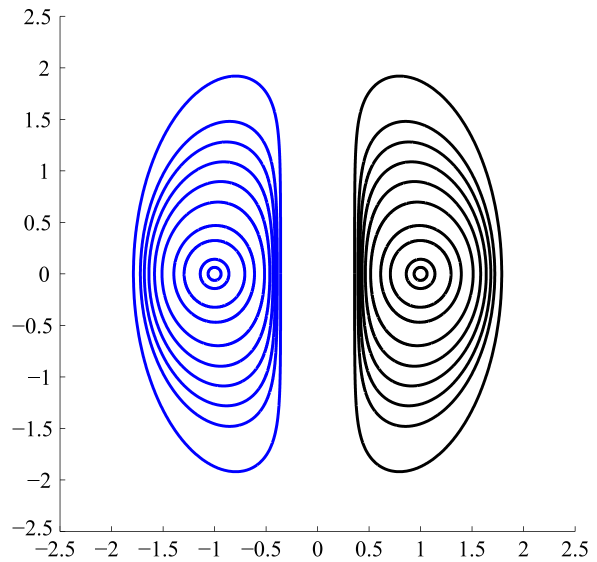

Fig. 4. Hollow vortex pairs of increasing area with vortex centroids fixed at $(\pm 1, 0)$. Different shapes correspond to different ρ with the corresponding values of vortex area and U listed in Table 1.

Table 1
Table of vortex area and speed U for various values of ρ.

ρ	Vortex area	Speed of translation U
0.001	0.012541	0.079577
0.005	0.062220	0.079566
0.025	0.300407	0.079307
0.05	0.579789	0.078598
0.1	1.103109	0.076337
0.15	1.608328	0.074717
0.2	2.118202	0.070361
0.25	2.648760	0.067176
0.3	3.213440	0.063990
0.4	4.499376	0.057681

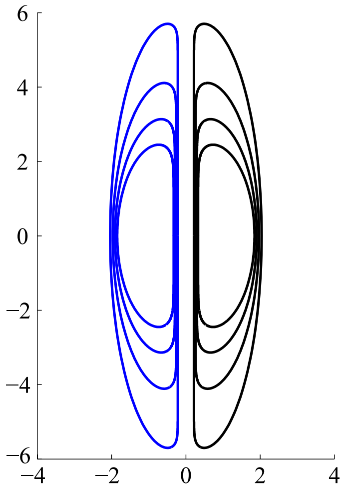

Fig. 5. Hollow vortex pairs for $\rho = 0.5, 0.6, 0.7$ and 0.8 with corresponding areas 6.485844, 8.808747, 12.471308 and 18.567377 and vortex centroids fixed at $(\pm 1, 0)$. As $\rho \to 1$ the area of the vortices grows without bound.

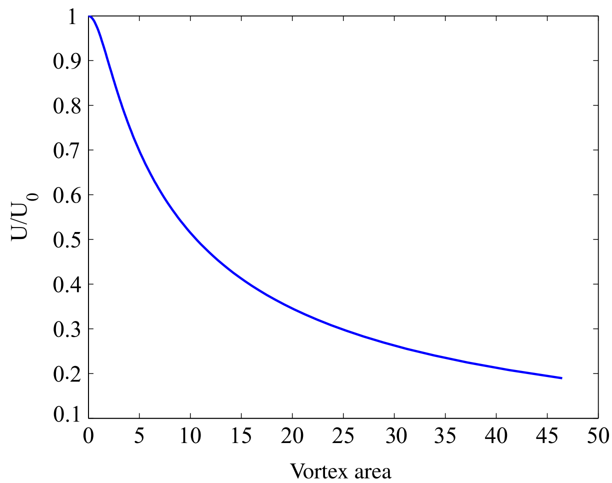

Fig. 6. Graph of U/U_0 against vortex area. The vortex pair travels more slowly as the area of each vortex increases.

7. Linear stability formulation

Baker et al. [6] found the equilibrium configuration for a singly periodic hollow vortex row and studied its linear stability. They introduced a novel formulation of the linear stability problem involving a change of independent variables from (x, y) to (ϕ, ψ) where ϕ and ψ are the velocity potential and the stream function of the steady state problem. The perturbed potential function is given as

$$\phi^T = \phi + \Phi(\phi, \psi, t), \tag{34}$$

with the perturbed vortex boundary given by

$$\psi = \psi_0 + \delta(\phi, t), \tag{35}$$

where δ and Φ are the linear perturbations. The equations for these perturbed quantities are found to be

$$\begin{aligned} &\frac{1}{q_0^2}\frac{\partial \delta}{\partial t} + \left.\frac{\partial \delta}{\partial \phi}\right|_{\psi=\psi_0} = \left.\frac{\partial \Phi}{\partial \psi}\right|_{\psi=\psi_0}, \\ &\frac{1}{q_0^2}\frac{\partial \Phi}{\partial t} + \frac{\partial}{\partial \psi}\left(\frac{1}{2}\frac{q^2}{q_0^2}\right)_{\psi=\psi_0}\delta + \frac{\partial \Phi}{\partial \phi} = 0. \end{aligned} \tag{36}$$

Llewellyn Smith & Crowdy [8] have recently used this formulation in their studies of the stability of other classes of hollow vortex equilibria. Here q is the magnitude of the fluid velocity and q_0 its value on the boundary.

To study the linear stability of the hollow vortex pair, we adopt a similar formulation. However, for our purposes, a second change of variable is convenient. Returning to the complex variable $\zeta = re^{i\theta}$, suppose the streamline $\psi = \psi_0$ corresponds to the circle $r = r_0$. On this boundary, ϕ is a function of θ and the normal derivative of ψ is a function of r, so we consider the perturbation equations written with θ and r as independent variables. An application of the chain rule to (36) gives

$$\begin{aligned} &\frac{1}{q_0^2}\frac{\partial \delta}{\partial t} + \frac{1}{\phi_\theta}\frac{\partial \delta}{\partial \theta} = \frac{1}{\psi_r}\frac{\partial \Phi}{\partial r}, \\ &\frac{1}{q_0^2}\frac{\partial \Phi}{\partial t} + \frac{1}{\phi_\theta}\frac{\partial \Phi}{\partial \theta} + \left(\frac{1}{\psi_r}\frac{\partial}{\partial r}\frac{1}{2}\frac{q^2}{q_0^2}\right)_{r=r_0}\delta = 0, \end{aligned} \tag{37}$$

where $\phi_\theta \equiv \partial\phi/\partial\theta$ and $\psi_r \equiv \partial\psi/\partial r$.

From the Cauchy–Riemann equations $\psi_r = -r^{-1}\phi_\theta$, so on writing all quantities in the form $\delta(t) = \hat{\delta}e^{\lambda t}$, for example, where $\hat{\delta}$ is a constant, the boundary conditions (37) transform to

$$\sigma\delta + Q\frac{\partial \delta}{\partial \theta} = -Qr\frac{\partial \Phi}{\partial r}, \qquad \sigma\Phi + Q\frac{\partial \Phi}{\partial \theta} = G\delta, \tag{38}$$

where all hats have been dropped, $\sigma = \lambda\Gamma/2\pi q_0$ is the non-dimensional growth rate, and

$$Q = \left.\frac{1}{\phi_\theta}\right|_{r=r_0}, \qquad G = Q\left[r\frac{\partial}{\partial r}\frac{1}{2}\frac{q^2}{q_0^2}\right]_{r=r_0}. \tag{39}$$

It is important to note that Eqs. (38) hold on any hollow vortex boundary corresponding to the image of circle $|\zeta| = r_0$.

It is helpful to rewrite the functions Q and G in terms of the analytic functions $W(\zeta)$ and $R(\zeta)$ introduced earlier in deriving the equilibrium states. After some straightforward manipulations it can be shown that

$$Q = \left.\frac{1}{\phi_\theta}\right|_{r=r_0} = \left.\frac{1}{\mathrm{Re}\,[i\zeta W'(\zeta)]}\right|_{r=r_0}. \tag{40}$$

If we write

$$S(\zeta) \equiv \frac{R(\zeta)}{q_0}, \qquad \frac{q^2}{q_0^2} = S(\zeta)\overline{S(\zeta)}, \tag{41}$$

then

$$\begin{aligned} r\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{q^2}{q_0^2}\right) &= \frac{1}{2}\left(\zeta\frac{\partial}{\partial \zeta} + \bar{\zeta}\frac{\partial}{\partial \bar{\zeta}}\right)S(\zeta)\overline{S(\zeta)} \\ &= \frac{1}{2}(\zeta S'(\zeta)\overline{S(\zeta)} + \overline{\zeta S'(\zeta)}S(\zeta)) \\ &= \mathrm{Re}\left[\frac{\zeta S'(\zeta)}{S(\zeta)}\right], \end{aligned} \tag{42}$$

where we have used the fact that $|S|^2 = 1$ on $r = r_0$. Hence

$$G = Q\,\mathrm{Re}\left[\frac{\zeta S'(\zeta)}{S(\zeta)}\right]_{r=r_0}. \tag{43}$$

For the hollow vortex pair, the boundary conditions (38) hold on both $r = 1$ and $r = \rho$. It follows that

$$\begin{aligned} &\sigma\delta + Q_1\frac{\partial \delta}{\partial \theta} = -Q_1 r\frac{\partial \Phi}{\partial r}, \qquad \sigma\Phi + Q_1\frac{\partial \Phi}{\partial \theta} = G_1\delta, \\ &\sigma\tilde{\delta} + Q_2\frac{\partial \tilde{\delta}}{\partial \theta} = -Q_2 r\frac{\partial \Phi}{\partial r}, \qquad \sigma\Phi + Q_2\frac{\partial \Phi}{\partial \theta} = G_2\tilde{\delta}, \end{aligned} \tag{44}$$

where

$$Q_1 = \frac{1}{\mathrm{Re}\,[\mathrm{i}\zeta W'(\zeta)]}\bigg|_{r=1}, \qquad G_1 = Q_1 \mathrm{Re}\left[\frac{\zeta S'(\zeta)}{S(\zeta)}\right]_{r=1},$$
$$Q_2 = \frac{1}{\mathrm{Re}\,[\mathrm{i}\zeta W'(\zeta)]}\bigg|_{r=\rho}, \qquad G_2 = Q_2 \mathrm{Re}\left[\frac{\zeta S'(\zeta)}{S(\zeta)}\right]_{r=\rho}. \tag{45}$$

From (11) and (14), we have

$$\mathrm{i}\zeta W'(\zeta) = \frac{\Gamma}{2\pi} - \frac{Ua}{\sqrt{\rho}}[L(\zeta/\sqrt{\rho},\rho) + L(\zeta\sqrt{\rho},\rho)]. \tag{46}$$

It is also known from (17) that

$$q_0 = q_1 = \frac{|A|}{\rho}, \tag{47}$$

so

$$S(\zeta) = \frac{R(\zeta)}{q_0} = \frac{A\rho}{|A|\zeta}\left[\frac{P(\zeta\mathrm{e}^{\mathrm{i}\Theta}/\sqrt{\rho},\rho)P(\zeta\mathrm{e}^{-\mathrm{i}\Theta}/\sqrt{\rho},\rho)}{P(\zeta\mathrm{e}^{\mathrm{i}\Theta}\sqrt{\rho},\rho)P(\zeta\mathrm{e}^{-\mathrm{i}\Theta}\sqrt{\rho},\rho)}\right]$$
$$= \frac{\mathrm{i}\rho}{\zeta}\left[\frac{P(\zeta\mathrm{e}^{\mathrm{i}\Theta}/\sqrt{\rho},\rho)P(\zeta\mathrm{e}^{-\mathrm{i}\Theta}/\sqrt{\rho},\rho)}{P(\zeta\mathrm{e}^{\mathrm{i}\Theta}\sqrt{\rho},\rho)P(\zeta\mathrm{e}^{-\mathrm{i}\Theta}\sqrt{\rho},\rho)}\right], \tag{48}$$

where we have used the fact that A is purely imaginary. It follows that

$$\frac{\zeta S'(\zeta)}{S(\zeta)} = -1 + K(\zeta\mathrm{e}^{\mathrm{i}\Theta}/\sqrt{\rho},\rho) + K(\zeta\mathrm{e}^{-\mathrm{i}\Theta}/\sqrt{\rho},\rho)$$
$$- K(\zeta\mathrm{e}^{\mathrm{i}\Theta}\sqrt{\rho},\rho) - K(\zeta\mathrm{e}^{-\mathrm{i}\Theta}\sqrt{\rho},\rho). \tag{49}$$

7.1. Numerical eigenvalue problem

The perturbation potential Φ is harmonic in the annulus $\rho < |\zeta| < 1$ so it can be written as

$$\Phi = \sum_{k=-\infty}^{\infty} \Phi_k \mathrm{e}^{\mathrm{i}k\theta} r^{|k|} + \sum_{k=-\infty,k\neq 0}^{\infty} \tilde{\Phi}_k \mathrm{e}^{\mathrm{i}k\theta}\left(\frac{\rho}{r}\right)^{|k|} \tag{50}$$

with the perturbations to the hollow vortex boundaries described by

$$\delta = \sum_{k=-\infty}^{\infty} \delta_k \mathrm{e}^{\mathrm{i}k\theta}, \qquad \tilde{\delta} = \sum_{k=-\infty}^{\infty} \tilde{\delta}_k \mathrm{e}^{\mathrm{i}k\theta}. \tag{51}$$

There are no logarithmic terms in Φ, as these would correspond to changes of circulation or sources/sinks in the system. On substitution of these expressions into the boundary conditions (44), we arrive at the matrix equations

$$-\sum_{m=-\infty}^{\infty} Q^{(1)}_{k-m}|m|[\Phi_m - \rho^{|m|}\tilde{\Phi}_m] - \mathrm{i}\sum_{m=-\infty}^{\infty} Q^{(1)}_{k-m} m\delta_m = \sigma\delta_k,$$
$$-\mathrm{i}\sum_{m=-\infty}^{\infty} Q^{(1)}_{k-m} m[\Phi_m + \rho^{|m|}\tilde{\Phi}_m] + \sum_{m=-\infty}^{\infty} G^{(1)}_{k-m}\delta_m$$
$$= \sigma[\Phi_k + \rho^{|k|}\tilde{\Phi}_k],$$
$$-\sum_{m=-\infty}^{\infty} Q^{(2)}_{k-m}|m|[\Phi_m\rho^{|m|} - \tilde{\Phi}_m] - \mathrm{i}\sum_{m=-\infty}^{\infty} Q^{(2)}_{k-m} m\tilde{\delta}_m = \sigma\tilde{\delta}_k,$$
$$-\mathrm{i}\sum_{m=-\infty}^{\infty} Q^{(2)}_{k-m} m[\Phi_m\rho^{|m|} + \tilde{\Phi}_m] + \sum_{m=-\infty}^{\infty} G^{(2)}_{k-m}\tilde{\delta}_m$$
$$= \sigma[\Phi_k\rho^{|k|} + \tilde{\Phi}_k]. \tag{52}$$

The truncated form of these equations yields a generalized eigenvalue problem

$$\mathbf{Ar} = \sigma\mathbf{Br}, \tag{53}$$

where

$$\mathbf{A} = \begin{pmatrix} -\mathcal{Q}^{(1)}|\mathcal{N}| & \mathcal{Q}^{(1)}|\mathcal{N}|\mathcal{R} & -\mathrm{i}\mathcal{Q}^{(1)}\mathcal{N} & 0 \\ -\mathrm{i}\mathcal{Q}^{(1)}\mathcal{N} & -\mathrm{i}\mathcal{Q}^{(1)}\mathcal{N}\mathcal{R} & \mathcal{G}^{(1)} & 0 \\ -\mathcal{Q}^{(2)}|\mathcal{N}|\mathcal{R} & \mathcal{Q}^{(2)}|\mathcal{N}| & 0 & -\mathrm{i}\mathcal{Q}^{(2)}\mathcal{N} \\ -\mathrm{i}\mathcal{Q}^{(2)}\mathcal{N}\mathcal{R} & -\mathrm{i}\mathcal{Q}^{(2)}\mathcal{N} & 0 & \mathcal{G}^{(2)} \end{pmatrix},$$
$$\mathbf{B} = \begin{pmatrix} 0 & 0 & \mathcal{I} & 0 \\ \mathcal{I} & \mathcal{R} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{I} \\ \mathcal{R} & \mathcal{I} & 0 & 0 \end{pmatrix}, \tag{54}$$

and

$$\mathbf{r} = [\Phi_{-N},\ldots,\Phi_N,\tilde{\Phi}_{-N},\ldots,\tilde{\Phi}_N,\delta_{-N},\ldots,\delta_N,$$
$$\tilde{\delta}_{-N},\ldots,\tilde{\delta}_N]^T. \tag{55}$$

The matrices in (54) have the elements

$$\mathcal{N}_{km} = k\delta_{km}, \qquad \mathcal{R}_{km} = \rho^{|k|}\delta_{km}, \tag{56}$$

$$\mathcal{Q}^{(1)}_{km} = \hat{Q}^{(1)}_{k-m}, \qquad \mathcal{Q}^{(2)}_{km} = \hat{Q}^{(2)}_{k-m},$$
$$\mathcal{G}^{(1)}_{km} = \hat{G}^{(1)}_{k-m}, \qquad \mathcal{G}^{(2)}_{km} = \hat{G}^{(2)}_{k-m} \tag{57}$$

where the function Q_1 has been expanded as a Fourier series according to

$$Q_1(\xi) = \sum_{j=-\infty}^{\infty} \hat{Q}^{(1)}_j \mathrm{e}^{\mathrm{i}j\xi}; \qquad \hat{Q}^{(1)}_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} Q_1(\xi)\mathrm{e}^{-\mathrm{i}j\xi}\,\mathrm{d}\xi, \tag{58}$$

with Q_2, G_1 and G_2 treated similarly. These quantities are readily computable using the Fast Fourier Transform.

The constant term Φ_0 can be set arbitrarily implying that there are $4N$ unknown coefficients $\{\Phi_m, \tilde{\Phi}_m\}$ and $2(2N+1) = 4N+2$ unknown coefficients $\{\delta_m, \tilde{\delta}_m\}$. This makes a total of $8N+2$ unknowns. On equating coefficients there are ostensibly $4(2N+1)$ equations, but the constant terms in the two pressure conditions can be ignored because one simply gives the linear perturbation to the Bernoulli constant while we can always set δ_0 to zero. This leaves $8N+2$ equations for the same number of unknowns.

A check on the numerical formulation is to verify that, as $\rho \to 0$, the linear stability spectrum tends to a union of two copies of the eigenfrequencies associated with an isolated circular hollow vortex. These nondimensional eigenfrequencies are found by solving for the perturbations of a circular hollow vortex:

$$\sigma^{\pm}_m = \mathrm{i}(m \pm |m|^{1/2}), \quad m \neq 0. \tag{59}$$

For sufficiently small ρ, the configuration is linearly stable. However, it is found that there is a loss of linear stability at the critical value $\rho = 0.022627$ when a duo of eigenvalues with imaginary part close to ± 2 develop non-zero real parts and therefore become unstable, leading to a quartet of eigenvalues. The area of each vortex at this critical value of ρ is 0.272931.

8. Discussion

A concise new representation of the counter-rotating hollow vortex pair originally derived by Pocklington has been presented and a full linear stability analysis of the equilibria has been carried out. The vortex pair is found to be linearly stable to infinitesimal perturbations provided the vortices are sufficiently small. The instability has the form typical of Hamiltonian systems in which eigenvalues on the imaginary axis split into two unstable modes with equal and opposite real parts.

As already mentioned, the linear stability properties of the uniform vortex patch solutions computed by Deem & Zabusky [3] and Pierrehumbert [4] have not yet been properly analysed in the literature, but are generally viewed as being linearly stable [1]. Although the vortex models are clearly very different, our results on the hollow vortex pair suggest the possibility that the vortex patch pair might similarly be linearly stable only when the area of the vortices is sufficiently small. This matter deserves fuller investigation.

Finally, by means of an analysis similar to that presented here, Crowdy & Green [9] have identified analytical solutions for steady von Kármán streets of hollow vortices. In the case where the two infinite rows of vortices making up the street are aligned vertically – the so-called "symmetric street" – the solutions amount to a singly-periodic array of hollow vortex pairs of precisely the kind studied here. The linear stability analysis of these new hollow vortex street solutions remains to be performed, but it should be amenable to study via an analysis similar to that performed here, combined with the ideas from Floquet theory recently used by Llewellyn Smith & Crowdy [8] to study the linear stability of the single vortex row found by Baker et al. [6].

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