

Nonunitary Scaling Theory of Non-Hermitian Localization

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Non-Hermiticity can destroy Anderson localization and lead to delocalization even in one dimension. However, a unified understanding of non-Hermitian delocalization has yet to be established. Here, we develop a scaling theory of localization in non-Hermitian systems. We reveal that non-Hermiticity introduces a new scale and breaks down the one-parameter scaling, which is the central assumption of the conventional scaling theory of localization. Instead, we identify the origin of unconventional non-Hermitian delocalization as the two-parameter scaling. Furthermore, we establish the threefold universality of non-Hermitian localization based on reciprocity; reciprocity forbids delocalization without internal degrees of freedom, whereas symplectic reciprocity results in a new type of symmetry-protected delocalization.

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Anderson localization [1] is the disorder-induced localization of coherent waves and plays an important role in transport phenomena of condensed matter [2,3], light [4], and cold atoms [5,6]. A unified understanding of Anderson localization is provided by the scaling theory [7,8]. On the basis of the one-parameter-scaling hypothesis of the conductance with respect to the system size, it describes the criticality of localization transitions in three dimensions and predicts the absence of delocalization in one and two dimensions. Symmetry further changes the universality class of localization. For example, time-reversal symmetry (reciprocity) in the presence of spin-orbit interaction enables delocalization even in two dimensions [9]; chiral (sublattice) symmetry enables delocalization of zero modes even in one dimension [10–14].

Meanwhile, the physics of non-Hermitian systems has attracted considerable interest in recent years [15–18]. Non-Hermiticity originates from exchanges of energy or particles with an environment and leads to rich properties unique to particle-number-nonconserving systems in dynamics [19–36] and topology [37–64]. Anderson localization was also investigated in non-Hermitian systems with asymmetric hopping [65–78] and gain or loss [79–84], the latter of which is directly relevant to random lasers [85]. Even in the presence of non-Hermiticity, random lasers in one dimension never exhibit delocalization similarly to the Hermitian case. By contrast, a non-Hermitian extension of the Anderson model with asymmetric hopping, which was first investigated by Hatano and Nelson [65], exhibits delocalization in one dimension. Importantly, this implies the breakdown of the conventional scaling theory of localization, which predicts the absence of delocalization in one dimension. In fact, since Anderson localization results from the destructive interference of coherent waves,

non-Hermiticity should lead to decoherence and destroy Anderson localization. However, it remains unclear how non-Hermiticity changes the scaling theory of localization, and a unified understanding of non-Hermitian localization has yet to be obtained.

In this Letter, we develop a scaling theory of localization in non-Hermitian systems. On the basis of the random-matrix approach for nonunitary scattering matrices, we reveal that non-Hermiticity introduces a new scale and breaks down the one-parameter-scaling hypothesis. Instead, we demonstrate the two-parameter scaling (Fig. 1), which is the origin of unconventional non-Hermitian delocalization. Furthermore, we establish the threefold universality of non-Hermitian localization according to reciprocity (Table I). While non-Hermitian systems exhibit unidirectional delocalization in the absence of symmetry, reciprocity forbids it without internal degrees of freedom, which explains the absence of delocalization in random lasers. We also find a new universality class of localization transitions: bidirectional delocalization protected by symplectic reciprocity.

Non-Hermitian delocalization.—In the conventional scaling theory of localization [8], we consider the dependence of the conductance G on the length scale L . A sufficiently small system is diffusive and described by Ohm's law (Boltzmann equation), leading to $G \propto L^{d-2}$ in d dimensions. For a sufficiently large system, on the other hand, the wave coherence is relevant and Anderson localization can occur, leading to $G \propto e^{-\alpha L}$ ($\alpha > 0$). The transition between these two regimes can be understood by the scaling function $\beta(G) := d \log G / d \log L$. In the localized regime, it is given as $\beta(G) = \log G < 0$ and hence the conductance G gets smaller with increasing the system length L . We have $\beta(G) = d - 2$ in the diffusive regime, which is positive (negative) for $d > 2$ ($d < 2$). Consequently, a localization

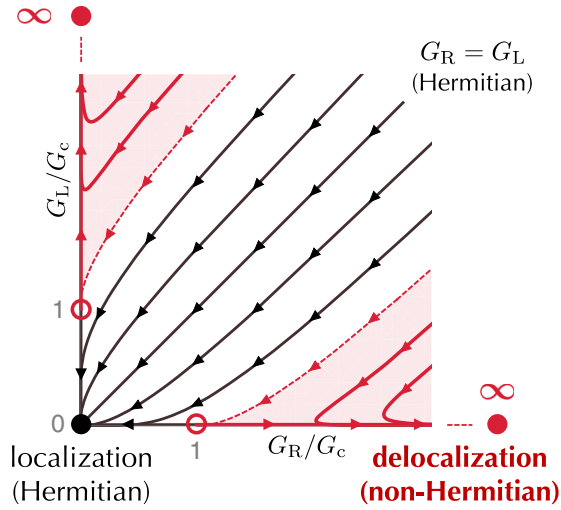


FIG. 1. Two-parameter scaling of non-Hermitian localization. The renormalization-group flow is shown according to the conductance G_R from the left to the right and the conductance G_L from the right to the left. The system size L increases along with the arrows. While localization with $(G_R, G_L) = (0, 0)$ (black dot) occurs in Hermitian systems ($G_R = G_L$), delocalization with $(G_R, G_L) = (\infty, 0), (0, \infty)$ (red dots) arises for sufficiently strong non-Hermiticity.

transition occurs in three dimensions at $G = G_c$ where $\beta(G_c) = 0$; by contrast, no transitions occur in one dimension since $\beta(G)$ is always negative and G monotonically decreases in both diffusive and localized regimes.

Non-Hermiticity gives rise to a new regime that has no analogs in particle-number-conserving systems. In fact, it describes coupling to an external environment and can lead to amplification (lasing), resulting in $G \propto e^{\gamma L}$ with the amplification rate $\gamma > 0$. In such a regime, we have $\beta(G) = \log G > 0$ in arbitrary dimensions, and hence delocalization is possible even in one dimension. The amplifying regime can arise from nonunitarity of scattering matrices. In Hermitian systems, unitarity is imposed on scattering matrices as a direct consequence of conservation of particle numbers, and the transmission amplitudes cannot exceed one. In non-Hermitian systems, by contrast, such a constraint is absent and the conductance G can be arbitrarily large, which enables the amplification as $G \propto e^{\gamma L}$.

TABLE I. Threefold universality of non-Hermitian localization based on reciprocity. The types of delocalization and the typical conductances for $L \gg \ell$ are shown according to non-Hermiticity γ and the mean free path $\ell > 0$.

Class	Symmetry	Delocalization	Conductances
A	No	Unidirectional	$e^{(\pm\gamma-1/\ell)L}$
AI [†]	$H^T = H$	No	$e^{-L/\ell}$
AII [†]	$\sigma_2 H^T \sigma_2^{-1} = H$	Bidirectional	$e^{(\gamma -1/\ell)L}$

The delocalization in the amplifying regime can also be understood by the Thouless criterion [7,86]. In the diffusive regime, it takes the Thouless time $t_{\text{Th}} \propto L^2$ for a particle to reach one end from the other in a system of size L^d . To realize this diffusive transport, t_{Th} should be smaller than the time scale $\Delta t \propto (\Delta E)^{-1}$ determined by the level spacing $\Delta E \propto L^{-d}$ of the spectrum. Because of $t_{\text{Th}}/\Delta t \propto L^{2-d}$, this is possible in three dimensions but impossible in one dimension. In the amplifying regime, on the other hand, particle inflow from the environment enables the ballistic transport, and the relevant time scale is $t_N \propto L$. Because of $t_N/\Delta t \propto L^{1-d}$, t_N is comparable to Δt even in one dimension, which results in the delocalization. Saliiently, an additional relevant scale accompanies the amplifying regime, which implies the breakdown of the one-parameter-scaling hypothesis [7,8], as discussed below.

Scaling equations.—To uncover universal behavior of Anderson localization in non-Hermitian systems, we revisit the Hatano-Nelson model [65] and derive the scaling equations for transport properties. We show that the scaling behavior should be understood in terms of two parameters rather than one parameter. On the basis of this understanding, we later discuss Anderson localization for other symmetry classes and find new universality classes. Our scaling theory also explains the different universality classes between the Hatano-Nelson model and random lasers.

The Hatano-Nelson model [65] reads

$$\hat{H} = \sum_n \left\{ -\frac{1}{2} (\hat{c}_{n+1}^\dagger J_R \hat{c}_n + \hat{c}_n^\dagger J_L \hat{c}_{n+1}) + \hat{c}_n^\dagger M_n \hat{c}_n \right\}, \quad (1)$$

where $\hat{c}_n$ ($\hat{c}_n^\dagger$) annihilates (creates) a fermion at site n , $J_R := J + \gamma/2$ ($J_L := J - \gamma/2$) describes the hopping from the left to the right (from the right to the left), and $M_n \in \mathbb{R}$ is the random potential at site n . The asymmetry γ of the hopping can be introduced, for example, in open photonic systems [31,74] and cold atoms with dissipation [52]. Whereas eigenstates are localized for weak γ , they can be delocalized for strong γ . In the literature, the complex spectrum [65, 67–69,76], the conductance [68,73], and the chiral transport [74,75] were investigated for this lattice model. Nevertheless, the scaling theory has not been fully formulated.

The nature of non-Hermitian delocalization should not depend on specific details of the model but solely on symmetry. To understand such a universal feature, we construct a continuum model from the Hatano-Nelson model. To this end, we focus on the narrow shell around the band center $\text{Re}E = 0$ and decompose the fermions by $\hat{c}_n = e^{ik_F n} \hat{\psi}_R + e^{-ik_F n} \hat{\psi}_L$ ($k_F = \pi/2$). Here, $\hat{\psi}_R$ and $\hat{\psi}_L$ are the right-moving and left-moving fermions on the two Fermi points (valleys), respectively. Assuming that $\hat{\psi}_R$ and $\hat{\psi}_L$ vary slowly in space, we have the continuum model $\hat{H} = \int dx (\hat{\psi}_R^\dagger \hat{\psi}_L^\dagger) h_A (\hat{\psi}_R \hat{\psi}_L)^T$ with

$$h_A = (-i\partial_x + i\gamma/2)\tau_3 + m_0(x) + m_1(x)\tau_1, \quad (2)$$

where Pauli matrices τ_i 's describe the two valley degrees of freedom. We assume that m_0 and m_1 are Gaussian disorder that satisfies $\langle m_i(x) \rangle = 0$ and $\langle m_i(x)m_j(x') \rangle = 2\mu_i\delta_{ij}\delta(x-x')$ with $\mu_i > 0$ and the ensemble average $\langle \star \rangle$. Although we begin with the Hatano-Nelson model, we emphasize that h_A does not depend on its specific details but universally on symmetry. Generic non-Hermitian systems without symmetry including h_A are defined to belong to class A in the 38-fold classification of internal symmetry [57,87,88].

Now, we formulate the scaling equations (functional renormalization group equations). The conductance G_R from the left to the right (G_L from the right to the left) is given by the corresponding transmission eigenvalue T_R (T_L) according to the Landauer formula [89]. Then, we consider the incremental changes of $T_{R/L}$, in addition to the reflection eigenvalue R_L from the left to the left (R_R from the right to the right), upon attachment of a thin slice in which the scattering can be treated perturbatively. Such attachment renormalizes the probability distribution of $T_{R/L}$ and $R_{L/R}$, resulting in its scaling (Fokker-Planck) equation according to the system size L [90]. It provides all the information about the transmission eigenvalues $T_{R/L}$ and the conductances $G_{R/L}$. In the Hermitian case, the scaling equations were obtained by Dorokhov, and by Mello, Pereyra, and Kumar [91–93].

For the continuum model h_A , we find that non-Hermiticity γ amplifies one of T_R and T_L and attenuates the other, but does not have significant influence on their phases. As a result, we have [90]

$$\frac{\langle dT_{R/L} \rangle}{dL} = \pm\gamma T_{R/L} - \frac{T_{R/L}(1 - R_{L/R})}{\ell}, \quad (3)$$

where $\ell := 1/2\mu_1$ is the mean free path determined by the disorder strength. The ensemble average $\langle \star \rangle$ is taken over the attached thin slice and the phases of the scattered waves for given $T_{R/L}$ and $R_{L/R}$. This scaling equation, Eq. (3), implies that the transmission amplitudes are given as $T_{R/L} = e^{\pm\gamma L} \tilde{T}$ with the transmission amplitude $\tilde{T}$ in Hermitian systems. For $L \gg \ell$, the conductance fluctuations become as large as the averages $\langle G \rangle$, which no longer represent the conductances of a single sample. In fact, the conductance distributions are broad and asymmetric, and follow the log-normal distributions. Consequently, the typical conductances are $G^{\text{typ}} := e^{(\log G)}$ instead of $\langle G \rangle$. Because of $\tilde{G}^{\text{typ}}/G_c \sim e^{-L/\ell}$ for $L \gg \ell$ in the Hermitian case [91–93], the typical conductances in the non-Hermitian case are $G_{R/L}^{\text{typ}}/G_c \sim e^{(\pm\gamma-1/\ell)L}$. Thus, either one of the two conductances exhibits delocalization. For $\gamma \geq 0$, for example, G_R^{typ} diverges for $L \rightarrow \infty$ above the

transition point $\gamma = \gamma_c := 1/\ell$, around which the critical behavior $|G_R^{\text{typ}} - G_c|/G_c \propto |\gamma - \gamma_c|$ appears.

Two-parameter scaling.—In Hermitian systems, the scaling equations and the conductance $\tilde{G}$ depend solely on L/ℓ . This confirms the one-parameter-scaling hypothesis, which underlies the absence of delocalization in one dimension [7,8]. However, the obtained scaling equation, Eq. (3), clearly indicates the emergence of the additional scale γ due to non-Hermiticity. In fact, non-Hermiticity leads to the distinction between G_R and G_L , which is impossible in Hermitian systems by conservation of particle numbers. From Eq. (3), we show in Fig. 1 the renormalization-group flow based on both G_R and G_L . In addition to the fixed point $(G_R, G_L) = (0, 0)$ for the localized phase, a pair of additional fixed points $(G_R, G_L) = (G_c, 0), (0, G_c)$ emerges away from $G_R = G_L$. As a result, delocalization with $(G_R, G_L) = (\infty, 0), (0, \infty)$ is possible for sufficiently strong non-Hermiticity. Therefore, the emergence of the new scale and the breakdown of the one-parameter scaling are the origin of the non-Hermitian delocalization in one dimension.

It is also notable that the average conductances are $\langle G_{R/L} \rangle/G_c \sim e^{(\pm\gamma-1/4\ell)L}$ since the Hermitian counterpart is $\langle \tilde{G} \rangle/G_c \sim e^{-L/4\ell}$ [91–93]. Hence, $\langle G_R \rangle$ exhibits critical behavior at $\gamma = 1/4\ell$, which is different from the critical point $\gamma = 1/\ell$ of the typical conductance G_R^{typ} [73]. Such a difference of the critical points is another manifestation of the breakdown of the one-parameter scaling. In fact, if the scaling equations are described solely by a single parameter ℓ , both $\langle G_R \rangle$ and G_R^{typ} are functions of L/ℓ , and hence their critical points should coincide with each other. The different critical points of $\langle G_R \rangle$ and G_R^{typ} imply the two different length scales ℓ and γ^{-1} .

Threefold universality by reciprocity.—Symmetry can further change the universality class of Anderson localization. In particular, reciprocity, defined by $\mathcal{T}H^T\mathcal{T}^{-1} = H$ with a unitary matrix $\mathcal{T}$, is fundamental symmetry relevant to localization. For example, reciprocity with $\mathcal{T}\mathcal{T}^* = +1$ (-1) enhances (suppresses) localization and shortens (lengthens) localization lengths in Hermitian wires in quasi-one dimension [91–93]. Moreover, symplectic reciprocity with $\mathcal{T}\mathcal{T}^* = -1$ enables delocalization even in two dimensions [9], although delocalization is forbidden without symmetry protection. Here, we uncover the threefold universality of non-Hermitian localization based on reciprocity (Table I). As demonstrated below, the influence of reciprocity is more dramatic than the Hermitian case.

We consider a non-Hermitian continuum model

$$h_{A\Gamma^\dagger} = -i\tau_3\partial_x + m_0(x) + [m_1(x) + i\gamma/2]\tau_1, \quad (4)$$

which respects $\tau_1 h_{A\Gamma^\dagger}^T \tau_1^{-1} = h$ and hence belongs to class $A\Gamma^\dagger$ (orthogonal class) [57,90]. Notably, the asymmetry between the valleys [i.e., $i(\gamma/2)\tau_3$ term in Eq. (2)] is forbidden because of reciprocity, which leads to $G_R = G_L$

even in non-Hermitian systems. Thus, the nonunitary fixed points away from $G_R = G_L$ in Fig. 1 cannot be reached, and the unidirectional delocalization is forbidden. In terms of the scaling equations, reciprocity-preserving non-Hermiticity is irrelevant by the ensemble average over disorder, whereas reciprocity-breaking non-Hermiticity gives rise to an additional scale. Consequently, the universality in class $\text{AI}^\dagger$ is the same as the Hermitian counterpart, which contrasts with class A. The continuum model in Eq. (4) describes disordered wires with gain or loss (i.e., complex on-site potential), including random lasers [85]. Reciprocity underlies the absence of delocalization in random lasers.

On the other hand, reciprocal systems with $\mathcal{T}\mathcal{T}^* = -1$ instead of $\mathcal{T}\mathcal{T}^* = +1$ are defined to belong to class $\text{AII}^\dagger$ (symplectic class) [57]. Although reciprocity imposes $G_R = G_L$ also in this case, an important distinction in the symplectic class is Kramers degeneracy, which gives rise to a new type of non-Hermitian delocalization protected by reciprocity. The corresponding continuum model is

$$h_{\text{AII}^\dagger} = [-i\partial_x + \Delta\sigma_1 + i(\gamma/2)\sigma_3]\tau_3 + m_0(x) + m_1(x)\tau_1, \quad (5)$$

which respects $(\sigma_2\tau_1)h_{\text{AII}^\dagger}^T(\sigma_2\tau_1)^{-1} = h_{\text{AII}^\dagger}$. Here, Pauli matrices σ_i 's describe the internal degrees of freedom such as spin. The scaling equations can be obtained in a similar manner to class A [90]. In this case, one of the Kramers pair is amplified to the right while the other to the left because of non-Hermiticity. We then have $G_R^{\text{typ}}/G_c = G_L^{\text{typ}}/G_c \sim (e^{\gamma L} + e^{-\gamma L})e^{-L/\ell} \sim e^{(|\gamma|-1/\ell)L}$ for $L \gg \ell$. Thus, the eigenstates are bidirectionally delocalized in contrast to classes A and $\text{AI}^\dagger$. Without symmetry, one of the transmitted channels dominates the other, and non-Hermitian delocalization is unidirectional. Hence, the bidirectional delocalization arises only in the presence of symplectic reciprocity. Despite $G_R = G_L$, the conductance of one channel serves as G_R and that of the corresponding Kramers partner serves as G_L in the two-parameter scaling shown in Fig. 1, the sum of which yields the total conductance.

While reciprocity is equivalent to time-reversal symmetry $\mathcal{T}H^*\mathcal{T}^{-1} = H$ in Hermitian systems, this is not the case in non-Hermitian systems. The corresponding symmetry classes with time-reversal symmetry are classes AI and AII [57]. The universality of non-Hermitian localization is also different depending on whether one imposes time-reversal symmetry or reciprocity. In fact, time-reversal symmetry does not change the universality of the non-Hermitian localization [90], whereas reciprocity can forbid or enhance it as discussed above. Reciprocity also leads to threefold universality of non-Hermitian random matrices [94].

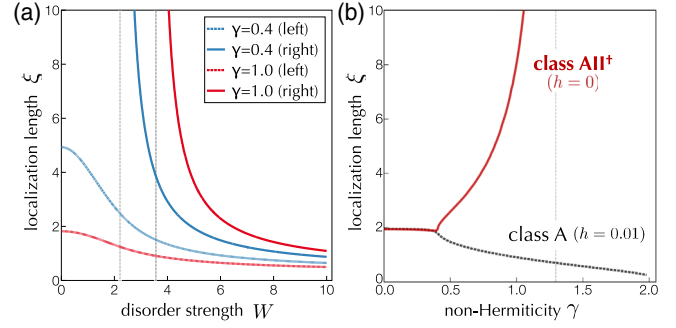


FIG. 2. Non-Hermitian localization on lattices ($L = 5000$, $J = 1.0$, $E = 0$). The disordered on-site potential is uniformly distributed over $[-W/2, W/2]$, and each datum is averaged over 1000 samples. (a) Hatano-Nelson model (class A). For $\gamma > 0$, the right localization length diverges at a transition point, whereas the left localization length remains finite. The transition points are $W_c = 2.22$ ($\gamma = 0.4$) and $W_c = 3.56$ ($\gamma = 1.0$). (b) Symplectic Hatano-Nelson model (class $\text{AII}^\dagger$; $\Delta = 0.2$, $W = 4.0$). Both right and left localization lengths diverge at the transition point $\gamma_c = 1.30$ (red solid curve). The delocalization vanishes in the presence of a reciprocity-breaking perturbation $h = 0.01$ (black dotted curve).

Lattice models.—To confirm our nonunitary scaling theory, we numerically investigate non-Hermitian lattice models by the transfer-matrix method [90,95]. In general, a wave function localized around site $n = n_0$ is proportional to $e^{-|n-n_0|/\xi_L(\xi_R)}$ for $n < n_0$ ($n > n_0$). While the two localization lengths ξ_L and ξ_R are equivalent in Hermitian systems, they are different in a similar manner to the conductances G_L and G_R . Figure 2(a) shows the localization lengths for the Hatano-Nelson model in Eq. (1). For $\gamma \geq 0$, the right localization length ξ_R diverges at a critical point, whereas the left localization length ξ_L remains finite, which is a signature of the unidirectional delocalization. Around the critical point, ξ_R diverges as $\xi_R \propto |\gamma - \gamma_c|^{-1}$.

A symplectic extension of the Hatano-Nelson model is given by Eq. (1) with $J_R := J - i\Delta\sigma_1 + \gamma\sigma_3/2$, $J_L := J + i\Delta\sigma_1 - \gamma\sigma_3/2$, and $M_n := m_n + h\sigma_3$. This lattice model with $h = 0$ corresponds to the continuum model in Eq. (5). In contrast to the original Hatano-Nelson model, we have $\xi_L = \xi_R$ for $h = 0$ because of reciprocity. As a result, both ξ_L and ξ_R diverge at a critical point [Fig. 2(b)], which is a signature of the bidirectional delocalization. Because of the reciprocity-protected nature, the delocalization vanishes even in the presence of a small reciprocity-breaking perturbation $h \neq 0$, which is unique to the symplectic class.

Chiral symmetry and sublattice symmetry.—In the presence of chiral or sublattice symmetry, zero modes can be delocalized even in Hermitian systems in one dimension, accompanied by Dyson's singularity [10]. Similarly to time-reversal symmetry and reciprocity, chiral symmetry and sublattice symmetry are distinct from each other in

non-Hermitian systems, the former (latter) of which corresponds to class AIII (AIII[†]) [57]. For example, a random hopping model with gain or loss respects chiral symmetry, while a random asymmetric hopping model respects sublattice symmetry. In the presence of chiral symmetry $\tau_1 h_{\text{AIII}}^\dagger \tau_1^{-1} = -h_{\text{AIII}}$, non-Hermiticity is found not to change the universality of the delocalization [90]; by contrast, in the presence of sublattice symmetry $\tau_1 h_{\text{AIII}^\dagger} \tau_1^{-1} = -h_{\text{AIII}^\dagger}$, non-Hermiticity enables the unidirectional delocalization in a similar manner to class A. In fact, the asymmetry between the valleys is allowed for sublattice symmetry, but forbidden for chiral symmetry.

Discussions.—Transport phenomena of disordered systems, including Anderson localization and transitions, enjoy universality in various scaling limits that is governed only by a few physical parameters. This is embodied by the one-parameter scaling of localization [7,8]. In this Letter, we have demonstrated that non-Hermiticity breaks it down and leads to the two-parameter scaling, which generally describes the unconventional non-Hermitian delocalization. While we limit ourselves to the single-channel case in this Letter, it is meaningful to consider the limit of thick wires in order to fully uncover universal properties—we leave this as a future problem.

In our nonunitary two-parameter scaling, the critical exponents are integers, which contrast with the more complicated exponents in the two-parameter scaling of the quantum Hall transition [96–99]. On the other hand, these two scaling theories share similarities from a topological perspective. In particular, the Hatano-Nelson model is characterized by a topological invariant unique to non-Hermitian systems [52,57]. In our continuum model, this topological invariant is $\text{sgn}\gamma$, similar to the Hall conductivity given by the Dirac mass term. An open problem is to formulate an effective field theory for the nonunitary two-parameter scaling, akin to the nonlinear sigma model augmented with a topological term for the quantum Hall transition [96–98]. In this regard, it is worth pointing out that topological field theory descriptions for non-Hermitian systems have been recently proposed [100].

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