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BOUNDS ON CONTINUOUS SCOTT RANK

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Abstract. An analog of Nadel's effective bound for the continuous Scott rank of metric structures, developed in [Adv. Math. 318 (2017), pp. 46–87], will be established: Let $\mathscr{L}$ be a language of continuous logic with code $\hat{\mathscr{L}}$. Let Ω be a weak modulus of uniform continuity with code $\hat{\Omega}$. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure. Let $\bar{\mathcal{D}}$ denote the completion structure of $\mathcal{D}$. Then $\mathrm{SR}_\Omega(\bar{D}) \leq \omega_1^{\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}}$, the Church-Kleene ordinal relative to $\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}$.

1. Introduction

The authors of [3] developed a new Scott analysis for continuous logic for metric structures. If $\mathscr{L}$ is a language of continuous logic, then an $\mathscr{L}$-structure of continuous logic is a Polish metric space endowed with a suitable interpretation for each symbol of $\mathscr{L}$. In this setting, a countable dense $\mathscr{L}$-pre-structure (Definition 2.6) completely determines the original structure through taking completions. The main goal of this paper is to establish a countable effective bound on the continuous Scott rank of a metric structure which depends on the definability content of the countable dense $\mathscr{L}$-pre-structure in a manner analogous to Nadel's effective bound for countable first order structure.

To motivate the ideas of the Scott analysis and the use of continuous logic to study Polish metric structures, a brief review of the classical Scott analysis will be given.

Let $\mathscr{L}$ be a first order language in the classical sense. Let $\mathcal{M}$ denote some $\mathscr{L}$-structure. Let $\bar{a}$ and $\bar{b}$ be two tuples in $\mathcal{M}$ of the same length. The Scott analysis begins by attempting to quantify how difficult it is to distinguish $\bar{a}$ and $\bar{b}$ in a manner expressible by $\mathscr{L}$. For instance, if there was an $\mathscr{L}$-automorphism of $\mathcal{M}$ taking $\bar{a}$ to $\bar{b}$, one would consider $\bar{a}$ and $\bar{b}$ indistinguishable by the first order expressive power of $\mathscr{L}$.

By recursion, one defines for each ordinal α, the back-and-forth relation $\sim_\alpha$ on finite tuples of elements of $\mathcal{M}$ as follows: Let $\bar{a} = (a_0, \ldots, a_{p-1})$ and $\bar{b} = (b_0, \ldots, b_{p-1})$ where $p \in \omega$.

- $\bar{a} \sim_0 \bar{b}$ if and only if the map taking a_i to b_i where $i < p$ is a partial $\mathscr{L}$-isomorphism.
- Suppose $\sim_\alpha$ has been defined. Then $\bar{a} \sim_{\alpha+1} \bar{b}$ if and only if $(\forall c)(\exists d)(\bar{a}c \sim_\alpha \bar{b}d) \wedge (\forall d)(\exists c)(\bar{a}c \sim_\alpha \bar{b}d)$.

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- Suppose α is a limit ordinal and for all $\beta < \alpha$, $\sim_\beta$ has been defined. Then $\bar{a} \sim_\alpha \bar{b}$ if and only if for all $\beta < \alpha$, $\bar{a} \sim_\beta \bar{b}$.

Intuitively, if $\neg(\bar{a} \sim_0 \bar{b})$ holds, then one can say that $\bar{a}$ and $\bar{b}$ have been distinguished and in fact, the two tuples fail to satisfy the same atomic formulas. If $\neg(\bar{a} \sim_1 \bar{b})$, then $\bar{a}$ and $\bar{b}$ have been distinguished by failing to share a property which is expressible by one existential quantification over atomic formulas. For each α, $\sim_\alpha$ is closely connected to the type satisfied by tuples via a formula of quantifier rank less than or equal to α.

Note that $\sim_\alpha$ is an equivalence relation on tuples from M. Each tuple is $\sim_\alpha$-related only to tuples of the same length and if $\alpha \leq \beta$, then $\sim_\beta \subseteq \sim_\alpha$. One definition of the Scott rank states that $\mathrm{SR}(M)$ is the least ordinal α so that for all $\beta \geq \alpha$, $\sim_\alpha = \sim_\beta$. Intuitively, the Scott rank of $\mathcal{M}$ is the least ordinal α so that every pair of tuples in $\mathcal{M}$ which can be distinguished by an infinitary $\mathscr{L}$-formula has been distinguished by an infinitary formula of rank less than or equal to α.

By a simple cardinality consideration, one has that $\mathrm{SR}(\mathcal{M}) < |\mathcal{M}|^+$. One can also view the back-and-forth process as a monotone operator which collects the distinguishable pairs: Let $\Gamma : \mathscr{P}(\mathcal{M}^{<\omega} \times \mathcal{M}^{<\omega}) \to \mathscr{P}(\mathcal{M}^{<\omega} \times \mathcal{M}^{<\omega})$ be defined by $\Gamma(X)$ as the set of $(\bar{a}, \bar{b}) \in \mathcal{M}^{<\omega} \times \mathcal{M}^{<\omega}$ so that one of the following holds:

(1) $|\bar{a}| \neq |\bar{b}|$, that is, the length of the two tuples are different.
(2) $\neg(\bar{a} \sim_0 \bar{b})$.
(3) $(\exists c)(\forall d)((\bar{a}c, \bar{b}d) \in X) \vee (\exists d)(\forall c)((\bar{a}c, \bar{b}d) \in X)$.

By recursion, define $\langle I^\alpha_\Gamma : \alpha \in \mathrm{ON} \rangle$ by: $I^0_\Gamma = \Gamma(\emptyset)$, $I^{<\alpha}_\Gamma = \bigcup_{\beta<\alpha} I^\beta_\Gamma$, and $I^\alpha_\Gamma = \Gamma(I^{<\alpha}_\Gamma)$. The closure ordinal $\|\Gamma\|$ of Γ is the least ordinal α so that $I^\alpha_\Gamma = I^{<\alpha}_\Gamma$.

Kripke-Platek set theory, KP, is a weak axiom system for set theory. KP contains some basic set theoretic axioms but has the axioms of separation and replacement restricted to Σ_0 formulas. An admissible set is a transitive set which satisfies KP. If x is a set, then an ordinal α is said to be x-admissible if and only if there is an admissible set $\mathcal{A}$ such that $x \in \mathcal{A}$ and $\alpha = \mathcal{A} \cap \mathrm{ON}$. (See [1] and [8] for more information about admissibility.) Effective bounds on Scott ranks are often obtained by ensuring that the relevant computations can be done correctly in suitable admissible sets.

If Γ is a monotone operator which is definable by a positive Σ-formula in an admissible set $\mathcal{A}$, then Gandy showed that $\|\Gamma\| \leq o(\mathcal{A}) = \mathcal{A} \cap \mathrm{ON}$, the ordinal height of $\mathcal{A}$. (See [1, Chapter VI] for more on inductive definability in admissible sets.) For the monotone operator Γ defined above, one can check that all distinguishable pairs of tuples from the Scott analysis appear in $I^{\|\Gamma\|}_\Gamma$ and Γ is positive Σ-definable in any admissible set containing the language $\mathscr{L}$ and the structure $\mathcal{M}$. If $\mathscr{L}$ and $\mathcal{M}$ are both countable (so they can essentially be coded by reals), then the minimal such admissible set is the initial segment of Gödel's relativized constructible universe, $L_{\omega_1^{\mathscr{L}\oplus\mathcal{M}}}(\mathscr{L} \oplus \mathcal{M})$, which has ordinal height $\omega_1^{\mathscr{L}\oplus\mathcal{M}}$. $\omega_1^{\mathscr{L}\oplus\mathcal{M}}$ is the Church-Kleene ordinal relative to $\mathscr{L} \oplus \mathcal{M}$ which is defined to be the least ordinal that does not have a presentation on ω recursive in $\mathscr{L} \oplus \mathcal{M}$. (It is also the least $\mathscr{L} \oplus \mathcal{M}$-admissible ordinal.) Thus one has obtained Nadel's [10] effective bound which asserts that if $\mathcal{M}$ is a countable $\mathscr{L}$-structure with $\mathscr{L}$ a countable first order language, then $\mathrm{SR}(\mathcal{M}) \leq \omega_1^{\mathscr{L}\oplus\mathcal{M}}$.

The above is the definition of Scott rank that [3] attempts to generalize. Another variation of the Scott rank, denoted SR^* which is used in [5], [4], and computable

model theory, focuses in on tuples rather than just the stabilization point of the back-and-forth relations. The same argument as above shows that if $\mathcal{M}$ is a countable structure in a countable language $\mathscr{L}$, then $\mathrm{SR}^*(\mathcal{M}) \leq \omega_1^{\mathscr{L}\oplus\mathcal{M}} + 1$. The distinction between recursive structures $\mathcal{M}$ and $\mathcal{N}$ in a recursive language $\mathscr{L}$ so that $\mathrm{SR}^*(\mathcal{M}) = \omega_1^{\emptyset} + 1$ (for example, the Harrison linear ordering) or $\mathrm{SR}^*(\mathcal{N}) = \omega_1^{\emptyset}$ (for example, the linear ordering of Makkai [9]) are of particular interest in computable model theory. The definition of Scott rank used in this article cannot make this distinction; however, for the purpose of finding an effective bound, this will not be relevant.

Polish metric spaces are complete separable metric spaces. By cardinality considerations, the classical first order Scott rank of a Polish metric space is less than $(2^{\aleph_0})^+$. Although Polish metric spaces may not be countable, they are entirely determined by their countable dense metric subspace. A natural question asked by Fokina, Friedman, Koerwien, and Nies [7] was whether the Scott rank of a Polish metric space is countable. The first author in [4] asked whether $\mathrm{SR}(\bar{\mathcal{M}}) \leq \omega_1^{\mathcal{M}}$, where $\bar{\mathcal{M}}$ is the completion of the countable metric space $\mathcal{M}$. This is the natural analog of Nadel's effective bound for Polish metric spaces.

However, these questions remain open (see [5] and [6]) despite some partial results. Fokina, Friedman, Koerwien, and Nies [7] showed that if $\mathcal{M}$ is a compact Polish metric space, then $\mathrm{SR}(\mathcal{M}) = \omega$ and, in fact, $\mathrm{SR}^*(\mathcal{M}) = \omega + 1$. (See [4] for a combinatorial proof of their result using the König's lemma.) Doucha [5] showed that if $\mathcal{M}$ is a Polish metric space, then $\mathrm{SR}^*(\mathcal{M}) \leq \omega_1$. The work [4] uses admissible sets and infinitary logic in countable admissible fragments to give another proof of Doucha's result and show that if $\mathcal{M}$ is a countable metric space so that the completion $\bar{\mathcal{M}}$ is a proper Polish metric space or a rigid Polish metric space, then $\mathrm{SR}^*(\bar{\mathcal{M}}) \leq \omega_1^{\mathcal{M}} + 1$. The results of [4] are proved by using winning strategies in approximation forms of the Ehrenfeucht-Fraïssé game in an ill-founded model of KP. The arguments are quite different from the classical method involving the monotone operator.

An alternative logic that appears more suitable for structures on Polish metric spaces is continuous logic for metric structures. The intended structures are Polish metric spaces with the functions and relations interpreted by continuous functions on the Polish metric spaces respecting certain predetermined moduli of continuity. The reader should consult [2] and [3] for more details on continuous logic for metric structures.

In continuous logic, the authors of [3] defined an analogous back-and-forth pseudodistance which depends on one additional object called a weak modulus of continuity Ω. For each language $\mathscr{L}$, weak modulus Ω, and separable $\mathscr{L}$-structure $\mathcal{N}$ of continuous logic, they defined a Scott rank $\mathrm{SR}_\Omega(\mathcal{N})$. From their Scott analysis, they derived a Scott sentence $\varphi_\mathcal{N}$ so that for all separable $\mathscr{L}$-structure $\mathcal{M}$, $\mathcal{M} \models (\varphi_\mathcal{N} = 0)$ if and only if $\mathcal{N}$ and $\mathcal{M}$ are $\mathscr{L}$-isomorphic in continuous logic, in the case that Ω is a universal weak modulus. See [3, Theorem 3.8 and 5.5] for these results and more details. These results give strong evidence that their theory can justly be called a "Scott analysis" for continuous logic.

The authors of [3] showed that every Polish metric structure in continuous logic has countable Scott rank by cardinality considerations. Every Polish metric structure in some language $\mathscr{L}$ is the completion $\mathscr{L}$-structure of a countable dense $\mathscr{L}$-prestructure. The main task addressed in this article is to investigate the connection

between the Scott rank of a Polish metric structure and any of its countable dense pre-structures.

Let $\mathscr{L}$ be a countable $\mathscr{L}$-structure. Let $\hat{\mathscr{L}}$ denote a real coding $\mathscr{L}$ which includes information about the associated modulus of the language. Let Ω be a weak modulus of continuity which is coded by a real $\hat{\Omega}$. Let $\mathcal{D}$ denote a countable $\mathscr{L}$-pre-structure. The main theorem is the following.

Theorem 4.5. *Let $\mathscr{L}$ be a countable language with code $\hat{\mathscr{L}}$. Let Ω be a weak modulus with code $\hat{\Omega}$. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure. Let $\bar{\mathcal{D}}$ denote the completion structure of $\mathcal{D}$. Then* $\mathrm{SR}_\Omega(\bar{\mathcal{D}}) \leq \omega_1^{\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}}$.

As an example, consider the class of metric structures consisting of the pure Polish metric spaces. That is, the language is $\mathscr{L} = \emptyset$, which consists of no additional non-logical symbols. Thus $\mathscr{L}$ has a code which is recursive. One can take Ω to be the universal weak modulus for this language, which exists and has a recursive code from inspecting the proof of [3, Proposition 5.3]. Now suppose $\mathcal{D}$ is a countable metric space. $\mathcal{D}$ is naturally a metric $\mathscr{L}$-pre-structure. Let $\bar{\mathcal{D}}$ be the completion of $\mathcal{D}$ as a metric $\mathscr{L}$-pre-structure. Then Theorem 4.5 states the continuous Scott rank of $\bar{\mathcal{D}}$, $\mathrm{SR}_\Omega(\bar{\mathcal{D}})$, is less than or equal to $\omega_1^{\mathcal{D}}$, the Church-Kleene ordinal relative to $\mathcal{D}$. This establishes the analog of Nadel's effective bound on the continuous Scott rank of Polish metric spaces.

The basic template for the proof of Theorem 4.5 is the same as the classical first order argument. Now one attempts to define various monotone operators on $\mathcal{D}^{<\omega} \times \mathcal{D}^{<\omega}$ that are positive Σ-definable in an admissible set containing $\mathcal{D}$, $\hat{\Omega}$, and $\hat{\mathscr{L}}$. However, for this to be meaningful, one needs to ensure the computation of the back-and-forth pseudo-distance within the desired admissible set evaluates to the correct or true computation as performed in the real world. This amounts to showing that the first back-and-forth function r_0 is computed correctly by the appropriate admissible set. This will be shown by producing a countable collection of basic formulas respecting the weak modulus Ω which has a code in every admissible set containing $\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}$ and such that this collection is dense in the collection of all basic formulas respecting the weak modulus under the uniform norm.

2. Continuous logic

See [2] for a more detailed exposition on continuous logic for metric structures.

Definition 2.1 ([3])**.** A modulus of arity n is a continuous function $\Delta : (\mathbb{R}^{\geq 0})^n \to \mathbb{R}^{\geq 0}$ such that $\Delta(\bar{0}) = 0$, and for all $\bar{r}, \bar{s} \in (\mathbb{R}^{\geq 0})^n$, $\Delta(\bar{r}) \leq \Delta(\bar{r} + \bar{s}) \leq \Delta(\bar{r}) + \Delta(\bar{s})$. ($+$ refers to coordinate-wise addition.) The latter succinctly states that Δ is non-decreasing and subadditive.

Let Δ be a modulus of arity n. Let (X_i, d_{X_i}) and (Z, d_Z) be metric spaces where $i < n$. Define $d^\Delta : (\prod_{i<n} X_i)^2 \to \mathbb{R}^{\geq 0}$ by

$$d^\Delta(\bar{x}, \bar{y}) = \Delta((d_{X_0}(\bar{x}(0), \bar{y}(0)), \ldots, d_{X_{n-1}}(\bar{x}(n-1), \bar{y}(n-1)))).$$

Let $f : \prod_{i<n} X_i \to Z$. f respects Δ if and only if $d_Z(f(\bar{x}), f(\bar{y})) \leq d^\Delta(\bar{x}, \bar{y})$.

Definition 2.2 ([3])**.** A weak modulus is a function $\Omega : (\mathbb{R}^{\geq 0})^\omega \to [0, \infty]$ which is non-decreasing, subadditive, lower semi-continuous in the product topology, separately continuous in each coordinate, and $\Omega(\bar{0}) = 0$.

For each $n \in \omega$, define $\Omega_n : (\mathbb{R}^{\geq 0})^n \to \mathbb{R}^{\geq 0}$ by

$$\Omega_n(x_0, \ldots, x_{n-1}) = \Omega(x_0, \ldots, x_{n-1}, 0, 0, 0 \ldots).$$

Fact 2.3 ([3, Lemma 2.3])**.** Let Ω be a weak modulus. For all $n \in \omega$, Ω_n is a modulus of arity n. For all $\bar{r} \in (\mathbb{R}^{\geq 0})^\omega$, $\Omega(\bar{r}) = \sup_{n \in \omega} \Omega_n(\bar{r} \restriction n)$.

Definition 2.4. A function $f : X^n \to \mathbb{R}$ respects the weak modulus Ω if and only if f respects the modulus Ω_n.

Definition 2.5. A relation symbol consists of a symbol R, a natural number $a(R)$, and a modulus Δ_R of arity $a(R)$. A function symbol consists of a symbol f, a natural number $a(f)$, and a modulus Δ_f of arity $a(f)$.

A language of continuous logic is a collection $\mathscr{L}$ of relation, function, and constant symbols along with a distinguished binary relation symbol d, which is intended to represent the distance function.

For convenience, one will assume all connectives, relation symbols, and the distance relation can only be interpreted to take value in the interval $[0, 1]$.

Definition 2.6. Let $\mathscr{L}$ be a language of continuous logic. An $\mathscr{L}$-pre-structure is a collection $\mathcal{M}$ consisting of the following: There is a (possibly incomplete) metric space M. For each relation symbol $R \in \mathscr{L}$, there is a continuous function $R^{\mathcal{M}} : M^{a(R)} \to [0, 1]$ which respects the modulus Δ_R. For each function symbol $f \in \mathscr{L}$, there is a continuous function $f^{\mathcal{M}} : M^{a(f)} \to M$ respecting the modulus Δ_f.

An $\mathscr{L}$-structure is an $\mathscr{L}$-pre-structure where M is a complete metric space.

Fact 2.7. If $\mathcal{D}$ is an $\mathscr{L}$-pre-structure, then there is a canonical $\mathscr{L}$-structure on $\bar{D}$, the completion of D, obtained by extending all the interpretation of symbols to the completion. This structure is denoted $\bar{\mathcal{D}}$.

Definition 2.8. Let $\mathscr{L}$ be a countable language of continuous logic. Fix an infinite set $\langle v_i : i \in \omega \rangle$ of variables.

The collection of $\mathscr{L}$-terms is the smallest set closed under the following:

1. Each v_i is a term.

2. If f is a function symbol and $t_0, \ldots, t_{a(f)-1}$ are terms, then $f(t_0, \ldots, t_{a(f)-1})$ is a term.

The atomic formulas are generated in the following way: If R is a relation symbol and $t_0, \ldots, t_{a(R)-1}$ are terms, then $R(t_0, \ldots, t_{a(R)-1})$ is an atomic formula. If t_1 and t_2 are terms, then $d(t_1, t_2)$ is an atomic formula.

The collection of $\mathscr{L}$-formulas, denoted $\mathscr{L}_{\omega,\omega}$, is the smallest collection closed under the following:

1. All atomic formulas are formulas.

2. If $u : [0, 1]^n \to [0, 1]$ is a continuous function and $\varphi_0, \ldots, \varphi_{n-1}$ are formulas, then $u(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.

3. If φ is a formula and v_i is a variable, then $\sup_{v_i} \varphi$ and $\inf_{v_i} \varphi$ are formulas.

Each term or formula has a canonically associated modulus. (See [3, Section 2.2] for more details.)

Definition 2.9. The collection of basic formulas is the smallest collection of formulas closed under 1 and 2 in the definition of $\mathscr{L}_{\omega,\omega}$.

Fact 2.10. If φ is a basic formula, then there are atomic formulas $\varphi_0, \ldots, \varphi_{k-1}$ and a continuous function $u : [0, 1]^k \to [0, 1]$ so that $\varphi = u(\varphi_0, \ldots, \varphi_{k-1})$.

Proof. This is proved by induction. □

Definition 2.11. Let $\mathscr{L}$ be a language of continuous logic. Let $\mathcal{D}$ be an $\mathscr{L}$-pre-structure.

The interpretation of the terms of $\mathscr{L}$ are defined as follows:

1. For each variable v_i, $v_i^{\mathcal{D}}(a_0, \dots, a_{n-1}) = a_i$, if $i < n$ and $a_0, \dots, a_{n-1} \in D$.
2. Suppose $t_0, \dots, t_{n-1}$ are terms mentioning variables $v_0, \dots, v_{k-1}$, f is an n-ary function symbol, $\bar{a} = (a_0, \dots, a_{k-1})$ is a tuple from D, and each $t_j^{\mathcal{D}}(\bar{a})$ has already been defined; then $(f(t_0, \dots, t_{n-1}))^{\mathcal{D}}(\bar{a}) = f^{\mathcal{D}}(t_0^{\mathcal{D}}(\bar{a}), \dots, t_{n-1}^{\mathcal{D}}(\bar{a}))$.

The interpretation of formulas is defined recursively as follows:

1. If R is an n-ary function symbol and $t_0, \dots, t_{n-1}$ are terms, then
$$(R(t_0, \dots, t_{n-1}))^{\mathcal{D}}(\bar{a}) = R^{\mathcal{D}}(t_0^{\mathcal{D}}(\bar{a}), \dots, t_{n-1}^{\mathcal{D}}(\bar{a})).$$
2. If $u : [0,1]^n \to [0,1]$ is a continuous function and $\varphi_0, \dots, \varphi_{n-1}$ are formulas such that $\varphi_i^{\mathcal{D}}(\bar{a})$ has been defined, then
$$(u(\varphi_0, \dots, \varphi_{n-1}))^{\mathcal{D}}(\bar{a}) = u(\varphi_0^{\mathcal{D}}(\bar{a}), \dots, \varphi_{n-1}^{\mathcal{D}}(\bar{a})).$$
3. Suppose φ is a formula such that $\varphi^{\mathcal{D}}(\bar{a})$ has been defined; then
$$(\sup_{v_i} \varphi)^{\mathcal{D}}(a_0, \dots, a_{k-1}) = \sup_{x \in D} \varphi^{\mathcal{D}}(a_0, \dots, a_{i-1}, x, a_{i+1}, \dots, a_{k-1}).$$

Definition 2.12. A continuous function $f : (\mathbb{R}^{\geq 0})^n \to \mathbb{R}^{\geq 0}$ is coded by a function $\hat{f} : \mathbb{Q}^n \to \mathbb{Q}^\omega$ with the property that for all $\bar{p} \in (\mathbb{Q}^{\geq 0})^n$, $\hat{f}(\bar{p})$ is a Cauchy sequence representing $f(\bar{p})$. Similar coding can be defined if $f : [0,1]^n \to [0,1]$.

Let $\mathscr{L}$ be a countable language for continuous logic. A code $\hat{\mathscr{L}}$ for $\mathscr{L}$ consists of the following objects:

1. The symbols of $\mathscr{L}$.
2. For each function symbol or relation symbol $P \in \mathscr{L}$, $\hat{\Delta}_P$.

If Ω is a weak modulus, the code for Ω is $\hat{\Omega}$ consisting of $\hat{\Omega}_n$ for each $n \in \omega$.

Each connective $u : [0,1]^n \to [0,1]$ has a code $\hat{u}$ as defined above. Using this and the recursive definition of formulas, one can obtain codes $\hat{\varphi}$ for each formula φ.

Let $\mathcal{D} = (D, d)$ (where d refers to the metric) be a countable $\mathscr{L}$-pre-structure. If $f : D^n \to \mathbb{R}$, then the code for f is $\hat{f} : D^n \to \mathbb{Q}^\omega$ so that for all $\bar{x} \in D^n$, $\hat{f}(\bar{x})$ is a Cauchy-sequence representing $f(\bar{x})$.

The code of $\mathcal{D}$, denoted $\hat{\mathcal{D}}$, consists of the underlying set D, $f^{\mathcal{D}}$, $\hat{R}^{\mathcal{D}}$, and $\hat{d}$.

If E is a countable collection of functions (which have a code as above), then a code for E is a function $\hat{E}$ on ω so that for all n, $\hat{E}(n)$ is a code for a function in E and for every function $f \in E$, there is some n so that $\hat{E}(n)$ is a code for f.

Remark 2.13. Throughout, some of the coding details will be left to the reader. For instance, the reader can check that in models of KP, values of functions on appropriate objects can be recovered from the appropriate codes of the function and the objects.

In some sense, information about the $\mathscr{L}$-structure $\bar{\mathcal{D}}$ is entirely contained in $\hat{\mathscr{L}} \oplus \mathcal{D}$. In the following, one will be concerned about analyzing the continuous Scott analysis of $\bar{\mathcal{D}}$. Unlike the first order case, the continuous Scott analysis has one additional parameter, a weak modulus Ω. Hence the real $\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}$ codes all the parameters in the continuous Scott analysis of $\mathcal{M}$ relative to the weak modulus Ω. The main concern is to find a bound on the continuous Scott rank of $\mathcal{M}$ relative to Ω using these parameters.

Recall that an admissible set is a transitive set which satisfies KP, a weak system of axioms for set theory. For any real r, there are countable admissible sets containing r, for instance $L_{\omega_1^r}(r)$.

Definition 2.14. Let $\mathscr{L}$ be a countable language. Let Ω be a weak modulus. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure (which one may assume has domain ω). Let $\mathcal{A}$ be a countable admissible set.

$\mathscr{L}$ (respectively, Ω) is said to belong to the admissible set $\mathcal{A}$ if and only if $\hat{\mathscr{L}} \in A$ (or $\hat{\Omega} \in A$, respectively).

$\mathcal{D}$ is said to belong to $\mathcal{A}$ if and only if $\mathcal{D} \in A$.

If $\varphi \in L_{\omega,\omega}$, then φ belongs to $\mathcal{A}$ if and only if $\hat{\varphi} \in A$.

Let $\mathscr{L}_{\omega,\omega}^{\mathcal{A}}$ denote the collection of all formulas of $\mathscr{L}_{\omega,\omega}$ that belong to $\mathcal{A}$.

The interpretation of $\mathscr{L}_{\omega,\omega}$ formulas evaluated at elements of $\mathcal{D}$ is the same as the interpretation in the completion $\bar{\mathcal{D}}$.

Fact 2.15. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure. Let $x_0, \dots, x_{n-1} \in D$. Let $\varphi \in \mathscr{L}_{\omega,\omega}$. Then $\varphi^{\mathcal{D}}(x_0, \dots, x_{n-1}) = \varphi^{\bar{\mathcal{D}}}(x_0, \dots, x_{n-1})$.

Proof. For all terms t and $\bar{a} \in D$, $t^{\mathcal{D}}(\bar{a}) = t^{\bar{D}}(\bar{a})$.

Fact 2.15 is proved by induction as follows.

Suppose the result holds for $\varphi_0, \dots, \varphi_{n-1}$ and $u : [0,1]^n \to [0,1]$ is a continuous function; then

$$\begin{aligned}(u(\varphi_0, \dots, \varphi_{n-1}))^{\bar{\mathcal{D}}}(\bar{a}) &= u(\varphi_0^{\bar{\mathcal{D}}}(\bar{a}), \dots, \varphi_{n-1}^{\bar{\mathcal{D}}}(\bar{a})) \\ &= u(\varphi_0^{\mathcal{D}}(\bar{a}), \dots, \varphi_{n-1}^{\mathcal{D}}(\bar{a})) = (u(\varphi_0, \dots, \varphi_{n-1}))^{\mathcal{D}}(\bar{a})\end{aligned}$$

by using the induction hypothesis.

Suppose the result holds for φ. Let $i < k$. Then

$$\begin{aligned}(\sup_{v_i} \varphi)^{\bar{\mathcal{D}}}(a_0, \dots, a_{k-1}) &= \sup_{b \in \bar{D}} \varphi^{\bar{\mathcal{D}}}(a_0, \dots, a_{i-1}, b, a_{i+1} \dots, a_k) \\ &= \sup_{b \in D} \varphi^{\bar{\mathcal{D}}}(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_{k-1}) \\ &= \sup_{b \in D} \varphi^{\mathcal{D}}(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_{k-1}) = (\sup_{v_i} \varphi)^{\mathcal{D}}(a_0, \dots, a_{k-1})\end{aligned}$$

using the continuity of φ and the induction hypothesis. A similar argument holds for inf.

By induction, the result has been shown. □

KP is capable of formulating the syntax and semantics of continuous logic using the codes of various continuous functions and Cauchy sequence representations of reals and elements of $\bar{D}$. The following is straightforward coding.

Fact 2.16. Let $\mathscr{L}$ be a countable language of continuous logic. Let $\mathcal{A}$ be a countable admissible set such that $\mathscr{L} \in \mathcal{A}$. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure in $\mathcal{A}$. Let $\varphi \in L_{\omega,\omega}^{\mathcal{A}}$. Let $\bar{a}$ be a tuple of elements in D. Let $(\varphi^{\mathcal{D}}(\bar{a}))^{\mathcal{A}}$ be the computation of $\varphi^{\mathcal{D}}(\bar{a})$ (considered as a $\mathbb{Q}$-Cauchy sequence) in the admissible set $\mathcal{A}$. Then $\varphi^{\mathcal{D}}(\bar{a}) = (\varphi^{\mathcal{D}}(\bar{a}))^{\mathcal{A}}$ and by Fact 2.15, this is equal to $\varphi^{\bar{\mathcal{D}}}(\bar{a})$.

3. Density of basic formulas respecting a weak modulus

The Scott analysis for continuous logic of [3] is formulated using the back-and-forth pseudo-distance functions $\langle r_\alpha : \alpha \in \mathrm{ON}\rangle$ from Definition 4.1. Like the classical Scott analysis, the continuous Scott analysis can be expressed through a certain monotone operator. The Scott rank is then the closure ordinal of this operator. To obtain an effective bound, one will show that there is an equivalent monotone operator in a suitable countable admissible set that correctly represents the true Scott analysis occurring in the real world.

This amounts to showing that one can compute each r_α correctly in an appropriate admissible set. From Definition 4.1, one can see that from r_α, one obtains $r_{\alpha+1}$ by a simple explicit procedure. Similarly, if β is a limit ordinal, one can obtain r_β by a simple explicit procedure from the collection $\{r_\gamma : \gamma < \beta\}$.

Thus one will need to show that there is an appropriate countable admissible set which can compute the initial function r_0 correctly. From Definition 4.1, r_0 is defined by taking a certain supremum over all basic $\mathscr{L}$-formulas respecting a modulus Ω. The collection of all such formulas is uncountable and so certainly does not belong to any countable admissible set. The main result of this section is to show that there is a countable collection of such formulas which is dense in the uniform norm among the collection of all basic $\mathscr{L}$-formulas respecting Ω and this collection is arithmetical in the code for $\mathscr{L}$ and Ω. See Fact 3.13 for the precise statement.

Definition 3.1. Suppose $n \in \omega$. A subset $A \subseteq (\mathbb{R}^{\geq 0})^n$ is said to be *nice* if and only if

1. For all $\bar{x} \in A$ and all $\bar{y} \in (\mathbb{R}^{\geq 0})^n$ such that for all $i < n$, $\bar{y}(i) \leq \bar{x}(i)$, then $\bar{y} \in A$.
2. There is an open set $U \subseteq (\mathbb{R}^{\geq 0})^n$ with $\bar{0} \in U$ and $U \subseteq A$.

The following will be useful notation throughout.

Definition 3.2. For $\bar{x} \in \mathbb{R}^k$, let $\pi(\bar{x}) \in (\mathbb{R}^{\geq 0})^k$ be defined by $\pi(\bar{x})(i) = |\bar{x}(i)|$ for all $i < k$.

Fact 3.3. Suppose $f : A \to \mathbb{R}^{\geq 0}$ is a function defined on a nice $A \subseteq (\mathbb{R}^{\geq 0})^n$, is non-decreasing, $f(\bar{0}) = 0$, continuous at $\bar{0}$, and has code $\hat{f}$. Then there is a function $g : (\mathbb{R}^{\geq 0})^n \to \mathbb{R}^{\geq 0}$ so that

(i) g is a modulus of arity n with the property that for all $\bar{x} \in A$, $g(\bar{x}) \leq f(\bar{x})$.

(ii) g is the largest modulus of arity n below f in the following sense: if h is a modulus of arity n below f in the sense that for all $\bar{x} \in A$, $h(\bar{x}) \leq f(\bar{x})$, then one has that for all $\bar{x} \in (\mathbb{R}^{\geq 0})^n$, $h(\bar{x}) \leq g(\bar{x})$.

(iii) $\hat{g}$, the code of g, is arithmetic in any function $\underline{f} : A \cap (\mathbb{Q}^{\geq 0})^n \to \mathbb{Q}^\omega$ such that for all $\bar{p} \in A \cap (\mathbb{Q}^{\geq 0})^n$, $\underline{f}(\bar{p})$ is a Cauchy sequence representing $f(\bar{p})$.

(Note that $\underline{f}$ is essentially a real and it codes f on $(\mathbb{Q}^{\geq 0})^n$ in a manner similar to how continuous functions are coded, but if f is not continuous, then f may not be recovered from $\underline{f}$.)

Proof. For notational simplicity, if $\bar{x}$ and $\bar{y}$ are elements of $(\mathbb{R}^{\geq 0})^n$, then one writes $\bar{x} \preceq \bar{y}$ if and only if for all $i < n$, $\bar{x}(i) \leq \bar{y}(i)$.

Define function g on $(\mathbb{R}^{\geq 0})^n$ as follows:

$$g(\bar{x}) = \inf\left\{f(\bar{x}_0) + \ldots + f(\bar{x}_{k-1}) : k \in \omega \wedge \bar{x}_0, \ldots, \bar{x}_{k-1} \in A \cap (\mathbb{Q}^{\geq 0})^n \wedge \bar{x} \preceq \sum_{i<k} \bar{x}_i\right\}$$

Note that $g(\bar{0}) = 0$ since $f(\bar{0}) = 0$ and for all $\bar{x} \in A \cap (\mathbb{Q}^{\geq 0})^n$, $g(\bar{x}) \leq f(\bar{x})$.

(Claim 1) g is non-decreasing. Suppose $\bar{x}, \bar{y} \in (\mathbb{R}^{\geq 0})^n$ is such that $\bar{x} \preceq \bar{y}$. Fix some $\bar{y}_0, \ldots, \bar{y}_{k-1} \in A \cap (\mathbb{Q}^{\geq 0})^n$ so that $\bar{y} \preceq \sum_{i<k} \bar{y}_i$. Then $\bar{x} \preceq \sum_{i<k} \bar{y}_k$ since $\bar{x} \preceq \bar{y}$. Since $\bar{y}_0, \ldots, \bar{y}_{k-1}$ with $\bar{y} \leq \sum_{i<k} \bar{y}_i$ were arbitrary, one has that $g(\bar{x}) \leq g(\bar{y})$ by the definition of g.

(Claim 2) g is subadditive. Fix $\bar{x}$ and $\bar{y}$. Suppose $\bar{x}_0, \ldots, \bar{x}_{k-1} \in A \cap (\mathbb{Q}^{\geq 0})^n$ are such that $\bar{x} \preceq \sum_{i<k} \bar{x}_i$. Suppose $\bar{y}_0, \ldots, \bar{y}_{p-1} \in A \cap (\mathbb{Q}^{\geq 0})^n$ are such that $\bar{y} \preceq \sum_{i<p} \bar{y}_i$. Let $\bar{z}_0, \ldots, \bar{z}_{k+p-1}$ enumerate $\bar{x}_0, \ldots, \bar{x}_{k-1}, \bar{y}_0, \ldots, \bar{y}_{p-1}$. Then $\bar{x} + \bar{y} \preceq \sum_{i<k+p} \bar{z}_i$. It follows from the definition of g that $g(\bar{x} + \bar{y}) \leq g(\bar{x}) + g(\bar{y})$.

(Claim 3) g is continuous. Fix $\epsilon > 0$. Since f is continuous at $\bar{0}$ and A is nice, there is some $\delta > 0$ so that for all $\bar{x} \in (\mathbb{R}^{\geq 0})^n$ with the property that for all $i < n$, $\bar{x}(i) < \delta$, one has that $\bar{x} \in A$ and $f(\bar{x}) < \epsilon$.

Fix $\bar{x} \in (\mathbb{R}^{\geq 0})^n$. Let $y \in (\mathbb{R}^{\geq 0})^n$ be such that for all $i < n$, $\pi(\bar{y} - \bar{x})(i) < \frac{\delta}{2}$. Choose a $\bar{z} \in (\mathbb{Q}^{\geq 0})^n$ such that $\pi(\bar{y} - \bar{x}) \preceq \bar{z}$ and $\bar{z}(i) < \delta$ for all $i < k$. By the choice of δ, $\bar{z} \in A \cap (\mathbb{Q}^{\geq 0})^n$. Without loss of generality, suppose that $g(\bar{y}) \geq g(\bar{x})$. Note that $\bar{y} \preceq \bar{x} + \pi(\bar{y} - \bar{x})$. Using the fact that g is non-decreasing, subadditive, g is less than f on $A \cap (\mathbb{Q}^{\geq 0})^n$, and by the choice of $\bar{z}$ and δ, one has that

$$\begin{aligned}|g(\bar{y}) - g(\bar{x})| = g(\bar{y}) - g(\bar{x}) &\leq g(\bar{x} + \pi(\bar{y} - \bar{x})) - g(\bar{x}) \leq g(\bar{x}) + g(\pi(\bar{y} - \bar{x})) - g(\bar{x}) \\ &= g(\pi(\bar{y} - \bar{x})) \leq g(\bar{z}) \leq f(\bar{z}) < \epsilon.\end{aligned}$$

The continuity of g has been established.

It has been shown that g is a modulus.

(Claim 4) For all $\bar{x} \in A$, $g(\bar{x}) \leq f(\bar{x})$. For each $j \in \omega$, let $\bar{x}_j \preceq \bar{x}$ be such that $\bar{x}_j \in (\mathbb{Q}^{\geq 0})^n$ and $(\bar{x} - \bar{x}_j)(i) < \frac{1}{j+1}$ for all $i < n$. Since $\bar{x} \in A$ and A is nice, $\bar{x}_j \in A \cap (\mathbb{Q}^{\geq 0})^n$ for all $j \in \omega$. Since $g(\bar{y}) \leq f(\bar{y})$ for all $\bar{y} \in A \cap (\mathbb{Q}^{\geq 0})^n$, the continuity of g and the fact that f is non-decreasing imply that $g(\bar{x}) = \lim_{j\to\infty} g(\bar{x}_j) = \lim_{j\to\infty} f(\bar{x}_j) \leq f(\bar{x})$.

(Claim 5) g is the largest n-ary modulus below f. Suppose h is modulus below f but there is some $\bar{x} \in (\mathbb{R}^{\geq 0})^n$ so that $g(\bar{x}) < h(\bar{x})$. Then there is some $\bar{x}_0, \ldots, \bar{x}_{k-1} \in A \cap (\mathbb{Q}^{\geq 0})^n$ so that $\bar{x} \preceq \sum_{i<k} \bar{x}_i$ and $\sum_{i<k} f(\bar{x}_i) < h(\bar{x})$. However since h is an n-ary modulus, one must have

$$\sum_{i<k} f(\bar{x}_i) < h(\bar{x}) \leq \sum_{i<k} h(\bar{x}_i) \leq \sum_{i<k} f(\bar{x}_i).$$

Contradiction.

Note that the definition of g depends only on the value of f on $A \cap (\mathbb{Q}^{\geq 0})^n$. This implies that there is a code $\hat{g}$ of g which is arithmetic in $\underline{f}$. □

Fact 3.4. Let $\varphi_0, \ldots, \varphi_{k-1}$ be atomic formulas of $\mathscr{L}_{\omega,\omega}$ with free variables $v_0, \ldots, v_{n-1}$. Let $\Delta_0, \ldots, \Delta_{k-1}$ be the canonical moduli of $\varphi_0, \ldots, \varphi_{k-1}$, respectively, as mentioned at the end of Definition 2.8. Assume that each Δ_i is not the constant 0

function. Let Ω be a weak modulus. Then there is a k-ary modulus Δ with a code which is arithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$ such that for all continuous functions $u : [0,1]^k \to [0,1]$, $u(\varphi_0, \dots, \varphi_k)$ respects Ω if and only if u respects Δ.

Proof. First, observe that since each φ_i is atomic, Δ_i is recursive in $\hat{\mathscr{L}}$. Suppose $u : [0,1]^k \to [0,1]$ is continuous and Δ^u is the least modulus of continuity for u as defined in [3, Section 2.1]. By definition, the canonical modulus for $u(\varphi_0, \dots, \varphi_{k-1})$ is $\Delta^u(\Delta_0, \dots, \Delta_{k-1})$. Suppose $u(\varphi_0, \dots, \varphi_{k-1})$ respects Ω_n. This means that for all $\bar{x} \in (\mathbb{R}^{\geq 0})^n$,

$$\Delta^u(\Delta_0(\bar{x}), \dots, \Delta_{k-1}(\bar{x})) \leq \Omega_n(\bar{x}).$$

Hence Δ^u must satisfy the following relation: For all $\bar{r} \in (\mathbb{R}^{\geq 0})^k$ such that there is some $\bar{x}$ with the property that for all $i < k$, $\Delta_i(\bar{x}) \geq \bar{r}(i)$,

$$\Delta^u(\bar{r}) \leq \inf\{\Omega_n(\bar{x}) : \bar{x} \in (\mathbb{R}^{\geq 0})^n \wedge (\forall i < k)(\Delta_i(\bar{x}) \geq \bar{r}(i))\}.$$

Let f be defined by

$$f(\bar{r}) = \inf\{\Omega_n(\bar{x}) : \bar{x} \in (\mathbb{R}^{\geq 0})^n \wedge (\forall i < k)(\Delta_i(\bar{x}) \geq \bar{r}(i))\}.$$

Using the assumption that each Δ_i is not constantly 0, f is defined on a nice $A \subseteq (\mathbb{R}^{\geq 0})^k$.

Note that $\Delta^u(\bar{r}) \leq f(\bar{r})$ for all $\bar{r} \in A$ if and only if $u(\varphi_0, \dots, \varphi_{k-1})$ respect Ω_n:

($\Rightarrow$) is clear.

($\Leftarrow$) Suppose there is some $\bar{r} \in A$ so that $f(\bar{r}) < \Delta^u(\bar{r})$. Then there is some $\bar{x}$ so that for all i, $\Delta_i(\bar{x}) \geq \bar{r}(i)$ and $\Omega_n(\bar{x}) < \Delta^u(\bar{r})$. Let $\bar{s}$ be such that $\bar{s}(i) = \Delta_i(\bar{x})$. Since for all $i < k$, $\bar{r}(i) \leq \bar{s}(i)$, one has that $\Delta^u(\bar{r}) \leq \Delta^u(\bar{s})$. But then $\Omega_n(\bar{x}) < \Delta^u(\bar{r}) \leq \Delta^u(\bar{s}) = \Delta^u(\Delta_0(\bar{x}), \dots, \Delta_{k-1}(\bar{x}))$. The canonical modulus for the formula does not respect Ω_n.

f is clearly non-decreasing where it is defined.

Next we show f is continuous at $\bar{0}$. Let $\epsilon > 0$. Since Ω_n is continuous, there is some $\delta > 0$ so that $\Omega_n(\bar{x}) < \epsilon$ whenever $\bar{x}$ has the property that for all $i < k$, $\bar{x}(i) < \delta$. Since each Δ_j is not constant in a neighborhood of $\bar{0}$, for each j, there is some $\bar{z}_j$ with $\bar{z}_j(i) < \frac{\delta}{n}$ for each $i < k$ and $\Delta_j(\bar{z}_j) > 0$. Let $\bar{z} = \sum_{j<k} \bar{z}_j$. Since each Δ_i is non-decreasing for $0 \leq i < k$, $\Delta_i(\bar{z}) > 0$. Let $\gamma = \min\{\Delta_i(\bar{z}) : i < k\}$. Suppose $\bar{r}$ is such that for all $i < k$, $\bar{r}(i) < \gamma$. Then

$$\Omega_n(\bar{z}) \in \{\Omega_n(\bar{x}) : \bar{x} \in (\mathbb{R}^{\geq 0})^n \wedge (\forall i < k)(\Delta_i(\bar{x}) \geq \bar{r}(i))\}.$$

Hence $f(\bar{r}) \leq \Omega_n(\bar{z}) < \epsilon$. f is continuous at $\bar{0}$.

One can find a function $\underline{f} : A \cap (\mathbb{Q}^{\geq 0})^n \to \mathbb{Q}^\omega$ with the properties in Fact 3.3 for this function f which is arithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$. (Note that in the definition of f, one obtains the same function if the infimum is taken over $\bar{x} \in (\mathbb{Q}^{\geq 0})^n$ with the required property above.) Fact 3.3 states there is a largest modulus Δ below f which is arithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$. This completes the proof. □

Definition 3.5. Let $I \subseteq \mathbb{R}$ be an open or closed interval. Let X be a compact metric space. Let $C(X, I)$ be the collection of continuous functions $f : X \to I$.

For $f \in C(X, I)$, let $\|f\| = \sup_{x \in X} |f(x)|$ be the uniform norm of f.

If $f, g \in C(X, I)$, then define $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$.

$L \subseteq C(X, I)$ is a lattice if and only if for all $f, g \in L$, $f \wedge g, f \vee g \in L$.

The following fact follows from the proof of the Stone-Weierstass theorem.

Fact 3.6. Let X be a compact metric space. Let $I \subseteq \mathbb{R}$ be an interval, $f \in C(X, I)$, and $L \subseteq C(X, I)$ be a lattice.

Suppose for all $x, y \in X$, there is some $g \in L$ so that $g(x) = f(x)$ and $g(y) = f(y)$. Then for all $\epsilon > 0$, there is some $h \in L$ so that $\|f - h\| < \epsilon$.

Proof. Fix $f \in C(X, I)$. Pick an $\epsilon > 0$. Fix $x \in X$. By the assumption, for each $y \in X$, choose functions $g_y^x \in L$ so that $f(x) = g_y^x(x)$ and $f(y) = g_y^x(y)$. Define $A_y^x = \{z \in X : g_y^x(z) < f(z) + \epsilon\}$. A_y^x is open using the continuity of g_y^x and f. Note that $y \in A_y^x$. $\bigcup_{y \in X} A_y^x$ covers X. By compactness, there is a finite set $F_x \subseteq X$ so that $\bigcup_{y \in F_x} A_y^x = X$.

Define $k_x = \bigwedge_{y \in F_x} g_y^x$. Since L is a lattice, $k_x \in L$. Note that $k_x(x) = f(x)$. For any $z \in X$, there is some $y \in F_x$ so that $z \in A_y^x$. Then $k_x(z) \leq g_x^y(z) < f(z) + \epsilon$. This show that for all $z \in X$, $k_x(z) < f(z) + \epsilon$.

For $x \in X$, let $B_x = \{z \in X : f(z) - \epsilon < k_x(z)\}$. Each B_x is open and $x \in B_x$. $\bigcup_{x \in X} B_x = X$. By compactness, there is a finite set $F \subseteq X$ so that $\bigcup_{x \in F} B_x = X$.

Let $h(z) = \bigvee_{x \in F} k_x(z)$. Again since L is a lattice, $h \in L$. Pick any $z \in X$. By the above, $f(z) - \epsilon < k_x(z) \leq h(z)$. Also by the above, $k_x(z) < f(z) + \epsilon$. Hence $\|f - h\| < \epsilon$. □

Fact 3.8 is the main technical approximation that will be needed. The following notation facilitates the exposition.

Definition 3.7. If Δ is a k-ary modulus, then let $\tilde{\Delta} : \mathbb{R}^n \to \mathbb{R}^{\geq 0}$ be defined by $\tilde{\Delta}(\bar{x}) = \Delta(\pi(\bar{x}))$. $\tilde{\Delta}$ is a continuous function that respects Δ using subadditivity.

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a continuous function and let Δ be a k-ary modulus. Let $U^{f,\bar{x},\Delta}(\bar{z}) = f(\bar{x}) + \tilde{\Delta}(\bar{z} - \bar{x})$. Let $L^{f,\bar{x},\Delta}(\bar{z}) = f(\bar{x}) - \tilde{\Delta}(\bar{z} - \bar{x})$. Note that f respects Δ if and only if for all $\bar{x}$, $L^{f,\bar{x},\Delta}(\bar{z}) \leq f(\bar{z}) \leq U^{f,\bar{x},\Delta}(\bar{z})$ for all $\bar{z}$.

Let Δ be a k-ary modulus of uniform continuity so that $\Delta[[0,1]^k] \subseteq [0,1]$. Let $\bar{x}, \bar{y} \in [0,1]^k$. Suppose $\tilde{\Delta}(\bar{y} - \bar{x}) > 0$. Let $a, b \in [0,1]$ with $a \leq b < a + \tilde{\Delta}(\bar{y} - \bar{x})$. Let $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}} : [0,1]^k \to [0,1]$ be defined by

$$\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}(\bar{z}) = \min\left\{1, a + \frac{b-a}{\tilde{\Delta}(\bar{y} - \bar{x})}\tilde{\Delta}(\bar{z} - \bar{x})\right\}.$$

Note that since $\tilde{\Delta}$ respects Δ, so does the $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}$.

If $\tilde{\Delta}(\bar{y} - \bar{x}) = 0$, then define

$$\tilde{\Delta}_{a,a}^{\bar{x},\bar{y}}(\bar{z}) = a.$$

This also respects Δ.

For $\bar{x}, \bar{y}, a, b$ satisfying the above conditions, $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}} : [0,1]^k \to [0,1]$, $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}(\bar{x}) = a$, $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}(\bar{y}) = b$, and this function respects Δ.

Note that if $\bar{x}, \bar{y} \in (\mathbb{Q} \cap [0,1])^k$ and $a, b \in \mathbb{Q} \cap [0,1]$, then the code for $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}$ is arithmetic in the code for Δ.

Fact 3.8. Let Δ be a k-ary modulus. Let $\bar{x}, \bar{y}, a, b$ and $\bar{x}', \bar{y}', a', b'$ satisfy the conditions in Definition 3.7. Let M be the maximum of Δ on $[0,1]^k$. Then

$$\|\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}} - \tilde{\Delta}_{a',b'}^{\bar{x}',\bar{y}'}\| \leq |a - a'| + \left|\frac{b-a}{\tilde{\Delta}(\bar{y} - \bar{x})}\right| \tilde{\Delta}(\bar{x} - \bar{x}') + M\left|\frac{b-a}{\tilde{\Delta}(\bar{y} - \bar{x})} - \frac{b'-a'}{\tilde{\Delta}(\bar{y}' - \bar{x}')}\right|.$$

The main observation is that for a fixed $\bar{x}, \bar{y}, a, b$, the latter expression in the above gets arbitrarily close to 0 as $\bar{x}', \bar{y}', a', b'$ get close to $\bar{x}, \bar{y}, a, b$, respectively. Therefore $\|\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}} - \tilde{\Delta}_{a',b'}^{\bar{x}',\bar{y}'}\|$ approaches 0 as $\bar{x}', \bar{y}', a', b'$ gets close to $\bar{x}, \bar{y}, a, b$.

Proof. For any $\bar{z} \in [0,1]^k$, one has by definition

$$|\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}(\bar{z}) - \tilde{\Delta}_{a',b'}^{\bar{x}',\bar{y}'}(\bar{z})| \leq \left| a + \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})}\tilde{\Delta}(\bar{z}-\bar{x}) - a' - \frac{b'-a'}{\tilde{\Delta}(\bar{y}'-\bar{x}')}\tilde{\Delta}(\bar{z}-\bar{x}') \right|.$$

Using the triangle inequality to extract out $|a-a'|$, one obtains

$$\leq |a-a'| + \left| \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})}\tilde{\Delta}(\bar{z}-\bar{x}) - \frac{b'-a'}{\tilde{\Delta}(\bar{y}'-\bar{x}')}\tilde{\Delta}(\bar{z}-\bar{x}') \right|.$$

By subtracting and adding the same expression, one has

$$= |a-a'| + \left| \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})}\tilde{\Delta}(\bar{z}-\bar{x}) - \frac{b-a}{\tilde{\Delta}(\bar{y}+\bar{x})}\tilde{\Delta}(\bar{z}-\bar{x}') + \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})}\tilde{\Delta}(\bar{z}-\bar{x}') - \frac{b'-a'}{\tilde{\Delta}(\bar{y}'-\bar{x}')}\tilde{\Delta}(\bar{z}-\bar{x}') \right|$$

Using the triangle inequality and factoring, one has

$$= |a-a'| + \left| \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})} \right| |\tilde{\Delta}(\bar{z}-\bar{x}) - \tilde{\Delta}(\bar{z}-\bar{x}')| + \tilde{\Delta}(\bar{z}-\bar{x}') \left| \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})} - \frac{b'-a'}{\tilde{\Delta}(\bar{y}'-\bar{x}')} \right|.$$

By assumption, $\tilde{\Delta}(\bar{z}-\bar{x}') \leq M$. Using the properties of Δ from Definition 2.1, one can show that $|\tilde{\Delta}(\bar{z}-\bar{x}) - \tilde{\Delta}(\bar{z}-\bar{x}')| = |\tilde{\Delta}(\bar{z}-\bar{x}) - \tilde{\Delta}(\bar{z}-\bar{x}')| \leq |\tilde{\Delta}(\bar{x}-\bar{x}')|$. Thus one has

$$\leq |a-a'| + \left| \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})} \right| \tilde{\Delta}(\bar{x}-\bar{x}') + M \left| \frac{b-a}{\tilde{\Delta}(\bar{y}-\bar{x})} - \frac{b'-a'}{\tilde{\Delta}(\bar{y}'-\bar{x}')} \right|.$$

The first statement has been verified.

For the main observation: Note that as a approaches a', the first term goes to 0. As $\bar{x}$ approaches $\bar{x}'$, the second term approaches 0. As a, b, $\bar{x}$, $\bar{y}$ approaches a', b', $\bar{x}'$, $\bar{y}'$, respectively, the third term goes to 0. □

Fact 3.9. Let $A \subseteq A' \subseteq C(X, I)$. Let $L(A)$ and $L(A')$ be the lattice generated by A and A', respectively. Suppose for all $\epsilon > 0$ and $f' \in A'$, there is some $f \in A$ so that $\|f - f'\| < \epsilon$. Then for all $\epsilon > 0$ and $f' \in L(A')$, there is some $f \in L(A)$ so that $\|f - f'\| < \epsilon$.

Proof. Observe that if $\|f - f'\| < \epsilon$ and $\|g - g'\| < \epsilon$, then $\|f \wedge g - f' \wedge g'\| < \epsilon$ and $\|f \vee g - f' \vee g'\| < \epsilon$. The result follows from this observation by induction. □

Definition 3.10. Let Δ is a k-ary modulus. Let D'_Δ be the smallest lattice containing $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}$ where $\bar{x}, \bar{y} \in (\mathbb{R} \cap [0,1])^k$ and $a, b \in \mathbb{R} \cap [0,1]$ satisfy the conditions in Definition 3.7 (with respect to Δ).

Let D_Δ be the smallest lattice containing $\tilde{\Delta}_{a,b}^{\bar{x},\bar{y}}$ where $\bar{x}, \bar{y} \in (\mathbb{Q} \cap [0,1])^k$ and $a, b \in \mathbb{Q} \cap [0,1]$ satisfying the conditions in Definition 3.7 (with the respect to Δ). D_Δ is a countable set. D_Δ has a code which is arithmetic in the code of Δ.

Fact 3.11. Let Δ be a modulus of arity k. If $u : [0,1]^k \to [0,1]$ is a function respecting Δ and $\epsilon > 0$, then there is an $h \in D_\Delta$ so that $\|u - h\| < \epsilon$.

Proof. Fix $\epsilon > 0$. Fix $\bar{x}, \bar{y} \in [0,1]^k$. First suppose that $\tilde{\Delta}(\bar{y} - \bar{x}) > 0$. Without loss of generality, suppose that $u(\bar{x}) \leq u(\bar{y})$. Since u respects the modulus Δ, $u(\bar{x}) \leq u(\bar{y}) < u(\bar{x}) + \tilde{\Delta}(\bar{y} - \bar{x})$. Hence $\tilde{\Delta}^{\bar{x},\bar{y}}_{u(\bar{x}),u(\bar{y})} \in D'_\Delta$. If $\tilde{\Delta}(\bar{y} - \bar{x}) = 0$, then $u(\bar{x}) = u(\bar{y})$. In this case, consider $\tilde{\Delta}^{\bar{x},\bar{y}}_{u(\bar{x}),u(\bar{x})} \in D'_\Delta$. This shows that Fact 3.6 can be used to find some $h \in D'_\Delta$ so that $\|u - h\| < \epsilon$.

Using Facts 3.8 and 3.9, one can find some $h \in D_\Delta$ so that $\|u - h\| < \epsilon$. □

Fact 3.12. Let Ω be a weak modulus. Let $\varphi_0, \ldots, \varphi_{k-1}$ be atomic formulas of $L_{\omega,\omega}$ with free variables among $v_0, \ldots, v_{n-1}$. Then there is a countable set $E(\varphi_0, \ldots, \varphi_{k-1})$ with the following properties:

1. All formulas $\psi \in E(\varphi_0, \ldots, \varphi_{k-1})$ are basic n-ary respecting Ω.
2. For any $\epsilon > 0$, any $\mathscr{L}$-structure $\mathcal{M}$, and any formula of the form $u(\varphi_0, \ldots, \varphi_{k-1})$ respecting Ω where $u : [0,1]^k \to [0,1]$ is continuous, there is some formula $\psi \in E(\varphi_0, \ldots, \varphi_{k-1})$ so that $\|\psi^{\mathcal{M}} - u(\varphi_0, \ldots, \varphi_{k-1})^{\mathcal{M}}\| < \epsilon$.
3. There is a real arithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$ which codes $E(\varphi_0, \ldots, \varphi_{k-1})$.

Proof. Let Δ_i be the canonical modulus for φ_i. Suppose some of the Δ_i are constantly 0. Without loss of generality, assume that there is some $j < k$ so that Δ_i is constantly 0 for all $i \geq j$. Fix an $\mathscr{L}$-structure $\mathcal{M}$. There are constants $b_j, \ldots, b_{k-1}$ in $[0,1]$ so that for each $i \geq j$, $\varphi_i^{\mathcal{M}}$ takes constant value b_j. Define $\tilde{u}(\varphi_0, \ldots, \varphi_{j-1}) = u(\varphi_0, \ldots, \varphi_{j-1}, b_j, \ldots, b_{k-1})$. $\tilde{u}(\varphi_0, \ldots, \varphi_{j-1})$ respects the weak modulus Ω and, $\tilde{u}(\varphi_0, \ldots, \varphi_{j-1})$ and $u(\varphi_0, \ldots, \varphi_{k-1})$ are equal when interpreted in $\mathcal{M}$.

From the above discussion, it suffices to consider the case when all the moduli $\Delta_0, \ldots, \Delta_{k-1}$ are not constantly zero.

Let Δ be the modulus from Fact 3.4. Let $E(\varphi_0, \ldots, \varphi_{k-1})$ be the collection of the formulas of the form $u(\varphi_0, \ldots, \varphi_{k-1})$ where $u \in D_\Delta$. □

Finally, the next fact shows that there is a countable dense set F^n_Ω of n-ary basic formulas respecting the weak modulus Ω which is dense among the collection of all basic n-ary formulas φ. Also F^n_Ω has a code arithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$.

Fact 3.13. There is a countable set F^n_Ω of n-ary basic formulas respecting the weak modulus Ω such that for all $\epsilon > 0$, $\mathscr{L}$-structures $\mathcal{M}$, and basic n-ary-formulas φ respecting Ω, there is some $\psi \in F^n_\Omega$ so that $\|\psi^{\mathcal{M}} - \varphi^{\mathcal{M}}\| < \epsilon$. Moreover, the code for F^n_Ω is arithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$.

Proof. Fix an $\mathscr{L}$-recursive enumeration of all finite tuples $(\varphi_0, \ldots, \varphi_{k-1})$ (where $k \in \omega$) of atomic formulas in the free variables $v_0, \ldots, v_{n-1}$. Let F^n_Ω be the union of all such $E(\varphi_0, \ldots, \varphi_{k-1})$. This works using Facts 2.10 and 3.12. □

4. Continuous Scott analysis

The following is the back-and-forth pseudo-distance.

Definition 4.1 ([3, Definition 3.1])**.** Let $\mathscr{L}$ be a language of continuous logic. Let $\mathcal{M}$ be an $\mathscr{L}$-structure. Let $\bar{a}, \bar{b}$ be tuples from M of the same length. Let Ω be a weak modulus.

Let

$$r_0(\bar{a}, \bar{b}) = \sup_{\varphi} |\varphi^{\mathcal{M}}(\bar{a}) - \varphi^{\mathcal{M}}(\bar{b})|,$$

where the supremum is taken over all basic $\mathscr{L}$-formulas respecting Ω.

Suppose r_β has been defined for all $\beta < \alpha$ and α is a limit ordinal; then let

$$r_\alpha(\bar{a}, \bar{b}) = \sup_{\beta<\alpha} r_\beta(\bar{a}, \bar{b}).$$

Suppose r_α has been defined, and let

$$r_{\alpha+1}(\bar{a}, \bar{b}) = \sup_{c,d\in M} \inf_{c',d'\in M} r_\alpha(\bar{a}c, \bar{b}d') \vee r_\alpha(\bar{a}c', \bar{b}d').$$

For each $n \in \omega$, let $r_{\alpha,n}$ be the restriction of r_α to $M^n \times M^n$.

Fact 4.2 ([3, Lemma 3.2 and 3.3])**.** Let α be an ordinal and $n \in \omega$. $r_{\alpha,n}$ is a pseudo-distance. Each $r_{\alpha,n}$ is a uniformly continuous function respecting the modulus Ω_n.

If $\alpha < \beta$ and $\bar{a}$ is a tuple in M, then $r_\alpha(\bar{a}, \bar{b}) \leq r_\beta(\bar{a}, \bar{b})$.

Definition 4.3 ([3, Definition 3.6])**.** The least ordinal α so that $r_\alpha = r_{\alpha+1}$ is the Scott rank of $\mathcal{M}$ with respect to Ω and is denoted $\mathrm{SR}_\Omega(\mathcal{M})$.

Fact 4.4. Let $\mathscr{L}$ be a countable language with code $\hat{\mathscr{L}}$. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure. Let $\bar{D}$ denote its completion structure. Let Ω be a weak modulus with code $\hat{\Omega}$. Let $\mathcal{A}$ be a countable admissible set containing $\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}$. Let $\bar{a}, \bar{b}$ be tuples from D of the same length. Let $(r_0(\bar{a}, \bar{b}))^{\mathcal{A}}$ be the computation of $r_0(\bar{a}, \bar{b})$ in $\mathcal{A}$. Then $r_0(\bar{a}, \bar{b}) = (r_0(\bar{a}, \bar{b}))^{\mathcal{A}}$.

Proof. Fact 2.16 implies that for any $\varphi \in L^{\mathcal{A}}_{\omega,\omega}$, $\varphi^{\bar{\mathcal{D}}}(\bar{a}) = (\varphi^{\bar{\mathcal{D}}}(\bar{a}))^{\mathcal{A}}$ and $\varphi^{\bar{\mathcal{D}}}(\bar{b}) = (\varphi^{\bar{\mathcal{D}}}(\bar{b}))^{\mathcal{A}}$. That is, the computation of formulas $\varphi \in L^{\mathcal{A}}_{\omega,\omega}$ is the same as in the real world.

Let n be the length $\bar{a}$. The computation of $r_0(\bar{a}, \bar{b})$ in $\mathcal{A}$ entails using only the basic n-ary formulas respecting Ω which belong to $\mathscr{L}^{\mathcal{A}}_{\omega,\omega}$ when evaluating the supremum. The set F^n_Ω from Fact 3.13 is contained in and is dense in the collection of all n-ary basic formulas respecting Ω under the uniform norm. The entire set F^n_Ω is hyperarithmetic in $\hat{\mathscr{L}} \oplus \hat{\Omega}$. So $F^n_\Omega \in \mathcal{A}$. In particular, each function of F^n_Ω is in $L^{\mathcal{A}}_{\omega,\omega}$.

Together, these facts show that the computation of $r_0(\bar{a}, \bar{b})$ in $\mathcal{A}$ gives the true value of $r_0(\bar{a}, \bar{b})$. □

Theorem 4.5. *Let $\mathscr{L}$ be a countable language with code $\hat{\mathscr{L}}$. Let Ω be a weak modulus with code $\hat{\Omega}$. Let $\mathcal{D}$ be a countable $\mathscr{L}$-pre-structure. Let $\bar{\mathcal{D}}$ denote its completion structure. Then* $\mathrm{SR}_\Omega(\bar{\mathcal{D}}) \leq \omega_1^{\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}}$.

Proof. Let $\mathcal{A}$ be any admissible set containing $\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D}$.

Fix a $q \in \mathbb{Q}^{>0}$. Define the operator $\Gamma_q : \mathscr{P}(D^{<\omega} \times D^{<\omega}) \to \mathscr{P}(D^{<\omega} \times D^{<\omega})$ by: $\Gamma_q(X)$ is the set of $(\bar{a}, \bar{b})$ so that one of the following hold:

1. $|\bar{a}| \neq |\bar{b}|$.
2. $r_0(\bar{a}, \bar{b}) > q$.
3. $(\exists c \in D)(\exists d \in D)(\forall c' \in D)(\forall d' \in D)((\bar{a}c', \bar{b}d) \in X \vee (\bar{a}c, \bar{b}d') \in X)$.

For each $q \in \mathbb{Q}$, Γ_q is defined by a positive Σ-formula with parameters from $\mathcal{A}$. By results of Gandy (see [1, Chapter VI, Corollary 2.8]), the closure ordinal $\|\Gamma_q\|$ of Γ_q is less than or equal to $o(\mathcal{A}) = A \cap \mathrm{ON}$, the ordinal height of $\mathcal{A}$.

Let $I^\infty_{\Gamma_q} = I^{\|\Gamma_q\|}_{\Gamma_q}$ be the least fixed point of Γ_q. By Fact 4.4, r_0 is computed correctly by $\mathcal{A}$. So $(\bar{a}, \bar{b}) \in I^\infty_{\Gamma_q}$ if and only if there is some α so that $r_\alpha(\bar{a}, \bar{b}) > q$.

Let $\gamma = \sup_{q\in\mathbb{Q}^{>0}} \|\Gamma_q\|$. By the above, $\gamma \leq o(\mathcal{A})$. So $r_\gamma(\bar{a},\bar{b}) = r_\beta(\bar{a},\bar{b})$ for all tuples $\bar{a},\bar{b}$ from D and $\beta \geq \gamma$. By Fact 4.2, each $r_{\beta,n}$ is a continuous function on $\bar{D}^n \times \bar{D}^n$. Since for each $\beta \geq \gamma$, $r_{\gamma,n} \restriction D^n \times D^n = r_{\beta,n} \restriction D^n \times D^n$, one has that $r_{\gamma,n} \restriction \bar{D}^n \times \bar{D}^n = r_{\beta,n} \restriction \bar{D}^n \times \bar{D}^n$ for all $\beta \geq \gamma$. Hence $r_\beta(\bar{a},\bar{b}) = r_\gamma(\bar{a},\bar{b})$ for all $\beta \geq \gamma$ and all tuples $\bar{a},\bar{b}$ of the same length from $\bar{D}$. $\mathrm{SR}_\Omega(\bar{\mathcal{D}}) \leq \gamma$ and in fact is equal. As $\gamma \leq o(\mathcal{A})$, one has $\mathrm{SR}_\Omega(\bar{\mathcal{D}}) \leq o(\mathcal{A})$.

Letting $\mathcal{A} = L_{\omega_1^{\hat{\mathscr{L}}\oplus\hat{\Omega}\oplus\mathcal{D}}}(\hat{\mathscr{L}} \oplus \hat{\Omega} \oplus \mathcal{D})$, which has ordinal height $o(\mathcal{A}) = \omega_1^{\hat{\mathscr{L}}\oplus\hat{\Omega}\oplus\mathcal{D}}$, completes the proof. □

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