Classification of simple algebras in the Deligne category $\mathrm{Rep}(S_t)$

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Abstract

We classify simple associative and Lie algebras inside the Deligne categories $\mathrm{Rep}(S_t)$, answering a question posed by Etingof.

Contents

Introduction

Deligne defined a family of categories $\mathrm{Rep}(S_t)$, where $t \in F$ is an arbitrary element of a ground field F of characteristic 0 [Del07]. These categories interpolate, in an appropriate sense, the categories of representations of symmetric groups as rigid symmetric monoidal F-linear categories.

In [Eti14] and [Eti16] Etingof laid out a program of "Representation Theory in Complex Rank" which hopes to classify certain algebraic structures internal to these Deligne categories (and their relatives for other families of groups) over the complex numbers and to interpolate more complicated objects in representation theory.

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One such problem proposed by Etingof was to characterize the simple algebras inside $\operatorname{Rep}(S_t)$ for all $t \in \mathbb{C}$, the conjectural characterization being that all such algebras should be interpolations of families of compatible simple S_n-algebras (that is, $\mathbb{C}$-algebras with S_n-equivariant multiplication, and no non-trivial S_n-invariant ideals). Where by algebras we could mean associative, commutative, or Lie.

The case of simple commutative algebras was studied by Sciarappa in [Sci15]. He characterized which families of simple commutative S_n-algebras have interpolations to the Deligne categories $\operatorname{Rep}(S_t)$, and showed that if t is transcendental, these form a complete list of the isomorphism classes of simple commutative algebras in $\operatorname{Rep}(S_t)$. However, his methods are unable to rule out the possibility of certain "exotic" simple commutative algebras existing at special algebraic values of t.

In [Har16] the first author gave a new interpretation for the Deligne categories $\operatorname{Rep}(S_t)$ at algebraic values of t in terms of model theory and the modular representation theory of symmetric groups, and using this sketched a proof that indeed Sciarappa's classification of simple commutative algebras holds at algebraic values of t too.

In this paper we complete this program by classifying the simple associative and Lie algebras inside $\operatorname{Rep}(S_t)$ for all values of $t \notin \mathbb{Z}_{\geq 0}$. The paper is structured as follows.

The first two sections are somewhat auxiliary. In the first section we review the definition and properties of the Deligne category $\operatorname{Rep}(S_t)$, with a focus on the model theoretic ultrafilter interpretation introduced in [Har16]. The second section contains some technical lemmas about subgroups of symmetric groups, S_n-algebras, and ultraproducts thereof.

In section 3 we give the classification of associative algebras in $\operatorname{Rep}(S_t)$. Theorem 3.1.2 says these are classified by a choice of a subgroup H of some symmetric group S_j, a projective representation of H, and an object of $\operatorname{Rep}(S_{t-j})$. In particular, these exactly correspond to families of simple associative S_N-algebras over $\mathbb{C}$ which have an interpolation in $\operatorname{Rep}(S_t)$. As in the commutative case, there are no "exotic" simple algebras appearing for special values of t.

In section 4 we focus on the case of Lie algebras, and here the situation is somewhat more complicated. Given an object V of dimension 0 in a rigid symmetric tensor category, one can construct a Lie algebra object $\mathfrak{psl}(V)$ of dimension -2 as a subquotient of $V \otimes V^*$. By combining this construction with induction from subgroups we see that there are many examples of simple Lie algebra objects in $\operatorname{Rep}(S_t)$ for t algebraic which do not come from interpolating simple S_n-Lie algebras over $\mathbb{C}$. However using the ultraproduct interpretation we can view these as interpolating S_n-Lie algebras in large positive characteristic. Theorem 4.2.5 says that this is all that can happen, and every simple Lie algebra object in $\operatorname{Rep}(S_t)$ indeed comes from either interpolating simple S_n-Lie algebras over $\mathbb{C}$, or by interpolating these algebras built from algebras of the form $\mathfrak{psl}(V)$ in positive characteristic.

In section 5 we provide a conjectural treatment of the same statement in the case of Lie superalgebras.

Acknowledgement

1 The Deligne category $\mathbf{Rep}(S_t)$.

1.1 Preliminary notations and definitions.

First we need to establish some notations which we will frequently use.

Definition 1.1.1. For a group H and a field k, denote by $\mathbf{Rep}(H; k)$ the category of finite-dimensional representations of H over k. By $\mathbf{Rep}(S_n)$ denote the category of finite-dimensional representations of S_n[1] over $\overline{\mathbb{Q}}$ i.e. $\mathbf{Rep}(S_n) = \mathbf{Rep}(S_n; \overline{\mathbb{Q}})$. By $\mathbf{Rep}_p(S_n)$ denote the category of finite-dimensional representations of S_n over $\overline{\mathbb{F}}_p$ i.e. $\mathbf{Rep}_p(S_n) = \mathbf{Rep}(S_n; \overline{\mathbb{F}}_p)$.

Note that for $p > n$ the latter category is semi-simple and the irreducible objects are the same as in characteristic 0. More precisely, irreducible representations over $\overline{\mathbb{F}}_p$ can be obtained as a reduction modulo p of irreducible $\overline{\mathbb{Q}}$-representations which sit inside irreducible $\mathbb{C}$-representations as a $\overline{\mathbb{Q}}$-lattice of the full rank. Below we will always work in the positive characteristic with $p > n$.

Definition 1.1.2. By $\mathcal{A}_n$ denote the subgroup of S_n consisting of even permutations.

Definition 1.1.3. For a Young diagram λ, by $l(\lambda)$ denote the number of rows of the diagram and by $|\lambda|$ the number of boxes.

The irreducible objects of both of these categories are in 1-1 correspondence with Young diagrams of weight n. So let us make the following definitions.

Definition 1.1.4. By $X(\lambda)$ denote the irreducible representation of $S_{|\lambda|}$ corresponding to the Young diagram λ. The field over which this representation is defined will be evident from the context.

Definition 1.1.5. Let X_k denote a vector space F^k with an action of S_k given by interchanging the basis vectors. Here also the field F will be evident from the context.

Below we will frequently use the following operation on Young diagrams:

Definition 1.1.6. For a Young diagram λ and an integer $n \geq \lambda_1 + |\lambda|$ denote by $\lambda|_n$ the Young diagram $(n - |\lambda|, \lambda_1, \ldots, \lambda_{l(\lambda)})$.

We will also need the following definition in the next subsection:

Definition 1.1.7. Denote by $FP_{n,m}$ a vector space over a field F with a basis given by all possible partitions of an $n + m$-element set. Graphically an element of the basis is represented by two rows of $\bullet$'s, the first of length n and the second of length m, where all $\bullet$'s belonging to the same part of the partition are connected by edges. So, in other words, it is a graph on $n + m$ vertices, the set of connected components of which corresponds to a partition of $n + m$ (This doesn't depend on the choice of the graph with the fixed set of the connected components).

Also define a map $\phi_t^{n,m,k} : FP_{m,k} \times FP_{n,m} \to FP_{n,k}$ for $t \in F$ as follows. Consider $\lambda \in FP_{n,m}$ and $\mu \in FP_{m,k}$. Take a vertical concatenation of the graphical representations of the corresponding partitions (the last one on top) and identify the rows of length m. After this we are left with a partition of three rows of $\bullet$'s of length n, m and k. Now let's denote by $l(\mu, \lambda)$ the number of connected components consisting purely of $\bullet$'s lying in the second row. Also consider a partition of rows n, k consisting of the same connected components as the partition of rows n, m, k but with elements of the second row deleted, denote it by $\mu \cdot \lambda$. Then $\phi_t^{n,m,k}(\mu, \lambda) = t^{l(\mu,\lambda)}\mu \cdot \lambda$.

Define $FP_n(t)$ to be $FP_{n,n}$ with a structure of an algebra given by the map $\phi_t^{n,n,n}$. This algebra is called the partition algebra and it was introduced by Purdon in [Pur91].

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[1]The symmetric group of rank n.

We need to introduce a few facts about symmetric monoidal categories:

Definition 1.1.8. For the object X of the symmetric monoidal cateogory, we will denote by σ_X the map from $X \otimes X$ to itself, given by the symmetric structure, i.e. a map permuting two copies of X, oftentimes, when the object we are referring to is obvious from the context, we will denote it by σ.

Also we will define the notion of a commutative/associative/Lie algebra in a symmetric monoidal category (we will ignore associativity and unity morphisms of the monoidal structure using MacLane strictness theorem and its consequences):

Definition 1.1.9. a) An associative algebra in a symmetric monidal category $\mathcal{C}$ is an object $X \in \mathcal{C}$ together with maps $m : X \otimes X \to X$, $e : 1_{\mathcal{C}} \to X$, such that the following equalities between morphisms hold $m \circ Id_X \otimes m = m \circ m \otimes Id_X : X \otimes X \otimes X \to X$ and $id_X = m \circ e \otimes Id_X = m \circ Id_X \otimes e : X \to X$.
b) A commutative algebra in a symmetric monidal category $\mathcal{C}$ is an associative algebra X such that $m \circ \sigma = m$.
c) A Lie algebra in a k-linear symmetric monidal category $\mathcal{C}$ is an object $X \in \mathcal{C}$ together with a map $[\ ,] : X \otimes X \to X$, such that $[\ ,] \circ \sigma = -[\ ,]$ and the Jacobi identity $[\ ,] \circ Id_X \otimes [\ ,] \circ (Id_X + (\sigma \otimes Id_X \circ Id_X \otimes \sigma) + (Id_X \otimes \sigma \circ \sigma \otimes Id_X)) = 0 : X \otimes X \otimes X \to X$ holds.

1.2 Definition and properties of $\mathrm{Rep}(S_t)$.

From now on let t by any element of $\mathbb{C}$.

Here we will briefly discuss the definition of $\mathrm{Rep}(S_t)$ and state some important properties it enjoys. For more about this see [Del07], [CO11], [Eti14].[2]

Note that our notion of a tensor category will differ slightly from [EGNO16]. We do not require our categories to be abelian, nevertheless all categories we are going to actually work with will be abelian.

First we need to define a preliminary skeletal category $\mathrm{Rep}_0(S_t)$:

Definition 1.2.1. $\mathrm{Rep}_0(S_t)$ is a skeletal tensor category. Its objects are elements of $\mathbb{Z}_{\geq 0}$, which can be graphically represented by rows of $\bullet$'s, and denoted by $[n]$.

The set of morphisms $\mathrm{Hom}_{\mathrm{Rep}_0(S_t)}([n],[m])$ is equal to $\mathbb{C}P_{n,m}$ and the composition maps are given by $\phi_t^{n,m,k}$.

Tensor product on objects is defined by the horizontal concatenation of rows and on morphisms by the horizontal concatenation of diagrams. All objects $[n]$ are self-dual.

Now we can define the Deligne category $\mathrm{Rep}(S_t)$ itself:

Definition 1.2.2. The Deligne category $\mathrm{Rep}(S_t)$ is the Karoubian envelope of the additive envelope of $\mathrm{Rep}_0(S_t)$. It is a rigid symmetric tensor category.

Definition 1.2.3. The object $[1]$ is called the fundamental representation and is denoted by $\mathcal{X}$. The object $[0]$ is called the trivial representation and is denoted by $\mathbb{C}$ (by a slight abuse of notation).

The important properties of $\mathrm{Rep}(S_t)$ are listed below:

Proposition 1.2.4. a) *For $t \notin \mathbb{Z}_{\geq 0}$ $Rep(S_t)$ is a semisimple category.*
b) *For $t \notin \mathbb{Z}_{\geq 0}$ simple objects of $Rep(S_t)$ are in 1-1 correspondence with Young diagrams of arbitrary size. They are denoted by $\mathcal{X}(\lambda)$. Moreover $\mathcal{X}(\lambda)$ is a direct summand in $[|\lambda|]$.*
c) *The categorical dimension of $\mathcal{X}$ is t and of $\mathbb{C}$ is 1.*
d) *All $\mathcal{X}(\lambda)$ are self-dual.*

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[2]For the general theory of tensor categories see [EGNO16].

Also we have an important universal property of the Deligne category:

Proposition 1.2.5. *(8.3 in [Del07]) For any $\mathbb{C}$-linear symmetric tensor category $\mathcal{T}$, the category of $\mathbb{C}$-linear symmetric tensor functors from $Rep(S_t)$ to $\mathcal{T}$ is equivalent to the category $\mathcal{T}_t^f$ of commutative Frobenius algebras in $\mathcal{T}$ of dimension t. The equivalence is obtained using the functor which sends a functor F to the object $F(\mathcal{X})$.*

Here by a commutative Frobenius algebra in a symmetric monoidal category we mean an object T with the following structure. First it is an associative commutative algebra with the corresponding algebraic structure given by $\mu_T, 1_T$. Second this object is rigid (dual objects exist). And finally if we define a map:

$$\text{Tr}: T \xrightarrow{1 \otimes coev_T} T \otimes T \otimes T^* \xrightarrow{\mu_T \otimes 1} T \otimes T^* \xrightarrow{ev_T} 1 \ ,$$

then the pairing $T \otimes T \xrightarrow{\mu_T} T \xrightarrow{\text{Tr}} 1$ is required to be non-degenerate, i.e. it corresponds to an isomorphism between T and T^* under the identification $\text{Hom}(T \otimes T, 1) = \text{Hom}(T, T^*)$.

1.3 Ultrafilters and ultraproducts.

It will be important for us that we can think of $\text{Rep}(S_t)$ for $t \notin \mathbb{Z}_{\geq 0}$ as a limit of categories of representations of S_n over $\overline{\mathbb{F}}_p$, when $n, p \mapsto \infty$. To formalize this statement we will need to introduce ultrafilters and ultraproducts below.

We will quickly define what ultrafilters and ultraproducts are, state their main properties and give some examples. The following discussion is taken from [Kal18]. For more details see [Sch10].

Definition 1.3.1. An ultrafilter $\mathcal{F}$ on a set X is a subset of 2^X satisfying the following properties:
- $X \in \mathcal{F}$;
- If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$;
- If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- For any $A \subset X$ either A or $X \backslash A$ belongs to A, but not both.

There is an obvious family of examples of ultrafilters: $\mathcal{F}_x = \{A \in 2^X | x \in A\}$ for $x \in X$. Such ultrafilters are called principal. Using Zorn's lemma one can show that non-principal ultrafilters $\mathcal{F}$ exist iff the cardinality of X is infinite. Also it follows that all cofinite sets belong to such an $\mathcal{F}$ (but not all sets belonging to $\mathcal{F}$ are cofinite). From now on we will denote by $\mathcal{F}$ a fixed non-principal ultrafilter on $\mathbb{N}$. Also by something being true for "almost all n", we will mean that it is true for all n in some $A \in \mathcal{F}$.

Remark 1. Note that by definition of an ultrafilter, if two statements hold for almost all n, then their conjunction holds for almost all n. Also note that if for almost all n the disjunction of a finite number of statements holds, then one of them holds for almost all n (if not then each of them holds on some subset $A \notin \mathcal{F}$ and the union of this subsets is not in $\mathcal{F}$). We will use these elementary observations quite frequently.

Let's now define a notion of an ultraproduct.

Definition 1.3.2. Suppose we have a collection of sets S_i labeled by natural numbers. Suppose that for almost all $x \in X$ one has $S_x \neq \emptyset$. Then $\prod_{\mathcal{F}} S_x$ is the quotient of $\prod_{x \in X} S_x$ by the following relation: $\{s_x\} \sim \{s'_x\}$ iff $s_x = s'_x$ for almost all x. If for almost all x one has $S_x = \emptyset$, then $\prod_{\mathcal{F}} S_x = \emptyset$. The set $\prod_{\mathcal{F}} S_x$ is called the ultraproduct of S_x.

Usually we will denote $\{s_x\} \in \prod_{\mathcal{F}} S_i$ by $\prod_{\mathcal{F}} s_x$.

First, let's note that an ultraproduct inherits any operation or any relation, which is defined for almost all n. Indeed to apply an operation to the elements of the ultraproduct

we can just apply the correspoding operations to the corresponding sequences of elements, and get a sequence of elements. In case of the relation doing the same thing we get a sequence of Boolean values, which are going to be the same for almost all n, since there are finite number of values.

The most important property of ultraproducts is the following:

Theorem 1.3.3. *Łoś's theorem* *(Theorem 2.3.2 in [Sch10])*

Suppose we have a collection of sequences of sets $S_i^{(k)}$ for $k = 1, \ldots, m$ and a collection of sequences of elements $f_i^{(r)}$ for $r = 1, \ldots, l$ and a formula of a first order language $\phi(x_1, \ldots, x_l, Y_1, \ldots, Y_m)$ depending on some parameters x_i and sets Y_j. Denote by $S^{(k)} = \prod_{\mathcal{F}} S_n^{(k)}$ and $f^{(r)} = \prod_{\mathcal{F}} f_n^{(r)}$. Then $\phi(f_n^{(1)}, \ldots, f_n^{(l)}, S_n^{(1)}, \ldots S_n^{(m)})$ is true for almost all n iff $\phi(f^{(1)}, \ldots, f^{(l)}, S^{(1)}, \ldots S^{(m)})$ is true.

In plain language this means that if we have a sequence of collections of sets with some algebraic structure given by maps between them, then, first, we have the corresponding maps between the ultraproducts of these sets. And, second, these maps satisfy a given set of axioms or properties for the ultraproducts iff they satisfy these axioms/properties for almost all n. Also frequently it is useful to think about an ultraproduct as a certain kind of a limit as $n \mapsto \infty$.

We give a number of examples of such constructions, which are going to be useful to us below:

Example 1.3.4. If S_i is a sequence of monoids/groups/rings/fields then $\prod_{\mathcal{F}} S_i$ with operations given by taking the ultraproduct of the operations in the corresponding sets of Hom_{Sets} gives us a structure of a monoid/group/ring/field by Łoś's theorem.

Example 1.3.5. If V_i are finite-dimensional vector spaces over a field k, then $\prod_{\mathcal{F}} V_i$ is not necessarily a finite-dimensional vector space, since the property of being finite-dimensional cannot be written in a first-order language. But if the dimensions of V_i are bounded, then they are the same for almost all i and hence V has the same dimension (for example, because the ultraproduct of bases is a basis).

Example 1.3.6. Take the ultraproduct of a countably infinite number of copies of $\overline{\mathbb{Q}}$. By Łoś's theorem $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$ is a field, which is algebraically closed. It has characteristic zero since $\forall k \in \mathbb{Z}$ such that $k \neq 0$ it follows that $k = \prod_{\mathcal{F}} k \neq 0$. Also it is easy to see that its cardinality is continuum. Hence by Steinitz's theorem[3] $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$. Note that there is no canonical isomorphism.

Example 1.3.7. Take the ultraproduct of $\overline{\mathbb{F}}_{p_n}$ for some sequence of distinct prime numbers p_n. As before, by Łoś's theorem $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$ is a field, which is algebraically closed. Also as before it has cardinality continuum. Now $k = \prod_{\mathcal{F}} k \neq 0$, since it is equal to zero for at most a finite number of k. Hence $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$, again not in a canonical way.

Example 1.3.8. Suppose $\mathcal{C}_i$ is a collection of small categories. We can define an ultraproduct category $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \mathcal{C}_i$ as a category with objects $Ob(\widehat{\mathcal{C}}) = \prod_{\mathcal{F}} Ob(\mathcal{C}_i)$ and $\text{Hom}_{\widehat{\mathcal{C}}}(\prod_{\mathcal{F}} X_i, \prod_{\mathcal{F}} Y_i) = \prod_{\mathcal{F}} \text{Hom}_{\mathcal{C}_i}(X_i, Y_i)$; composition maps are given by the ultraproducts of the composition maps, i.e. $(\prod_{\mathcal{F}} f_i) \circ (\prod_{\mathcal{F}} g_i) = \prod_{\mathcal{F}}(f_i \circ g_i)$. By Łoś's theorem this data satisfies the axioms of a category. If the categories $\mathcal{C}_i$ have some structures, for example the structures of an abelian/monoidal/tensor category, then $\widehat{\mathcal{C}}$ also has these structures[4].

Usually $\widehat{\mathcal{C}}$ is too big and it is interesting to consider a certain full subcategorie $\mathcal{C}$ in there, or, equivalently, consider ultraproducts only of some sequences of objects of $\mathcal{C}_i$, for example bounded in some sense.

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[3]This theorem tells us that two uncountable algebraically closed fields are isomorphic iff their characteristic and cardinality are the same. It is proven in [Ste10].

[4]But the finite-length property, for example, does not survive.

This construction obviously extends to essentially small categories up to an equivalence of such a category with respect to all relevant structures. All categories which we will consider are essentially small, and for all the questions we are going to discuss the fact that ultaproducts are defined up to an equivalence does not matter, so we won't bother mentioning this later.

1.4 $\mathbf{Rep}(S_t)$ as an ultraproduct.

Here, we will show how to construct $\mathrm{Rep}(S_t)$ using ultraproducts, and discuss some important consequences of this construction. See [Del07],[Har16][5].

We want to apply the last example of the previous section to $\mathcal{C}_i = \mathbf{Rep}(S_{n_i}, \mathbb{K}_i)$ – the tensor category of finite-dimensional representations of S_{n_i} over $\mathbb{K}_i$. As was stated before (Definition 1.1.1) we will denote by $\mathbf{Rep}(S_n) = \mathbf{Rep}(S_n, \overline{\mathbb{Q}})$ and by $\mathbf{Rep}_p(S_n) = \mathbf{Rep}(S_n, \overline{\mathbb{F}}_p)$. We have the following result (Introduction of [Del07] or Theorem 1.1 in [Har16]):

Theorem 1.4.1. *a) Suppose $t \in \mathbb{C}$ is transcendental. Consider $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \mathbf{Rep}(S_n)$. Denote by $X_i = \overline{\mathbb{Q}}^i$ – the fundamental representation of S_i and $X_t = \prod_{\mathcal{F}} X_i$. Fix an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$ such that $\prod_{\mathcal{F}} i = t$. Then the full subcategory of the $\prod_{\mathcal{F}} \overline{\mathbb{Q}}$-linear category $\widehat{\mathcal{C}}$ generated by X_t under taking tensor products, direct sums and direct summands is equivalent to the $\mathbb{C}$–linear category $Rep(S_t)$, in a way consistent with the above isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{Q}} \simeq \mathbb{C}$.*

b) Suppose $t \in \mathbb{C}$ is algebraic but not integer, with a minimal polynomial $q(x) \in \mathbb{Z}[x]$. Fix a sequence of distinct primes p_n and sequence of integers t_n tending to infinity such that $q(t_n) = 0$ in $\mathbb{F}_{p_n}$. Moreover fix an isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$ such that $\prod_{\mathcal{F}} t_i = t$. Set $\widehat{\mathcal{C}} = \prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(S_{t_n})$. Denote by $X_{t_i} = \overline{\mathbb{F}}_{p_i}^{t_i}$ the fundamental representation of S_{t_i} and set $X_t = \prod_{\mathcal{F}} X_{t_i}$. Then the full subcategory of the $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$-linear category $\widehat{\mathcal{C}}$ generated by X_t under taking tensor products, direct sums and direct summands is equivalent to the $\mathbb{C}$-linear category $Rep(S_t)$, in a way consistent with the above isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} \simeq \mathbb{C}$.

Proof. **a)** First let us prove that it is indeed possible to fix such an isomorphism. The ultraproduct $\prod_{\mathcal{F}} i$ is an element of $\mathbb{C}$. Suppose it is algebraic over $\mathbb{Q}$, then it should satisfy a monic equation f with coefficients in $\mathbb{Q}$. Then by Łoś's theorem for almost all i we have $f(i) = 0$, but since this is true for an infinite number of distinct i's, it follows that $f = 0$. Hence by contradiction we conclude that $\prod_{\mathcal{F}} i$ is transcendental. Now by fixing an automorphism of $\mathbb{C}$ over $\mathbb{Q}$ we may send this transcendental number to t.

So we have a tensor category $\widehat{\mathcal{C}}$ linear over $\mathbb{C}$, with an object $\prod_{\mathcal{F}} X_i$ of dimension t. Since every X_i is a commutative Frobenius algebra, it follows by Łoś's theorem that X_t is also a commutative Frobenius algebra. Hence by Proposition 1.2.5 we obtain a monoidal symmetric functor $F : \mathrm{Rep}(S_t) \to \widehat{\mathcal{C}}$. Since $\mathrm{Rep}(S_t)$ is generated by $\mathcal{X}$ under taking tensor products, direct sums and direct summands, it follows that the image of $\mathrm{Rep}(S_t)$ under F is the full subcategory $\mathcal{C}$ in $\widehat{\mathcal{C}}$ generated by X_t under taking tensor products, direct sums and direct summands. So we know that $F : \mathrm{Rep}(S_t) \to \mathcal{C}$ is essentially surjective. Now it is enough to prove that it is fully faithful.

Note that it is enough to prove that

$$\prod_{\mathcal{F}} \mathrm{Hom}_{S_n}(X_n^{\otimes r}, X_n^{\otimes s}) = \mathrm{Hom}_{\mathrm{Rep}(S_t)}([r], [s]) \ ,$$

and that the composition maps are the same. Indeed both categories can be obtained as the Karoubian envelope of the additive envelope of the categories consisting of all $[s]$ or $X_t^{\otimes r}$ respectively.

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[5]For the similar discussion about $\mathrm{Rep}(GL_t)$ see [Del07], [Har16], [Kal18].

But this follows from Theorem 2.6 in [CO11]. Indeed there it is stated that there is an isomorphism between $\overline{\mathbb{Q}}P_{r,s}$ and $\mathrm{Hom}_{S_n}(X_n^{\otimes r}, X_n^{\otimes s})$ for $n > r + s$. So for almost all n we have $\mathrm{Hom}_{S_n}(X_n^{\otimes r}, X_n^{\otimes s}) = \overline{\mathbb{Q}}P_{n,m}$. Also Proposition 2.8 in the same article states that under this isomorphism the composition rule on $\mathrm{Hom}_{S_n}(X_n^{\otimes r}, X_n^{\otimes s})$ transforms into the composition rule on $\overline{\mathbb{Q}}P_{n,m}$ in the definition of $\mathrm{Rep}_0(S_t)$. So it follows that indeed $\prod_{\mathcal{F}} \mathrm{Hom}_{S_n}(X_n^{\otimes r}, X_n^{\otimes s}) = \mathrm{Hom}_{\mathrm{Rep}(S_t)}([r], [s])$, and the composition rule is the same.

b) First, again, we need to explain how we can fix such an isomorphism. Let us prove that there is indeed an infinite number of pairs t_n and p_n such that $q(t_n) = 0 \mod p_n$. It is enough to show that there are infinite number of primes dividing the numbers $q(n)$ (if in this case the sequence t_n is bounded, it follows that some $q(t_n)$ is divisible by an infinite number of prime numbers, which is absurd). Suppose it is not so, and there are only k such primes. Fix C such that we have $q(n) < C \cdot n^{\deg(q)}$ for all positive integer n. Denote by Q the number of integers of the form $q(n)$ for $n \in \mathbb{Z}_{\geq 0}$ such that $q(n) < N$. By the above inequality this number is at least $\frac{1}{C} \cdot N^{\frac{1}{\deg(q)}}$. On the other hand the number P of numbers less than N divisible only by k fixed primes is less or equal to $\log_2(N)^k$, since each prime number is at least 2. Hence for big enough N we have $P < Q$, which contradicts the hypothesis[6].

So we indeed can choose such unbounded sequences t_n and p_n. Now by Łoś's theorem it follows that $\prod_{\mathcal{F}} t_n$ is a root of q in $\mathbb{C}$, so by composing with an automorphism of $\mathbb{C}$ we may assume that under an isomorphism $\mathbb{C} \simeq \prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n}$, $\prod_{\mathcal{F}} t_n$ maps to t.

The rest of the proof is the same since the representation theory of S_n is the same in zero characteristic and in characteristic $p > n$, and $p_n > t_n$ for almost all n. $\qquad\square$

Remark 2. Generally proving something in the case of algebraic t is harder than in the case of the transcendental t. Thus we will for the most time think about the transcendental case as a subcase of the algebraic case using the following formalism. By $\overline{\mathbb{F}}_0$ we will mean $\overline{\mathbb{Q}}$, and so the case $t_n = n$, $p_n = 0$ gives us transcendental t.

Also we will always assume that the sequences p_n and t_n are the sequences from Theorem 1.4.1b) corresponding to the given t.

Remark 3. Note that for any finite group G we can characterize $\mathrm{Rep}(S_t) \boxtimes \mathbf{Rep}(G; \mathbb{C})$[7] as a full subcategory in $\prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(S_{t_n}) \boxtimes \mathbf{Rep}_{p_n}(G)$, which consists of sequences of objects $\prod_{\mathcal{F}} V_n = \prod_{\mathcal{F}} \sum_k U_{n,k} \otimes Y_k$, where Y_k runs over all the irreducible objects of G (with $p_n > |G|$), such that each $\prod_{\mathcal{F}} U_{n,k}$ is an object of $\mathrm{Rep}(S_t)$. Indeed on the level of abelian categories it follows from the fact that $\mathbf{Rep}_{p_n}(G)$ splits into a finite sum of categories of vector spaces, and it's easy to check that the tensor structure agrees (it's the same for almost all n).

Now we can understand how to obtain the objects $\mathcal{X}(\lambda)$ as ultraproducts. Indeed $FP_n(t)$ for any $t \neq 0, \ldots, 2n$ is a semisimple algebra with the same collection of idempotents given by the specialization of idempotents from the same algebra, but there we treat t as a formal variable[8]. So from Proposition 3.25 in [CO11] it follows that the idempotent corresponding to $\mathcal{X}(\lambda)$ in $FP_n(t)$ is the same one which corresponds to the irreducible representation of S_k with $k > 2n$ given by the Young diagram $(k - |\lambda|, \lambda_1, \ldots, \lambda_{l(\lambda)})$.

So using the notation introduced in Definition 1.1.2 the following Corollary holds.

Corollary 1.4.2. *For all $t \notin \mathbb{Z}_{\geq 0}$, the irreducible object $\mathcal{X}(\lambda)$ of $\mathrm{Rep}(S_t)$ can be obtained as an ultraproduct of irreducible objects of $\mathbf{Rep}_{p_n}(S_{t_n})$ as $\mathcal{X}(\lambda) = \prod_{\mathcal{F}} X_{t_n}(\lambda|_{t_n})$.*

Remark 4. Note that this means that if V as an object of $\mathrm{Rep}(S_t)$ equals to the ultraproduct $V = \prod_{\mathcal{F}} V_n$, then almost all V_n cannot contain the sign representation. Indeed, since

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[6]This proof is also written by the first author in his paper, see the proof of Prop. 2.2 in [Har16].

[7]In this paper we use $\boxtimes$ to denote a Deligne tensor product of locally finite abelian categories, for the definition see 1.11 in [EGNO16].

[8]See discussion in chapter 3.3 in [CO11].

$V = \oplus \mathcal{X}(\lambda_i)$, we have for almost all n, $V_n = \oplus X(\lambda_i|_n)$, hence the height of the Young diagrams appearing in V_n is bounded for almost all n. But if V_n would contain the sign representation for almost all n it would mean that each V_n would contain the diagram of height t_n which contradicts the boundness of heights of the Young diagrams.

We also need to describe the generalizations of the induction and the restriction functors. First let's define the latter using the universal property of $\text{Rep}(S_t)$.

Definition 1.4.3. Consider the category $\text{Rep}(S_{t-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$ for an integer k, and in it the object $\mathcal{X} \otimes \mathbb{C} \oplus \mathbb{C} \otimes X_k$. This object is a commutative Frobenius algebra, and has dimension t, so by the universal property we have a functor $\text{Rep}(S_t) \to \text{Rep}(S_{t-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$. This functor is called the restriction functor and is denoted by $\text{Res}_{S_{t-k} \times S_k}^{S_t}$.

Now we want to describe it in terms of ultraproducts.

Proposition 1.4.4. *For all $t \notin \mathbb{Z}_{\geq 0}$, the functor $\text{Res}_{S_{t-k} \times S_k}^{S_t}$ is equal to $\prod_{\mathcal{F}} \text{Res}_{S_{t_n-k} \times S_k}^{S_{t_n}}$, where the latter functors are the regular restriction functors for the finite groups.*

Proof. Recall that the Littlewood-Richardson coefficient $c_{\nu,\mu}^{\lambda}$ equals to the number of Littlewood-Richardson tables of the skew shape λ/ν and of weight μ. We will use the fact that we have:

$$\text{Res}_{S_{|\lambda|-k} \times S_k}^{S_{|\lambda|}}(X(\lambda)) = \bigoplus_{|\nu|=|\lambda|-k, \ |\mu|=k} c_{\nu,\mu}^{\lambda} X(\nu) \otimes X(\mu) .$$

A priori the above ultraproduct functor is a not a functor between $\text{Rep}(S_t)$ and $\text{Rep}(S_{t-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$, but between the bigger categories of unrestricted ultraproducts. Indeed this functor acts bewteen:

$$\prod_{\mathcal{F}} \text{Res}_{S_{t_n-k} \times S_k}^{S_{t_n}} : \prod_{\mathcal{F}} \mathbf{Rep}_{p_n}(S_{t_n}) \to \prod_{\mathcal{F}} \left(\mathbf{Rep}_{p_n}(S_{t_n-k}) \boxtimes \mathbf{Rep}_{p_n}(S_k) \right) .$$

Now by Remark 3 after the Theorem 1.4.1 we know how to characterize the category $\text{Rep}(S_{t-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$ inside the image. So we need to check that if we restrict it to $\text{Rep}(S_t)$, we will indeed get objects of $\text{Rep}(S_{t-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$.

So consider $\mathcal{X}(\lambda) = \prod_{\mathcal{F}} X(\lambda|_n)$. Now

$$\left(\prod_{\mathcal{F}} \text{Res}_{S_{t_n-k} \times S_k}^{S_{t_n}} \right) (\mathcal{X}(\lambda)) = \prod_{\mathcal{F}} \text{Res}_{S_{t_n-k} \times S_k}^{S_{t_n}}(X(\lambda|_n)) = \prod_{\mathcal{F}} \bigoplus_{|\mu|=k,|\nu|=t_n-k} c_{\nu,\mu}^{\lambda|_n} X(\nu) \otimes X(\mu) ,$$

where the c's are the Richardson-Littlewood coefficients.

So if t_n is sufficiently big, the gap between the first and the second rows of $\lambda|_{t_n}$ is bigger than k. For such t_n the skew shapes μ/ν for admissible ν are all disconnected – there is a part above the first row and the part in the first row. Note that we also can put any sequence of numbers in the part lying in the first row. Hence if we denote by $M(\mu)$ the set of weights μ' (not necessarily partitions) such that $0 \leq \mu_i' \leq \mu_i$ it follows that the previous expression equals

$$\prod_{\mathcal{F}} \bigoplus_{|\mu|=k,\mu' \in M(\mu),\nu} c_{\nu,\mu'}^{\lambda} X(\nu|_{t_n}) \otimes X(\mu) = \bigoplus_{|\mu|=k,\mu' \in M(\mu),\nu} c_{\nu,\mu'}^{\lambda} \mathcal{X}(\nu) \otimes X(\mu) ,$$

where $c_{\nu,\mu'}^{\lambda}$ is the number of skew-shapes λ/ν of weight μ'. Indeed in this formula we just first summed over the possible choices of fixing the length and the content of the first row (by fixing μ') and then the rest. So the image of $\mathcal{X}(\lambda)$ under the ultraproduct functor indeed lies in $\text{Rep}(S_{t-k}) \boxtimes \mathbf{Rep}(S_k; \mathbb{C})$.

Now since $\prod_{\mathcal{F}} \text{Res}_{S_{t_n-k} \times S_k}^{S_{t_n}}(\mathcal{X}) = \mathcal{X} \otimes \mathbb{C} \oplus \mathbb{C} \otimes X_k$, it follows that the ultraproduct functor sends $\mathcal{X}$ to $\mathcal{X} \otimes \mathbb{C} \oplus \mathbb{C} \otimes X_k$. So by universality we conclude that the functors are the same. $\square$

Corollary/Definition 1.4.5. *There is a functor from $Rep(S_{t-k}) \boxtimes \boldsymbol{Rep}(S_k; \mathbb{C})$ to $Rep(S_t)$ biadjoint to $Res_{S_{t-k} \times S_k}^{S_t}$, denoted by $Ind_{S_{t-k} \times S_k}^{S_t}$ and it is equal to $\prod_{\mathcal{F}} Ind_{S_{t_n-k} \times S_k}^{S_{t_n}}$.*

Proof. It can be proven in the same way as above that the ultraproduct of the induction functors defines a functor into the Deligne category.

After we know this, by Łoś's theorem it follows that this functor is biadjoint to the restriction functor, since it is true in the finite rank. $\qquad\square$

Note that this allows us to define the restriction and the induction functors for any subgroup of S_k in the following way:

$$\operatorname{Ind}_{S_{t-k} \times G}^{S_t} = \operatorname{Ind}_{S_{t-k} \times S_k}^{S_t} \circ \operatorname{Ind}_{S_{t-k} \times G}^{S_{t-k} \times S_k},$$

where the later functor is defined to be $\operatorname{Ind}_{S_{t-k} \times G}^{S_{t-k} \times S_k} = \left(\operatorname{Id} \boxtimes Ind_G^{S_k} \right)$.

The same thing holds for restrictions.

2 Technical results on representations of S_N.

In this section we prove some technical lemmas which we will use extensively in our proofs of the classification. The reader can skip this section at first, and then go back when the need arises.

2.1 Facts about small-index subgroups in S_N.

In this subsection we will prove that under some restrictions on the index of a subgroup of S_N it is conjugate to either $\mathcal{A}_n \times H$ or $S_n \times H$, where H is a subgroup of S_{N-n}.

So suppose $N > 10$, r an integer less then $N/2$, and G a subgroup of S_N of index less than $\binom{N}{r}$. First following Theorem 5.2 in [DM96] we have the following proposition:

Proposition 2.1.1. *Under the above assumptions, up to a conjugation, G contains the group $\mathcal{A}_{N-j}$ with $j < r$, where $\mathcal{A}_{N-j}$ is the group of even permutations of the first $N - j$ elements.*

Now we have the second result:

Proposition 2.1.2. *Suppose G is a subgroup of S_N which contains $\mathcal{A}_{N-j}$ and $N > 2j+7$, then G is conjugate to either $S_{N-j'} \times H$ or $\mathcal{A}_{N-j'} \times H$ for some $H \subset S_{j'}$ and $j' \leq j$.*

Proof. Let's consider the standard action of S_N on N elements. Consider the orbit of the first element under the action of G. By assumption it contains the first $N - j$ elements. Up to taking a group conjugate to G (we conjugate by an element fixing the first $N - j$ elements) we may assume that the orbit of the first element under G is equal to the first $N - j'$ elements for $j' \leq j$. We want to prove that G contains $\mathcal{A}_{N-j'}$. To do this, it is enough to prove that any 3-cycle consisting of the first $N - j'$ elements belongs to G.

Let's denote by B the set of the first $N - j$ elements and by C the set of the $j - j'$ elements directly after B. So we need to consider 3-cycles of four types.

The first case is a 3-cycle consisting solely of elements of B. It is trivial by assumption.

The second case is a 3-cycle permuting elements x, y, z such that $x, y \in B$ and $z \in C$. Also let's denote the first element by 1. Since z belongs to the orbit of 1 under G, it follows that $\exists g \in G$ such that $g(z) = 1$. Now since $|B| = N - j$ is bigger than j by at least 6, it follows by the pigeonhole principle that there exist two elements $a, b \in B$ such that $g(a), g(b) \in B$ and all $a, b, g(a), g(b), x, y$ are distinct. Now consider a double transposition τ which interchanges $a \leftrightarrow x$ and $b \leftrightarrow y$, it belongs to $\mathcal{A}_{N-j}$ and hence to G. Now consider a 3-cycle π permuting $g(a), g(b)$ and 1. It also belongs to $\mathcal{A}_{N-j}$ and

hence G. Now $\tau g^{-1} \pi g \tau \in G$ is a 3-cycle permuting x, y, z. Indeed if $c \in N$ is not equal to a, b, x, y, z then both π and τ act trivially, hence c maps to c under the above map. The elements a and m first map to x, y accordingly, then under g they map to something on which π acts trivially, so they are mapped back and then back to a and b. Now (x, y, z) first map to (a, b, z) then to $(g(a), g(b), 1)$, then to $(g(b), 1, g(a))$, then to (b, z, a) and to (y, z, x).

The third case is $x \in B$ and $y, z \in C$. Again suppose $g \in G$ maps z to 1. Again by the pigeonhole principle there are $a, b, c \in B$ such that $g(a), g(b), g(c) \in B$ and $a, b, c, g(a), g(b), g(c), x$ are distinct. By τ denote the double transposition interchanging $a \leftrightarrow x$ and $b \leftrightarrow c$, as before $\tau \in G$ by the assumptions. Now using two previous cases it follows that a 3-cycle π which permutes $g(a), 1, g(z)$ belongs to G. Hence by the same logic as above $\tau g^{-1} \pi g \tau \in G$ is the required 3-cycle.

The final case is $x, y, z \in C$. As before, fix $g \in G$ mapping z to 1. By the above cases there is a 3-cycle $\pi \in G$ permuting $g(y), g(z), 1$. Then $g^{-1} \pi g$ is the required cycle.

Hence $\mathcal{A}_{N-j'} \subset G'$, where G' is a group conjugate to G. Since the orbit of 1 consists of the first $N - j'$ elements, it follows that $G' \subset S_{N-j'} \times S_{j'}$. By the above discussion we are limited to the two cases: $G' = \mathcal{A}_{N-j'} \times H$ or $G' = S_{N-j'} \times H$, where $H \subset S_{j'}$. $\qquad \square$

Now we are ready to state the main theorem of this section:

Theorem 2.1.3. *Suppose $G \subset S_N$ has index less than $\binom{N}{r}$ for $N > 2r + 8$. Then G is conjugate either to $S_{N-j} \times H$ or $\mathcal{A}_{N-j} \times H$ for some $H \subset S_j$ and $j \leq r$.*

Proof. Using Proposition 2.1.1 we conclude that the conjugate group G' contains $\mathcal{A}_{N-j'}$ for $j' < r$. Now using the Proposition 2.1.2 we conclude that the conjugate group G'' is equal to either $S_{N-j} \times H$ or $\mathcal{A}_{N-j} \times H$ for some $H \subset S_j$ and $j \leq j' \leq r$, since $N > 2r + 7 \geq 2j' + 7$. $\qquad \square$

2.2 Lemmas on ultraproducts of representations of S_{t_n}.

Lemma 2.2.1. *Suppose V is an object of $Rep(S_t)$ such that $V = \prod_{\mathcal{F}} V_n$ and $V_n = Ind_{G_n}^{S_{t_n}}(W_n)$ for some subgroup $G_n \subset S_{t_n}$. Then it follows that $G_n = S_{t_n - j} \times H$ for some $j \in \mathbb{Z}_{>0}$ and $H \subset S_j$, for almost all n. Also $W = \prod_{\mathcal{F}} W_n$ is an object of $Rep(S_{t-j}) \boxtimes$* **Rep**$(H; \mathbb{C})$, *hence $V = Ind_{S_{t-j} \times H}^{S_t}(W)$.*

Proof. Suppose V is equal to the sum of $l(V)$ simple objects of $Rep(S_t)$ such that each one is a subobject of $[m]$ with $m \leq m(V)$. Then for almost all n we have V_n being equal to the sum of $l(V)$ irreducible representations included in V^m for $m \leq m(V)$. Hence for almost all n we have $\dim V_n \leq l(V) \cdot (t_n)^{m(V)}$. But since $V_n = \mathrm{Ind}_{G_n}^{S_{t_n}}(W_n)$ for $G_n \subset S_{t_n}$, we know that $\dim V_n = \dim W_n \cdot |S_{t_n}|/|G_n| \geq |S_{t_n}|/|G_n|$. Hence we obtain the following inequality:

$$l(V) \cdot (t_n)^{m(V)} \geq |S_{t_n}|/|G_n| .$$

So we have a subgroup $G_n \subset S_{t_n}$ with the index bounded by $l(V) \cdot t_n^{m(V)}$. Since $\binom{t_n}{m(V)+1}$ is a polynomial of degree $m(V) + 1$ with the highest term being equal to $\frac{t_n^{m(V)+1}}{(m(V)+1)!}$ it follows that all but finite number of t_n we have $\binom{t_n}{m(V)+1} \geq l(V) \cdot (t_n)^{m(V)}$. Hence for almost all n, G_n satisfies the condition of Theorem 2.1.3 with $N = t_n$ and $r = m(V) + 1$. Thus for almost all n we have, after a conjugation, $G'_n = S_{t_n - j_n} \times H_n$ or $G'_n = \mathcal{A}_{t_n - j_n} \times H_n$ for $j \leq m(V) + 1$ and $H_n \subset S_{j_n}$.

For conjugate subgroups G and G' the objects $\mathrm{Ind}_G^{S_N}(U)$ and $\mathrm{Ind}_{G'}^{S_N}(U)$ are isomorphic for the correct choice of the action of G and G' on U. Hence we may suppose that $V_n = \mathrm{Ind}_{G'_n}^{S_{t_n}}(W_n)$. But there is a finite number of subgroups H in S_j for $j \leq m(V) + 1$, hence there is a finite number of ways to choose G'_n for every n. Thus (note that here

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we use Remark 1 in 1.3) for almost all n we have the same $j_n = j$ and $H_n = H \subset S_j$, and $G_n = S_{t_n-j} \times H$ or $G_n = \mathcal{A}_{t_n-j} \times H$ for almost all n. First we need to rule out the possibility $G_n = \mathcal{A}_{t_n-j} \times H$.

So suppose $G_n = \mathcal{A}_{t_n-j} \times H$. Then

$$V_n = \mathrm{Ind}_{G_n}^{S_{t_n}}(W_n) = \mathrm{Ind}_{S_{t_n-j} \times S_j}^{S_{t_n}}(U_n) \ ,$$

where U_n is an $S_{t_n-j} \times S_j$-module with an action of S_{t_n-j} induced from the action of $\mathcal{A}_{t_n-j}$. But any such representation is equivalent to itself tensored with the sign representation, hence if a partition λ appears in the decomposition, so does its conjugate λ^*.

However for any partition we have that $l(\lambda) \times l(\lambda^*) \geq |\lambda|$, so in particular $\max\{l(\lambda), l(\lambda^*)\} \geq \sqrt{|\lambda|}$. Therefore any representation induced from $\mathcal{A}_{t_n-j}$ to S_{t_n-j} contains an irreducible component corresponding to a partition of length at least $\sqrt{t_n - j}$. So

$$U_n = \sum_i X(\lambda_i) \otimes X(\mu_i) \ ,$$

where $|\lambda_i| = t_n - j$, $|\mu_i| = j$ and at least one of λ_i is of length at least $\sqrt{t_n - j}$.

So we have:

$$V_n = \bigoplus_{i,\zeta} X(\zeta)^{\oplus c_{\lambda_i,\mu_i}^{\zeta}}, \tag{1}$$

where ζ are partitions of t_n and c's are the Littlewood-Richardson coefficients.

Suppose λ_j is of length at least $\sqrt{t_n - j}$. Then there is ζ such that $c_{\lambda_j,\mu_j}^{\zeta} \neq 0$ and hence ζ contains λ_i and thus $l(\zeta) \geq l(\lambda_j) \geq \sqrt{t_n - j}$. So the lengths of Young diagrams appearing in V_n are unbounded. But this contradicts V being an object of the Deligne category, because all the simple objects appearing in V lie in $\mathcal{X}^{\otimes m}$ for some bounded m, and hence the length of the Young diagrams appearing in V_n should be bounded for almost all n. Hence $G_n = S_{t_n-j} \times H$.

So $V_n = \mathrm{Ind}_{S_{t_n-j} \times H}^{S_{t_n}}(W_n)$. The last thing to check is that $\prod_{\mathcal{F}} W_n$ is an object of the Deligne category. By the Remark 3 after Theorem 1.4.1, writing $W_n = \oplus W_n^k \otimes U_k$, where U_k is all possible irreducible representations of H, it's enough to check that each sequence W_n^k gives an object of $\mathrm{Rep}(S_t)$. As we know from our previous discussions this is true iff the number of irreducible representations in the sequence W_n^k for each k is bounded and the number of boxes in the corresponding Young diagrams in all the rows except the first one is bounded too. But note that when we induce, each representation in W_n^k gives us at least one irreducible representation in the resulting object, so if the number of irreducible representation is unbounded here it is also unbounded in V_n. Also if the number of boxes in all the rows except the first one is unbounded, then it follows that the number of boxes in the irreducible components of V_n is also unbounded. Indeed by Littlewood-Richardson rule we only add boxes to diagrams when applying induction. Hence for V to lie in $\mathrm{Rep}(S_t)$, W also should lie in $\mathrm{Rep}(S_{t-j}) \boxtimes \mathbf{Rep}(H; \mathbb{C})$. So we are done and $V = \mathrm{Ind}_{S_{t-j} \times H}^{S_t}(W)$. $\qquad\square$

The next lemma concerns the projective representations of S_n. Denote by $\widehat{S}_n$ the double cover of S_n. We may regard projective representations of S_n as a linear representations of $\widehat{S}_n$. We will need the following result ([KT12]):

Theorem 2.2.2. *Suppose $n > 12$ and $p \neq 2$, then any irreducible projective representation of S_n, which is faithful as a $\widehat{S}_n$-representation has dimension at least:*

$$\min\left(2^{\lfloor \frac{n-1-\kappa_n}{2} \rfloor}, 2^{\lfloor \frac{n-2-\kappa_{n-1}}{2} \rfloor}(n - 2 - \kappa_n - 2\kappa_{n-1})\right) \ ,$$

where κ_n is 1 if $p|n$ and 0 otherwise.

Under the same assumptions, any irreducible projective representation of $\mathcal{A}_n$, which is faithful as a $\widehat{\mathcal{A}}_n$-representation has dimension at least:

$$\min\left(2^{\lfloor\frac{n-2-\kappa_n}{2}\rfloor}, 2^{\lfloor\frac{n-3-\kappa_{n-1}}{2}\rfloor}(n-2-\kappa_n-2\kappa_{n-1})\right) ,$$

where κ_n is the same.

Since the only nontrivial normal subgroups of $\widehat{S}_n$ are $\widehat{\mathcal{A}}_n$ and the central subgroup, it follows that any non-linear representation of S_n satisfies the condition of the above theorem. Now we can apply this to obtain the following lemma:

Lemma 2.2.3. *Suppose W_n is a sequence of projective representations of S_{t_n} (or $\mathcal{A}_{t_n}$) for some unbounded sequence t_n, such that $\dim W_n \leq M t_n^L$. Then almost all W_n are actually linear representations of S_{t_n} (or $\mathcal{A}_{t_n}$).*

Proof. Suppose the action of S_{t_n} on W_n is non-linear for almost all n. Then by the above theorem it follows that for almost all n we have $\dim W_n \geq 2^{\lfloor\frac{t_n-4}{2}\rfloor}$ and hence $\dim W_n \geq 2^{t_n-5}$. So we get that for almost all n $M t_n^l \geq 2^{t_n-5}$, which is a contradiction since this inequality holds only for a finite number of n.

The same proof with $\dim W_n \geq 2^{t_n-6}$ instead of $\dim W_n \geq 2^{t_n-5}$ holds for $\mathcal{A}_{t_n}$. $\square$

To prove the last lemma we will need to use another result, namely Lemma 2.8 from [Eti14]:

Lemma 2.2.4. *For each $C > 0$ and $k \in \mathbb{Z}_+$ there exists $N(C, k) \in \mathbb{Z}_+$ such that for each $m > N(C, k)$, if $X(\mu)$ is an irreducible representation of S_m which has dimension $\dim X(\mu) \leq C m^k$, then either the first row or the first column of μ has length $\geq m - k$.*

Now to state our lemma we will need the following definitions:

Definition 2.2.5. a) For an object W of a symmetric tensor category define $\mathfrak{gl}(W)$ to be the object $W \otimes W^*$. It has a structure of an associative algebra given by $1 \otimes ev \otimes 1 : \mathfrak{gl}(W) \otimes \mathfrak{gl}(W) \to \mathfrak{gl}(W)$, and thus has a structure of a Lie algebra given by a commutator.
b) For an object W of a symmetric tensor category define $\mathfrak{sl}(W)$ to be the Lie algebra given by the kernel of the map $ev : \mathfrak{gl}(W) \to \mathbf{1}$. In case of the category $Vect$ this algebra is simple iff the map $\mathbf{1} \to \mathbf{1}$ given by the composition of the evaluation and the coevaluation maps for W is not zero.
c) For an object W of a symmetric tensor category such that the above map $\mathbf{1} \to \mathbf{1}$ is zero, define $\mathfrak{psl}(W)$ to be the cokernel of the map $coev : \mathbf{1} \to \mathfrak{sl}(W)$. In case of the category $Vect$ this algebra is simple.
d) For an object W of a symmetric tensor category equipped with a (skew-)symmetric non-degenerate bilinear form (an isomorphism $\psi : W \to W^*$), define $\mathfrak{so}(W)(\mathfrak{sp}(W))$ to be the Lie subalgebra in $\mathfrak{gl}(W)$ given by the kernel of $\sigma \circ \psi \otimes \psi^{-1} + Id$. In the case of the category $Vect$ this algebra is simple.

Lemma 2.2.6. *Suppose V is an object of $Rep(S_t)$ given by the ultraproduct of $V_n \in \mathbf{Rep}_{p_n}(S_{t_n})$, almost all of which are isomorphic to $End(W_n)$ (or $\mathfrak{sl}(W_n)$, $\mathfrak{psl}(W_n)$, $\mathfrak{so}(W_n)$, $\mathfrak{sp}(W_n)$ if these objects are defined), for some $W_n \in \mathbf{Rep}(S_{t_n})$. Then there exist $W_n' \in \mathbf{Rep}(S_{t_n})$ such that $End(W_n) \simeq End(W_n')$ (or the corresponding Lie algebras are defined and isomorphic) and $W = \prod_{\mathcal{F}} W_n'$ is an object of $Rep(S_t)$. Hence $V = W \otimes W^*$, $\mathfrak{sl}(W)$, $\mathfrak{psl}(W)$, $\mathfrak{so}(W)$,*

Proof. Suppose $V = \sum_{i=1}^M \mathcal{X}(\lambda_i)$, where all $\mathcal{X}(\lambda_i)$ lie in $[k]$ for $k \leq L$. Then for almost all n $(n > 3)$

$$\dim W_n \leq \dim V_n \leq M(t_n)^L,$$

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so by Lemma 2.2.4 there is a number $N(M, L)$ such that for any $n > N(M, L)$ any irreducible representation $X(\mu)$ appearing in W_n is such that either the first row or the first column of μ has length $t_n - L$.

Hence for almost all n we have $W_n = \oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$, with each $\mu_j^{(n)}$ having the first row of length at least $t_n - L$, and $\mathcal{E}_j^{(n)}$ being either a one-dimensional trivial or a sign representation. Let's denote by W_n' a representation equal to $W_n' = \oplus X(\mu_j^{(n)})$.

We need to check several cases.

- $V_n = \operatorname{End}(W_n)$

Since the number of summands in $V_n = W_n \otimes W_n$ is bounded and is bigger or equal than the number of summands in W_n it follows that the number of summands in the latter representation is bounded. Hence, since there is a finite number of ways to put L boxes into the rows of a Young diagram, it follows that for almost all n we have $W_n = \oplus X(\mu_j|_n) \otimes \mathcal{E}_j^{(n)}$, for some partitions μ_j of weight at most L. But now since there are also a finite number of ways to assign to each j either a trivial or a sign representation it follows that for almost all n the same assigment is used and so we can write $W_n = \oplus X(\mu_j|_n) \otimes \mathcal{E}_j$. Now, if one of $\mathcal{E}_j$ is the sign representation and another $\mathcal{E}_i$ is the trivial representation then the ultraproduct of $X(\mu_j|_n) \otimes X(\mu_i|_n) \otimes \mathcal{E}_j \otimes \mathcal{E}_i = (\oplus X(\eta_j|_n)) \otimes sgn$ is not an object of $\operatorname{Rep}(S_t)$ since the number of rows is unbounded. But this contradicts V being the object of $\operatorname{Rep}(S_t)$, hence $\mathcal{E}_j$ are all trivial or all sign representations. In the latter case taking W_n' instead of W_n does not change $W_n \otimes W_n = W_n' \otimes W_n'$ (since $W_n' \otimes W_n' = W_n \otimes W_n \otimes sgn^{\otimes 2}$). But hence $W = \prod_{\mathcal{F}} W_n' = \oplus \mathcal{X}(\mu_j)$ is well defined.

- $\mathfrak{sl}$ or $\mathfrak{psl}$

In the first case we subtract one trivial representation of S_{t_n} and in the second case two trivial representations, so it follows that the number of summands in W_n is still bounded. Using the same reasoning as above it also follows that $\mathcal{E}_j$ are all trivial or are all sign representations since subtracting trivial representations from $W_n \otimes W_n$ cannot delete a representation with a big number of rows. So it follows that we again can take W_n' and get the same $\operatorname{End}(W_n) = \operatorname{End}(W_n')$ and hence the same Lie algebra.

- $\mathfrak{so}$

Let's write W_n as $W_n = \oplus X(\nu_j^{(n)}) \otimes U_j$. There all $\nu_j^{(n)}$ are different for different j and U_j are trivial (but not necessarily one-dimensional) representations of S_{t_n}. Now, an invariant symmetric bilinear form on W_n is given by an isomorphism $\phi : W_n \to W_n^*$. Since all irreducible representations of S_n are real and self-dual, by Schur's lemma it follows that an isomorphism $\phi : W_n \to W_n^*$ decomposes to the sum of isomorphisms $\phi_j : X(\nu_j^{(n)}) \otimes U_j \to X^*(\nu_j^{(n)}) \otimes U_j^*$ given by the tensor product of the isomorphism $X(\nu_j^{(n)}) \to X^*(\nu_j^{(n)})$ and an isomorphism $\psi_j : U_j \to U_j^*$. So our symmetric invariant bilinear form is the sum of products of the symmetric invariant forms on $X(\nu_j^{(n)})$ and invariant forms on U_j. Thus the forms on U_j also should be symmetric. But then up to change of basis we can assume that the invariant bilinear form pairs $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$ to itself in the decomposition $W_n = \oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$. Hence $\mathfrak{so}(W_n)$ contains copies of all tensor products of $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)} \otimes \oplus X(\mu_i^{(n)}) \otimes \mathcal{E}_i^{(n)}$ for $i \neq j$. This means first that the number of summands in W_n is bounded. And that the previous argument can be again used to prove that all $\mathcal{E}_j$ are either trivial or sign. So we can again take W_n' instead of W_n to obtain the same Lie algebra.

- $\mathfrak{sp}$

Here the discussion in the previous paragraph can be repeated, but the invariant bilinear form on U_j should be skew-symmetric. Hence all U_j are even-dimensional. Again up to the change of basis in U_j, we can write down W_n as $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)} \otimes \overline{\mathbb{F}}_{p_n}^2$ where invariant form is given by sum of products of invariant forms on $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)}$ with standard skew-symmetric form on $\overline{\mathbb{F}}_{p_n}^2$. Thus again it follows that $\mathfrak{sp}(W_n)$ contains copies

of $\oplus X(\mu_j^{(n)}) \otimes \mathcal{E}_j^{(n)} \otimes \oplus X(\mu_i^{(n)}) \otimes \mathcal{E}_i^{(n)}$ for $i \neq j$. So all the previous arguments can be repeated. $\qquad\square$

3 Classification of simple associative algebras in $\mathrm{Rep}(S_t)$ and applications.

3.1 Classification.

First we need to understand the classification of simple associative algebras for $\mathbf{Rep}_p(S_N)$. There is the following way of constructing such algebras. Fix $G \subset S_N$ and a simple associative algebra $\mathrm{Mat}_m(\overline{\mathbb{F}}_p)$ with an action of G. From this information we can construct the algebra $\mathrm{Fun}_G(S_N, \mathrm{Mat}_m(\overline{\mathbb{F}}_p)) \in \mathbf{Rep}_p(S_N)$, which is equal to $\mathrm{Ind}_G^{S_N}(\mathrm{Mat}_m(\overline{\mathbb{F}}_p))$ as a representation. We have the following theorem (see for example [Eti17], where it is formulated for any group):

Theorem 3.1.1. *Fix an algebraically closed field k. Any simple associative algebra in $Rep(S_N, k)$ is isomorphic to $Fun_G(S_N, Mat_m(k))$ and all such algebras are simple. Moreover G is defined up to conjugation in S_N and the action of G on $Mat_m(k)$ up to conjugation in $Aut(Mat_m(k))$.*

Now by Łoś's theorem simple associative algebras in $\mathrm{Rep}(S_t)$ are given by ultraproducts of simple associative algebras in $\mathbf{Rep}_{p_n}(S_{t_n}) = \mathcal{C}_n$ such that their ultraproduct as objects of $\mathcal{C}_n$ lies in $\mathrm{Rep}(S_t)$.

So suppose $A \in \mathrm{Rep}(S_t)$ is a simple associative algebra in $\mathrm{Rep}(S_t)$, which is equal to the ultraproduct of $\mathrm{Ind}_{G_n}^{S_{t_n}}(B_n)$, where B_n are matrix algebras. Then we can apply Lemma 2.2.1 to conclude that for almost all n we have $G_n = S_{t_n-j} \times H$, $B = \prod_{\mathcal{F}} B_n$ is an object of $\mathrm{Rep}(S_{t-j}) \boxtimes \mathbf{Rep}(H; \mathbb{C})$ and $A = \mathrm{Ind}_{S_{t-j} \times H}^{S_t}(B)$.

For the next step, we need to understand which sequences of B_n are admissible and wat can we obtain as the result of taking their ultraproduct. We know that $B_n = \mathrm{Mat}_{m_n}(\overline{\mathbb{F}}_{p_n})$ with a structure of a representation of $S_{t_n-j} \times H$. Let's slightly change the notation and denote $B_n = \mathrm{End}(V_n)$, where V_n are some finite-dimensional spaces over $\overline{\mathbb{F}}_{p_n}$. Since $S_{t_n-j} \times H$ acts by algebra automorphisms on B_n, we have a homomorphism $S_{t_n-j} \times H \to \mathrm{Aut}(B_n) = PGL(V_n)$. So we have a structure of a projective representation of $S_{t_n-j} \times H$ on V_n. But note that $\dim V_n \leq \dim B_n$ which is bounded by some $M(t_n-j)^L$ since $B = \prod_{\mathcal{F}} B_n$ is an object of the Deligne category. So Lemma 2.2.3 can be applied, and hence we conclude that the structure of the representation of S_{t_n-j} is linear and not projective.

Thus each V_n is a representation of S_{t_n-j} together with a projective action of H. Now we want to prove that $\prod_{\mathcal{F}} V_n$ as a representation of S_{t_n-j} is a well-defined object of $\mathrm{Rep}(S_{t-j})$. But this follows from Lemma 2.2.6. So indeed we have $V = \prod_{\mathcal{F}} V_n$ an object of $\mathrm{Rep}(S_{t-j})$.

Now we are ready to state the classification theorem:

Theorem 3.1.2. *Suppose A is a simple associative algebra in $Rep(S_t)$, then it is isomorphic to $Ind_{S_{t-j} \times H}^{S_t}(B)$, where $j \in \mathbb{Z}_+$, $H \subset S_j$ and B equals to $V \otimes V^*$, where V is an object of $Rep(S_{t-j})$, together with an action of H on $V \otimes V^*$ by algebra automorphisms. Any algebra obtained in this way is a simple associative algebra in $Rep(S_t)$.*

Moreover, H is defined uniquely up to conjugation in S_j, and the structure of a H-representation on $V \otimes V^*$ is defined uniquely up to conjugation inside $Aut_{Ass-alg}(V \otimes V^*)$.

Proof. From the above discussion it follows that $A = Ind_{S_{t-j} \times H}^{S_t}(B)$, for $B = \prod_{\mathcal{F}} B_n$ and that $B_n = V_n \otimes V_n^*$ for V_n with a projective action of H and linear action of S_{t_n-j} which commute with each other. Moreover $\prod_{\mathcal{F}} V_n = V$ is a well-defined object of $\mathrm{Rep}(S_{t-j})$.

Suppose $V = \bigoplus \mathcal{X}(\lambda_j) \otimes \mathbb{C}^{k_j}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$, then a projective action of H on each V_n is given by the map $\rho_n : H \to \left(\bigoplus GL(\mathbb{C}^{k_j}) \right) / (\{c \cdot Id\})$. But there is a finite number of such maps up to conjugacy, hence for almost all n they are the same and we get the projective action of H on V itself. Which is the same as the action of H on $V \otimes V^*$ by algebra automorphisms.

Also any such algebra is simple by Łoś's theorem .

Now we only need to check the uniqueness statement. Suppose we have two algebras $\mathrm{Ind}_{S_{t-j} \times H}^{S_t}(V \otimes V^*)$ and $\mathrm{Ind}_{S_{t-j'} \times H'}^{S_t}(W \otimes W^*)$. These algebras are isomorphic iff almost all algebras in the corresponding ultraproducts are isomorphic. But by Theorem 3.0.1 it follows that this is only possible iff $S_{t_n-j} \times H$ and $S_{t_n-j'} \times H'$ are conjugate in S_{t_n} for almost all n, $\mathrm{End}(V_n) = \mathrm{End}(W_n)$ and the actions of $S_{t_n-j} \times H$ and $S_{t_n-j'} \times H'$ are conjugate in $\mathrm{Aut}(\mathrm{End}(V_n))$. So it follows that $j = j'$ (for $t_n > 2\max(j, j')$) for almost all n and hence H is conjugate to H' inside S_j (since the conjugation should leave S_{t_n-j} invariant). Also it follows that V_n and W_n must have the same dimension, and since the action of S_{t_n} on them is the same up to conjugation, we can assume that $V_n = W_n$ and they lead to the same object of $\mathrm{Rep}(S_t)$. Hence the last requirement is that the actions of H and H' on $\mathrm{End}(V_n)$ are conjugate. Hence by Łoś's theorem the statement of our Theorem follows.

$\square$

Remark 5. This gives us a classification of simple commutative algebras in $\mathrm{Rep}(S_t)$ given in [Sci15], [Har16] as a special case, where we restrict ourselves to B being 1-dimensional.

3.2 Application: functors between the Deligne categories.

In this section we will show how our result about the classification of commutative algebras helps us classify symmetric tensor functors $\mathrm{Rep}(S_t) \to \mathrm{Rep}(S_{t'})$ and also their generalization $\mathrm{Rep}(S_t) \to \mathrm{Rep}(S_{t'_1}) \boxtimes \cdots \boxtimes \mathrm{Rep}(S_{t'_k})$.

We will start with functors $\mathrm{Rep}(S_t) \to \mathrm{Rep}(S_{t'})$, for $t, t' \notin \mathbb{Z}_{\geq 0}$. From Proposition 1.2.5 we know that all such functors are classified by commutative Frobenius algebras of dimension t in $\mathrm{Rep}(S_{t'})$. We will start with the following lemma.

Lemma 3.2.1. *Any commutative Frobenius algebra $A \in \mathrm{Rep}(S_t)$ is isomorphic to the direct sum of simple commutative algebras.*

Proof. Using previous notation, we know from Łoś's theorem that A corresponds to a sequence of objects $A_n \in \mathcal{C}_n$, with almost all of them being commutative Frobenius algebras. Now note that such an algebra cannot have a non-trivial radical. Indeed since $\sqrt{0}$ lies in every maximal ideal, it also lies in the kernel of Tr. But then for any element $N \in \sqrt{0}$, we have $\mathrm{Tr}(a \cdot N) = 0$, hence the form is degenerate, which is a contradiction. So almost all A_n are semisimple as commutative algebras in the category of vector spaces. Also all of them are finite-dimensional, since they are objects of $\mathcal{C}_i$.

Thus it follows that the action of S_{t_n} on such an A_n arises from the action of S_{t_n} on $\mathrm{mspec}(A_n)$. And now if the action of S_{t_n} on $\mathrm{mspec}(A_n)$ has l_n orbits with stabilizers $H_1, \ldots, H_{l_n}$, it follows that $A_n = \bigoplus \mathrm{Fun}_{H_j}(S_{t_n}, \overline{\mathbb{F}_{p_n}})$.

So almost all A_n are semisimple commutative algebras as objects of $\mathcal{C}_i$, so by Łoś's theorem the same holds for A. $\square$

We also need another lemma.

Lemma 3.2.2. *Any semisimple commutative algebra $A \in \mathrm{Rep}(S_t)$ is a Frobenius algebra.*

Proof. We know that $A = \bigoplus_{i=1}^N A_i$, where A_i are simple algebras. Note that Tr_A is equal to $(\mathrm{Tr}_{A_1}, \ldots, \mathrm{Tr}_{A_N})$, so if we prove that each A_i is a Frobenius algebra it will follow that A is too.

So consider $A = \text{Ind}_{S_{t-j} \times H}^{S_t}(\mathbf{1})$. We want to prove that $\text{Tr} \circ \mu$ is a non-degenerate form on A. To do that it is enough to prove that for n such that $t_n > j$ the corresponding form on $A_n = \text{Ind}_{S_{t_n - j} \times H}^{S_{t_n}}(\mathbf{1}) = \text{Fun}_{S_{t_n - j} \times H}(S_{t_n}, \mathbf{1})$ is non-degenerate.

Consider functions h_i which are zero everywhere except one conjugacy class of $S_{t_n - j} \times H$, where they are equal to 1. Such functions give us a basis of A_n. Now obviously $h_i h_j = \delta_{ij} h_i$. Hence $\text{Tr}(h_i) = 1$, and $\text{Tr}(h_i h_j) = \delta_{ij}$. So the form is indeed non-degenerate, and we are done. $\qquad \square$

From these two lemmas and Proposition 1.2.5 it follows:

Proposition 3.2.3. *All $\mathbb{C}$-linear symmetric tensor functors between the Deligne categories $Rep(S_t) \to Rep(S_{t'})$ for $t, t' \notin \mathbb{Z}_{\geq 0}$ are in 1-1 correspondence with semisimple commutative algebras in $Rep(S_{t'})$ of dimension t.*

Proof. From Proposition 1.2.5 we know that such functors are in 1-1 correpondence with commutative Frobenius algebras of dimension t, but from Lemmas 3.1.1 and 3.1.2 we know that any commutative Frobenius algebra is a semisimple commutative algebra and vice versa. $\qquad \square$

Now since the dimension of $\text{Ind}_{S_{t_n - k} \times H}^{S_{t_n}}(\mathbb{C})$ is $\binom{t_n}{k} \frac{k!}{|H|}$, it follows that dimensions of simple commutative algebras in $Rep(S_t)$ are multiples of $\binom{t}{k}$ for some integer k, and thus all possible dimensions of commutative Frobenius algebras are positive integer linear combinations of $\binom{t}{k}$. Or in other words they can be described as $f(t)$, where $f \in R_+$ and R_+ is the subset in the algebra of integer valued polynomials, given by all positive integer linear combination of binomial coefficients. Hence we have the following Corollary.

Corollary 3.2.4. *Symmetric monoidal functors between the Deligne categories $Rep(S_t) \to Rep(S_{t'})$ with $t, t' \notin \mathbb{Z}_{\geq 0}$ exist iff $t = f(t')$ for some $f \in R_+$.*

Remark 6. One can obtain a similar description for $\mathbb{C}$-linear symmetric tensor functors $Rep(S_t) \to Rep(S_{t'_1}) \boxtimes \cdots \boxtimes Rep(S_{t'_k})$. Such functors are in 1-1 correspondence with finite sums of external tensor products of simple commutative algebras in $Rep(S_{t'_i})$. And such a functor exists iff t is the positive integer linear combination of products of binomial coefficients in $t'_1, \ldots, t'_n$.

To strengthen the result we can solve a simple number-theoretic problem.

Lemma 3.2.5. *Consider a rational number $\frac{r}{s}$ with r, s – coprime positive integers and $s > 1$. Let $p_1, \ldots, p_m$ denote all prime numbers which divide s. Then the set $R_+(\frac{r}{s}) = \{f(\frac{r}{s}) | \ f \in R_+\}$ is equal to $\mathbb{Z}[1/s] = \{\frac{n}{p_1^{k_1} \ldots p_m^{k_m}} | k_i \in \mathbb{Z}_{\geq 0}\}$.*

Proof. First note that $R_+(\frac{p}{q})$ is closed under multiplication. It is enough to prove that the product of $\binom{t}{i}$ and $\binom{t}{j}$ can be expressed as a positive integer linear combination of $\binom{t}{k}$. This fact follows from the identity $(1 + z)^t (1 + w)^t = (1 + (z + w + zw))^t$. Indeed, expanding this we get:

$$\sum_{i,j} \binom{t}{i} \binom{t}{j} z^i w^j = \sum_k \binom{t}{k} (z + w + zw)^k \ ,$$

so it follows that $\binom{t}{i} \binom{t}{j}$ is equal to the coefficient of $z^i w^j$ in $\sum_k \binom{t}{k} (z + w + zw)^k$, which is going to be a positive integer linear combination of $\binom{t}{k}$ since all coefficients in the expansion of $(z + w + zw)^k$ are positive integers.

Now consider $\binom{\frac{r}{s}}{k} = \frac{r(r-s)(r-2s) \cdots (r-(k-1)s)}{k! s^k}$. If we consider any prime number q which does not divide s, then $s \cdot l$ is divisible by p^d iff l is divisible by p^d. Hence in the sequence

$r, r-s, r-2s, \ldots$ the numbers divisible by p are p terms apart, divisible by $p^2 - p^2$ terms apart and so on. Hence the valuation $\nu_p(\prod_{j=0}^{k}(r-js)) \geq \nu_p(\prod_{j=0}^{k}(1+j))$.

Thus if we write $k! = h(k)g(k)$, where $h(k)$ is the part which contains the product of all prime numbers which do not divide s, and $g(k)$ is the remainder, then we have: $\frac{r(r-s)\ldots(r-(k-1)s)}{k!} = \frac{N}{g(k)}$ for some integer N. So it follows that:

$$\binom{\frac{r}{s}}{k} = \frac{N}{g(k)s^k} \in \mathbb{Z}[1/s].$$

Since $\mathbb{Z}[1/s]$ is obviously closed under addition, it follows that $R_+(\frac{r}{s}) \subset \mathbb{Z}[1/s]$, so we need to show the other inclusion. Consider first k such that $r < s(k-1)$, then $\binom{\frac{r}{s}}{k}$ turns out to be negative. So we have an element $\frac{N}{M}$ with $N < 0$ and N, M – coprime in $R_+(\frac{r}{s})$. We also have $1 \in R_+(\frac{r}{s})$. Since N and M are coprime it follows that there are integers a, b such that $aN + bM = 1$. Note that if $a < 0$ and $b > 0$, then the sum is at least $N + M > 1$. The same problem arises with $a > 0$ and $b < 0$. So a, b have the same sign. So making them positive we can arrange for $aN + bM = \pm 1$. So it follows that $a\frac{N}{M} + b = \frac{\pm 1}{M} \in R_+$ and since R_+ is closed under multiplication $\frac{1}{M^2} \in R_+(\frac{r}{s})$. Note that since M was divisible at least by s, it is divisible by all primes $p_1, \ldots, p_m$. Hence by multiplying by an integer we conclude that $\frac{1}{p_i} \in R_+(\frac{r}{s})$, hence by taking products an multiplying by positive integers all $\frac{n}{p_1^{k_1}\ldots p_m^{k_m}}$ with $n \geq 0$ are in $R_+(\frac{r}{s})$.

So, since $-N - 1 \geq 0$, it follows that $\frac{-N-1}{M}$ is in $R_+(\frac{r}{s})$, thus $\frac{-1}{M}$ is in $R_+(\frac{r}{s})$ and by the same logic all $\frac{-1}{p_i} \in R_+(\frac{p}{q})$. Thus $\mathbb{Z}[1/s] \subset R_+(\frac{r}{s})$ and we conclude that $R_+(\frac{r}{s}) = \mathbb{Z}[1/s]$. $\qquad\square$

Remark 7. Note that since r, s are coprime, it follows that $\mathbb{Z}[\frac{1}{s}] = \mathbb{Z}[\frac{r}{s}]$. Also it is easy to see that if $l \in \mathbb{Z}_{<0}$, then $\mathbb{Z}[l] = \mathbb{Z}$ and $R_+(l) = \mathbb{Z}$.

Now we can apply this lemma to get an interesting result about functors into the Deligne categories.

Corollary 3.2.6. *A symmetric monoidal functor between the Deligne categories $Rep(S_t) \to Rep(S_{t'})$ with $t' \in \mathbb{Q}\backslash\mathbb{Z}_{\geq 0}$ exists iff $t \in \mathbb{Z}[t']$.*

4 Classification of simple Lie algebras in $\mathrm{Rep}(S_t)$.

First we need to state the classification theorem for Lie algebras in characteristic p. See chapter 4 of [Str04].

Theorem 4.0.1. *Suppose $\mathfrak{g}$ is a simple finite-dimensional Lie algebra over an algebraically closed field of characteristic $p > 5$. Then it is either of classical or Cartan type.*

Now we need to explain what classical and Cartan type means. First, classical type Lie algebras can be obtained in the following way. Take any Dynkin diagram C and define a Lie algebra $\mathfrak{g}_C$ as a vector space spanned by Chevalley basis corresponding to C with the ordinary Chevalley relations taken modulo p. It turns out that this algebra is simple for any C except A_{kp-1} for a positive integer k. In this case we also need to take quotient by 1-dimensional center of $\mathfrak{sl}_{kp}$ spanned by scalar matrices and we get a simple algebra $\mathfrak{psl}_{kp}$.

Algebras of Cartan type form four series of simple Lie algebras, namely $W(m, \underline{n})$, $S(m, \underline{n})$, $H(m, \underline{n})$ and $K(m, \underline{n})$, where $m \in \mathbb{Z}_{>0}$ and $\underline{n} \in \mathbb{Z}_{>0}^m$ (in the last case m is odd, in the second to last case it is even). We will discuss some of their properties in the next subsection.

The result analogous to Theorem 3.0.1 also holds in the case of Lie algebras, we only need to exchange word "associative" to "Lie" in the statement of the Theorem ([Eti17]).

Theorem 4.0.2. *Fix an algebraically closed field k. Any simple Lie algebra in $Rep(S_N, k)$ is isomorphic to $Fun_G(S_N, \mathfrak{h})$, for a Lie algebra $\mathfrak{h}$ simple in the category of vector spaces. All such algebras are simple. Moreover G is defined up to conjugation in S_N and the action of G on $\mathfrak{h}$ up to conjugation in $Aut(Mat_m(k))$.*

We can now state the following Proposition:

Proposition 4.0.3. *Any simple Lie algebra $\mathfrak{g} \in Rep(S_t)$ is equal to $Ind_{S_{t-j} \times H_j}^{S_t} \mathfrak{h}$, for some $j \in \mathbb{Z}_{>0}$ and $H \subset S_j$, where $\mathfrak{h}$ is a simple Lie algebra given by the ultraproduct of simple Lie algebras $\mathfrak{h}_n \in \mathcal{C}_n$ which remain simple under the forgetful functor $Res : \mathcal{C}_n \to Vec$.*

Proof. Since we know that $\mathfrak{g} = \prod_{\mathcal{F}} \mathfrak{g}_n$ and $\mathfrak{g}_n = Ind_{G_n}^{S_{t_n}}(\mathfrak{h}_n)$, where $\mathfrak{h}_n$ is a simple Lie algebra as an object of the category of vector spaces, the result follows from Lemma 2.2.1 and Łoś's theorem . $\square$

4.1 Ultraproducts of Lie algebras of Cartan type

We want to rule out the case of almost all $\mathfrak{h}_n$ being of Cartan type.

To do this let's first explain what $W(m, \underline{n})$ actually is (see chapter 4.2 in [Str04] for details). First, we need to define $\mathcal{O}(m)$ and $\mathcal{O}(m, \underline{n})$.

Definition 4.1.1. By $\mathcal{O}(m)$ denote a commutative algebra over $\overline{\mathbb{F}}_q$ with a basis $x_1^{(a_1)} \dots x_m^{(a_m)}$, for $a_i \in \mathbb{Z}_{\geq 0}$ with multiplication defined by:

$$x_1^{(a_1)} \dots x_m^{(a_m)} \cdot x_1^{(b_1)} \dots x_m^{(b_m)} = \binom{a_1 + b_1}{a_1} \dots \binom{a_m + b_m}{a_m} x_1^{(a_1+b_1)} \dots x_m^{(a_m+b_m)} .$$

By $\mathcal{O}(m, \underline{n})$ denote a subalgebra of $\mathcal{O}(m)$ spanned by $x_1^{(a_1)} \dots x_m^{(a_m)}$ with $0 \leq a_i < p^{n_i}$.

Using this, the Witt algebra $W(m, \underline{n})$ can be obtained in the following way.

Definition 4.1.2. By $W(m, \underline{n})$ denote a simple Lie algebra given as follows:

$$W(m, \underline{n}) = \sum_{i=1}^m \mathcal{O}(m, \underline{n}) \partial_i .$$

All other simple algebras $S(m, \underline{n})$, $H(m, \underline{n})$ and $K(m, \underline{n})$ can be realized as subalgebras in $W(m, \underline{n})$. We will need an important proposition about the automorphism groups of such algebras, see chapter 7.3 of [Str04].

Proposition 4.1.3. *There is an isomorphism $\phi : Aut_{\mathcal{C}}(\mathcal{O}(m, \underline{n})) \to Aut(W(m, \underline{n}))$ from a certain subgroup $Aut_{\mathcal{C}}(\mathcal{O}(m, \underline{n})) \subset Aut(\mathcal{O}(m, \underline{n}))$, to the group of automorphisms of Witt algebra given by:*

$$\sigma \to (D \mapsto \sigma \circ D \circ \sigma^{-1}) ,$$

where D is an arbitary element of $W(m, \underline{n})$. Moreover it restricts to give an isomorphism between $S(m, \underline{n})$, $H(m, \underline{n})$ and $K(m, \underline{n})$ and certain subgroups of $Aut_{\mathcal{C}}(\mathcal{O}(m, \underline{n}))$.

So from this proposition it follows what we need to understand the structure of the group $Aut_{\mathcal{C}}(\mathcal{O}(m, \underline{n}))$. This is done in [Wil71]. See Corollary 1 and Theorem 2.

Proposition 4.1.4. *Take any isomorphism $\sigma \in Aut_{\mathcal{C}}(\mathcal{O}(m, \underline{n}))$, denote by y_i the images of $x_i = x_i^{(1)}$ under this morphism. By $\bar{y}_i$ denote the linear part of y_i. It follows that the map $x_i \mapsto \bar{y}_i$ defines an element of $GL(m, \overline{\mathbb{F}}_p)$. Also there is an exact sequence:*

$$0 \to \mathcal{B} \to Aut_{\mathcal{C}}(\mathcal{O}(m, \underline{n})) \to GL(m, \overline{\mathbb{F}}_p) ,$$

where $\mathcal{B}$ is solvable and the last morphism is as described above.

Since the automorphism groups of all Cartan type Lie algebras are subgroups in $\mathrm{Aut}_{\mathcal{C}}(\mathcal{O}(m,\underline{n}))$[9] it follows that such an exact sequence holds for any $\mathrm{Aut}(X(m,\underline{n}))$ with $X = W, S, H, K$. Namely we have:

$$0 \to \mathcal{B}_X \to \mathrm{Aut}(X(m,\underline{n})) \to GL(m, \overline{\mathbb{F}}_p) \ , \tag{2}$$

for different $\mathcal{B}_X$.

Also we will need to know the dimension formulas for Cartan type Lie algebras, they are summarized in the following proposition (see [Str04] section 4.2).

Proposition 4.1.5. *The dimension of Cartan type algebras are given by the formulas* $\dim(W(m,\underline{n})) = mp^{\sum n_i}$, $\dim(S(m,\underline{n})) = (m-1)(p^{\sum n_i} - 1)$, $\dim(H(m,\underline{n})) = p^{\sum n_i} - 2$ *and* $\dim(K(m,\underline{n})) = p^{\sum n_i}$ *or* $p^{\sum n_i} - 1$ *depending on* $m \mod p$.

Now we have everything we need to move on. So let us prove the following proposition.

Proposition 4.1.6. *In Proposition 4.0.3 almost all $\mathfrak{h}_n$ are of classical type.*

Proof. Suppose that almost all $\mathfrak{h}_n$ in Proposition 4.0.3 are of Cartan type. Let's denote $\mathfrak{h}_n = X_n(m_n, \underline{N_n})$ ($X_n = W, S, H, K$), then we have a homomorphism $S_{t_n-j} \to \mathrm{Aut}(\mathfrak{h}_n)$. Hence, because of (2), we have $S_{t_n-j} \to GL(m, \overline{\mathbb{F}}_{p_n})$. There are two possibilities here. Either for almost all n this homomorphism is trivial or not.

First suppose it is trivial for almost all n. Then for almost all n, $S_{t_n-j} \to \mathcal{B}_n$, but since the latter group is solvable, it follows that for almost all n the kernel of this morphism contains $\mathcal{A}_n$. But then $\mathfrak{h}_n$ contains only one-dimensional representations of S_{t_n-j}. But since the dimension of $\mathfrak{h}_n$ is bigger than $p_n - 3$ it follows that the length of $\mathfrak{h}_n$ as a representation of S_{t_n-j} is unbounded, hence its ultraproduct does not define an object of the Deligne category.

The only other option is that this morphism is non-trivial for almost all n. Note that this morphism cannot have $\mathcal{A}_n$ as its kernel or the previous argument can be repeated. Hence, since the lowest dimension of a S_{t_n-j}-representation which is not trivial or sign is t_n-j-1, it follows that $m_n \geq t_n-j-1$, and thus the dimension of $\mathfrak{h}_n$ is at least $p_n^{t_n-j-1}-3$. So it grows exponentially. But the dimension of any sequence of representations defining an element of the Deligne category grows polynomially. So again ultraproduct of $\mathfrak{h}_n$ does not belong to $\mathrm{Rep}(S_{t-j})$.

Thus the result follows. $\qquad\square$

4.2 Ultraproducts of classical Lie algebras

So, now we can assume that for almost all $\mathfrak{h}_n$ are of classical type.

Here we again have two possibilities. Either the size of the Dynkin diagram coresponding to $\mathfrak{h}_n$ is bounded for almost all n or it is not. Let's start with the first case.

Proposition 4.2.1. *If the size of the Dynkin diagram corresponding to $\mathfrak{h}_n$ is bounded, then $\mathfrak{h}$ has trivial action of S_{t-j}.*

Proof. Since there is a finite number of Dynkin diagrams of bounded size, it follows that for almost all n the corresponding Dynkin diagram is the same, so as a Lie algebra in the category of vector spaces, $\mathfrak{h}_n$ is of the same type (again see Remark 1 in section 1.3). But then its automorphism group is a subgroup in some $GL_N(\overline{\mathbb{F}}_{p_n})$. So since the lowest dimension of an irreducible S_{t_n-j}-representation which is not trivial or sign is $t_n - j - 1$ it follows that almost all $\mathfrak{h}_n$ are sums of one-dimensional representations of S_{t_n-j}. But it cannot contain the sign representations for almost all n, or the ultraproduct wouldn't

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[9]Theorem 7.3.2 [Str04].

lie in $\text{Rep}(S_{t-j})$. Hence for almost all n, $\mathfrak{h}_n$ is a trivial representation of S_{t_n-j}. Thus the corresponding ultra-product is the same classical Lie algebra corresponding to the Dynkin diagram with a trivial action of S_t, i.e. equal to the sum of the copies of the unit object in $\text{Rep}(S_{t-j})$. □

In the second case the size of the Dynkin diagram is unbounded. But the number of infinite series of Dynkin diagrams is finite, so we may assume that for almost all n the type of the Dynkin diagram is the same, and it is either A, B, C or D. To proceed further we need to know something about the automorphism groups of these algebras. This information can be found in [Sel60], it is summarized in the following Proposition.

Proposition 4.2.2. *The group of automorphisms of the Lie algebra of type A_{n-1} (both in the case $p|n$ and $p \nmid n$) is the semi-direct product of $PSL(n)$ by $\mathbb{Z}/2\mathbb{Z}$, where the second group acts by $X \mapsto -X^t$. We will denote the generator of this group by τ.*
The group of automorphisms of the Lie algebra of type B_n is $PSp(n)$.
The group of automorphisms of the Lie algebra of type C_n is $PSO(2n+1)$.
The group of automorphisms of the Lie algebra of type D_n is $PO(2n)$, for $n > 4$.

We have an additinal complication in the A_{n-1} case, so let us deal with this case first.

Proposition 4.2.3. *If $\mathfrak{h}_n$ is a simple Lie algebra of type A for almost all n, then, for almost all n, S_{t_n-j} maps into the subgroup $PSL(n)$ of automorphisms of $\mathfrak{h}_n$.*

Proof. Suppose the map $S_{t_n-j} \to \mathbb{Z}/2\mathbb{Z}$ obtained as a composition of maps $S_{t_n-j} \to \text{Aut}(\mathfrak{h}_n)$ and $\text{Aut}(\mathfrak{h}_n) \to \mathbb{Z}/2\mathbb{Z}$ is non-trivial for almost all n. Note that the map $\mathcal{A}_{t_n-j} \to \mathbb{Z}/2\mathbb{Z}$ obtained by restriction of this map is trivial, so $\mathcal{A}_{t_n-j}$ maps into $PSL(N_n)$. Suppose this map is non-trivial. By Lemma 2.2.3 it follows that this map gives us a linear representation of $\mathcal{A}_{t_n-j}$ on $V = \overline{\mathbb{F}_{p_n}}^{N_n}$. By choosing a bilinear $\mathcal{A}_{t_n-j}$-invariant form on V, we can suppose that $V^* \simeq V$ as a representation of $\mathcal{A}_{t_n-j}$. But then since τ acts on an element of $V \otimes V^*$ as $\tau(v \otimes w) = -w \otimes v$, it follows that τ commutes with the action of $\mathcal{A}_{t_n-j}$. So the action of τ on the isomorphic image of $\mathcal{A}_{t_n-j}$ is actually trivial, and hence the image of S_{t_n-j} is actually a direct product of $\mathcal{A}_{t_n-j}$ and $\mathbb{Z}/2\mathbb{Z}$, which is absurd. So $\mathcal{A}_{t_n-j}$ is in the kernel of the map $S_{t_n-j} \to \text{Aut}(\mathfrak{h}_n)$. So for almost all n, $\mathfrak{h}_n$ decomposes as the sum of one-dimensional representations of S_{t_n-j}, but their ultraproduct lies in the Deligne category, so almost all of them can not contain any sign representations. Hence the action of S_{t_n-j} on $\mathfrak{h}_n$ is actually trivial. So we get a contradiction. □

From this proposition it follows that in each case S_{t_n-j} maps into a projective group of the corresponding group of linear transformations of a vector space. But from Lemma 2.2.3 it follows that in each case we have an honest map from S_{t_n-j} to the corresponding group of linear transformations, i.e. a represnetation of S_{t_n-j} on the corresponding vector space.
We have the following proposition:

Proposition 4.2.4. *If the size of the Dynkin diagram corresponding to $\mathfrak{h}_n$ is unbounded, then for almost all n, $\mathfrak{h}_n = \mathfrak{r}(V_n)$ for the same $\mathfrak{r}$ ($\mathfrak{r} = \mathfrak{sl}, \mathfrak{psl}, \mathfrak{sp}, \mathfrak{so}$). Also there exist V'_n such that $\mathfrak{r}(V_n) = \mathfrak{r}(V'_n)$ and such that $V = \prod_{\mathcal{F}} V'_n$ is an object of $Rep(S_{t-j})$, hence $\mathfrak{h} = \mathfrak{r}(V)$.*

Proof. We have established the first part of the proposition. Also from the discussion above it follows that we have an action of S_{t_n-j} on V_n, which leaves the correponding bilinear form invariant. Now using Lemma 2.2.6 we conclude that such V'_n indeed exist. □

Now we can formulate the following classification theorem.

Theorem 4.2.5. *Every simple Lie algebra in the category $Rep(S_t)$ is isomorphic to the one given by $Ind_{S_{t-j} \times H}^{S_t}(\mathfrak{h})$, where $j > 0$ is an integer, H a subgroup of S_j which acts on a Lie algebra $\mathfrak{h}$ in $Rep(S_{t-j})$ by Lie algebra automorphisms, and $\mathfrak{h}$ is of one of the following kinds:*

    · An exceptional Lie algebra which is equal to the sum of copies of the unit object of $Rep(S_{t-j})$.

    · $\mathfrak{sl}(V)$ for any V of non-zero dimension, or $\mathfrak{psl}(V)$ for any V of dimension zero.

    · $\mathfrak{so}(V)$ or $\mathfrak{sp}(V)$ for any V with a (skew)-symmetric non-degenerate bilinear form.

    Finally, such a simple Lie algebra is determined uniquely by the above data up to conjugation of H inside S_j and conjugation of the action of H inside of $Aut(\mathfrak{h})$.

Proof. We have already checked that $Ind_{S_{t-j} \times H}^{S_t}(\mathfrak{h})$ defined using the data described above gives us a simple Lie algebra, since the resulting algebra $\mathfrak{g}$ is an ultraproduct of the Lie algebras which are simple due to Theorem 4.0.2.

Now from Propositions 4.0.3, 4.1.6, 4.2.1 and 4.2.4 we conclude that any simple Lie algebra can be obtained in this way. Indeed from these propositions we know that such $\mathfrak{h}$ exists and is either a trivial representation of S_{t-j} or it is given by $\mathfrak{x}(V)$. Now note that if $\mathfrak{h} = \mathfrak{sl}(V)$, the dimension of almost all V_n are not divisible by p_n, or the algebra $\mathfrak{sl}(V_n)$ would not be simple for almost all n, hence the dimension of V, which can be obtained through the isomorphism $\prod_{\mathcal{F}} \overline{\mathbb{F}}_{p_n} = \mathbb{C}$ is non-zero. In the case of $\mathfrak{psl}(V)$ on the other hand it is divisible by p_n for almost all n and hence the dimension of V is zero. In the case $\mathfrak{x} = \mathfrak{so}$, $\mathfrak{sp}$ the S_{t_n-j}-module V_n has an S_{t_n-j}-invariant (skew)-symmetric bilinear form for almost all n, hence it gives us a (skew)-symmetric bilinear form on V in $Rep(S_t)$. So indeed every simple Lie algebra can be obtained in the specified way.

The proof of the uniqueness is the same as in Theorem 3.0.2.

$\square$

Remark 8. This theorem can likely be extended to the degenerate case when $t \in \mathbb{Z}_{\geq 0}$. $Rep(S_t)$ is not abelian for $t \in \mathbb{Z}$ so instead one should work with the abelian envelope $Rep^{ab}(S_t)$ defined in [CO14]. This abelian envelope has a lot of nice properties, for example it is a highest-weight category (see [BEAH17]), and it still can be similarly interpreted via an ultrafilter construction as outlined in [Har16]. The main difference is that the categories $Rep^{ab}(S_t)$ are not semisimple so some care needs to be taken in some of the arguments. However the main technical step of ruling out the Cartan type Lie algebras (Proposition 4.1.6) by looking at their dimension growth goes through as is.

5   A conjecture concerning the classification of simple Lie superalgebras in $\mathbf{Rep}(S_t)$

We will state a conjectural extension of the main results of this paper to the setting of Lie superalgebras, and outline a possible approach to generalize the methods in this paper. The textbook reference about the theory of Lie superalgebras is [Mus12], it contains the classification of simple Lie superalgebras over $\mathbb{C}$ and their construction (Chapters 1-2 and 4). See the original paper of Kac [Kac77] for the classification.[10] How these results generalize to the modular case with big enough p can be found in [BGL$^+$15] Section 2.3, [Lei07] and [LBG09] Section 10.

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[10]The classification problem of finite dimensional simple Lie superalgebras in zero and positive characteristic has a long history, which we in no way attempt to review; thus many important references are not given here.

5.1 Lie superalgebras in tensor categories and their simplicity in $Vect$

Below we assume that for every Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the $\mathfrak{g}_1$ component is non-zero. If $\mathfrak{g}_1$ is zero, then $\mathfrak{g}$ is a regular Lie algebra and the result of the previous section applies.

First we will need some definitions.

Definition 5.1.1. Fix V, W to be non-zero objects of a symmetric rigid tensor category.
a) Define the Lie superalgebra $\mathfrak{gl}(V|W)$ to be the object $(V \oplus W) \otimes (V^* \oplus W^*)$ with the $\mathbb{Z}/2\mathbb{Z}$-grading given by $\mathfrak{gl}(V|W)_0 = V \otimes V^* \oplus W \otimes W^*$ and $\mathfrak{gl}(V|W)_1 = V \otimes W^* \oplus W \otimes V^*$. The superbracket $[\ ,\ ]_{i,j} : \mathfrak{gl}(V|W)_i \otimes \mathfrak{gl}(V|W)_j \to \mathfrak{gl}(V|W)_{i+j}$ is given by $\mu - (-1)^{ij}\mu \circ \sigma$, where μ is the associative algebra multiplication and $\sigma = \sigma_{\mathfrak{gl}(V|W)}$ (see Def. 1.1.8).
b) Define the Lie superalgebra $\mathfrak{sl}(V|W)$ to be the subalgebra in $\mathfrak{gl}(V|W)$ given by the kernel of the map $str : \mathfrak{gl}(V|W) \to \mathfrak{gl}(V|W)_0 \xrightarrow{(ev_V \ , \ -ev_W)} \mathbf{1}$.
c) Consider the map $l : \mathbf{1} \xrightarrow{coev_V \oplus coev_W} \mathfrak{gl}(V|W)_0$. The image of this map lies in $\mathfrak{sl}(V|W)$ iff $\dim V = \dim W$. In this case define the Lie superalgebra $\mathfrak{psl}(V|W)$ to be the cokernel of $l : \mathbf{1} \to \mathfrak{sl}(V|W)$.
d) Fix a bilinear form on $V \oplus V^*$ specified by the identity map $\psi : V \oplus V^* \to (V \oplus V^*)^* = V^* \oplus V$. Define the Lie superalgebra $\mathfrak{p}(V)$ to be the subalgebra in $\mathfrak{sl}(V|V^*)$ preserving this form, i.e. the kernel of the map $\mathfrak{gl}(V|V^*) \xrightarrow{1 + \sigma \circ (\psi \otimes \psi^{-1})} \mathfrak{gl}(V|V^*)$.
e) Consider a morphism $c : V \oplus V \to V \oplus V$, given by the matrix $\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$. This morphism can also be considered as an element (i.e. a map $\mathbf{1} \to \mathfrak{gl}(V|V)$) of $\mathfrak{gl}(V|V)$ using evaluation and coevaluation maps. The Lie superalgebra $\hat{\mathfrak{q}}(V)$ is defined as the centralizer of this element, i.e. as the kernel of the map $\mathfrak{gl}(V|V) \to \mathfrak{gl}(V|V) \otimes \mathbf{1} \xrightarrow{Id \otimes c} \mathfrak{gl}(V|V) \otimes \mathfrak{gl}(V|V) \xrightarrow{[\ ,\ ]} \mathfrak{gl}(V|V)$. Next we define the Lie superalgebra $\tilde{\mathfrak{q}}(V)$ as the kernel of the map $\hat{\mathfrak{q}}(V) \to \mathbf{1}$ given by the restriction of the map $\mathfrak{gl}(V|V)_1 \xrightarrow{(ev_V \ , \ ev_V)} \mathbf{1}$. Then there is a non-zero map $l : \mathbf{1} \xrightarrow{coev \oplus coev} \tilde{\mathfrak{q}}(V)_0$. The cokernel of this map it the Lie superalgebra $\mathfrak{q}(V)$.
f) Suppose there is a symmetric non-degenerate bilinear form on V and a skew-symmetric non-degenerate bilinear form on W. Denote the corresponding maps $\psi_V : V \to V^*$ and $\psi_W : W \to W^*$. Then $\mathfrak{osp}(V|W)$ is the following subalgebra in $\mathfrak{gl}(V|W)$. We define $\mathfrak{osp}(V|W)_0$ to be equal to the kernel of the map

$$\mathfrak{gl}(V|W)_0 \xrightarrow{(Id + \sigma \circ (\psi_V \otimes \psi_V^{-1})) \oplus (Id + \sigma \circ (\psi_W \otimes \psi_W^{-1}))} \mathfrak{gl}(V|W)_0 \ ,$$

and $\mathfrak{osp}(V|W)_1$ to be equal to the kernel of the map

$$\mathfrak{gl}(V|W)_1 = V \otimes W^* \oplus W \otimes V^* \xrightarrow{(\sigma \circ (\psi_V \otimes 1) \ , \ 1 \otimes \psi_W)} W^* \otimes V^* \ .$$

Remark 9. These definitions mimic the standard definitions of the above Lie superalgebras in an element-free fashion. It is straightforward to check that this definition agrees with the usual definitions for the category of supervector spaces, and that the superbracket descends onto the various kernels and cokernels used in the definition.

Now we want to know when exactly these superalgebras are simple for the categories $Vect_{k_0}$ and $Vect_{k_p}$. This is explained by the various classification results. Here k_0 stands for an algebraically closed field of characteristic 0 and k_p stands for an algebraically closed field of characteristic p.

Proposition 5.1.2. *(Theorem 1.3.1 in [Mus12]) Suppose V, W are non-zero objects of $Vect_{k_0}$.*

a) *The Lie superalgebra* $\mathfrak{sl}(V|W)$ *is simple in* $Vect_{k_0}$ *iff* $\dim(V) \neq \dim(W)$.

b) *In* $Vect_{k_0}$ *if* $\dim(V) = \dim(W) > 1$ *the Lie superalgebra* $\mathfrak{psl}(V|W)$ *is defined and is simple.*

c) *The Lie superalgebra* $\mathfrak{osp}(V|W)$ *is simple in* $Vect_{k_0}$.

d) *The Lie superalgebra* $\mathfrak{q}(V)$ *is simple in* $Vect_{k_0}$ *iff* $\dim(V) \geq 2$.

e) *The Lie superalgebra* $\mathfrak{p}(V)$ *is simple in* $Vect_{k_0}$ *iff* $\dim(V) \geq 2$.

Proposition 5.1.3. *(Section 10 in [LBG09], Section 6 in [BGL08], Section 4.1 in [BGLL13])*
Suppose V, W *are non-zero objects of* $Vect_{k_p}$.

a) *The Lie superalgebra* $\mathfrak{sl}(V|W)$ *is simple in* $Vect_{k_p}$ *iff* $\dim(V) \neq \dim(W)$ *mod* p *(See Section 10 in [LBG09]).*

b) *In* $Vect_{k_p}$ *if* $\dim(V) = \dim(W)$ *mod* p *and* $\dim(V), \dim(W) > 1$, *the Lie superalgebra* $\mathfrak{psl}(V|W)$ *is defined and is simple (See Section 10 in [LBG09]).*

c) *The Lie superalgebra* $\mathfrak{osp}(V|W)$ *is simple in* $Vect_{k_p}$. *(See Section 10 in [LBG09]).*

d) *The Lie superalgebra* $\mathfrak{q}(V)$ *is simple in* $Vect_{k_p}$ *iff* $\dim(V) \geq 2$.

e) *The Lie superalgebra* $\mathfrak{p}(V)$ *is simple in* $Vect_{k_p}$ *iff* $\dim(V) \geq 2$.

We have some results about the classification of all such superalgebras.

Theorem 5.1.4. *(Theorem 1.3.1 in [Mus12], Section 4.2 in [Kac77]) Let* $\mathfrak{g}$ *be a finite dimensional simple Lie superalgebra over* k_0. *Then it is either given by one of the examples of Proposition 5.1.2 or by one of the exceptional Lie superalgebras* $\mathfrak{d}(2,1;\alpha)$ *for* $\alpha \in k_0$, $\alpha \neq 0, 1$, $\mathfrak{f}(4)$ *or* $\mathfrak{g}(3)$ *or by one of the Cartan type superalgebras* $W(n)$, $S(n)$, $\tilde{S}(n)$ *and* $H(n)$.

Remark 10. The Lie superalgebras $\mathfrak{d}(2,1;\alpha)$ form a one-parametric series of superalgebras of the same dimension.

Conjecture 5.1.5. *(Conjecture 1 in [Lei07]) Let* $\mathfrak{g}$ *be a finite dimensional simple Lie superalgebra over* k_p *with* $p \geq 7$. *Then it is either given by one of the examples of Proposition 5.1.3 or by one of the exceptional Lie superalgebras or by a certain algebra of Cartan type.*

5.2 Lie superalgebras in $\mathrm{Rep}(S_t)$

Using Definition 5.1.1 we can construct the Lie superalgebras in the category $\mathrm{Rep}(S_t)$ as follows. Fix an integer $j > 0$, a subgroup $H \subset S_j$ and a Lie superalgebra $\mathfrak{h}$ in $\mathrm{Rep}(S_{t-j})$ of one of the following kinds:

· An exceptional or a Cartan type Lie superalgebra which as an object of $\mathrm{Rep}(S_{t-j})$ is equal to the sum of the copies of the unit object.

· $\mathfrak{sl}(V|W)$ for V, W such that $\dim(V) \neq \dim(W)$ and $V, W \neq 0$.

· $\mathfrak{psl}(V|W)$ for V, W such that $\dim(V) = \dim(W)$ and both objects are not 0 or **1**.

· $\mathfrak{osp}(V|W)$ for any pair of non-zero objects V, W with a non-degenerate billinear form, which is symmetric and skew-symmetric respectively.

· $\mathfrak{q}(V)$ for V not 0 or **1**.

· $\mathfrak{p}(V)$ for V not 0 or **1**.

Also fix an action of H on $\mathfrak{h}$ by Lie superalgebra automorphisms. Then we can denote $\mathfrak{g}$ in $\mathrm{Rep}(S_t)$ as $\mathrm{Ind}_{S_{t-j} \times H}^{S_t}(\mathfrak{h})$. This is obviously a Lie superagbera since it can be presented as an ultraproduct of Lie superalgebras. The fact that this is simple follows from the statement simmilar to Prop. 4.0.2 for the case of Lie superalgebras, which also follows from the general statement for operads given in [Eti17].

Now using this we can state a conjecture similar to Theorem 4.2.5.

Conjecture 5.2.1. *Any simple Lie superalgebra in $Rep(S_t)$ is isomorphic to the one obtained as $Ind_{S_{t-j} \times H}^{S_t}(\mathfrak{h})$ from the data $j, H, \mathfrak{h}$ as described above.*

Such a simple Lie superalgebra is determined uniquely by the above data up to conjugation of H inside S_j and conjugation of the action of H inside of $Aut(\mathfrak{h})$.

Sketch of Proof. Below is a rough sketch of the proof for transcendental t. In the case of algebraic t the proof might be similar but relies on Conjecture 5.1.5.

The steps of the proof are the same as in Theorem 4.2.5.

First using the analogue of Proposition 4.0.3 it is easy to see that indeed $\mathfrak{g} = Ind_{S_{t-j} \times H}^{S_t}(\mathfrak{h})$ for some simple Lie superalgebra $\mathfrak{h}$ which is given by the ultraproduct of simple Lie superalgebras $\mathfrak{h}_n$ which remain simple when considered as an object of $Vect$.

The next step is a little bit more vague, since we need to rule out the possibility of almost all $\mathfrak{h}_n$ being of Cartan type with a non-trivial action of S_{t_n-j}(analogue of Proposition 4.1.6). This requires some work in the case of algebraic t, but is straightforward for transcendental t. Indeed in this case all $\mathfrak{h}_n$ are Lie superalgebras over a field of characteristic 0 and the dimension of Cartan type superalgebras grows exponentially with n (hence their rank is bounded, hence almost all actions of S_{t_n-j} are trivial).

Next as an analogue of Proposition 4.2.1 it is easy to show that if the dimensions of $\mathfrak{h}_n$ are bounded, then the action of S_{t_n-j} will be trivial for almost all n.

Another step is to show that $S_{t_n-j} \times H$ acts by inner automorphisms of $\mathfrak{h}_n$ for almost all n. This is done in a way similar to Proposition 4.2.3 since the groups of outer automorphisms of these superalgebras are some small groups.

The last step is the analogue of Proposition 4.2.4 which is based on the Lemma 2.2.6. To prove this one needs to extend Lemma 2.2.6 to cover some more examples relevant for the superalgebras case, which can be done in a similar way.

After all this is done the claim will follow in the same way as in Theorem 4.2.5. $\qquad \square$

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