
Unifying Clustered and Non-stationary Bandits

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Abstract

Non-stationary bandits and clustered bandits lift the restrictive assumptions in contextual bandits and provide solutions to many important real-world scenarios. Though they have been studied independently so far, we point out the essence in solving these two problems overlaps considerably. In this work, we connect these two strands of bandit research under the notion of test of homogeneity, which seamlessly addresses change detection for non-stationary bandit and cluster identification for clustered bandit in a unified solution framework. Rigorous regret analysis and extensive empirical evaluations demonstrate the value of our proposed solution, especially its flexibility in handling various environment assumptions, e.g., a clustered non-stationary environment.

1 Introduction

Most existing contextual bandit algorithms impose strong assumptions on the mapping between context and reward (Abbasi-Yadkori et al., 2011; Chu et al., 2011; Li et al., 2010): typically it is assumed that the expected reward associated with a particular action is determined by a *time-invariant function* of the context vector. This overly simplified assumption restricts the application of contextual bandits in many important real-world scenarios, where a learner has to serve a population of users with possible mutual dependence and changing interest. This directly motivates recent efforts that postulate more general reward assumptions (Wu et al., 2016; Filippi et al., 2010; Mailard and Mannor, 2014; Kleinberg et al., 2008); among

them, *non-stationary bandits* (Wu et al., 2018; Slivkins and Upfal, 2008; Cao et al., 2019; Besson and Kaufmann, 2019; Russac et al., 2019; Chen et al., 2019) and *clustered bandits* (Gentile et al., 2014; Li et al., 2016; Gentile et al., 2017; Li et al., 2019) received much attention.

In non-stationary bandits, the reward mapping function becomes time-variant. A typical non-stationary setting is the abruptly changing environment, a.k.a, a piecewise stationary environment, in which the environment undergoes abrupt changes at unknown time points but remains stationary between two consecutive change points (Yu and Mannor, 2009; Garivier and Moulines, 2011). A working solution needs to either properly discount historical observations (Hartland et al., 2006; Garivier and Moulines, 2011; Russac et al., 2019) or detect the change points and reset the model estimation accordingly (Yu and Mannor, 2009; Cao et al., 2019; Wu et al., 2018). In clustered bandits, grouping structures of bandit models are assumed, e.g., users in a group share the same bandit model. But instead of assuming an explicit dependency structure, e.g., leveraging existing social network among users (Cesa-Bianchi et al., 2013; Wu et al., 2016), clustered bandit algorithms aim to simultaneously cluster and estimate the bandit models during the sequential interactions with users (Gentile et al., 2014; Li et al., 2016; Gentile et al., 2017; Li et al., 2019). Its essence is thus to measure the relatedness between different bandit models. Typically, confidence bound of model parameter estimation (Gentile et al., 2014) or reward estimation (Gentile et al., 2017) is used for this purpose.

So far these two problems have been studied in parallel; but the key principles to solve them overlap considerably. On the one hand, mainstream solutions for piecewise stationary bandits detect change points in the underlying reward distribution by comparing the observed rewards (Cao et al., 2019) or the quality of estimated rewards (Yu and Mannor, 2009; Wu et al., 2018) in a window of consecutive observations. If change happens in the window, the designed statistics

of interest would exceed a threshold with a high probability. This is essentially sequential hypothesis testing of a model’s fitness (Siegmund, 2013). On the other hand, existing solutions for clustered bandits evaluate if two bandit models share the same set of parameters (Gentile et al., 2014; Li et al., 2016) or the same reward estimation on a particular arm (Gentile et al., 2017). This can also be understood as a goodness-of-fit test between models.

In this work, we take the first step to unify these two parallel strands of bandit research under the notion of *test of homogeneity*, and study non-stationarity in linear bandit with time-varying arm set, which distinguishes us from most existing work. We address both problems by testing whether the collection of observations in a bandit model follows the same distribution as that of new observations (i.e., change detection in non-stationary bandit algorithms) or of those in another bandit model (i.e., cluster identification in clustered bandit algorithms). Built upon our solution framework, bandit models can operate on individual users with much stronger flexibility, so that new bandit learning problems can be created and addressed. This enables us to study a new and challenging bandit problem in a *clustered non-stationary* environment, where the learner has to reset individual models when a change of reward distribution is detected, and merge them when they are determined as identical. This task of doing change detection while clustering is novel and important by itself (Mazhar et al., 2018), and has never been considered in bandit problem where the observations are non-IID in nature. Since our solution automatically detects changes and clustering structure, it has a much weaker assumption about the environment (e.g., it can be clustered, or non-stationary, or both). Furthermore, our solution enables data sharing across both users and time, when such structure exists in the environment, thus greatly reducing sample complexity in learning bandit models. Our rigorous regret analysis and extensive empirical evaluations demonstrate the value of this unified solution, especially its advantages in handling various environment assumptions.

2 Related work

Our work is closely related to the studies in non-stationary bandits and clustered bandits. In this section, we discuss the most representative solutions in each direction and highlight their connections.

Non-stationary bandits. Instead of assuming a time-invariant environment, the reward mapping is allowed to change over time in this problem setting. Commonly imposed assumptions include slowly-varying environment (Besbes et al., 2019; Che-

ung et al., 2019) and abruptly-changing environment (Moulines, 1985; Wu et al., 2018; Auer et al., 2019). We focus on the latter setting, which is also known as a piecewise stationary environment in literature (Yu and Mannor, 2009; Garivier and Moulines, 2011). In a non-stationary setting, the main focus is to eliminate the distortion from out-dated observations, which follow a different reward distribution than that of the current environment. Popular solutions for the piecewise stationary environment actively detect change points and reset bandit models accordingly (Yu and Mannor, 2009; Cao et al., 2019; Besson and Kaufmann, 2019; Wu et al., 2018; Hariri et al., 2015; Auer et al., 2019; Chen et al., 2019). It should be noted that this paper studies non-stationarity in linear bandit with time-varying arm set (Wu et al., 2018; Cheung et al., 2019; Russac et al., 2019; Zhao et al., 2020), which is different from the solutions for non-stationary MAB problem (Yu and Mannor, 2009; Cao et al., 2019; Besson and Kaufmann, 2019; Hariri et al., 2015; Auer et al., 2019) or the non-stationary contextual MAB (Agarwal et al., 2014; Luo et al., 2018; Chen et al., 2019). Therefore, their results do not apply to the setting considered in this paper. The closest work to our setting is Wu et al. (2018), which maintains a pool of base linear bandit models and adaptively adds or selects from them via a change detector, which monitors how well each base bandit model predicts the new observations. This in essence boils down to a likelihood-ratio test for change in the bandit parameter. To the best of our knowledge, all the other studies for non-stationary linear bandit assume a slowly-varying environment and adopts strategies like sliding window (Cheung et al., 2019), decaying weight (Russac et al., 2019) or periodical restart (Zhao et al., 2020) to eliminate the distortion from out-dated observations.

Clustered bandits. When serving a population of users, the vanilla linear bandit usually models the preference of each individual user in isolation, neglecting the correlation between users. In order to improve sample efficiency, such user correlation can be utilized to enable collaboration among each individual bandit models (Li, 2016; Gentile et al., 2014; Li et al., 2016; Gentile et al., 2017; Cesa-Bianchi et al., 2013; Wu et al., 2016). Besides leveraging explicit structure among users, such as social networks (Buccapatnam et al., 2013; Cesa-Bianchi et al., 2013; Wu et al., 2016; Yang et al., 2020), recent efforts focus on online clustering of bandits via the interactions with users (Gentile et al., 2014; Li et al., 2016; Gentile et al., 2017; Li et al., 2019). For example, Gentile et al. (2014) assumed that observations from different users in the same cluster share the same underlying bandit parameter. Thus, they estimate the clustering structure among users based on the difference between their

estimated bandit parameters. Li et al. (2016) used a similar idea to cluster items (arms) as well. Gentile et al. (2017) further studied arm-dependent clustering of users, by the projected difference between models on each arm. Li et al. (2019) proposed a phase-based algorithm to relax the uniform user frequency assumption in the analysis of Gentile et al. (2014). Essentially, these algorithms measure the relatedness between users by evaluating the homogeneity of observations associated with individual models, though they have used various measures for this purpose.

3 Methodology

In this section, we first formulate the problem setup studied in this paper. Then we describe two key components pertaining to non-stationary bandits and clustered bandits, and pinpoint the essential equivalence between them under the notion of homogeneity test, which becomes the cornerstone of our unified solution. Based on our construction of homogeneity test, we explain the proposed solution, followed by our theoretical analysis of the resulting upper regret bound of the proposed solution.

3.1 Problem formulation

To offer a unified approach that addresses the two target problems, we formulate a general bandit learning setting that encompasses both non-stationarity in individual models and existence of clustering structure.

Consider a learner that interacts with a set of n users, $\mathcal{U} = \{1, \dots, n\}$. At each time $t = 1, 2, \dots, T$, the learner receives an arbitrary user indexed by $i_t \in \mathcal{U}$, together with a set of available arms $C_t = \{\mathbf{x}_{t,1}, \mathbf{x}_{t,2}, \dots, \mathbf{x}_{t,K}\}$ to choose from, where $\mathbf{x}_{t,j} \in \mathbb{R}^d$ denotes the context vector associated with the arm indexed by j at time t (assume $\|\mathbf{x}_{t,j}\| \leq 1$ without loss of generality), and K denotes the size of arm pool C_t . After the learner chooses an arm $\mathbf{x}_t$, its reward $y_t \in \mathbb{R}$ is fed back from the user i_t . We follow the linear reward assumption (Abbasi-Yadkori et al., 2011; Chu et al., 2011; Li et al., 2010) and use $\theta_{i_t,t}$ to denote the parameter of the reward mapping function in user i_t at time t (assume $\|\theta_{i_t,t}\| \leq 1$). Under this assumption, the reward at time t is $y_t = \mathbf{x}_t^\top \theta_{i_t,t} + \eta_t$, where η_t is Gaussian noise drawn from $N(0, \sigma^2)$. Interaction between the learner and users repeats, and the learner's goal is to maximize the accumulated reward it receives from all users in $\mathcal{U}$ up to time T .

Denote the set of time steps when user $i \in \mathcal{U}$ is served up to time T as $\mathcal{N}_i(T) = \{1 \leq t \leq T : i_t = i\}$. Among time steps $t \in \mathcal{N}_i(T)$, user i 's parameter $\theta_{i,t}$ changes abruptly at arbitrary time steps $\{c_{i,1}, \dots, c_{i,\Gamma_i(T)-1}\}$,

but remain constant between any two consecutive change points. $\Gamma_i(T)$ denotes the total number of stationary periods in $\mathcal{N}_i(T)$. The set of unique parameters that $\theta_{i,t}$ takes for any user at any time is denoted as $\{\phi_k\}_{k=1}^m$ and their frequency of occurrences in T is $\{p_k\}_{k=1}^m$. Note that we do not impose any assumption on the distribution over the user, nor on the distribution over the unique bandit parameter appearing in each round. Also note that the ground-truth linear parameters, the set of change points, the number and frequencies of unique parameters are unknown to the learner. Moreover, the number of users, i.e., n , and the number of unique bandit parameters across users, i.e., m , are finite but arbitrary.

Our problem setting defined above is general. The non-stationary and clustering structure of an environment can be specified by different associations between $\{\theta_{i,t}\}_{i=1}^n$ and $\{\phi_k\}_{k=1}^m$ across users over time $t = 1, 2, \dots, T$. For instance, by setting $n > m$ and $\Gamma_i(T) = 1, \forall i \in \mathcal{U}$, the problem reduces to the clustered bandits problem, which assumes sharing of bandit models among users with stationary reward distributions. By setting $n = 1, m > 1$ and $\Gamma_i(T) > 1, \forall i \in \mathcal{U}$, it reduces to the piecewise stationary bandits problem, which only concerns users with non-stationary reward distributions in isolation.

To make our solution compatible with existing work in non-stationary bandits and clustered bandits, we also follow the three commonly made assumptions about the environment.

Assumption 1 (Change detectability) *For any user $i \in \mathcal{U}$ and any change point c in user i , there exists $\Delta > 0$ such that at least ρ portion of arms satisfy: $|\mathbf{x}^\top \theta_{i,c-1} - \mathbf{x}^\top \theta_{i,c}| > \Delta$ (Wu et al., 2018).*

Assumption 2 (Separateness among $\{\phi_k\}_{k=1}^m$) *For any two different unique parameters $\phi_i \neq \phi_j$, we have $\|\phi_i - \phi_j\| \geq \gamma > 0$ (Gentile et al., 2014, 2017; Li et al., 2019).*

Assumption 3 (Context regularity) *At each time t , arm set C_t is generated i.i.d. from a sub-Gaussian random vector $X \in \mathbb{R}^d$, such that $\mathbb{E}[XX^\top]$ is full-rank with minimum eigenvalue $\lambda' > 0$; and the variance ς^2 of the random vector satisfies $\varsigma^2 \leq \frac{\lambda'^2}{8 \ln 4K}$ (Gentile et al., 2014, 2017; Li et al., 2019).*

The first assumption establishes the detectability of change points in each individual bandit models over time. The second assumption ensures separation within the global unique parameter set shared by all users, and the third assumption specifies the property of context vectors. Based on these assumptions, we establish the problem setup in this work and illustrate

it on the left side of Figure 1.

3.2 Test statistic for homogeneity

As discussed in Section 2, the key problem in non-stationary bandits is to detect changes in the underlying reward distribution, and the key problem in clustered bandits is to measure the relatedness between different models. We view both problems as testing homogeneity between two sets of observations to unify these two seemingly distinct problems. For change detection, we test homogeneity between recent and past observations to evaluate whether there has been a change in the underlying bandit parameters for these two consecutive sets of observations. For cluster identification, we test homogeneity between observations of two different users to verify whether they share the same bandit parameter. On top of the test results, we operate the bandit models accordingly for model selection, model aggregation, arm selection, and model update.

We use $\mathcal{H}_1 = \{(\mathbf{x}_i, y_i)\}_{i=1}^{t_1}$ and $\mathcal{H}_2 = \{(\mathbf{x}_j, y_j)\}_{j=1}^{t_2}$ to denote two sets of observations, where $t_1, t_2 \geq 1$ are their cardinalities. $(\mathbf{X}_1, \mathbf{y}_1)$ and $(\mathbf{X}_2, \mathbf{y}_2)$ denote design matrices and feedback vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, where each row of $\mathbf{X}$ is the context vector of a selected arm and the corresponding element in $\mathbf{y}$ is the observed reward for this arm. Under linear reward assumption, $\forall (\mathbf{x}_i, y_i) \in \mathcal{H}_1, y_i \sim N(\mathbf{x}_i^\top \theta_1, \sigma^2)$, and $\forall (\mathbf{x}_j, y_j) \in \mathcal{H}_2, y_j \sim N(\mathbf{x}_j^\top \theta_2, \sigma^2)$. The test of homogeneity between $\mathcal{H}_1$ and $\mathcal{H}_2$ can thus be formally defined as testing whether $\theta_1 = \theta_2$.

Because θ_1 and θ_2 are not observable, the test has to be performed on their estimates, for which maximum likelihood estimator (MLE) is a typical choice. Denote MLE for θ on a dataset $\mathcal{H}$ as $\vartheta = (\mathbf{X}^\top \mathbf{X})^- \mathbf{X}^\top \mathbf{y}$, where $(\cdot)^-$ stands for generalized matrix inverse. A straightforward approach to test homogeneity between $\mathcal{H}_1$ and $\mathcal{H}_2$ is to compare $\|\vartheta_1 - \vartheta_2\|$ against the estimation confidence on ϑ_1 and ϑ_2 . The clustering methods by Gentile et al. (2014, 2017) essentially followed this idea. However, theoretical guarantee on the false negative probability of this method only exists when the minimum eigenvalues of $\mathbf{X}_1^\top \mathbf{X}_1$ and $\mathbf{X}_2^\top \mathbf{X}_2$ are lower bounded by a predefined threshold. In other words, when one does not have sufficient observations in either $\mathcal{H}_1$ or $\mathcal{H}_2$, this test will not be effective.

To address this limitation, we choose the test statistic that has been proved to be *uniformly most powerful* for this type of problems (Chow, 1960; Cantrell et al., 1991; Wilson, 1978):

$$s(\mathcal{H}_1, \mathcal{H}_2) = \frac{\|\mathbf{X}_1(\vartheta_1 - \vartheta_{1,2})\|^2 + \|\mathbf{X}_2(\vartheta_2 - \vartheta_{1,2})\|^2}{\sigma^2} \quad (1)$$

where $\vartheta_{1,2}$ denotes the estimator using data from both

$\mathcal{H}_1$ and $\mathcal{H}_2$. The knowledge about σ^2 can be relaxed by replacing it with empirical estimate, which leads to Chow test that has an F-distribution (Chow, 1960).

When $s(\mathcal{H}_1, \mathcal{H}_2)$ is above a threshold v , it suggests the pooled estimator deviates considerably from the individual estimators on two datasets. Thus, we conclude $\theta_1 \neq \theta_2$; otherwise, we conclude $\mathcal{H}_1$ and $\mathcal{H}_2$ are homogeneous. The choice of v is critical, as it determines the type-I and type-II error probabilities of the test. Upper bounds of these two error probabilities are given below and their proofs are provided in the appendix.

Theorem 3.1 *The test statistic $s(\mathcal{H}_1, \mathcal{H}_2)$ follows a non-central χ^2 distribution $s(\mathcal{H}_1, \mathcal{H}_2) \sim \chi^2(df, \psi)$, where the degree of freedom $df = \text{rank}(\mathbf{X}_1) + \text{rank}(\mathbf{X}_2) - \text{rank}\left(\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}\right)$, and the non-centrality parameter $\psi = \frac{\begin{bmatrix} \mathbf{X}_1 \theta_1 \\ \mathbf{X}_2 \theta_2 \end{bmatrix}^\top \begin{bmatrix} \mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} (\mathbf{x}_1^\top \mathbf{x}_1 + \mathbf{x}_2^\top \mathbf{x}_2)^- \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \theta_1 \\ \mathbf{X}_2 \theta_2 \end{bmatrix}}{\sigma^2}$.*

Lemma 3.2 *When $\theta_1 = \theta_2$, $\psi = 0$; the type-I error probability can be upper bounded by:*

$$P(s(\mathcal{H}_1, \mathcal{H}_2) > v | \theta_1 = \theta_2) \leq 1 - F(v; df, 0),$$

where $F(v; df, 0)$ denotes the cumulative density function of distribution $\chi^2(df, 0)$ evaluated at v .

This lemma states that given two datasets $\mathcal{H}_1$ and $\mathcal{H}_2$ (hence the degree-of-freedom df is determined), the type-I error probability of this test only depends on the specified threshold v .

Lemma 3.3 *When $\theta_1 \neq \theta_2$, $\psi \geq 0$; the type-II error probability can be upper bounded by,*

$$P(s(\mathcal{H}_1, \mathcal{H}_2) \leq v | \theta_1 \neq \theta_2) \leq \begin{cases} F(v; d, \psi), & \text{if } \mathbf{X}_1 \text{ and } \mathbf{X}_2 \text{ are full-rank.} \\ F(v; df, 0), & \text{otherwise.} \end{cases}$$

$$\text{where } \psi = \frac{\|\theta_1 - \theta_2\|^2 / \sigma^2}{1/\lambda_{\min}(\mathbf{X}_1^\top \mathbf{X}_1) + 1/\lambda_{\min}(\mathbf{X}_2^\top \mathbf{X}_2)}.$$

Compared with the type-I error probability, this lemma shows that the type-II error probability also depends on the ground-truth parameters (θ_1, θ_2) and the variance of noise σ^2 .

These error probabilities are the key concerns in our problem: in change detection, they correspond to the early and late detection of change points (Wu et al., 2018); and in cluster identification, they correspond to missing a user model in the neighborhood and placing a wrong user model in the neighborhood (Gentile et al., 2014). Given it is impossible to completely eliminate these two types of errors in a non-deterministic

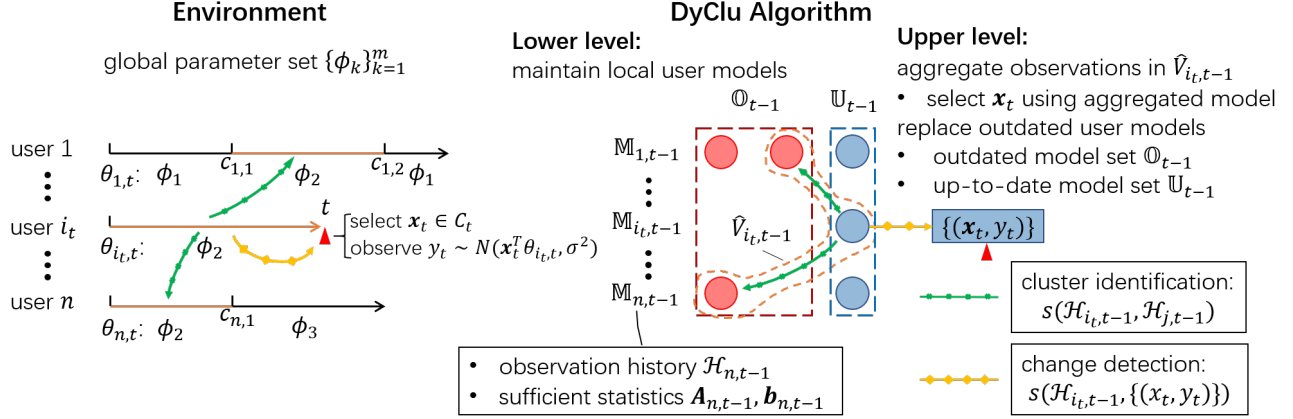


Figure 1: Online bandit learning in a non-stationary and clustered environment. The environment setting is shown on the left side of the figure, where each user’s reward mapping function undergoes a piecewise stationary process; and the reward mapping functions are globally shared across users. The proposed DyClu algorithm is illustrated on the right side of the figure. The model has a two-level hierarchy: at the lower level, individual users’ bandit models are dynamically maintained; and at the upper level, a unified test of homogeneity is performed for the purpose of change detection and cluster identification among the lower-level user models.

algorithm, the uniformly most powerful property of the test defined in Eq (1) guarantees its sensitivity is optimal at any level of specificity.

3.3 Algorithm

In the environment specified in Section 3.1, the user’s reward mapping function is piecewise stationary (e.g., the line segments on each user’s interaction trace in Figure 1), which calls for the learner to actively detect changes and re-initialize the estimator to avoid distortion from outdated observations (Yu and Mannor, 2009; Cao et al., 2019; Besson and Kaufmann, 2019; Wu et al., 2018). A limitation of these methods is that they do not attempt to reuse outdated observations because they implicitly assume each stationary period has a unique parameter. Our setting relaxes this by allowing existence of identical reward mappings across users and time (e.g., the orange line segments in Figure 1), which urges the learner to take advantage of this situation by identifying and aggregating observations with the same parameter to obtain a more accurate reward estimation.

Since neither the change points nor the grouping structure is known, in order to reuse past observations while avoiding distortion, the learner needs to accurately detect change points, stores observations in the interval between two consecutive detections together, and then correctly identify intervals with the same parameter as the current one. In this paper, we propose to unify these two operations using the test in Section 3.2, which leads to our algorithm Dynamic Clustering of Bandits, or DyClu in short. DyClu forms a two-level

hierarchy as shown in Figure 1: at the lower level, it stores observations in each interval and their sufficient statistics in a user model; at the upper level, it detects change in user’s reward function to decide when to create new user models and clusters individual user models for arm selection. Detailed steps of DyClu are explained in Algorithm 1.

The lower level of DyClu manages observations associated with each user $i \in \mathcal{U}$ in user models, denoted by $\mathbb{M}_{i,t}$. Each user model $\mathbb{M}_{i,t} = (\mathbf{A}_{i,t}, \mathbf{b}_{i,t}, \mathcal{H}_{i,t})$ stores:

1. $\mathcal{H}_{i,t}$: a set of observations associated with user i since the initialization of $\mathbb{M}_{i,t}$ up to time t , where each element is a context vector and reward pair $(\mathbf{x}_k, y_k)$.
2. Sufficient statistics: $\mathbf{A}_{i,t} = \sum_{(\mathbf{x}_k, \cdot) \in \mathcal{H}_{i,t}} \mathbf{x}_k \mathbf{x}_k^\top$ and $\mathbf{b}_{i,t} = \sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t}} \mathbf{x}_k y_k$.

Every time DyClu detects change in a user’s reward mapping function, a new user model is created to replace the previous one (line 15 in Algorithm 1). We refer to the replaced user models as outdated models and the others up-to-date ones. We denote the set of all outdated user models at time t as $\mathbb{O}_t$ and the up-to-date ones as $\mathbb{U}_t$. In Figure 1, the row of circles next to $\mathbb{M}_{1,t-1}$ represents all the user models for user 1, red ones denote outdated models and the blue one denotes up-to-date model.

The upper level of DyClu is responsible for managing the user models via change detection and model clustering. It replaces outdated models in each user

Algorithm 1 Dynamic Clustering of Bandits (DyClu)

- 1: **Input:** sliding window size τ , $\delta, \delta_e \in (0, 1)$, threshold for change detection and neighbor identification v^e and v^c , and regularization parameter λ
- 2: **Initialization:** for each user model $\mathbb{M}_{i,0}, \forall i \in \mathcal{U}$: $\mathbf{A}_{i,0} = \mathbf{0} \in \mathbb{R}^{d \times d}$, $\mathbf{b}_{i,0} = \mathbf{0} \in \mathbb{R}^d$, $\mathcal{H}_{i,0} = \emptyset$, $\hat{e}_{i,0} = 0$; the set of outdated user models $\mathbb{O}_0 = \emptyset$, and up-to-date user models $\mathbb{U}_0 = \{\mathbb{M}_{i,0}\}_{i \in \mathcal{U}}$
- 3: **for** $t = 1, 2, \dots, T$ **do**
- 4: Observe user $i_t \in \mathcal{U}$, and set of available arms $C_t = \{x_{t,1}, \dots, x_{t,K}\}$
- 5: Choose arm $\mathbf{x}_t \in C_t$ by Eq 2:
$$\arg \max_{\mathbf{x} \in C_t} \mathbf{x}^\top \hat{\theta}_{\hat{V}_{i_t,t-1}} + CB_{\hat{V}_{i_t,t-1}}(\mathbf{x})$$
- 6: Observe reward y_t from user i_t
- 7: Compute $e_{i_t,t} = \mathbf{1}\{S(\mathcal{H}_{i_t,t-1}, (\mathbf{x}_t^\top, y_t)) > v^e\}$
- 8: Update $\hat{e}_{i_t,t} = \sum_{i_t(\tau) < j \leq t: i_j = i_t} e_{i_t,j}$
- 9: **if** $\hat{e}_{i_t,t} \leq 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}}$ **then**
- 10: **if** $e_{i_t,t} = 0$ **then**
- 11: $\mathbb{M}_{i_t,t} : \mathcal{H}_{i_t,t} = \mathcal{H}_{i_t,t-1} \cup (\mathbf{x}_t, y_t)$, $\mathbf{A}_{i_t,t} = \mathbf{A}_{i_t,t-1} + \mathbf{x}_t \mathbf{x}_t^\top$, $\mathbf{b}_{i_t,t} = \mathbf{b}_{i_t,t-1} + \mathbf{x}_t y_t$
- 12: **end if**
- 13: **else**
- 14: $\mathbb{O}_t = \mathbb{O}_{t-1} \cup \mathbb{M}_{i_t,t-1}$, $\hat{e}_{i_t,t} = 0$
- 15: Replace $\mathbb{M}_{i_t,t-1}$ with $\mathbb{M}_{i_t,t} = (A_{i_t,t} = \mathbf{0}, b_{i_t,t} = \mathbf{0}, \mathcal{H}_{i_t,t} = \emptyset)$ in $\mathbb{U}_t$
- 16: **end if**
- 17: Compute $\hat{V}_{i_t,t} = \{\mathbb{M} \in \mathbb{U}_t \cup \mathbb{O}_t : S(\mathcal{H}_{i_t,t}, \mathcal{H}) \leq v^c\}$ and update $\hat{V}_{i_t,t}$ for $i \neq i_t$ accordingly.
- 18: **end for**

and aggregates models across users and time for arm selection.

• **Change detection.** A one-sample homogeneity test is used to construct a test variable $e_{i_t,t} = \mathbf{1}\{s(\mathcal{H}_{i_t,t-1}, \{(\mathbf{x}_t, y_t)\}) > v^e\}$ to measure whether the user model $\mathbb{M}_{i_t,t-1}$ is ‘admissible’ to the new observation $(\mathbf{x}_t, y_t)$. v^e is a chosen threshold for change detection. To make more reliable change detection, we use the empirical mean of $e_{i_t,t}$ in a sliding window of size $\min(|\mathcal{H}_{i_t,t-1}|, \tau)$ as the test statistic, denoted as $\hat{e}_{i_t,t} = \frac{1}{\min(|\mathcal{H}_{i_t,t-1}|, \tau)} \sum_k e_{i_t,k}$. Lemma 3.4 specifies the upper bound of early detection probability using $\hat{e}_{i_t,t}$, which is used for selecting threshold for it.

Lemma 3.4 *From Lemma 3.2, type-1 error probability $P(e_{i_t,t} = 1) = 1 - F(v^e; 1, 0)$, and thus $\mathbb{E}[e_{i_t,t}] = 1 - F(v^e; 1, 0)$. Applying Hoeffding inequality gives,*

$$P(\hat{e}_{i_t,t} > 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}}) \leq \delta_e$$

At any time step t , DyClu only updates $\mathbb{M}_{i_t,t-1}$ when $e_{i_t,t} = 0$ (line 10-12 in Algorithm 1). This guarantees that if the underlying reward distribution has changed, with a high probability we have $e_{i_t,t} = 1$, and thus the

user model $\mathbb{M}_{i_t,t-1}$ will not be updated. This prevents any distortion in $\mathcal{H}_{i_t,t}$ by observations from different reward distributions.

When $\hat{e}_{i_t,t}$ exceeds the threshold specified by Lemma 3.4, DyClu will inform the lower level to move $\mathbb{M}_{i_t,t-1}$ to the outdated model set $\mathbb{O}_t = \mathbb{O}_{t-1} \cup \{\mathbb{M}_{i_t,t-1}\}$; and then create a new model $\mathbb{M}_{i_t,t} = (A_{i_t,t} = \mathbf{0}, b_{i_t,t} = \mathbf{0}, \mathcal{H}_{i_t,t} = \emptyset)$ for user i_t as shown in line 13-16 in Algorithm 1.

• **Clustering of user models.** In this step, DyClu finds the set of “neighborhood” user models $\hat{V}_{i_t,t}$ of current user model $\mathbb{M}_{i_t,t}$, where $\hat{V}_{i_t,t-1} = \{\mathbb{M} = (\mathbf{A}, \mathbf{b}, \mathcal{H}) \in \mathbb{U}_t \cup \mathbb{O}_t : s(\mathcal{H}_{i_t,t}, \mathcal{H}) \leq v^c\}$. Basically, DyClu executes homogeneity test between $\mathbb{M}_{i_t,t}$ and all other user models $\mathbb{M} \in \mathbb{U}_t \cup \mathbb{O}_t$ (both outdated and up-to-date) with a given threshold v^c (line 17 in Algorithm 1). Lemma 3.2 and 3.3 again specify error probabilities of each decision.

When selecting an arm for user i_t at time t , DyClu aggregates the sufficient statistics of user models in neighborhood $\hat{V}_{i_t,t-1}$. Then it adopts the popular UCB strategy by Auer (2002); Li et al. (2010) to balance exploitation and exploration. Specifically, DyClu selects arm $\mathbf{x}_t$ that maximizes the UCB score computed by aggregated sufficient statistics as follows (line 5 in Algorithm 1),

$$\mathbf{x}_t = \arg \max_{\mathbf{x} \in C_t} \mathbf{x}^\top \hat{\theta}_{\hat{V}_{i_t,t-1}} + CB_{\hat{V}_{i_t,t-1}}(\mathbf{x}) \quad (2)$$

In Eq (2), $\hat{\theta}_{\hat{V}_{i_t,t-1}} = \mathbf{A}_{\hat{V}_{i_t,t-1}}^{-1} \mathbf{b}_{\hat{V}_{i_t,t-1}}$ is the ridge regression estimator using aggregated statistics $\mathbf{A}_{\hat{V}_{i_t,t-1}} = \lambda \mathbf{I}_d + \sum_{(\mathbf{A}_j, \mathbf{b}_j, \mathcal{H}_j) \in \hat{V}_{i_t,t-1}} \mathbf{A}_j$ and $\mathbf{b}_{\hat{V}_{i_t,t-1}} = \sum_{(\mathbf{A}_j, \mathbf{b}_j, \mathcal{H}_j) \in \hat{V}_{i_t,t-1}} \mathbf{b}_j$; the confidence bound of reward estimation for arm $\mathbf{x}$ is $CB_{\hat{V}_{i_t,t-1}}(\mathbf{x}) = \alpha_{\hat{V}_{i_t,t-1}} \sqrt{\mathbf{x}^\top \mathbf{A}_{\hat{V}_{i_t,t-1}}^{-1} \mathbf{x}}$, where $\alpha_{\hat{V}_{i_t,t-1}} = \sigma \sqrt{d \log(1 + \frac{\sum_{(\mathbf{A}_j, \mathbf{b}_j, \mathcal{H}_j) \in \hat{V}_{i_t,t-1}} |\mathcal{H}_j|}{d\lambda})} + 2 \log \frac{1}{\delta} + \sqrt{\lambda}$.

3.4 Regret analysis

Denote $R_T = \sum_{t=1}^T \theta_{i_t}^\top \mathbf{x}_t^* - \theta_{i_t}^\top \mathbf{x}_t$ as the accumulative regret, where $\mathbf{x}_t^* = \arg \max_{\mathbf{x}_t, j \in C_t} \theta_{i_t}^\top \mathbf{x}_{t,j}$ is the optimal arm at time t . Our regret analysis relies on the high probability results by Abbasi-Yadkori et al. (2011) and decomposition of “good” and “bad” events according to change detection and clustering results. The full proof, along with ancillary results and discussions, are given in the appendix.

Theorem 3.5 *Under Assumptions 1, 2 and 3, the regret of DyClu is upper bounded by:*

$$R_T = O\left(\sigma d \sqrt{T \log^2 T} \left(\sum_{k=1}^m \sqrt{p_k}\right) + \sum_{i \in \mathcal{U}} \Gamma_i(T) \cdot C\right)$$

where $C = \frac{1}{1-\delta_e} + \frac{\sigma^2}{\gamma^2 \lambda^2} \log \frac{d}{\delta'}$, with a probability at least $(1-\delta)(1-\frac{\delta_e}{1-\delta_e})(1-\delta')$.

Note that the first term matches the regret of the ideal case that the learner knows the exact change points and clustering structure of each user and time step, while the second term corresponds to the additional regret due to the interplay between errors in change detection and clustering, which is unique to our problem. To better understand this result, we discuss in the following paragraph how it compares with state-of-the-art bandit solutions in settings like non-stationary environment only or clustered environment only.

Case 1: Setting $m = 1$, $n = 1$ and $\Gamma_1(T) = 1$ reduces the problem to the basic linear bandit setting, because the environment consists of only one user with a stationary reward distribution for the entire time of interaction. With only one user who has a stationary reward distribution, we have $\sum_{k=1}^1 \sqrt{p_k} = 1$ where p_k is frequency of occurrences of ϕ_k in T as defined in Section 3.1. In addition, since there is only one stationary period, the added regret caused by late detection does not exist; and the added regret due to the failure in clustering can be bounded by a constant, which only depends on environment variables (see Lemma D.2 in appendix for details). The upper regret bound of DyClu then becomes $O(\sigma d \sqrt{T \log^2 T})$, which achieves the same order of regret as that in LinUCB (Abbasi-Yadkori et al., 2011). **Case 2:** Setting $\Gamma_i(T) = 1, \forall i \in \mathcal{U}$ reduces the problem to the clustered bandit setting (Gentile et al., 2014), because all users in the environment have a stationary reward distribution of their own. Similar to Case 1, the added regret caused by late detection becomes zero and the added regret due to the failure in clustering is bounded by a constant, which leads to the upper regret bound of $O(\sigma d \sqrt{T \log^2 T} (\sum_{k=1}^m \sqrt{p_k}))$. DyClu achieves the same order of regret as that in CLUB (Gentile et al., 2014). **Case 3:** Setting $n = 1$ reduces the problem to a piecewise stationary bandit setting, because the environment consists of only one user with piecewise stationary reward distributions. For the convenience of comparison, we can rewrite the upper regret bound of DyClu in the form of $O(\sum_{k \in [m]} R_{Lin}(|N_k^\phi(T)|) + \Gamma_1(T))$, where $R_{Lin}(t) = O(d \sqrt{t \log^2 t})$ (Abbasi-Yadkori et al., 2011) and $N_k^\phi(T) = \{1 \leq t' \leq T : \theta_{i_{t'}, t'} = \phi_k\}$ is the set of time steps up to time T when the user being served has the bandit parameter equal to ϕ_k . Detailed derivation of this is given in appendix (Section D.1). Note that the upper regret bound of dLinUCB (Wu et al., 2018) for this setting is $O(\Gamma_1(T) R_{Lin}(S_{max}) + \Gamma_1(T))$, where S_{max} denotes the maximum length of

stationary periods. The regret of DyClu depends on the number of unique bandit parameters in the environment, instead of the number of stationary periods as in dLinUCB, because DyClu can reuse observations from previous stationary periods. This suggests DyClu has a tighter regret bound if different stationary periods share the same unique bandit parameters; for example, in situations where a future reward mapping function switches back to a previous one.

4 Experiments

We investigate the empirical performance of DyClu by comparing with a list of state-of-the-art baselines for both non-stationary bandits and clustered bandits on synthetic and real-world recommendation datasets.

4.1 Experiment setup and baselines

- **Synthetic dataset.** We create a set of unique bandit parameters $\{\phi_k\}_{k=1}^m$ and arm pool $\{\mathbf{x}_j\}_{j=1}^K$ ($K = 1000$), where ϕ_k and $\mathbf{x}_j$ are first sampled from $N(\mathbf{0}_d, \mathbf{I}_d)$ with $d = 25$ and then normalized so that $\forall k, j, \|\phi_k\| = 1$ and $\|\mathbf{x}_j\| = 1$. When sampling $\{\phi_k\}_{k=1}^m$, the separation margin γ is set to 0.9 and enforced via rejection sampling. n users are simulated. In each user, we sample a series of time intervals from (S_{min}, S_{max}) uniformly; and for each time interval, we sample a unique parameter from $\{\phi_k\}_{k=1}^m$ as the ground-truth bandit parameter for this period. This creates asynchronous changes and clustering structure in users' reward functions. The users are served in a round-robin fashion. At time step $t = 1, 2, \dots, T$, a subset of arms are randomly chosen and disclosed to the learner. Reward of the selected arm is generated by the linear function governed by the corresponding bandit parameter and context vector, with additional Gaussian noise sampled from $N(0, \sigma^2)$.

- **LastFM dataset.** The LastFM dataset is extracted from the music streaming service Last.fm (Cesa-Bianchi et al., 2013), which contains 1892 users and 17632 items (artists). "Listened artists" of each user are treated as positive feedback. We followed Wu et al. (2018) to preprocess the dataset and simulate a clustered non-stationary environment by creating 20 "hybrid users". We first discard users with less than 800 observations and then use PCA to reduce the dimension of TF-IDF feature vector to $d = 25$. We create hybrid users by sampling three real users uniformly and then concatenating their associated data points together. Hence, data points of the same real user would appear in different hybrid users, which is analogous to stationary periods that share the same unique bandit parameters across different users and time.

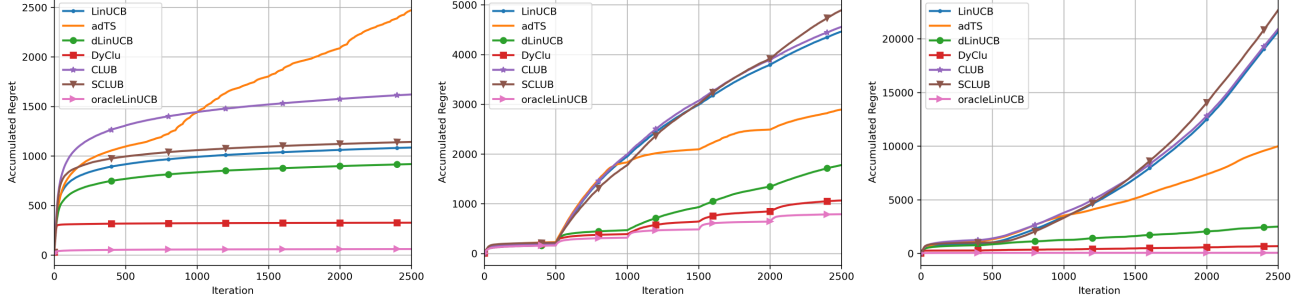


Figure 2: Accumulated regret on synthetic datasets with three different environment settings. Environment 1: $n = 100$ users share a global set of $m = 5$ unique bandit parameters, and each user remains stationary all the time. Environment 2: $n = 20$ user with fixed stationary period length 500; each period sample a unique bandit parameter. Environment 3: $n = 100$ users share a global set of $m = 5$ unique bandit parameters, and each user changes in a asynchronous manner.

• **Baselines.** We compare DyClu with a set of state-of-the-art bandit algorithms: linear bandit LinUCB by Abbasi-Yadkori et al. (2011), non-stationary bandit dLinUCB by Wu et al. (2018) and adTS by Slivkins and Upfal (2008), as well as clustered bandit CLUB by Gentile et al. (2014) and SCLUB by Li et al. (2019). For experiments on synthetic dataset, we also include oracle-LinUCB for comparison, which runs an instance of LinUCB for each unique bandit parameter. Comparing with it helps us understand the added regret due to errors in change detection and clustering.

• **Hyper-parameters.** We set the same regularization parameter $\lambda = 0.1$ for all the algorithms, and set the same sliding window size $\tau = 20$ for both dLinUCB and DyClu on synthetic dataset and $\tau = 50$ on LastFM dataset. The thresholds v^e and v^c for DyClu are essentially the upper-tail critical values of chi-square distributions $\chi^2(1)$ and $\chi^2(d)$, which directly control the type-I error probability for change detection and clustering, i.e. $1 - F(v^e; 1)$ and $1 - F(v^c; d)$ respectively. Their values affect the second term in the regret upper bound given in Theorem 3.5 (see Lemma D.1 and Lemma D.2 in appendix for details). In all our experiments, v^e and v^c are selected such that the corresponding significance level equals to 0.01, e.g., to make $F(v^c; 25) = 0.01$, we set $v^c = 44.314$.

4.2 Experiment results

• **Empirical comparisons on synthetic dataset.** We compare accumulated regret of all bandit algorithms under three environment settings, and the results are reported in Figure 2. Environment 1 simulates the clustered bandit setting in Gentile et al. (2014), where *no* change in the reward function is introduced. DyClu outperformed other baselines, including CLUB and SCLUB, demonstrating the quality of its identified clustering structure. Specifically,

compared with adTS that incurs high regret as a result of too many false detections, the change detection in DyClu has much less false positives, as there is no change in each user’s reward distribution. Environment 2 simulates the piecewise stationary bandit setting in Wu et al. (2018). Algorithms designed for stationary environment, e.g., CLUB, SCLUB, and LinUCB suffer from a linear regret after the first change point. DyClu achieved the best performance, with a wide margin from the second best, dLinUCB, which is designed for this environment. It shows the power of our change detection method against dLinUCB’s. Environment 3 combines previous two settings with both non-stationarity and clustering structure. DyClu again outperformed others. It is worth noting that regret of all algorithms increased compared with Environment 1 due to the nonstationarity, but the increase in DyClu is the smallest. And in all settings, DyClu’s performance is closest to the oracle-LinUCB’s, which shows that DyClu can correctly cluster and aggregate observations from the dynamically changing users.

• **Sensitivity to environment settings.** According to our regret analysis, the performance of DyClu depends on environment parameters like the number of unique bandit parameters m , the number of stationary periods $\Gamma_i(T)$ for $i \in \mathcal{U}$, and variance of Gaussian noise σ^2 . We investigate their influence on DyClu and baselines, by varying these parameters while keeping the others fixed. The accumulated regret under different settings are reported in Table 1. DyClu outperformed other baselines in all 9 different settings, and the changes of its regret align with our theoretical analysis. A larger number of unique parameters m leads to higher regret of DyClu as shown in setting 1, 2 and 3, since observations are split into more clusters with smaller size each. In addition, larger number of stationary periods incurs more errors in change detec-

Table 1: Comparison of accumulated regret under different environment settings.

	n	m	S_{min}	S_{max}	T	σ	oracle.	LinU.	adTS	dLinU.	CLUB	SCLUB	DyClu
1	100	10	400	2500	2500	0.09	115	19954	9872	2432	20274	19989	853
2	100	50	400	2500	2500	0.09	489	20952	9563	2420	21205	21573	1363
3	100	100	400	2500	2500	0.09	873	21950	10961	2549	22280	22262	1958
4	100	10	200	400	2500	0.09	112	39249	36301	10831	39436	43836	3025
5	100	10	800	1000	2500	0.09	113	34385	13788	3265	34441	33514	1139
6	100	10	1200	1400	2500	0.09	112	24769	8124	2144	24980	23437	778
7	100	10	400	2500	2500	0.12	166	22453	10567	3301	22756	22683	1140
8	100	10	400	2500	2500	0.15	232	19082	10000	5872	19427	20664	1487
9	100	10	400	2500	2500	0.18	307	23918	11255	9848	24050	23677	1956

tion, leading to an increased regret. This is confirmed by results in setting 4, 5 and 6. Lastly, as shown in setting 7, 8 and 9, larger Gaussian noise leads to higher regret, as it slows down convergence of reward estimation and change detection.

• **Empirical comparisons on LastFM.** We report normalized accumulative reward (ratio between baselines and uniformly random arm selection strategy (Wu et al., 2019)) on LastFM in Figure 3. In this environment, realizing both non-stationarity and clustering structure is important for an online learning algorithm to perform well. DyClu’s improvement over other baselines confirms its quality in partitioning and aggregating relevant data points across users. The advantage of DyClu is more apparent at the early stage of learning, where each local user model has not collected sufficient amount of observations for individualized reward estimation; and thus change detection and clustering are more difficult there.

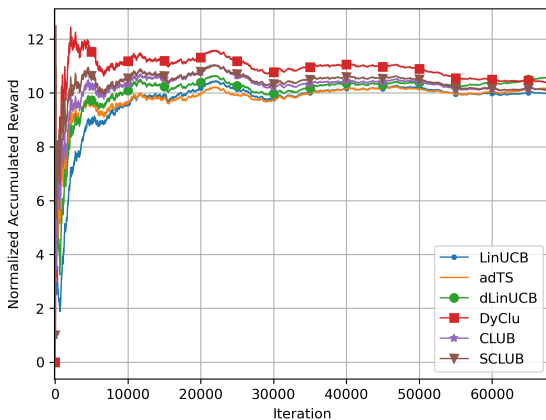


Figure 3: Comparison of accumulated reward normalized by a random policy on LastFM dataset.

5 Conclusion

In this work, we unify the efforts in non-stationary bandits and clustered bandits via homogeneity test, and also propose a new bandit problem setting that generalizes both. Our solution adaptively detects changes in the underlying reward distribution and clusters bandit models for aggregated arm selection. The resulting upper regret bound matches with the ideal algorithm’s only up to a constant; and extensive empirical evaluations validate its effectiveness in a non-stationary and clustered environment.

There are still several directions left open in this research: 1) Assumption 1 and 2 are mainly needed for controlling the type-II error of change detection and clustering, to ensure the estimator used for arm selection have no heterogeneous observations (i.e., contamination). These assumptions are arguably rigid, considering if the difference in bandit parameters is small or negligible, aggregating such heterogeneous observations may not be detrimental. To the best of our knowledge, no existing work for linear bandit addressed this issue (Gentile et al., 2014; Li et al., 2019; Wu et al., 2018). Pursuing this direction requires balancing the quantity vs. quality trade-off of contaminated observations. 2) Our current analysis for the test statistic assumes Gaussian reward noise, and it would be interesting to consider the more general sub-Gaussian noise that is commonly assumed in bandit literature. 3) Despite the existence of multiple users, all computations are done in a centralized manner; to make it more practical, asynchronous and distributed model update is more desired.

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A Notations used in this paper

Here we list the notations used in this paper and their descriptions in Table 2.

Table 2: Notations used in this paper.

Notation	Description
1 $\mathcal{U}$	the set of all users, with its cardinality denoted by n
2 C_t	the set of available arms at time t , with its cardinality denoted by K
3 (x_t, y_t)	context vector of the selected arm and its observed reward at time t
4 $\theta_{i,t}$	bandit parameter of user i at time t
5 d	dimension of context vector and bandit parameter
6 $\{\phi_k\}_{k=1}^m$	a set of m unique bandit parameters
7 $k_{i,t}$	index of unique bandit parameter associated with user i at time t
8 η_t	Gaussian noise in the reward observed at time t
9 σ^2	variance of the Gaussian noise in reward
10 $\mathcal{N}_i(T)$	the set of time steps user $i \in \mathcal{U}$ is served up to time T
11 $\Gamma_i(T)$	the total number of stationary periods in $\mathcal{N}_i(T)$
12 $c_{i,j}$	the time step when the j 'th change point of user i occurs
13 $N_k^\phi(t)$	the set of time steps up to time t that ϕ_k occurs
14 p_k	frequency of total occurrences of ϕ_k
15 $\mathcal{H}$	a set of observations
16 $\mathbf{X}, \mathbf{y}$	design matrix and feedback vector of $\mathcal{H}$
17 ϑ	MLE on dataset $\mathcal{H}$
18 $s(\mathcal{H}_1, \mathcal{H}_2)$	homogeneity test statistic between $\mathcal{H}_1$ and $\mathcal{H}_2$
19 $\chi^2(df, \psi)$	noncentral χ^2 distribution with degree-of-freedom df and noncentrality parameter ψ
20 $F(v; df, \psi)$	the cumulative density function of $\chi^2(df, \psi)$ evaluated at v
21 $\lambda_{\min}(\cdot)$	function that returns minimum eigenvalue of a matrix
22 λ	regularization parameter
23 $\mathbf{A}, \mathbf{b}$	sufficient statistics of $\mathcal{H}$
24 $\mathbb{M}_{i,t}$	user model of user i at time t
25 $\tilde{k}_{i,t}$	index of the unique bandit parameter associated with observations in $\mathbb{M}_{i,t}$
26 $\mathbb{U}_t, \mathbb{O}_t$	sets of up-to-date and outdated user models at time t
27 $\hat{V}_{i,t}$	estimated neighborhood of user i at time t
28 $\hat{N}_{\tilde{k}_{i,t}}^\phi(t)$	the set of time steps associated with observations in $\hat{V}_{i,t}$
29 $e_{i,t,t}$	indicator variable of the one-sample homogeneity test at time t
30 $\hat{e}_{i,t}$	empirical mean of $e_{i,t,t}$ in a sliding window, with its size denoted by τ
31 $\hat{\theta}_{\hat{V}_{i,t}}$	ridge regression estimator by aggregating observations in $\hat{V}_{i,t}$
32 $CB_{\hat{V}_{i,t}}(\mathbf{x})$	confidence bound for reward estimation on $\mathbf{x}$ using $\hat{\theta}_{\hat{V}_{i,t}}$
33 v^e, v^c	thresholds for homogeneity test in change detection and cluster identification
34 $R_{Lin}(\cdot)$	high probability regret upper bound of standard LinUCB Abbasi-Yadkori et al. (2011)

B Proof of lemmas for homogeneity test in Section 3.2

The statistical test introduced in Section 3.2 falls under the category of χ^2 test of homogeneity. Specifically, it is used to test whether the parameters of linear regression models associated with two datasets are the same, assuming equal variance. The test statistic follows the noncentral χ^2 -distribution $s(\mathcal{H}_1, \mathcal{H}_2) \sim \chi^2(df, \psi)$, where $df = \text{rank}(\mathbf{X}_1) + \text{rank}(\mathbf{X}_2) - \text{rank}\left(\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}\right)$ denotes the degree of freedom, and non-centrality parameter

$\psi = \frac{1}{\sigma^2} \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_2 \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_2 \end{bmatrix}$. Its proof is beyond the scope of this paper. We refer the interested readers to statistics or econometrics literature like Chow (1960); Cantrell et al. (1991).

Proof of Lemma 3.2.

When datasets $\mathcal{H}_1$ and $\mathcal{H}_2$ are homogeneous, which means $\theta_1 = \theta_2$, the non-centrality parameter becomes:

$$\begin{aligned} \psi &= \frac{1}{\sigma^2} \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix} \\ &= \frac{1}{\sigma^2} \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix} - \frac{1}{\sigma^2} \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix} \\ &= \frac{1}{\sigma^2} \left[\theta_1^\top (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2) \theta_1 - \theta_1^\top (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2) (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2) \theta_1 \right] \\ &= \frac{1}{\sigma^2} \left[\theta_1^\top (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2) \theta_1 - \theta_1^\top (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2) \theta_1 \right] = 0 \end{aligned}$$

Therefore, when $\theta_1 = \theta_2$, the test statistic $s(\mathcal{H}_1, \mathcal{H}_2) \sim \chi^2(df, 0)$. The type-I error probability can be upper bounded by $P(s(\mathcal{H}_1, \mathcal{H}_2) > v | \theta_1 = \theta_2) \leq 1 - F(v; df, 0)$, which concludes the proof of Lemma 3.2.

Proof of Lemma 3.3.

Similarly, using the cumulative density function of $\chi^2(df, \psi)$, we can show that the type-II error probability $P(s(\mathcal{H}_1, \mathcal{H}_2) \leq v | \theta_1 \neq \theta_2) \leq F(v; df, \psi)$. As mentioned in Section 3.2, the value of ψ depends on the unknown parameters θ_1 and θ_2 . From the definition of ψ , we know that $\theta_1 = \theta_2$ is only a sufficient condition for $\psi = 0$. The necessary and sufficient condition for $\psi = 0$ is that $\begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_2 \end{bmatrix}$ is in the column space of $\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$, e.g., there exists θ such that $\begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1\theta \\ \mathbf{X}_2\theta \end{bmatrix}$. Only when both $\mathbf{X}_1$ and $\mathbf{X}_2$ have a full column rank, $\theta_1 = \theta_2$ becomes the necessary and sufficient condition for $\psi = 0$. This means when either $\mathbf{X}_1$ or $\mathbf{X}_2$ is rank-deficient, there always exists θ_1 and θ_2 , and $\theta_1 \neq \theta_2$, that make $\psi = 0$. For example, assuming $\mathbf{X}_1$ is rank-sufficient and $\mathbf{X}_2$ is rank-deficient, then $\psi = 0$ as long as $\theta_1 - \theta_2$ is in the null space of $\mathbf{X}_2$.

To obtain a non-trivial upper bound of the type-II error probability, or equivalently a non-zero lower bound of the non-centrality parameter ψ , both $\mathbf{X}_1$ and $\mathbf{X}_2$ need to be rank-sufficient. Under this assumption, we can rewrite ψ in the following way to derive its lower bound.

Denote $\epsilon = \theta_2 - \theta_1$. Then $\theta_2 = \theta_1 + \epsilon$. We can decompose $\sigma^2\psi$ as:

$$\begin{aligned} \sigma^2\psi &= \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2(\theta_1 + \epsilon) \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2(\theta_1 + \epsilon) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \left(\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \left(\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} 0 \\ \mathbf{X}_2\epsilon \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \mathbf{X}_2\epsilon \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \left(\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \mathbf{X}_2\epsilon \end{bmatrix}^\top \left[\mathbf{I}_{t_1+t_2} - \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \left(\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{X}_2^\top \end{bmatrix} \right] \begin{bmatrix} 0 \\ \mathbf{X}_2\epsilon \end{bmatrix} \end{aligned}$$

Since $\begin{bmatrix} \mathbf{X}_1\theta_1 \\ \mathbf{X}_2\theta_1 \end{bmatrix}$ is in the column space of $\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$, the first term in the above result is zero. The second and third terms can be shown equal to zero as well using the property that matrix product is distributive w.r.t. matrix

addition, which leaves us only the last term. Therefore, by substituting $\epsilon = \theta_2 - \theta_1$ back, we obtain:

$$\begin{aligned}\psi &= \frac{1}{\sigma^2} (\theta_1 - \theta_2)^\top \mathbf{X}_2^\top \mathbf{X}_2 (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} \mathbf{X}_1^\top \mathbf{X}_1 (\theta_1 - \theta_2) \\ &\geq \frac{1}{\sigma^2} \|\theta_1 - \theta_2\|^2 \lambda_{\min} \left(\mathbf{X}_2^\top \mathbf{X}_2 (\mathbf{X}_1^\top \mathbf{X}_1 + \mathbf{X}_2^\top \mathbf{X}_2)^{-1} \mathbf{X}_1^\top \mathbf{X}_1 \right) \\ &\geq \frac{\|\theta_1 - \theta_2\|^2 / \sigma^2}{\frac{1}{\lambda_{\min}(\mathbf{X}_1^\top \mathbf{X}_1)} + \frac{1}{\lambda_{\min}(\mathbf{X}_2^\top \mathbf{X}_2)}}\end{aligned}$$

The first inequality uses the Rayleigh-Ritz theorem, and the second inequality uses the relation $\mathbf{Y}(\mathbf{X} + \mathbf{Y})^{-1}\mathbf{X} = (\mathbf{X}^{-1} + \mathbf{Y}^{-1})^{-1}$, where $\mathbf{X}$ and $\mathbf{Y}$ are both invertible matrices. This relation can be derived by taking inverse on both sides of the equation $\mathbf{X}^{-1}(\mathbf{X} + \mathbf{Y})\mathbf{Y}^{-1} = \mathbf{X}^{-1}\mathbf{X}\mathbf{Y}^{-1} + \mathbf{X}^{-1}\mathbf{Y}\mathbf{Y}^{-1} = \mathbf{Y}^{-1} + \mathbf{X}^{-1}$.

Discussions The results above show that given two datasets $\mathcal{H}_1$ and $\mathcal{H}_2$, the type-I error probability of the homogeneity test only depends on the selection of threshold v , while the type-II error probability also depends on the ground-truth parameters (θ_1, θ_2) and the variance of noise σ^2 . If either $\mathbf{X}_1$ or $\mathbf{X}_2$ is rank-deficient, the type-II error probability will be trivially upper bounded by $F(v; df, 0)$, which means for a smaller upper bound of type-I error probability (i.e., $1 - F(v; df, 0)$), the upper bound of type-II error probability (i.e., $F(v; df, 0)$) will be large. Intuitively, for a certain level of type-I error, to ensure a smaller type-II error probability in the worst case, we at least need both $\mathbf{X}_1$ and $\mathbf{X}_2$ to be rank-sufficient and the value of $\frac{\|\theta_1 - \theta_2\|^2 / \sigma^2}{\frac{1}{\lambda_{\min}(\mathbf{X}_1^\top \mathbf{X}_1)} + \frac{1}{\lambda_{\min}(\mathbf{X}_2^\top \mathbf{X}_2)}}$ to be large. Similar idea is also found in Wu et al. (2018); Gentile et al. (2014, 2017), where they require the confidence bounds of the estimators (which is essentially equivalent to the condition on minimum eigenvalue $\lambda_{\min}(\mathbf{X}_1^\top \mathbf{X}_1)$ and $\lambda_{\min}(\mathbf{X}_2^\top \mathbf{X}_2)$ in our analysis) to be small enough, w.r.t. $\|\theta_1 - \theta_2\|^2$, to ensure their change detection or cluster identification is accurate. Here we unify the analysis of these two tasks with this homogeneity test.

C Proof of Lemma 3.4

Note that early detection corresponds to type-I error of the homogeneity test in Lemma 3.2, e.g., when change has not happened (thus $\mathcal{H}_{i_t, t-1}$ and $(\mathbf{x}_t, y_t)$ are homogeneous), but the test statistic exceeds the threshold v^e : $e_{i_t, t} = \mathbf{1}\{s(\mathcal{H}_{i_t, t-1}, \{(\mathbf{x}_t, y_t)\}) > v^e\} = 1$. Therefore, we have $\mathbb{E}[e_{i_t, t}] \leq 1 - F(v^e; 1, 0)$. Then we can use Hoeffding inequality given in Lemma E.1 to upper bound the early detection probability using $\hat{e}_{i_t, t}$.

As the test variable $e_{i_t, t} \in \{0, 1\}$, it is $\frac{1}{2}$ -sub-Gaussian. By applying Hoeffding inequality, we have:

$$P(\tau \hat{e}_{i_t, t} - \tau \mathbb{E}[e_{i_t, t}] \geq h) \leq \exp\left(-\frac{2h^2}{\tau}\right)$$

Then setting $\delta_e = \exp(-\frac{2h^2}{\tau})$ gives $h = \sqrt{\frac{\tau \log 1/\delta_e}{2}}$. Substituting this back and re-arrange the inequality gives us:

$$P\left(\hat{e}_{i_t, t} < \mathbb{E}[e_{i_t, t}] + \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) > 1 - \delta_e$$

Since $\mathbb{E}[e_{i_t, t}] \leq 1 - F(v^e; 1, 0)$, we have:

$$P\left(\hat{e}_{i_t, t} < 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) > 1 - \delta_e$$

when change has not happened.

D Proof of Theorem 3.5

In this section, we give the full proof of the upper regret bound in Theorem 3.5. We first define some additional notations necessary for the analysis and arrange the proof into three parts: 1) proof of Eq (3); 2) proof of Lemma D.1; and 3) proof of Lemma D.2. Specifically, Eq (3) provides an intermediate upper regret bound with three terms, and Lemma D.1 and Lemma D.2 further bound the second and third terms to obtain the final result in Theorem 3.5.

Consider a learner that has access to the ground-truth change points and clustering structure, or equivalently, the learner knows the index of the unique bandit parameter each observation is associated with (but it does not know the value of the parameter). For example, when serving user i_t at some time step t , the index of user i_t 's unique bandit parameter for the moment is $k_{i_t,t}$, such that $\theta_{i_t,t} = \phi_{k_{i_t,t}}$. Then since this learner knows $k_{i_t',t'}$ for $t' \in [t]$, it can precisely group the observations associated with each unique bandit parameter ϕ_k for $k \in [m]$. Recall that we denote $N_k^\phi(t) = \{1 \leq t' \leq t : \theta_{i_{t'},t'} = \phi_k\}$ as the set of time steps up to time t when the user being served has the bandit parameter equal to ϕ_k , e.g., all the observations obtained at time steps in $N_k^\phi(t)$ have the same unique bandit parameter ϕ_k . Then an ideal reference algorithm would be the one that aggregates these observations to compute UCB score for each unique bandit parameter, and then select arm using the UCB score associated with the true bandit parameter in each time step. The regret of this ideal reference algorithm can be upper bounded by $\sum_{k=1}^m R_{Lin}(|N_k^\phi(T)|)$ where $R_{Lin}(|N_k^\phi(T)|) = O\left(d\sqrt{|N_k^\phi(T)| \log^2 |N_k^\phi(T)|}\right)$ (Abbasi-Yadkori et al., 2011).

However in our learning environment, such knowledge is not available to the learner; as a result, the learner does not know $N_{k_{i_t,t}}^\phi(t-1)$ when interacting with user i_t at time t ; instead, it uses observations in the estimated neighborhood $\hat{V}_{i_t,t-1}$ as shown in Algorithm 1 (line 17). Denote the set of time steps associated with observations in $\hat{V}_{i_t,t-1}$ as $\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1)$, where $\tilde{k}_{i_t,t-1}$ is the index of the unique parameter associated with observations in $\mathcal{H}_{i_t,t-1}$. $\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1)$ is essentially an estimate of $N_{k_{i_t,t}}^\phi(t-1)$, obtained by running cluster identification w.r.t. $\mathcal{H}_{i_t,t-1}$ (whose true unique bandit parameter index is denoted by $\tilde{k}_{i_t,t-1}$). We define a ‘good event’ as $\{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) = N_{k_{i_t,t}}^\phi(t-1)\}$, which matches with the reference algorithm, and since there is a non-zero probability of errors in both change detection and cluster identification, we also have ‘bad event’ $\{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1)\}$, which incurs extra regret.

Recall the estimated neighborhood $\hat{V}_{i_t,t-1} = \{\mathbb{M} \in \mathbb{U}_{t-1} \cup \mathbb{O}_{t-1} : S(\mathcal{H}_{i_t,t-1}, \mathcal{H}) \leq v^c\}$. If $k_{i_t,t} \neq \tilde{k}_{i_t,t-1}$, it means there is a mismatch between user model $\mathbb{M}_{i_t,t-1}$ and the current ground-truth user parameter $\theta_{i_t,t}$, but the change detection module has failed to detect this. Then the obtained neighborhood is incorrect even if the cluster identification model made no mistake. Therefore, the bad event can be further decomposed into $\left(\{\tilde{k}_{i_t,t-1} \neq k_{i_t,t}\} \cap \{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1)\}\right) \cup \left(\{\tilde{k}_{i_t,t-1} = k_{i_t,t}\} \cap \{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1)\}\right)$. The first part is a subset of event $\{\tilde{k}_{i_t,t-1} \neq k_{i_t,t}\}$, which suggests late detection happens at time t . The second part indicates incorrectly estimated cluster for user i_t at time t .

Discussions Before moving on, we would like to provide some explanations about the use of $\{\tilde{k}_{i_t,t-1} \neq k_{i_t,t}\}$ to denote the event that the user’s underlying bandit parameter has changed, but the learner failed to detect it, i.e., late detection. Recall that $\tilde{k}_{i_t,t-1}$ is the index of the unique bandit parameter associated with observations in $\mathbb{M}_{i_t,t-1}$, i.e. $\mathcal{H}_{i_t,t-1}$, while $k_{i_t,t}$ is the index of the unique parameter that governs observation $(\mathbf{x}_t, y_t)$ from user i_t at time t . Our change detection mechanism in Algorithm 1 (line 9) is expected to replace model $\mathbb{M}_{i_t,t-1}$ if change has happened at time t , thus ensuring $\{\tilde{k}_{i_t,t} = k_{i_t,t}\}$ again. However, when it fails to detect the change, it will cause $\{\tilde{k}_{i_t,t} \neq k_{i_t,t}\}$, which means DyClu has failed to update the user model $\mathbb{M}_{i_t,t}$ to reflect the new behavior or preference that user i_t has switched to at time t .

With detailed proof in Section D.1, following the decomposition discussed above, we can obtain:

$$\begin{aligned} R_T \leq & O\left(\sigma d \sum_{k \in [m]} \sqrt{|N_k^\phi(T)| \log^2(|N_k^\phi(T)|)}\right) + 2 \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1}\{\tilde{k}_{i_t,t-1} \neq k_{i_t,t}\} \\ & + 2 \sum_{t=1}^T \mathbf{1}\{\tilde{k}_{i_t,t-1} = k_{i_t,t}\} \cap \{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1)\} \end{aligned} \quad (3)$$

with a probability at least $1 - \delta$.

In this upper regret bound, the first term matches the regret of the reference algorithm that has access to the exact change points and clustering structure of each user and time step. We can rewrite it using the frequency

of unique model parameter ϕ_k as $O(\sigma d \sqrt{T \log^2 T} (\sum_{k=1}^m \sqrt{p_k}))$ similar to Section A.4 in Gentile et al. (2014). The second term is the added regret caused by the late detection of change points; and the third term is the added regret caused by the incorrect cluster identification for arm selection. The latter two terms can be further bounded by the following lemmas. Their proofs are later given in Section D.2 and Section D.3 respectively.

Lemma D.1 *Under Assumption 1 and 3, by setting the sliding window size $\tau \geq \frac{2 \log 1/\delta_e}{\Delta^2/\sigma^2}$, where $\psi^e = \frac{1}{1+1/\left[\frac{\lambda'}{4} S_{\min} - 8 \left(\log \frac{d S_{\min}}{\delta'} + \sqrt{S_{\min} \log \frac{d S_{\min}}{\delta'}} \right) \right]}$, the second term in Eq (3) can be upper bounded by:*

$$2 \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1} \left\{ \tilde{k}_{i_t, t-1} \neq k_{i_t} \right\} \leq 2 \sum_{i \in \mathcal{U}} \left(\Gamma_i(T) - 1 \right) \left(\tau + \frac{2}{1 - \delta_e} \right)$$

with a probability at least $1 - \frac{\delta_e}{1 - \delta_e}$.

Lemma D.2 *Define function $g(\psi; d, v) = F(v; \psi, d)$, and $g^{-1}(\cdot | d, v)$ as its inverse function. Under Assumption 2 and 3, the third term in Eq (3) can be upper bounded by:*

$$2 \sum_{t=1}^T \mathbf{1} \left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \leq 2 \sum_{i \in \mathcal{U}} \Gamma_i(T) O \left(\frac{2\psi^c \sigma^2}{\gamma^2 \lambda'^2} \log \frac{d}{\delta'} \right)$$

with a probability at least $1 - \delta'$, where $\psi^c = g^{-1} \left(\frac{p(1-F(v^c; d, 0))}{1-p}; d, v^c \right)$ is a constant.

Combining results in Eq (3), Lemma D.1 and Lemma D.2, we obtain the upper regret bound:

$$\begin{aligned} R_T &\leq O \left(\sigma d \sqrt{T \log^2 T} \left(\sum_{k=1}^m \sqrt{p_k} \right) \right) + 2 \sum_{i \in \mathcal{U}} \left(\Gamma_i(T) - 1 \right) \left(\tau + \frac{2}{1 - \delta_e} \right) + 2 \sum_{i \in \mathcal{U}} \Gamma_i(T) O \left(\frac{2\psi^c \sigma^2}{\gamma^2 \lambda'^2} \log \frac{d}{\delta'} \right) \\ &= O \left(\sigma d \sqrt{T \log^2 T} \left(\sum_{k=1}^m \sqrt{p_k} \right) + \sum_{i \in \mathcal{U}} \Gamma_i(T) \cdot C \right) \end{aligned}$$

where $C = \frac{1}{1 - \delta_e} + \frac{\sigma^2}{\gamma^2 \lambda'^2} \log \frac{d}{\delta'}$, with a probability at least $(1 - \delta)(1 - \frac{\delta_e}{1 - \delta_e})(1 - \delta')$.

D.1 Proof of Eq (3)

Recall that we define a ‘good’ event as $\left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) = N_{k_{i_t, t}}^\phi(t-1) \right\}$, which means at time t , DyClu selects an arm using the UCB score computed with observations associated with $\phi_{k_{i_t, t}}$. And the ‘bad’ event is defined as its complement: $\left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\}$, which can be decomposed and then contained as shown below:

$$\begin{aligned} &\left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \\ &= \left(\left\{ \tilde{k}_{i_t, t-1} \neq k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \right) \cup \left(\left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \right) \\ &\subseteq \left\{ \tilde{k}_{i_t, t-1} \neq k_{i_t, t} \right\} \cup \left(\left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \right) \end{aligned}$$

where the event $\left\{ \tilde{k}_{i_t, t-1} \neq k_{i_t, t} \right\}$ means at time step t there is a late detection, and the event $\left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\}$ means there is no late detection, but the cluster identification fails to correctly cluster user models associated with $\phi_{k_{i_t, t}}$ together (for example, there might be models not belonging to this cluster, or models failed to be put into this cluster).

Under the ‘good’ event, arm $\mathbf{x}_t$ is selected using the UCB strategy by aggregating all existing observations associated with $\phi_{k_{i_t, t}}$, which is the unique bandit parameter for user i_t at time t . To simplify the

notations, borrowing the notation used in Eq (2), we denote $\hat{\theta}_{N_{k_{i_t,t}}^\phi(t-1)} = \mathbf{A}_{N_{k_{i_t,t}}^\phi(t-1)}^{-1} \mathbf{b}_{N_{k_{i_t,t}}^\phi(t-1)}$, where $\mathbf{A}_{N_{k_{i_t,t}}^\phi(t-1)} = \lambda \mathbf{I} + \sum_{j \in N_{k_{i_t,t}}^\phi(t-1)} \mathbf{x}_j \mathbf{x}_j^\top$ and $\mathbf{b}_{N_{k_{i_t,t}}^\phi(t-1)} = \sum_{j \in N_{k_{i_t,t}}^\phi(t-1)} \mathbf{x}_j y_j$, as the ridge regression estimator constructed using these observations, and $CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}) = \alpha_{N_{k_{i_t,t}}^\phi(t-1)} \sqrt{\mathbf{x}^\top \mathbf{A}_{N_{k_{i_t,t}}^\phi(t-1)}^{-1} \mathbf{x}}$, where $\alpha_{N_{k_{i_t,t}}^\phi(t-1)} = \sigma \sqrt{d \log(1 + \frac{|N_{k_{i_t,t}}^\phi(t-1)|}{d\lambda}) + 2 \log \frac{1}{\delta} + \sqrt{\lambda}}$ is the corresponding reward estimation confidence bound on $\mathbf{x}$.

Then we can upper bound the instantaneous regret r_t as follows,

$$\begin{aligned} r_t &= \langle \theta_{i_t,t}, \mathbf{x}_t^* \rangle - \langle \theta_{i_t,t}, \mathbf{x}_t \rangle \leq \langle \theta_{i_t,t}, \mathbf{x}_t \rangle - \langle \theta_{i_t,t}, \mathbf{x}_t \rangle \\ &= \langle \tilde{\theta}_{i_t,t} - \hat{\theta}_{\hat{V}_{i,t-1}}, \mathbf{x}_t \rangle + \langle \hat{\theta}_{\hat{V}_{i,t-1}} - \theta_{i_t,t}, \mathbf{x}_t \rangle \\ &\leq \begin{cases} 2CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t), & \text{if } \{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) = N_{k_{i_t,t}}^\phi(t-1)\} \\ 2, & \text{otherwise.} \end{cases} \end{aligned}$$

The first inequality is because $\langle \tilde{\theta}_{i_t,t}, \mathbf{x}_t \rangle$ is optimistic, where $\mathbf{x}_t \in C_t$ and $\tilde{\theta}_{i_t,t} \in \left\{ \theta \in \mathbb{R}^d : \|\hat{\theta}_{\hat{V}_{i,t-1}} - \theta\|_{\mathbf{A}_{\hat{V}_{i,t-1}}^{-1}} \leq \alpha_{N_{k_{i_t,t}}^\phi(t-1)} \right\}$. For the second inequality, we split it into two cases according to the occurrence of the ‘good’ or ‘bad’ events. Recall that $\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1)$ denotes the set of time steps associated with observations in $\hat{V}_{i,t-1}$. Then under the ‘good’ event $\{\hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) = N_{k_{i_t,t}}^\phi(t-1)\}$, with probability at least $1 - \delta$, we have $\langle \tilde{\theta}_{i_t,t} - \hat{\theta}_{\hat{V}_{i,t-1}}, \mathbf{x}_t \rangle \leq CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t)$ and $\langle \hat{\theta}_{\hat{V}_{i,t-1}} - \theta_{i_t,t}, \mathbf{x}_t \rangle \leq CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t)$, so that $r_t \leq 2CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t)$. Under the ‘bad’ event when wrong cluster is used for arm selection, we simply bound r_t by 2 because $\|\theta_{i_t,t}\| \leq 1$ and $\|\mathbf{x}_t\| \leq 1$.

Then the accumulated regret R_T can be upper bounded by:

$$\begin{aligned} R_T &= \sum_{t=1}^T r_t \leq 2 \sum_{t=1}^T \mathbf{1} \left\{ \hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) = N_{k_{i_t,t}}^\phi(t-1) \right\} CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t) + 2 \sum_{t=1}^T \mathbf{1} \left\{ \hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1) \right\} \\ &\leq \sum_{t=1}^T 2CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t) + 2 \sum_{t=1}^T \mathbf{1} \left\{ \hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1) \right\} \\ &\leq \sum_{t=1}^T 2CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t) + 2 \sum_{t=1}^T \mathbf{1} \left\{ \tilde{k}_{i_t,t-1} \neq k_{i_t,t} \right\} + 2 \sum_{t=1}^T \left(\mathbf{1} \left\{ \tilde{k}_{i_t,t-1} = k_{i_t,t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1) \right\} \right) \\ &\leq \sum_{t=1}^T 2CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t) + 2 \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1} \left\{ \tilde{k}_{i,t-1} \neq k_{i,t} \right\} \\ &\quad + 2 \sum_{t=1}^T \left(\mathbf{1} \left\{ \tilde{k}_{i_t,t-1} = k_{i_t,t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t,t-1}}^\phi(t-1) \neq N_{k_{i_t,t}}^\phi(t-1) \right\} \right) \end{aligned}$$

The first term is essentially the upper regret bound of the reference algorithm mentioned earlier in this section, which can be further upper bounded with probability at least $1 - \delta$ by:

$$\sum_{t=1}^T 2CB_{N_{k_{i_t,t}}^\phi(t-1)}(\mathbf{x}_t) = \sum_{k \in [m]} \sum_{t \in N_k^\phi(T)} 2CB_{N_k^\phi(t-1)}(x_t) \leq \sum_{k \in [m]} R_{Lin}(|N_k^\phi(T)|)$$

where $R_{Lin}(|N_k^\phi(T)|)$ is the high probability upper regret bound in Abbasi-Yadkori et al. (2011) (Theorem 3),

which is defined as:

$$\begin{aligned} R_{Lin}(|N_k^\phi(T)|) &= 4\sqrt{d|N_k^\phi(T)| \log(\lambda + \frac{|N_k^\phi(T)|}{d})} \left(\sigma\sqrt{2\log \frac{1}{\delta} + d\log(1 + \frac{|N_k^\phi(T)|}{d\lambda})} + \lambda^{1/2} \right) \\ &= O\left(\sigma d\sqrt{|N_k^\phi(T)| \log^2 |N_k^\phi(T)|} + \sigma\sqrt{d|N_k^\phi(T)| \log \frac{|N_k^\phi(T)|}{\delta}} \right) \end{aligned}$$

D.2 Proof of Lemma D.1

Now we have proved the intermediate regret upper bound in Eq (3). In this section, we continue to upper bound its second term $2\sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1}\{\tilde{k}_{i,t-1} \neq k_{i,t}\}$, which essentially counts the total number of late detections in each user, e.g., there is a mismatch between $\mathbb{M}_{i,t-1}$ and the current ground-truth bandit parameter $\theta_{i,t}$, but the learner fails to detect this. To prove this lemma, we need the following lemmas that upper bound the probability of late detections.

As opposed to early detection in Lemma 3.4, late detection corresponds to type-II error of homogeneity test in Lemma 3.3. Therefore we have the following lemma.

Lemma D.3 *When change has happened ($\tilde{k}_{i,t-1} \neq k_{i,t}$), we have*

$$P(e_{i,t} = 1) \geq \rho(1 - \delta') [1 - F(v^e; 1, \psi^e)]$$

$$\text{where } \psi^e = \frac{\Delta^2/\sigma^2}{1 + 1/\left(\frac{\lambda'}{4} S_{min} - 8\left(\log \frac{dS_{min}}{\delta'} + \sqrt{S_{min} \log \frac{dS_{min}}{\delta'}}\right)\right)}.$$

Proof of Lemma D.3.

Combining Lemma 3.3, Assumption 1 and 3, we can lower bound the probability that $e_{i,t} = 1$ when change has happened as:

$$\begin{aligned} P(e_{i,t} = 1) &= P(s(\mathcal{H}_{i,t-1}, \{\mathbf{x}_t, y_t\}) > v^e) \\ &\geq 1 - F\left(v^e; 1, \frac{[\mathbf{x}_t^\top (\theta_{i,t,c} - \theta_{i,t,c-1})]^2 / \sigma^2}{1 + \mathbf{x}_t^\top (\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)^{-1} \mathbf{x}_t}\right) \\ &\geq 1 - F\left(v^e; 1, \frac{[\mathbf{x}_t^\top (\theta_{i,t,c} - \theta_{i,t,c-1})]^2 / \sigma^2}{1 + \frac{\|\mathbf{x}_t\|^2}{\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)}}\right) \\ &\geq \rho \left[1 - F\left(v^e; 1, \frac{\Delta^2/\sigma^2}{1 + \frac{1}{\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)}}\right) \right] \end{aligned}$$

Since the minimum length of stationary period is S_{min} , by applying Lemma E.2, we can obtain the following lower bound on minimum eigenvalue when change happens as:

$$\lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top \right) \geq \frac{\lambda'}{4} S_{min} - 8 \left(\log \frac{dS_{min}}{\delta'} + \sqrt{S_{min} \log \frac{dS_{min}}{\delta'}} \right)$$

with probability at least $1 - \delta'$.

Denote $\psi^e = \frac{\Delta^2/\sigma^2}{1 + 1/\left(\frac{\lambda'}{4} S_{min} - 8\left(\log \frac{dS_{min}}{\delta'} + \sqrt{S_{min} \log \frac{dS_{min}}{\delta'}}\right)\right)}$. We obtain the following lower bound on the probability of detection:

$$P(e_{i,t} = 1) \geq \rho(1 - \delta') [1 - F(v^e; 1, \psi^e)]$$

when change has happened.

Lemma D.4 When change has happened ($\tilde{k}_{i_t, t-1} \neq k_{i_t, t}$),

$$P\left(\hat{e}_{i_t, t} \geq 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) \geq 1 - \delta_e$$

if the size of sliding window $\tau \geq \frac{2 \log 1/\delta_e}{\{[1 - F(v^e; 1, \psi^e)]\rho(1 - \delta') - 1 + F(v^e; 1, 0)\}^2}$.

Proof of Lemma D.4.

Similarly to the proof of Lemma 3.4, applying Hoeffding inequality given in Lemma E.1, we have:

$$\begin{aligned} P\left(\hat{e}_{i_t, t} \leq \mathbb{E}[e_{i_t, t}] - \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) &\leq \delta_e \\ P\left(\hat{e}_{i_t, t} \geq \mathbb{E}[e_{i_t, t}] - \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) &\geq 1 - \delta_e \end{aligned}$$

From Lemma D.3, when change has happened, $\mathbb{E}[e_{i_t, t}] \geq \rho(1 - \delta')[1 - F(v^e; 1, \psi^e)]$, with ψ^e being a variable dependent on environment as defined in Lemma D.3. By substituting this into the above inequality, we have:

$$P\left(\hat{e}_{i_t, t} \geq \rho(1 - \delta')[1 - F(v^e; 1, \psi^e)] - \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) \geq 1 - \delta_e$$

Then by rearranging terms above, we can find that if the sliding window size τ is selected to satisfy:

$$\tau \geq \frac{2 \log 1/\delta_e}{\{[1 - F(v^e; 1, \psi^e)]\rho(1 - \delta') - 1 + F(v^e; 1, 0)\}^2}$$

we can obtain:

$$\begin{aligned} P\left(\hat{e}_{i_t, t} \geq 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) &\geq 1 - \delta_e \\ P\left(\hat{e}_{i_t, t} \leq 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}}\right) &\leq \delta_e \end{aligned}$$

when change has happened ($\tilde{k}_{i_t, t-1} \neq k_{i_t, t}$).

Proof of Lemma D.1.

With results from Lemma D.4, our solution to further upper bound the number of late detections in each stationary period is similar to Wu et al. (2018) (Theorem 3.2). We include the proof here for the sake of completeness.

Denote the probability of detection when change has happened as $p_d = P(\hat{e}_{i_t, t} \geq 1 - F(v^e; 1, 0) + \sqrt{\frac{\log 1/\delta_e}{2\tau}})$, and from Lemma D.4, we have $p_d \geq 1 - \delta_e$. The probability distribution over the number of late detections when change has happened follows a geometric distribution: $P(n_{\text{late}} = k) = (1 - p_d)^{k-1} p_d$. Then by applying Chebyshev's inequality, we have $P(n_{\text{late}} \leq \frac{2}{1 - \delta_e}) \geq 1 - \frac{\delta_e}{1 - \delta_e}$.

Now we can upper bound the number of late detections $\sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1}\{\tilde{k}_{i_t, t-1} \neq k_{i_t, t}\}$ in user i . In Assumption 1 we have assumed that the total number of change points of user i is $\Gamma_i(T) - 1$. Therefore, $\sum_{t \in \mathcal{N}_i(T)} \mathbf{1}\{\tilde{k}_{i_t, t-1} \neq k_{i_t, t}\} \leq (\Gamma_i(T) - 1)(\tau + \frac{2}{1 - \delta_e})$ with probability at least $1 - \frac{\delta_e}{1 - \delta_e}$. Then we can upper bound the second term in Eq (3) by:

$$2 \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1}\{\tilde{k}_{i_t, t-1} \neq k_{i_t, t}\} \leq 2 \sum_{i \in \mathcal{U}} (\Gamma_i(T) - 1)(\tau + \frac{2}{1 - \delta_e})$$

D.3 Proof of Lemma D.2

The third term $\sum_{t=1}^T \mathbf{1} \left\{ \tilde{k}_{i,t-1} = k_{i,t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i,t-1}}^\phi(t-1) \neq N_{\tilde{k}_{i,t-1}}^\phi(t-1) \right\}$ counts the total number of times that there is no late detection, but cluster identification module fails to correctly cluster user models. We upper bound this using a similar idea as Gentile et al. (2017), but is based on the properties of homogeneity test. For the proof of Lemma D.2, we need the following lemmas related to probability of errors of cluster identification.

Lemma D.5 *When the underlying bandit parameters $\phi_{\tilde{k}_{i,t-1}}$ and $\phi_{\tilde{k}_{j,t-1}}$ of two observation history $\mathcal{H}_{i,t-1}$ and $\mathcal{H}_{j,t-1}$ are the same, the probability that cluster identification fails to cluster them together corresponds to the type-I error probability given in Lemma 3.2, and it can be upper bounded by:*

$$P\left(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) > v^c \mid \phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}}\right) \leq 1 - F(v^c; df, 0)$$

where $df = \text{rank}(\mathbf{X}_1) + \text{rank}(\mathbf{X}_2) - \text{rank}\left(\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}\right)$.

Corollary D.5.1 (Lower bound $P(N_{\tilde{k}_{i,t}}^\phi(t-1) \subseteq \hat{N}_{\tilde{k}_{i,t-1}}^\phi(t-1))$) *Since $N_{\tilde{k}_{i,t}}^\phi(t-1)$ denotes the set of time indices associated with all observations whose underlying bandit parameter is $\phi_{\tilde{k}_{i,t}}$, and $\hat{N}_{\tilde{k}_{i,t-1}}^\phi(t-1)$ denotes those in the estimated neighborhood $\hat{V}_{i,t-1}$, when there is no late detection, i.e., we have $\tilde{k}_{i,t-1} = k_{i,t}$. It naturally follows Lemma D.5 that:*

$$P\left(N_{\tilde{k}_{i,t}}^\phi(t-1) \subseteq \hat{N}_{\tilde{k}_{i,t-1}}^\phi(t-1)\right) \geq F(v^c; df, 0)$$

Lemma D.6 *When the underlying bandit parameters $\phi_{\tilde{k}_{i,t-1}}$ and $\phi_{\tilde{k}_{j,t-1}}$ of two observation sequence $\mathcal{H}_{i,t-1}$ and $\mathcal{H}_{j,t-1}$ are not the same, the probability that cluster identification module clusters them together corresponds to the type-II error probability given in Lemma 3.3, which can be upper bounded by:*

$$P\left(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c \mid \phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}\right) \leq F(v^c; d, \psi^c)$$

under the condition that both $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ and $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ are at least $\frac{2\psi^c \sigma^2}{\gamma^2}$.

Proof of Lemma D.6.

Recall that type-II error probability of the homogeneity test can be upper bounded by $P(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c \mid \phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}) \leq F(v^c; df, \psi)$ as discussed in Section 3.3. If either design matrix of the two datasets is rank-deficient, the noncentrality parameter ψ is lower bounded by 0 (lower bound achieved when the difference between two parameters lies in the null space of rank-deficient design matrix). Therefore, a nontrivial upper bound of type-II error probability only exists when the design matrices of both datasets are rank-sufficient. In this case, combining Lemma 3.3 and Assumption 2 gives:

$$\begin{aligned} & P(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1})) \\ & \leq F\left(v^c; d, \frac{\|\phi_{\tilde{k}_{i,t-1}} - \phi_{\tilde{k}_{j,t-1}}\|^2 / \sigma^2}{1/\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top) + 1/\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)}\right) \\ & \leq F\left(v^c; d, \frac{\gamma^2 / \sigma^2}{1/\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top) + 1/\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)}\right) \end{aligned}$$

Define $\psi^c > 0$; then by rearranging terms we obtain the conditions that, when both:

$$\lambda_{\min}\left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top\right) \geq \frac{2\psi^c \sigma^2}{\gamma^2} \quad \text{and} \quad \lambda_{\min}\left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top\right) \geq \frac{2\psi^c \sigma^2}{\gamma^2}$$

we have

$$P(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c \mid \phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}) \leq F(v^c; d, \psi^c)$$

Lemma D.7 *If the cluster identification module clusters observation history $\mathcal{H}_{i,t-1}$ and $\mathcal{H}_{j,t-1}$ together, the probability that they actually have the same underlying bandit parameters is denoted as $P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c)$.*

$$P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) \geq F(v^c; df, 0)$$

under the condition that both $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ and $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ are at least $\frac{2\psi^c \sigma^2}{\gamma^2}$, where $\psi^c = g^{-1}(\frac{p(1-F(v^c; d, 0))}{1-p}; d, v^c)$.

Proof of Lemma D.7.

Discussions Compared with the type-I and type-II error probabilities given in Lemma D.5 and D.6, the probability $P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c)$ also depends on the population being tested on. Two extreme examples would be 1) testing on a population that all user models have the same bandit parameter, and 2) every user model has an unique bandit parameter. Then in the former case $P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) = 1$ and in the latter case $P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) = 0$.

Denote the events $\{\phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}\} \cap \{S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) > v^c\}$ as True Positive (TP), $\{\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}}\} \cap \{S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c\}$ as True Negative (TN), $\{\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}}\} \cap \{S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) > v^c\}$ as False Positive (FP), and $\{\phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}\} \cap \{S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c\}$ as False Negative (FN) of cluster identification, respectively. We can rewrite the probabilities in Lemma D.5, D.6 and D.7 as:

$$\begin{aligned} P(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) > v^c | \phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}}) &= \frac{\text{FP}}{\text{TN} + \text{FP}} \leq 1 - F(v^c; df, 0) \\ P(S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c | \phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}) &= \frac{\text{FN}}{\text{FN} + \text{TP}} \leq F(v^c; df, \psi^c) \\ P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) &= \frac{\text{TN}}{\text{TN} + \text{FN}} = \frac{1}{1 + \frac{\text{FN}}{\text{TN}}} \end{aligned}$$

We can upper bound $\frac{\text{FN}}{\text{TN}}$ by:

$$\frac{\text{FN}}{\text{TN}} \leq \frac{\text{TP} + \text{FN}}{\text{TN} + \text{FP}} \cdot \frac{F(v^c; df, \psi^c)}{F(v^c; df, 0)}$$

where $\frac{\text{TP} + \text{FN}}{\text{TN} + \text{FP}}$ denotes the ratio between the number of positive instances ($\phi_{\tilde{k}_{i,t-1}} \neq \phi_{\tilde{k}_{j,t-1}}$) and negative instances ($\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}}$) in the population, which can be upper bounded by $\frac{1-p}{p}$ where p denotes the lower bound of the portion that each unique bandit parameter occurs in all stationary periods, i.e. $p = 1$ means the same unique bandit parameter occurs in all stationary periods.

It is worth noting that when either design matrix of $\mathcal{H}_{i,t-1}$ or $\mathcal{H}_{j,t-1}$ does not have full column rank, $P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) \geq 1 / (1 + \frac{1-p}{p} \cdot \frac{F(v^c; df, 0)}{F(v^c; df, 0)}) \geq p$, which is then trivially lower bounded by the percentage of negative instances in the population.

Under the conditions given in Lemma D.6, we have:

$$P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) \geq 1 / \left(1 + \frac{1-p}{p} \cdot \frac{F(v^c; d, \psi^c)}{F(v^c; d, 0)}\right)$$

Then by setting $\psi^c = g^{-1}(\frac{p(1-F(v^c; d, 0))}{1-p}; d, v^c)$, we have:

$$P(\phi_{\tilde{k}_{i,t-1}} = \phi_{\tilde{k}_{j,t-1}} | S(\mathcal{H}_{i,t-1}, \mathcal{H}_{j,t-1}) \leq v^c) \geq 1 / \left(1 + \frac{1-p}{p} \cdot \frac{F(v^c; d, \psi^c)}{F(v^c; d, 0)}\right) = F(v^c; df, 0)$$

Corollary D.7.1 (Lower bound $P(\hat{N}_{\tilde{k}_{i,t}, t-1}^\phi(t-1) \subseteq N_{\tilde{k}_{i,t}, t}^\phi(t-1))$) *It naturally follows Lemma D.7 that:*

$$P(\hat{N}_{\tilde{k}_{i,t}, t-1}^\phi(t-1) \subseteq N_{\tilde{k}_{i,t}, t}^\phi(t-1)) \geq F(v^c; df, 0)$$

under the condition that both $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ and $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ are at least $\frac{2\psi^c \sigma^2}{\gamma^2}$, where $\psi^c = g^{-1}(\frac{p(1-F(v^c; d, 0))}{1-p}; d, v^c)$.

Proof of Lemma D.2.

From Corollary D.5.1 and D.7.1, when both $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ and $\lambda_{\min}(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{j,t-1}} \mathbf{x}_k \mathbf{x}_k^\top)$ are at least $\frac{2\psi^c \sigma^2}{\gamma^2}$, with probability at least $F(v^c; df, 0)$, we have event $\left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) = N_{k_{i_t, t}}^\phi(t-1) \right\}$. Therefore, the third term in Eq (3) is upper bounded by:

$$\begin{aligned} & 2 \sum_{t=1}^T \mathbf{1} \left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \\ & \leq 2 \sum_{t=1}^T \mathbf{1} \left\{ \exists \mathcal{H} \in \mathbb{U}_{t-1} \cup \mathbb{O}_{t-1} : \lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}} \mathbf{x}_k \mathbf{x}_k^\top \right) < \frac{2\psi^c \sigma^2}{\gamma^2} \right\} \\ & \leq 2 \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1} \left\{ \lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i, t-1}} \mathbf{x}_k \mathbf{x}_k^\top \right) < \frac{2\psi^c \sigma^2}{\gamma^2} \right\} \end{aligned}$$

Essentially, it counts the number of time steps in total when minimum eigenvalue of a user model $\mathbb{M}$'s correlation matrix is smaller than $\frac{2\psi^c \sigma^2}{\gamma^2}$. We further decompose the summation by considering each stationary period of each user.

$$\begin{aligned} & 2 \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1} \left\{ \lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i, t-1}} \mathbf{x}_k \mathbf{x}_k^\top \right) < \frac{2\psi^c \sigma^2}{\gamma^2} \right\} \\ & = 2 \sum_{i \in \mathcal{U}} \sum_{s \in [0, c_{i,1}, \dots, c_{i, \Gamma_i(T)-1}]} \sum_{t \in S_{i,s}} \mathbf{1} \left\{ \lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i, t-1}} \mathbf{x}_k \mathbf{x}_k^\top \right) < \frac{2\psi^c \sigma^2}{\gamma^2} \right\} \end{aligned}$$

where $S_{i,s}$ denotes the s 'th stationary period of user i .

Borrowing the notation from Gentile et al. (2017), denote A_t as a correlation matrix constructed through a series of rank-one updates using context vectors from $\{C_t\}_{t \in S}$, where S denotes the set of time steps we performed model update. Note that the choice of which context vector to select from C_t for $t \in S$ can be arbitrary. Then we denote the maximum number of updates it takes until $\lambda_{\min}(A_t)$ is lower bounded by η as $\text{HD}(\{C_t\}_{t \in S}, \eta) = \max \{t \in S : \exists \mathbf{x}_1 \in C_1, \dots, \mathbf{x}_t \in C_t : \lambda_{\min}(A_t) \leq \eta\}$, where $A_t = \sum_{u \in S: u \leq t} \mathbf{x}_u \mathbf{x}_u^\top$. Therefore, we obtain:

$$\begin{aligned} & \sum_{i \in \mathcal{U}} \sum_{t \in \mathcal{N}_i(T)} \mathbf{1} \left\{ \lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i, t-1}} \mathbf{x}_k \mathbf{x}_k^\top \right) < \frac{2\psi^c \sigma^2}{\gamma^2} \right\} \\ & = \sum_{i \in \mathcal{U}} \sum_{s \in [0, c_{i,1}, \dots, c_{i, \Gamma_i(T)-1}]} \sum_{t \in S_{i,s}} \mathbf{1} \left\{ \lambda_{\min} \left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}_{i, t-1}} \mathbf{x}_k \mathbf{x}_k^\top \right) < \frac{2\psi^c \sigma^2}{\gamma^2} \right\} \\ & \leq \sum_{i \in \mathcal{U}} \sum_{s \in [0, c_{i,1}, \dots, c_{i, \Gamma_i(T)-1}]} \text{HD} \left(\{C_t\}_{t \in S_{i,s}}, \frac{2\psi^c \sigma^2}{\gamma^2} \right) \end{aligned}$$

Then similar to Gentile et al. (2017) (Lemma 1), by applying Lemma E.2 we can upper bound the third term in Eq (3):

$$\begin{aligned} & 2 \sum_{t=1}^T \mathbf{1} \left\{ \tilde{k}_{i_t, t-1} = k_{i_t, t} \right\} \cap \left\{ \hat{N}_{\tilde{k}_{i_t, t-1}}^\phi(t-1) \neq N_{k_{i_t, t}}^\phi(t-1) \right\} \\ & \leq 2 \sum_{i \in \mathcal{U}} \sum_{s \in [0, c_{i,1}, \dots, c_{i, \Gamma_i(T)-1}]} \text{HD} \left(\{C_t\}_{t \in S_{i,s}}, \frac{2\psi^c \sigma^2}{\gamma^2} \right) \\ & \leq 2 \sum_{i \in \mathcal{U}} \Gamma_i(T) O \left(\frac{2\psi^c \sigma^2}{\gamma^2 \lambda'^2} \log \frac{d}{\delta'} \right) \end{aligned}$$

with probability at least $1 - \delta'$.

E Technical lemmas

Here are some of the technical lemmas needed for the proofs in this paper.

Lemma E.1 (Hoeffding inequality) *Suppose that we have independent variables $x_i, i = 1, \dots, n$, and x_i has mean μ_i and sub-Gaussian parameter σ_i . Then for all $h \geq 0$, we have*

$$P\left(\sum_{i=1}^n (x_i - \mu_i) \geq h\right) \leq \exp\left(-\frac{h^2}{2 \sum_{i=1}^n \sigma_i^2}\right)$$

Lemma E.2 (Lemma 1 of Gentile et al. (2017)) *Under Assumption 3 that, at each time t , arm set C_t is generated i.i.d. from a sub-Gaussian random vector $X \in \mathbb{R}^d$, such that $\mathbb{E}[XX^\top]$ is full-rank with minimum eigenvalue $\lambda' > 0$; and the variance ς^2 of the random vector satisfies $\varsigma^2 \leq \frac{\lambda'^2}{8 \ln 4K}$. Then we have the following lower bound on minimum eigenvalue of the correlation matrix of observation history $\mathcal{H}$:*

$$\lambda_{\min}\left(\sum_{(\mathbf{x}_k, y_k) \in \mathcal{H}} \mathbf{x}_k \mathbf{x}_k^\top\right) \geq \frac{\lambda'}{4} |\mathcal{H}| - 8 \left(\log \frac{d|\mathcal{H}|}{\delta'} + \sqrt{|\mathcal{H}| \log \frac{d|\mathcal{H}|}{\delta'}} \right)$$

with probability at least $1 - \delta'$.