

Stability of Current Density Impedance Imaging II

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Abstract

This paper is a continuation of the authors earlier work on stability of Current Density Impedance Imaging (CDII) [R. Lopez, A. Moradifam, Stability of Current Density Impedance Imaging, SIAM J. Math. Anal., to appear (2020).] We show that CDII is stable with respect to errors in both measurement of the magnitude of the current density vector field in the interior and the measurement of the voltage potential on the boundary. This completes the authors study of the stability of Current Density Independence Imaging which was previously shown only by numerical simulations.

1 Introduction

Let σ be the isotropic conductivity of an object $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where Ω is a bounded open region in with connected boundary. Suppose J is the current density vector field generated by imposing a given boundary voltage f on $\partial\Omega$. Then the corresponding voltage potential u satisfies the elliptic equation

$$\nabla \cdot (\sigma \nabla u) = 0, \quad u|_{\partial\Omega} = f. \quad (1)$$

By Ohm's law $J = -\sigma \nabla u$, and u is the unique minimizer of the weighted least gradient problem

$$I(w) = \min_{w \in BV_f(\Omega)} \int_{\Omega} a |\nabla w| dx, \quad (2)$$

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where $a = |J|$, and $BV_f(\Omega) = \{w \in BV(\Omega), w|_{\partial\Omega} = f\}$, see [21, 23, 25, 26, 27].

Note that the weighted least gradient problem (2) is not strictly convex, and hence in general it may not have a unique minimizer. See [12] where the second author and his collaborators showed that for $a \in C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$, the least gradient problem (2) could have infinitely many minimizers. On the other hand, since any stability result trivially implies uniqueness, it is clear that one needs additional assumptions to prove any stability result. Indeed stability analysis of CDII is a challenging problem. In [17] and [18] the authors proved interesting local stability results for CDII. Recently in [15] we proved the first global stability results on CDII. Indeed we proved the following theorems.

Theorem 1.1 ([15]). *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$ and corresponding current density vector fields J and $\tilde{J}$, respectively. Suppose the level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.4 in [15]. Then*

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)}^{\frac{1}{2}}, \quad (3)$$

where C is independent of $\tilde{u}$ and $\tilde{\sigma}$.

Theorem 1.2 ([15]). *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (25). In addition suppose u satisfies (17), the level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.4 in [15], and the level sets of u are well-structured in the sense of Definition 4.2 in [15]. Then*

$$\|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}}, \quad (4)$$

for some constant C is independent of $\tilde{u}$ and $\tilde{\sigma}$.

Theorem 1.3 ([15]). *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (25). If u satisfies (17), the level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.4 in [15], and the level sets of u are well-structured in the sense of Definition in [15], then*

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \leq C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)}^{\frac{1}{4}}, \quad (5)$$

for some constant C independent of $\tilde{\sigma}$.

Similar results were also proved in dimension $n = 2$. A natural question which remains open is how the presence of errors in measurements of the boundary voltage f together with errors in measurements of $|J|$ affect reconstruction of the conductivity σ in the interior? In this paper, we generalize our approach in [15] to prove that in dimensions $n = 2, 3$ the following stability result holds

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \leq C_1 \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\Omega)}^{\frac{1}{4}}, \quad (6)$$

for some constants C_1, C_2 independent of $\tilde{\sigma}$ (see Theorems 4.5 and 4.6 for precise statements of the results). The proofs are generalizations of the arguments developed in [15].

The paper is organized as follows. In Section 2, under very weak assumptions, we will prove that the structure of level sets of the least gradient problem (2) is stable. In Section 3, we will provide stability results for minimizers of (2) in L^1 . In Section 4, we will prove stability of minimizers of (2) in $W^{1,1}$, and shall use them to prove Theorems 4.5 and 4.6 which are the main results of this paper.

2 Stability of level sets

In this section, we show that the structure of the level sets of minimizers of the least gradient problem (2) is stable. Throughout the paper, we will assume that $a, \tilde{a} \in C(\Omega)$ with

$$0 < m \leq a(x), \tilde{a}(x) \leq M \quad \forall x \in \Omega \quad \text{and} \quad |f(y)|, |\tilde{f}(y)| \leq M \quad \forall y \in \partial\Omega \quad (7)$$

for some positive constants m, M . Although in section 2 it is only necessary that $a \geq 0$, we will need a to be bounded away from zero in the above sense in sections 3 and 4.

Remark 2.1. In general the least gradient problem (2) may not have a minimizer [6, 30]. Throughout the paper we shall assume that (2) has a solution. For sufficient conditions for the existence of minimizers of weighted least gradient problems we refer to [9, 12, 20]. Note also that any voltage potential u solving the equation (1) is also a minimizer of (2).

Theorem 2.2 ([20]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and assume that $a \in C(\overline{\Omega})$ is a non-negative function, and $f \in L^1(\partial\Omega)$. Then there exists a divergence free vector field $J \in (L^\infty(\Omega))^n$ with $|J| \leq a$ a.e. in Ω such that every minimizer w of (2) satisfies*

$$J \cdot \frac{Dw}{|Dw|} = |J| = a, \quad |Dw| - \text{a.e. in } \Omega, \quad (8)$$

where $\frac{Dw}{|Dw|}$ is the Radon-Nikodym derivative of Dw with respect to $|Dw|$.

Remark 2.3. Throughout the paper we will assume that $\partial\Omega$ is Lipschitz at the very least.

Lemma 2.4. *Let $f, \tilde{f} \in L^1(\partial\Omega)$. Suppose u solves (1) for $u|_{\partial\Omega} = f$, and $\tilde{u}$ solves (1) for $\tilde{u}|_{\partial\Omega} = \tilde{f}$. Then there exists $C(m, M, \Omega, f) > 0$ such that*

$$\max \left\{ \int_{\Omega} |D\tilde{u}|, \int_{\Omega} |Du| \right\} \leq C. \quad (9)$$

Proof. Fix $w \in BV_f(\Omega)$ and let $\tilde{w} \in BV_{\tilde{f}}(\Omega)$. Then in view of (7) we have

$$\begin{aligned} m \int_{\Omega} |D\tilde{u}| dx &\leq \int_{\Omega} \tilde{a} |D\tilde{u}| dx \leq \int_{\Omega} \tilde{a} |D\tilde{w}| dx \leq M \int_{\Omega} |D\tilde{w}| \\ &\leq M \int_{\Omega} |Dw| + M \int_{\Omega} |D(w - \tilde{w})| \\ &\leq M \int_{\Omega} |Dw| + MC_1 \|f - \tilde{f}\|_{L^1(\partial\Omega)} \\ &\leq M \int_{\Omega} |Dw| + M^2 C_1 |\Omega| =: C(m, M, \Omega, f), \end{aligned}$$

where we have used Theorem 2.16 in [9] to get the fifth inequality above. Similarly we can establish an analogous estimate for u and show that $\int_{\Omega} |Du| \leq C$, where C is the constant appearing in the above estimates. Hence

$$\max \left\{ \int_{\Omega} |D\tilde{u}|, \int_{\Omega} |Du| \right\} \leq C,$$

for some $C(m, M, \Omega, f)$ independent of $\tilde{u}$, u , and $\tilde{f}$. $\square$

Lemma 2.5. *Let $f, \tilde{f} \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are the corresponding minimizers of (2) with the weights a and $\tilde{a}$, respectively. Then*

$$\left| \int_{\Omega} a |Du| dx - \int_{\Omega} \tilde{a} |D\tilde{u}| dx \right| \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)} + C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}, \quad (10)$$

for some constants $C_i = C(m, M, \Omega, f)$ independent of u , $\tilde{u}$, and $\tilde{f}$.

Proof. Let $w \in BV(\Omega)$ such that $w|_{\partial\Omega} = f - \tilde{f}$. Suppose $u, \tilde{u}$ are the minimizers of (2) with the weights a and $\tilde{a}$ and boundary data f and $\tilde{f}$, respectively. Note:

$$u - w \in BV_{\tilde{f}}(\Omega), \quad \tilde{u} + w \in BV_f(\Omega)$$

We have

$$\int_{\Omega} a|Du|dx - \int_{\Omega} \tilde{a}|D\tilde{u}|dx = \int_{\Omega} (a - \tilde{a})(|Du| + |D\tilde{u}|)dx + \int_{\Omega} \tilde{a}|Du| - a|D\tilde{u}|dx \quad (11)$$

Hence

$$\begin{aligned} \int_{\Omega} a|Du|dx - \int_{\Omega} \tilde{a}|D\tilde{u}|dx &\leq (\|Du\|_{L^1(\Omega)} + \|D\tilde{u}\|_{L^1(\Omega)}) \|a - \tilde{a}\|_{L^\infty(\Omega)} + \int_{\Omega} \tilde{a}|Du| - a|D\tilde{u}|dx \\ &\leq 2C\|a - \tilde{a}\|_{L^\infty(\Omega)} + \int_{\Omega} \tilde{a}|Du| - a|D\tilde{u}|dx \end{aligned}$$

Where we have applied Lemma 2.5 to the first term. Focusing on the second term, we have

$$\begin{aligned} \int_{\Omega} \tilde{a}|Du| - a|D\tilde{u}|dx &= \int_{\Omega} \tilde{a}|Du| - a|D(\tilde{u} + w) - Dw|dx \\ &\leq \int_{\Omega} \tilde{a}|Du| - a|D(\tilde{u} + w)| + a|Dw|dx \\ &\leq \int_{\Omega} \tilde{a}|Du| - a|Du| + a|Dw|dx \\ &\leq \|Du\|_{L^1(\Omega)}\|a - \tilde{a}\|_{L^\infty(\Omega)} + M \int_{\Omega} |Dw|dx \end{aligned}$$

This comes from the triangle inequality and the fact that u is a minimizer for (2) on $BV_f(\Omega)$. Now, by invoking the extension Theorem 2.16 in [9] we get:

$$\|Dw\|_{L^1(\Omega)} \leq C'\|f - \tilde{f}\|_{L^1(\partial\Omega)} \quad (12)$$

and subsequently

$$\int_{\Omega} a|Du|dx - \int_{\Omega} \tilde{a}|D\tilde{u}|dx \leq 2C\|a - \tilde{a}\|_{L^\infty(\Omega)} + MC'\|f - \tilde{f}\|_{L^1(\partial\Omega)} \quad (13)$$

Similarly, we can prove

$$\int_{\Omega} \tilde{a}|D\tilde{u}|dx - \int_{\Omega} a|Du|dx \leq 2C\|a - \tilde{a}\|_{L^\infty(\Omega)} + MC'\|f - \tilde{f}\|_{L^1(\partial\Omega)},$$

and hence (10) follows. $\square$

Let ν_Ω denote the outer unit normal vector to $\partial\Omega$. Then for every $T \in (L^\infty(\Omega))^n$ with $\nabla \cdot T \in L^n(\Omega)$ there exists a unique function $[T, \nu_\Omega] \in L^\infty(\partial\Omega)$ such that

$$\int_{\partial\Omega} [T, \nu_\Omega]u \, d\mathcal{H}^{n-1} = \int_{\Omega} u \nabla \cdot T \, dx + \int_{\Omega} T \cdot Du \, dx, \quad u \in C^1(\bar{\Omega}). \quad (14)$$

Moreover, for $u \in BV(\Omega)$ and $T \in (L^\infty(\Omega))^n$ with $\nabla \cdot T \in L^n(\Omega)$, the linear functional $u \mapsto (T \cdot Du)$ gives rise to a Radon measure on Ω , and (14) holds for every $u \in BV(\Omega)$ (see [1, 2] for a proof). We shall need the weak integration by parts formula (14).

Lemma 2.6. *Let $f, \tilde{f} \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (2) with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2. Suppose $0 \leq \sigma(x) \leq \sigma_1$ in Ω for some constant $\sigma_1 > 0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du|$. Then*

$$\int_{\Omega} |J| |\tilde{J}| - J \cdot \tilde{J} dx \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)} + C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}, \quad (15)$$

where $C_i = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$ and $\tilde{f}$.

Proof. We have

$$\begin{aligned} \int_{\Omega} |J| |\tilde{J}| - J \cdot \tilde{J} dx &= \int_{\Omega} \sigma |\tilde{J}| |Du| - \sigma \tilde{J} \cdot Du dx \\ &\leq \sigma_1 \int_{\Omega} |\tilde{J}| |Du| - \tilde{J} \cdot Du dx \\ &= \sigma_1 \left(\int_{\Omega} |\tilde{J}| |Du| dx - \int_{\partial\Omega} f [\tilde{J}, \nu_\Omega] dx \right) \\ &= \sigma_1 \left(\int_{\Omega} |\tilde{J}| |Du| dx + \int_{\partial\Omega} (\tilde{f} - f) [\tilde{J}, \nu_\Omega] dx - \int_{\partial\Omega} \tilde{f} [\tilde{J}, \nu_\Omega] dx \right) \\ &\leq \sigma_1 \left(\int_{\Omega} |\tilde{J}| |Du| - \tilde{J} \cdot D\tilde{u} dx + \|[\tilde{J}, \nu_\Omega]\|_{L^\infty(\partial\Omega)} \|f - \tilde{f}\|_{L^1(\partial\Omega)} \right) \\ &\leq \sigma_1 \left(\int_{\Omega} |\tilde{J}| |Du| - |\tilde{J}| |D\tilde{u}| dx + \|\tilde{a}\|_{L^\infty(\Omega)} \|f - \tilde{f}\|_{L^1(\partial\Omega)} \right) \\ &\leq \sigma_1 \left(\int_{\Omega} |\tilde{J}| |Du| - |\tilde{J}| |D\tilde{u}| dx + M \|f - \tilde{f}\|_{L^1(\partial\Omega)} \right) \end{aligned}$$

where we have used the integration by parts formula (14) to get the second inequality above. On the other hand, it follows from Lemma 2.5 that

$$\begin{aligned} \sigma_1 \int_{\Omega} |\tilde{J}| |Du| - |\tilde{J}| |D\tilde{u}| dx &= \sigma_1 \int_{\Omega} |\tilde{J}| |Du| - |J| |Du| + |J| |Du| - |\tilde{J}| |D\tilde{u}| dx \\ &= \sigma_1 \left(\int_{\Omega} (a - \tilde{a}) |Du| dx + \int_{\Omega} a |Du| - \tilde{a} |D\tilde{u}| dx \right) \\ &\leq \sigma_1 \|Du\|_{L^1(\Omega)} \|a - \tilde{a}\|_{L^\infty(\Omega)} \\ &\quad + \sigma_1 C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)} + \sigma_1 C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{\Omega} |J||\tilde{J}| - J \cdot \tilde{J} dx &\leq \sigma_1 (\|Du\|_{L^1(\Omega)} + C_1) \|a - \tilde{a}\|_{L^\infty(\Omega)} \\ &\quad + \sigma_1 (M + C_2) \|f - \tilde{f}\|_{L^1(\partial\Omega)}, \end{aligned}$$

which yields the desired result. $\square$

Roughly speaking, Lemma 2.6 implies that as $a \rightarrow \tilde{a}$ and $f \rightarrow \tilde{f}$, $\frac{Du}{|Du|}(x)$ becomes parallel to $\frac{D\tilde{u}}{|D\tilde{u}|}(x)$ at points where the two gradients do not vanish. We are now ready to prove the main result of this section.

Theorem 2.7. *Let $f, \tilde{f} \in L^1(\partial\Omega)$, and assume there exist u and $\tilde{u}$ which are minimizers of (2) with the weights a and $\tilde{a}$ and boundary data f and $\tilde{f}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2. Suppose $0 \leq \sigma(x) \leq \sigma_1$ in Ω for some constant $\sigma_1 > 0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du|$. Then*

$$\|J - \tilde{J}\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}^{\frac{1}{2}}, \quad (16)$$

where $C_i = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$ and $\tilde{f}$.

Proof. The second line following from the argument outlined in the beginning of Theorem 2.5 in [15] we have:

$$\begin{aligned} \|J - \tilde{J}\|_{L^1(\Omega)} &= \int_{\Omega} \left(|J - \tilde{J}|^2 \right)^{\frac{1}{2}} dx \\ &\leq \int_{\Omega} \left| |J| - |\tilde{J}| \right| dx + \int_{\Omega} \left(2(|J||\tilde{J}| - J \cdot \tilde{J}) \right)^{\frac{1}{2}} dx \\ &= \int_{\Omega} |a - \tilde{a}| dx + \int_{\Omega} \left(2(|J||\tilde{J}| - J \cdot \tilde{J}) \right)^{\frac{1}{2}} dx \\ &\leq |\Omega| \|a - \tilde{a}\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{2}} \left(\int_{\Omega} 2(|J||\tilde{J}| - J \cdot \tilde{J}) dx \right)^{\frac{1}{2}} \\ &\leq |\Omega| \|a - \tilde{a}\|_{L^\infty(\Omega)} + (2|\Omega|)^{\frac{1}{2}} (C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)} + C_2 \|f - \tilde{f}\|_{L^\infty(\partial\Omega)})^{\frac{1}{2}} \\ &\leq |\Omega| \|a - \tilde{a}\|_{L^\infty(\Omega)} \\ &\quad + (2|\Omega|)^{\frac{1}{2}} \left[(C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)})^{\frac{1}{2}} + (C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)})^{\frac{1}{2}} \right] \\ &\leq \left[|\Omega| (2M)^{\frac{1}{2}} + (2C_1 |\Omega|)^{\frac{1}{2}} \right] \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \\ &\quad + [2C_2 |\Omega|]^{\frac{1}{2}} \|f - \tilde{f}\|_{L^1(\partial\Omega)}^{\frac{1}{2}}, \end{aligned}$$

where we have used the Holder's inequality and Lemma 2.6. $\square$

Remark 2.8. In view of Theorem 2.2, $\frac{Du}{|Du|}$ and $\frac{D\tilde{u}}{|D\tilde{u}|}$ are parallel to J and $\tilde{J}$, respectively. So Theorem 2.7 roughly implies that if $\tilde{a}$ is close to a and $\tilde{f}$ is close to f , then the structure of level sets of $\tilde{u}$ is close to that of u .

3 L^1 stability of the minimizers

In this section, we establish stability of minimizers of the least gradient problem (2) in L^1 . In general (2) does not even have unique minimizers, so in order to prove any stability results further assumptions on the weights $a, \tilde{a}$, and on the corresponding minimizers are expected.

Definition 3.1. Fix the positive constants $\sigma_0, \sigma_1 \in \mathbb{R}$. We say that $u \in C^1(\bar{\Omega})$ is admissible if it solves the conductivity equation (1) for some $\sigma \in C(\Omega)$ with

$$0 < \sigma_0 \leq \sigma \leq \sigma_1,$$

and $m \leq |J| = |\sigma \nabla u| \leq M$, where m and M are positive constants as in (7). We shall denote the corresponding induced current by $J = -\sigma \nabla u$.

We will first prove our results in dimension $n = 2$ and then extend them to dimensions $n = 3$.

Let $u \in C^1(\Omega)$ with $|\nabla u| > 0$ in Ω . Then it follows from the regularity result of De Giorgi (see, e.g, Theorem 4.11 in [9]) that all level sets of u are C^1 curves. We will assume that the length of level sets of u in Ω is uniformly bounded, i.e.

$$\sup_{t \in \mathbb{R}} \int_{\{u=t\} \cap \Omega} 1 dl = L_M < \infty. \quad (17)$$

Theorem 3.2. Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$, and corresponding current density vector fields J and $\tilde{J}$, respectively. If u satisfies (17), then

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2 \|f - \tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}}, \quad (18)$$

for some constants $C_i(m, M, \sigma_0, \sigma_1, f, u, L_M)$ independent of $\tilde{u}$, $\tilde{\sigma}$, and $\tilde{f}$.

Proof. Since u is admissible,

$$|\nabla u(x)| = \frac{|J(x)|}{\sigma(x)} \geq \frac{m}{\sigma_1} > 0, \quad \forall x \in \Omega.$$

Using the coarea formula we get

$$\frac{m}{\sigma_1} \int_{\Omega} |u - \tilde{u}| dx \leq \int_{\Omega} |\nabla u| |u - \tilde{u}| dx = \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dS dt. \quad (19)$$

Since $|\nabla u| > 0$ in Ω , it follows from the regularity result of De Giorgi (Theorem 4.11 in [9]) that all level sets of u are C^1 curves. Now let Γ_t be a connected component of $\{x \in \Omega : u(x) = t\} \subset \Omega$, and $\gamma : [0, L] \rightarrow \Gamma_t$ to be a path parametrized by the arc length with $\gamma(0) \in \partial\Omega$. We will henceforth denote $\gamma(0)$ by x_t^0 . Define

$$h(s) := u(\gamma(s)) - \tilde{u}(\gamma(s)).$$

Since $\nabla u(\gamma(s)) \cdot \gamma'(s) = 0$ on Γ_t , we have

$$\begin{aligned} h'(s) &= \nabla u(\gamma(s)) \cdot \gamma'(s) - \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) \\ &= \left(\frac{\sigma}{\tilde{\sigma}}(\gamma(s)) \nabla u(\gamma(s)) - \nabla \tilde{u}(\gamma(s)) \right) \cdot \gamma'(s). \end{aligned}$$

We can rewrite the above equality as

$$h'(s) = \frac{J(\gamma(s)) - \tilde{J}(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s).$$

Note that

$$h(0) = u(\gamma(0)) - \tilde{u}(\gamma(0)) = f(x_t^0) - \tilde{f}(x_t^0).$$

Consequently, we have that

$$h(s) - h(0) = \int_0^s \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau$$

and, moreover,

$$h(s) = \int_0^s \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau + f(x_t^0) - \tilde{f}(x_t^0).$$

Now let x_t^* be a point on Γ_t where the maximum distance between u and $\tilde{u}$ along the path γ occurs, i.e.

$$|u(x_t^*) - \tilde{u}(x_t^*)| = \max_{x \in \Gamma_t} |u(x) - \tilde{u}(x)|.$$

Then $x_t^* = \gamma(s_0)$ for some $s_0 \in [0, L]$, and

$$\begin{aligned}
|u(x_t^*) - \tilde{u}(x_t^*)| = |h(s_0)| &= \left| \int_0^{s_0} \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau + f(x_t^0) - \tilde{f}(x_t^0) \right| \\
&\leq \int_0^{s_0} \frac{1}{\tilde{\sigma}(\gamma(\tau))} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)| \\
&\leq \frac{1}{\sigma_0} \int_0^{s_0} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)|.
\end{aligned}$$

In particular for every $x \in \Gamma_t$

$$|u(x) - \tilde{u}(x)| \leq |u(x_t^*) - \tilde{u}(x_t^*)| \leq \frac{1}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)|,$$

where L denotes the entire length of Γ_t . Hence

$$\begin{aligned}
\int_{\Gamma_t} |u(x) - \tilde{u}(x)| dl &\leq |u(x_t^*) - \tilde{u}(x_t^*)| \int_{\Gamma_t} 1 dl \\
&\leq L_M |u(x_t^*) - \tilde{u}(x_t^*)| \\
&\leq \frac{L_M}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + L_M |f(x_t^0) - \tilde{f}(x_t^0)| \\
&= \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)|,
\end{aligned}$$

and therefore

$$\int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dl \leq \frac{L_M}{\sigma_0} \int_{\{u=t\} \cap \Omega} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)|. \quad (20)$$

Since $u \in C(\overline{\Omega})$ solves (1), by maximum and minimum principles for solutions to elliptic equations,

$$\max_{\Omega} u = \max_{\partial\Omega} f := C_f$$

$$\min_{\Omega} u = \min_{\partial\Omega} f := c_f$$

and hence $c_f \leq u \leq C_f$, with $-M \leq c_f, C_f \leq M$. Thus we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dl dt &= \int_{c_f}^{C_f} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dl dt \\
&\leq L_M \int_{c_f}^{C_f} \left(\int_{\{u=t\} \cap \Omega} \frac{1}{\sigma_0} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)| \right) dt \\
&\leq \frac{L_M}{\sigma_0} \int_{c_f}^{C_f} \int_{\{u=t\} \cap \Omega} |J - \tilde{J}| dl dt + L_M \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \int_{c_f}^{C_f} dt \\
&= \frac{L_M}{\sigma_0} \int_{\Omega} |\nabla u| |J - \tilde{J}| dx + L_M (C_f - c_f) \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\leq \frac{L_M}{\sigma_0} \|\nabla u\|_{L^\infty(\Omega)} \int_{\Omega} |J - \tilde{J}| dx + L_M (C_f - c_f) \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\leq \frac{L_M}{\sigma_0} \|\nabla u\|_{L^\infty(\Omega)} \left[C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}^{\frac{1}{2}} \right] \\
&\quad + L_M (C_f - c_f) \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\leq \frac{L_M M C_1}{\sigma_0^2} \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \\
&\quad + \left[\frac{L_M M C_2}{\sigma_0^2} |\partial\Omega|^{\frac{1}{2}} + L_M (C_f - c_f) (2M)^{\frac{1}{2}} \right] \|f - \tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}},
\end{aligned}$$

where we have used (20) and Theorem 2.7. Hence (18) follows. Note that $C_i(m, M, \sigma_0, \sigma_1, f, u, L_M)$ are independent of $\tilde{u}$, $\tilde{\sigma}$, and $\tilde{f}$. $\square$

Next we generalize Theorem 3.2 to dimension $n = 3$. In order to do this, we need the following additional assumption on level sets of u .

Definition 3.3. Let $u \in C^1(\bar{\Omega})$ be admissible. We say that level sets of u can be foliated to one-dimensional curves if for almost every $t \in \text{range}(u)$, every connected component Γ_t of $\{u = t\}$ (equipped with the metric induced from the Euclidean metric in $\mathbb{R}^3$) there exists a function $g_t(x) \in C^1(\Gamma_t)$ such that $0 < c_g \leq |\nabla g_t|$, $|g_t| \leq C_g$, for some constants c_g and C_g independent of t (where ∇g_t is being taken on the tangent space of Γ_t). Moreover, every connected component of $\{u = t\} \cap \{g_t = r\} \cap \Omega$ is a C^1 curve reaching the boundary $\partial\Omega$ for almost every $t \in \text{range}(u)$ and all $r \in \mathbb{R}$. Similar to the case $n = 2$, we assume that the length of connected components of $\{u = t\} \cap \{g_t = r\} \cap \Omega$ are uniformly bounded by some constant L_M .

See Remark 3.5 in [15] for a discussion about sufficient conditions that guarantee the assumptions in Definition 3.3 to hold.

Definition 3.4. Let $t \in \text{range}(u)$ and suppose Γ_t^i , $i \in I$, are C^1 connected components of $\{u = t\}$, where I is countable. Then there exists functions $g_t^i : \Gamma_t^i \rightarrow R$ whose level sets foliate Γ_t^i into one dimensional curves in the sense of Definition 3.3. We define $g_t : \{u = t\} \rightarrow R$ be the function with

$$g_t|_{\Gamma_t^i} = g_t^i, \quad i \in I. \quad (21)$$

We shall use this notation throughout the paper.

Theorem 3.5. Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ and corresponding current density vector fields J and $\tilde{J}$, respectively. Suppose the level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.3. Then

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2 \|f - \tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}}, \quad (22)$$

where $C(m, M, \sigma_0, \sigma_1, f, u, L_M, c_g, C_g, g)$ is independent of $\tilde{u}$, $\tilde{\sigma}$, and $\tilde{f}$.

Proof. The proof is similar to the proof of Theorem 3.2, and we provide the details for the sake of the reader. Since u is admissible,

$$\frac{m}{\sigma_1} \int_{\Omega} |u - \tilde{u}| dx \leq \int_{\Omega} |\nabla u| |u - \tilde{u}| dx = \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dS dt. \quad (23)$$

The level sets of u can be foliated into one-dimensional curves by level sets of some function g in the sense of Definition 3.3. Thus

$$\begin{aligned} \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dS dt &= \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} \frac{|\nabla g_t|}{|\nabla g_t|} |u - \tilde{u}| dS dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} \frac{1}{|\nabla g_t|} |u - \tilde{u}| dl dr dt \\ &\leq \frac{1}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |u - \tilde{u}| dl dr dt. \end{aligned}$$

Similar to the two dimensional case, we parameterize every connected component Γ_t of $\{u = t\} \cap \{g = r\} \cap \Omega$ by arc length, $\gamma : [0, L] \rightarrow \Gamma_t$ with $\gamma(0) = x_t^0 \in \partial\Omega$, and let $h(s) = u(\gamma(s)) - \tilde{u}(\gamma(s))$. Let x_t^* be the point that maximizes $|u - \tilde{u}|$ on Γ_t and suppose $\gamma(s_0) = x_t^*$ for some $s_0 \in (0, L)$, where L is the length of Γ_t . Then by an argument similar to the one in the proof of Theorem 3.2 we get

$$|u(x_t^*) - \tilde{u}(x_t^*)| \leq \frac{1}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)|,$$

and consequently

$$\int_{\Gamma_t} |u(x) - \tilde{u}(x)| dl \leq \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)|.$$

Hence

$$\int_{\{u=t\} \cap \{g=r\} \cap \Omega} |u - \tilde{u}| dl \leq \frac{L_M}{\sigma_0} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)|. \quad (24)$$

Using this estimate and the coarea formula we have

$$\begin{aligned} \frac{m}{\sigma_1} \int_{\Omega} |u - \tilde{u}| dx &\leq \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dS dt \\ &\leq \frac{1}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |u - \tilde{u}| dl dr dt \\ &\leq \frac{1}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{L_M}{\sigma_0} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)| \right) dr dt \\ &= \frac{L_M}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |J - \tilde{J}| dl dr dt \\ &\quad + \frac{L_M}{c_g} \int_{\min f}^{\max f} \int_{\min g}^{\max g} |f(x_t^0) - \tilde{f}(x_t^0)| dr dt \\ &\leq \frac{L_M C_g}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |\nabla g_t| |J - \tilde{J}| dS dt \\ &\quad + \frac{2ML_M}{c_g} (2\|g\|_{L^\infty(\Omega)}) \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \\ &= \frac{L_M C_g}{c_g \sigma_0} \int_{\Omega} |\nabla u| |J - \tilde{J}| dx + \frac{4ML_M \|g\|_{L^\infty(\Omega)}}{c_g} \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \\ &\leq \frac{L_M C_g}{c_g \sigma_0} \|\nabla u\|_{L^\infty(\Omega)} \left(C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2 \|f - \tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}} \right) \\ &\quad + \frac{4ML_M \|g\|_{L^\infty(\Omega)}}{c_g} \|f - \tilde{f}\|_{L^\infty(\partial\Omega)} \\ &\leq \frac{L_M C_g M C_1}{c_g \sigma_0^2} \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \\ &\quad + \left[\frac{L_M C_g C_2}{c_g \sigma_0} + \frac{4ML_M \|g\|_{L^\infty(\Omega)}}{c_g} (2M|\Omega|)^{\frac{1}{2}} \right] \|f - \tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}}, \end{aligned}$$

where we have applied Theorem 1.3. □

4 $W^{1,1}$ stability of the minimizers

In this section, we prove stability of minimizers of (2) in $W^{1,1}$. As mentioned in Section 3, in general (2) does not even have unique minimizers, so in order to prove stability results in $W^{1,1}$, it is natural to expect stronger assumptions on the minimizers.

Lemma 4.1. *Let $n = 2, 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f, \tilde{u}|_{\partial\Omega} = \tilde{f}$ the respective traces of functions $f, \tilde{f} \in H^3(\Omega)$ and corresponding conductivities σ and $\tilde{\sigma}$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ with*

$$\|\sigma\|_{C^2(\Omega)}, \|\tilde{\sigma}\|_{C^2(\Omega)} \leq \sigma_2 \quad (25)$$

for some $\sigma_2 \in \mathbb{R}$. Let

$$G(x) := \frac{\tilde{J}(x) - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega, \quad (26)$$

with $G = (G_1, G_2)$ for $n = 2$ and $G = (G_1, G_2, G_3)$ for $n = 3$. Then

$$\|\nabla G_i\|_{L^1(\Omega)} \leq C_1 \|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}}, \quad (27)$$

for some constant C_1 which depends only on $\Omega, \sigma_0, \sigma_2$ and $\|f\|_{L^\infty(\Omega)}$.

Proof. The proof is similar to that of Lemma 4.1 in [15] and we omit it. $\square$

Next we prove that u and $\tilde{u}$ are close in $W^{1,1}(\Omega)$. In order to do so, we need additional assumptions on the structure of level sets of u .

Definition 4.2. *Suppose u is admissible, $n = 2$, and $x \in \Omega$. Pick $h \in \mathbb{R}^2$ with $|h| = 1$, and $t \in \mathbb{R}$ small enough such that $x + th \in \Omega$. Let Γ and Γ_t be the level sets of u passing through x and $x + th$, respectively. Parametrize Γ and Γ_t by the arc length such that $\gamma(0), \gamma_t(0) \in \partial\Omega$, and denote these parametrizations by γ and γ_t , respectively.*

Similarly in dimension $n = 3$, let u be admissible and suppose level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.3. Pick $x \in \Omega$ and $h \in \mathbb{R}^3$ with $|h| = 1$, and choose t small enough such that $x + th \in \Omega$. Let Γ and Γ_t be the unique curves in

$$\{\{u = \tau\} \cap \{g_\tau = r\} \mid \tau, r \in \mathbb{R}\}$$

which pass through x and $x + th$, respectively, and let γ and γ_t be the parametrization of these curves with respect to arc length with $\gamma(0), \gamma_t(0) \in \partial\Omega$.

We say that level sets of u are well structured if the following conditions are satisfied

(a) There exists $K \geq 0$ such that

$$\left| \frac{\gamma'_t(s) - \gamma'(s)}{t} \right| \leq K \quad (28)$$

for every $s \in [0, L]$, $t \in \mathbb{R}$, $x \in \Omega$ and $h \in S^{n-1}$. In particular,

$$\gamma'_t(s) \rightarrow \gamma'(s) \quad \text{as } t \rightarrow 0, \quad (29)$$

where $\gamma'(s) = \frac{d\gamma(s)}{ds}$ and $\gamma'_t(s) = \frac{d\gamma_t(s)}{ds}$.

(b) There exists a bounded function $F_{x,h}(s) = F(x, h; s) \in L^\infty(\Omega \times S^{n-1} \times [0, L_M])$ such that

$$\lim_{t \rightarrow 0} \frac{\gamma_t(s) - \gamma(s)}{t} = F_{x,h}(s) \quad (30)$$

for every $s \in [0, L]$, $x \in \Omega$ and $h \in S^{n-1}$.

See Remark 4.3 in [15] for a discussion on sufficient conditions which guarantee the assumptions of Definition 4.2 to hold.

Theorem 4.3. Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (25). If u satisfies (17), and the level sets of u are well-structured in the sense of Definition 4.2, then

$$\|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}}, \quad (31)$$

for some constant $C(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. Fix $x \in \Omega$ and $h \in \mathbb{R}^2$ with $|h| = 1$. Then

$$\mathcal{L}(x, h) := (\nabla \tilde{u}(x) - \nabla u(x)) \cdot h = \lim_{t \rightarrow 0} \frac{[\tilde{u}(x + th) - u(x + th)] - [\tilde{u}(x) - u(x)]}{t}.$$

First we estimate the above limit. Since all level sets of u reach the boundary $\partial\Omega$, there exist $z, z_t \in \partial\Omega$ such that

$$u(x) = u(z),$$

$$u(x + th) = u(z_t).$$

Thus

$$[\tilde{u}(x + th) - u(x + th)] - [\tilde{u}(x) - u(x)] = [\tilde{u}(x + th) - u(z_t)] - [\tilde{u}(x) - u(z)]$$

$$= [\tilde{u}(x+th) - \tilde{u}(z_t)] - [\tilde{u}(x) - \tilde{u}(z)] + [\tilde{u}(z_t) - u(z_t)] - [\tilde{u}(z) - u(z)]$$

Let γ and γ_t be the curves passing through x and $x+th$, described in Definition 4.2 with $\gamma(0) = z$ and $\gamma_t(0) = z_t$. Suppose $\gamma(s_0) = x$ and reparamterize γ_t so that $\gamma_t(s_0) = x+th$. Then we have

$$\begin{aligned} [\tilde{u}(x+th) - \tilde{u}(z)] - [\tilde{u}(x) - \tilde{u}(z)] &= [\tilde{u}(\gamma_t(s_0)) - \tilde{u}(\gamma_t(0))] - [\tilde{u}(\gamma(s_0)) - \tilde{u}(\gamma(0))] \\ &= \int_0^{s_0} \nabla \tilde{u}(\gamma_t(s)) \cdot \gamma'_t(s) ds - \int_0^{s_0} \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) ds. \end{aligned}$$

Hence

$$\mathcal{L}(x, h) = \lim_{t \rightarrow 0} \frac{1}{t} \left(\int_0^{s_0} \nabla \tilde{u}(\gamma_t(s)) \cdot \gamma'_t(s) ds - \int_0^{s_0} \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) ds \right) \quad (32)$$

$$+ \lim_{t \rightarrow 0} \frac{1}{t} ([\tilde{u}(z_t) - u(z_t)] - [\tilde{u}(z) - u(z)]) \quad (33)$$

Now, we can focus on the second term here by noticing

$$[\tilde{u}(z_t) - u(z_t)] - [\tilde{u}(z) - u(z)] = [\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)].$$

Also, we denote the tangential direction along $\partial\Omega$ at z by θ_z and we get,

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{[\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)]}{t} \\ &= \lim_{t \rightarrow 0} \left(\frac{[\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)]}{|z_t - z|} \right) \lim_{t \rightarrow 0} \frac{|z_t - z|}{t} \\ &\leq |F_{x,h}(0)| \lim_{t \rightarrow 0} \left(\frac{[\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)]}{|z_t - z|} \right) \\ &= |F_{x,h}(0)| \frac{\partial}{\partial \theta_z} (\tilde{f} - f) \\ &\leq \|F\|_{L^\infty(\Omega \times S^{n-1} \times [0, L_M])} \|\nabla(f - \tilde{f})\|_{L^\infty(\partial\Omega)} \\ &\leq \|F\|_{L^\infty(\Omega \times S^{n-1} \times [0, L_M])} \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}. \end{aligned} \quad (34)$$

We can now shift our focus onto the first term (32). Substituting $\nabla \tilde{u}$ by $\frac{\tilde{J}}{\tilde{\sigma}}$ and using the fact that J is perpendicular to γ' and γ'_t we get

$$\lim_{t \rightarrow 0} \frac{1}{t} \left(\int_0^{s_0} \frac{\tilde{J}(\gamma_t(s)) - J(\gamma_t(s))}{\tilde{\sigma}(\gamma_t(s))} \cdot \gamma'_t(s) ds - \int_0^{s_0} \frac{\tilde{J}(\gamma(s)) - J(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s) ds \right).$$

Now define

$$G(x) := \frac{J(\tilde{x}) - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega.$$

Hence we get

$$\lim_{t \rightarrow 0} \frac{1}{t} \left(\int_0^{s_0} G(\gamma_t(s)) \cdot \gamma'_t(s) ds - \int_0^{s_0} G(\gamma(s)) \cdot \gamma'(s) ds \right).$$

This term can be bounded in the same way as in the proof of Theorem 4.4 in [15], so we omit the calculation as it is identical. Hence we have

$$\begin{aligned} |\nabla \tilde{u}(x) - \nabla u(x)| &\leq \sup_{h \in \mathbb{R}^n, |h|=1} \mathcal{L}(x, h) \\ &\leq \frac{K}{\sigma_0} \int_0^L |\tilde{J}(\gamma(s)) - J(\gamma(s))| dl \\ &\quad + \|F\|_{L^\infty} \int_0^L |\nabla G_1(\gamma(s))| + |\nabla G_2(\gamma(s))| dl \\ &\quad + \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_\Gamma |\nabla \tilde{u}(x) - \nabla u(x)| dl &\leq \frac{KL_M}{\sigma_0} \int_\Gamma |\tilde{J}(x) - J(x)| dl \\ &\quad + L_M \|F\|_{L^\infty} \int_\Gamma |\nabla G_1(x)| + |\nabla G_2(x)| dl \\ &\quad + L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)}, \end{aligned}$$

and consequently

$$\begin{aligned} \int_{\{u=\tau\} \cap \Omega} |\nabla \tilde{u}(x) - \nabla u(x)| dl &\leq \frac{KL_M}{\sigma_0} \int_{\{u=\tau\} \cap \Omega} |\tilde{J}(x) - J(x)| dl \quad (35) \\ &\quad + L_M \|F\|_{L^\infty} \int_{\{u=\tau\} \cap \Omega} |\nabla G_1(x)| + |\nabla G_2(x)| dl \\ &\quad + L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)}. \end{aligned}$$

Using (35) and the coarea formula we have

$$\begin{aligned}
\frac{m}{\sigma_1} \|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} &\leq \int_{\Omega} |\nabla u| |\nabla \tilde{u} - \nabla u| dx \\
&= \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} |\nabla \tilde{u} - \nabla u| dld\tau \\
&\leq \frac{KL_M}{\sigma_0} \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} |\tilde{J} - J| dld\tau \\
&+ L_M \|F\|_{L^\infty} \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} |\nabla G_1| + |\nabla G_2| dld\tau \\
&+ L_M \|F\|_{L^\infty} (2M) \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \frac{KL_MM}{(\sigma_0)^2} \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} \frac{|\tilde{J} - J|}{|\nabla u|} dld\tau \\
&+ \frac{L_M \|F\|_{L^\infty} M}{\sigma_0} \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} \frac{|\nabla G_1| + |\nabla G_2|}{|\nabla u|} dld\tau \\
&+ 2ML_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&= \frac{KL_MM}{(\sigma_0)^2} \int_{\Omega} |\tilde{J} - J| dx \\
&+ \frac{L_M \|F\|_{L^\infty} M}{\sigma_0} \int_{\Omega} |\nabla G_1| + |\nabla G_2| dx \\
&+ 2ML_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \frac{KL_MM}{(\sigma_0)^2} \|J - \tilde{J}\|_{L^1(\Omega)} \\
&+ \frac{2L_MC_1 \|F\|_{L^\infty} M}{\sigma_0} \|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}} \\
&+ 2ML_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)}
\end{aligned}$$

where we have used Lemma 4.1 to obtain the last inequality. Applying Theorem 2.7, and noting that

$$\|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}} \leq (2M|\Omega|)^{\frac{1}{2}},$$

where M is defined in (7), we arrive at (31). $\square$

Now we prove three dimensional version of this theorem.

Theorem 4.4. *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (25). In addition suppose u satisfies (17), the level sets of u can be foliated to one-dimensional*

curves in the sense of Definition 3.3, and the level sets of u are well-structured in the sense of Definition 4.2. Then

$$\|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}}, \quad (36)$$

for some constant $C_i(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M, c_g, C_g)$ is independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. With an argument similar to the one used in the proof of Theorem 4.3 we get

$$\begin{aligned} \int_{U_{\tau,r}} |\nabla \tilde{u}(x) - \nabla u(x)| dl &\leq \frac{KL_M}{\sigma_0} \int_{U_{\tau,r}} |\tilde{J}(x) - J(x)| dl \\ &\quad + L_M \|F\|_{L^\infty} \int_{U_{\tau,r}} |\nabla G_1(x)| + |\nabla G_2(x)| + |\nabla G_3(x)| dl \\ &\quad + L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \end{aligned} \quad (37)$$

where $U_{\tau,r} := \{u = \tau\} \cap \{g_\tau = r\} \cap \Omega$ and $G = (G_1, G_2, G_3)$ is defined in (26).

It follows from (37) and the coarea formula that

$$\begin{aligned} \frac{m}{\sigma_1} \|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} &\leq \int_{\Omega} |\nabla u| |\nabla \tilde{u} - \nabla u| dx \\ &= \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} |\nabla \tilde{u} - \nabla u| dS d\tau \\ &= \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} \frac{|\nabla g_\tau|}{|\nabla g_\tau|} |\nabla \tilde{u} - \nabla u| dS d\tau \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{1}{|\nabla g_\tau|} |\nabla \tilde{u} - \nabla u| dl dr d\tau \\ &\leq \frac{KL_M}{\sigma_0 c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} |\tilde{J} - J| dl dr dt \\ &\quad + \frac{L_M \|F\|_{L^\infty}}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} |\nabla G_1| + |\nabla G_2| + |\nabla G_3| dl dr dt \\ &\quad + 2\|g\|_{L^\infty(\Omega)} L_M \|F\|_{L^\infty} (2M) \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{|\tilde{J} - J|}{|\nabla u| |\nabla g_{\tau}|} dl dr dt \\
&+ \frac{L_M M \|F\|_{L^\infty} C_g}{\sigma_0 c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u| |\nabla g_t|} dl dr dt \\
&+ 4M \|g\|_{L^\infty(\Omega)} L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&= \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} \frac{|\tilde{J} - J|}{|\nabla u|} dS dt \\
&+ \frac{L_M M \|F\|_{L^\infty} C_g}{\sigma_0 c_g} \int_{\mathbb{R}} \int_{\{u=\tau\} \cap \Omega} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u|} dS dt \\
&+ 4M \|g\|_{L^\infty(\Omega)} L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&= \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \int_{\Omega} |\tilde{J} - J| dx \\
&+ \frac{L_M M \|F\|_{L^\infty} C_g}{\sigma_0 c_g} \int_{\Omega} |\nabla G_1| + |\nabla G_2| + |\nabla G_3| dx \\
&+ 4M \|g\|_{L^\infty(\Omega)} L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \frac{KL_M MC_g}{(\sigma_0)^2} \|J - \tilde{J}\|_{L^1(\Omega)} \\
&+ \frac{2L_M C_1 M \|F\|_{L^\infty(\Omega)} C_g}{\sigma_0} \|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}} \\
&+ 4M \|g\|_{L^\infty(\Omega)} L_M \|F\|_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)},
\end{aligned}$$

where we have used (4.1) to obtain the last inequality. Applying Theorem 2.7, and noting that

$$\|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}} \leq (2M|\Omega|)^{\frac{1}{2}},$$

we obtain the inequality (31). $\square$

Now, we are ready to prove our main stability results.

Theorem 4.5. *Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (25). If u satisfies (17) and level sets of u are well-structured in the sense of Definition 4.2, then*

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}},$$

for some constants $C_i(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, f, L_M)$ independent of $\tilde{\sigma}$.

Proof. Using Theorem 4.3 we have

$$\begin{aligned}
\int_{\Omega} |\sigma - \tilde{\sigma}| dx &= \int_{\Omega} \left| \frac{|J|(|\nabla \tilde{u}| - |\nabla u|)}{|\nabla u||\nabla \tilde{u}|} + \frac{|J| - |\tilde{J}|}{|\nabla \tilde{u}|} \right| dx \\
&\leq \int_{\Omega} \frac{|J|}{|\nabla u||\nabla \tilde{u}|} ||\nabla u| - |\nabla \tilde{u}|| dx + \int_{\Omega} \frac{1}{|\nabla \tilde{u}|} \left| |J| - |\tilde{J}| \right| dx \\
&\leq \int_{\Omega} \frac{|J|}{|\nabla u||\nabla \tilde{u}|} |\nabla u - \nabla \tilde{u}| dx + \int_{\Omega} \frac{1}{|\nabla \tilde{u}|} \left| |J| - |\tilde{J}| \right| dx \\
&\leq \frac{M\sigma_1^2}{m^2} \left(C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}} \right) \\
&\quad + \frac{\sigma_1|\Omega|}{m} \|a - \tilde{a}\|_{L^\infty(\Omega)} \\
&\leq \left[\frac{M\sigma_1^2 C_1}{m^2} + \frac{\sigma_1|\Omega|(2M)^{\frac{3}{4}}}{m} \right] \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} \\
&\quad + \frac{M\sigma_1^2 C_2}{m^2} \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}}
\end{aligned}$$

□

Theorem 4.6. *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (25). If u satisfies (17), the level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.3, and the level sets of u are well-structured in the sense of Definition 4.2, then*

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}},$$

for some constants $C_i(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, f, L_M, g)$ independent of $\tilde{\sigma}$.

Proof. The proof follows from Theorem 4.4 and a calculation similar to that of the proof of Theorem 4.5. □

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