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Convergence of fixed-point algorithms for elastic demand dynamic user equilibrium

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ABSTRACT

In this paper we present sufficient conditions for convergence of projection and fixed-point algorithms used to compute dynamic user equilibrium with elastic travel demand (E-DUE). The assumption of strongly monotone increasing path delay operators is not needed. In its place, we assume path delay operators are merely weakly monotone increasing, a property assured by Lipschitz continuity, while inverse demand functions are strongly monotone decreasing. Lipschitz continuity of path delay is a very mild regularity condition. As such, nonmonotone delay operators may be weakly monotone increasing and satisfy our convergence criteria, provided inverse demand functions are strongly monotone decreasing. We illustrate convergence for nonmonotone path delays via a numerical example.

1. Introduction

Friesz and Meimand (2014) and Han et al. (2015a) give differential variational inequality (DVI) formulations of the elastic demand dynamic user equilibrium (E-DUE) problem. These build on the variational inequality formulation of fixed demand dynamic user equilibrium found in Friesz et al. (1993) and Friesz et al. (2001), and extended in Friesz et al. (2011) and Friesz et al. (2013). The DVI formulation of DUE does not make the underlying model better or worse, but it does signal that a substantial literature for numerical analyses is available and helpful when studying DUE. In particular, (Friesz et al., 2011, 2013; Friesz and Han, 2019; Han et al., 2015a,b; Han et al., 2019) present and test fixed-point DUE algorithms that employ a minimum norm projection to assure feasibility. This method of computation has proven to be reliable in practice, often converging to dynamic user equilibria despite the prospect of nonmonotone path delay operators. The lack of monotonicity of path delay operators has been demonstrated by (Smith and Ghali, 1990; Ghali and Smith, 1993; Mounce, 2006), and others. In the classical theory of convergence for fixed-point algorithms, sufficient conditions assuring convergence generally include the requirement that the relevant operator be strongly monotone; this requirement is often imposed to prove the convergence of other algorithms, as well. As such, DUE computation has lacked a satisfactory theory of convergence.

The dynamic network loading (DNL) phase of DUE calculations gives rise to the path delay operator, which is analogous to the pay-off function in classical Nash games, and plays a pivotal role in DTA and DUE problems. The properties of the delay operator are critical to the existence and computation of DUE models. However, it is widely recognized that solutions of the DNL model, or the delay operator derived therefrom, are not available in closed form; instead, they have to be numerically determined for each instant of time and traffic conditions via some sort of computational procedure. As a result, the mathematical properties of the path delay operator remain largely unknown. This has significantly impacted the

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Table 1
a history of monotonicity assumptions.

Paper	DUE type	Algorithm	Convergence Criteria
Huang and Lam (2002)	SRDT DUE	route-swapping	CC
Lo and Szeto (2002)	RC DUE	alternating-direction	CC
Szeto and Lo (2004)	SRDT E-DUE	alternating-direction	CC
Szeto and Lo (2006)	RC BR-DUE	route-swapping	CC
Mounce and Carey (2011)	RC-DUE	route-swapping	LC & M
Friesz et al. (2011)	SRDT	fixed-point	LC & SM
Tian et al. (2012)	SRDT DUE	route-swapping	M
Long et al. (2013)	RC DUE	extra-gradient	PM
Han et al. (2015b)	SRDT BR-DUE	projection	SSQM & D-property
Han et al. (2015a)	SRDT E-DUE	proximal-point	MSWM
Thong et al. (2020)	SRDT DUE	FBF	PM

computation of DUE due to the lack of provable convergence theories, which generally require certain forms of generalized monotonicity of the delay operators. Table 1 shows some relevant computational algorithms for DUE and their convergence conditions, most especially the relevant continuity and monotonicity assumptions regarding the delay operator. The reader is referred to El Farouq (2001) and Han et al. (2015a) for definitions of different types of generalized monotonicity.

In order to provide the context for this paper, we want to take a moment to review the DUE algorithms and their associated convergence criteria that have appeared in the technical literature heretofore. To do this we first remark that nearly all proofs of convergence for DUE algorithms make use of Lipschitz continuity. We will need some abbreviations and acronyms. In particular, the following apply: (1) Lipschitz continuous = LC, (2) monotone = M, (3) strongly monotone = SM, (4) pseudomonotone = PM, (5) semi-strictly quasimonotone = SSQM, (6) mixed strongly-weakly monotone = MSWM, (7) co-coercive = CC, (8) simutaneous route-and-departure-time choice = SRDT, (9) route choice = RC, (10) bounded rationality = BR, (11) elastic demand = E, and (12) forward-backward-forward = FBF.

If a model is not explicitly indicated as incorporating elastic demand, it is based on inelastic demand. Consequently, we have the Table 1. Note that, in the above table, reference is made to the D-property; it is the property of boundedly rational user equilibria sometimes imposed in modeling; it requires effective delay to be strictly greater than minimum delay and is dicussed by Han et al. (2015b), as well as (Han et al., 2019) and (Thong et al., 2020). Note that (Han et al., 2019) and (Thong et al., 2020) refer to the D-property as the “P-property”. Note also that co-coercive operators are monotone, but not necessarily strongly monotone. Further note that none of the generalizations of monotonicity presented in Table 1 may be satisfied by a delay operator that is nonmonotone. From Table 1, the ubiquitous nature of monotonicity in investigations of DUE algorithm convergence is clear. In this paper, we introduce the notion of weakly monotone operators for the study of DUE algorithm convergence, taking great care to demonstrate that weakly monotone operators form a very broad class, one that includes operators neither satisfying the definition of monotonicity nor the various definitions of generalized monotonicity.

Our presentation in this paper shows that the distinct notions of weak monotonicity and strong monotonicity allow a proof of convergence for E-DUE fixed-point algorithms when there are appropriate relationships among key parameters. This paper does not present extensive numerical tests or comparisons of algorithms; rather, a single directly relevant numerical example is provided. Pertinent numerical experiments have been previously reported in Han et al. (2015a). It is the intent of this manuscript to provide a proof of convergence for fixed-point DUE algorithms when the notion of a strongly monotone decreasing operator is applied to inverse demand functions, and path delay operators are viewed as weakly monotone. The concept of weak monotonicity is not widely familiar and has seldom been previously used in the study of traffic equilibria. In fact operators that are weakly monotone need not be monotone. Indeed, a weakly monotone operator or function may be locally monotone increasing and locally monotone decreasing in adjacent but not identical locales. Friesz et al. (2011), in an appendix, give what is likely the first example of such a function in the context of dynamic user equilibrium. It is our contention that path delay operators pertinent to DUE modeling may be characterized as weakly monotone in good conscience, provided they are Lipschitz continuous.

2. Mathematical notation and background

Throughout this paper, the time interval of analysis is a single commuting period expressed as $[t_0, t_f] \subset \Re$ where $t_f > t_0$, and both t_0 and t_f are fixed, with $t_0 < t_f$. We let $\mathcal{P}$ be the set of all paths utilized by travelers. For each $p \in \mathcal{P}$, we associate the path departure rate (in vehicles per unit time) $h_p(\cdot): \ [t_0, t_f] \to \Re_+$, which is a function of departure time $t \in [t_0, t_f]$, where $\Re_+$ denotes the set of non-negative real numbers. Each path departure rate $h_p(t)$ is interpreted as a path flow measured at the entrance of the first link of the relevant path. We next define $h(\cdot) = \{h_p(\cdot) : p \in \mathcal{P}\}$ to be a vector of departure rates, which is viewed as a vector-valued function of t, the departure time.[1]

[1] For notational convenience, when no confusion will result, we will sometimes use h instead of $h(\cdot)$ to denote the vector of path departure rates.

We let $L^2[t_0, t_f]$ be the space of square-integrable functions defined on the interval $[t_0, t_f]$, and $L^2_+[t_0, t_f]$ its subset consisting of non-negative functions. We stipulate that each path departure rate is square integrable: $h_p(\cdot) \in L^2_+[t_0, t_f]$ and $h(\cdot) \in (L^2_+[t_0, t_f])^{|\mathcal{P}|}$, where $\left(L^2[t_0, t_f]\right)^{|\mathcal{P}|}$ is the $|\mathcal{P}|$-fold product of the Hilbert space $L^2[t_0, t_f]$, and $\left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|}$ is its subset consisting of non-negative path departure vectors. The inner product on the Hilbert space $\left(L^2[t_0, t_f]\right)^{|\mathcal{P}|}$ is defined as

$$\left\langle h^1, h^2\right\rangle = \int_{t_0}^{t_f} \left(h^1(t)\right)^T h^2(t)\, dt = \sum_{p\in\mathcal{P}} \int_{t_0}^{t_f} h^1_p(t)\cdot h^2_p(t)\, dt \tag{1}$$

where the superscript T denotes the transpose of vectors. Moreover, the norm

$$\|u\|_{L^2} = \langle u, u\rangle^{1/2} \tag{2}$$

is induced by the inner product (1).

Here, as in all DUE modeling, the single most crucial ingredient is the path delay operator, which maps a given vector of departure rates h to a vector of path travel times. More precisely, we let

$$D_p(t, h) \qquad \forall t \in [t_0, t_f], \quad \forall p \in \mathcal{P}$$

be the path travel time of a driver departing at time t and following path p, given the departure rates associated with all the paths in the network, which are expressed by h above. We then define the path delay operator $D(\cdot)$ by letting $D(h) = \{D_p(\cdot, h) : p \in \mathcal{P}\}$, which is a vector of time-dependent path travel times $D_p(t, h)$. Moreover, we use $D(\cdot)$ to denote an operator, defined on $\left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|}$, that maps a vector valued function $h(\cdot)$ to another vector-valued function $\{D_p(\cdot, h) : p \in \mathcal{P}\}$. In summary,

$$D : \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|} \to \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|} \tag{3}$$

$$h(\cdot) = \{h_p(\cdot),\ p \in \mathcal{P}\} \mapsto D(h) = \{D_p(\cdot, h),\ p \in \mathcal{P}\} \tag{4}$$

The *effective* path delay operator Ψ is similarly defined, except that the effective path delay embodies arrival penalties, in addition to path travel time. As such, the effective path delay is a more general notion of "travel cost" than path delay. The effective delay operator is defined as

$$\Psi : \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|} \to \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|} \tag{5}$$

$$h(\cdot) = \{h_p(\cdot),\ p \in \mathcal{P}\} \mapsto \Psi(h) = \{\Psi_p(\cdot, h),\ p \in \mathcal{P}\} \tag{6}$$

where

$$\Psi_p(t, h) = D_p(t, h) + F\left[t + D_p(t, h) - A\right)\big] \qquad \forall t \in [t_0, t_f], \quad \forall p \in \mathcal{P} \tag{7}$$

where A is the desired arrival time and $A < t_f$. In (7), the term $F\left[t + D_p(t, h) - A\right)\big]$ assesses a nonnegative penalty whenever

$$t + D_p(t, h) \neq A \tag{8}$$

since $t + D_p(t, h)$ is the clock time at which departing traffic arrives at the destination of path $p \in \mathcal{P}$. Note that, for convenience, A is assumed to be independent of path or origin-destination pair. However, that assumption is easy to relax, and the consequent generalization is a trivial extension of our presentation.

We interpret $\Psi_p(t, h)$ as the perceived travel cost of drivers departing at time t following path p given the vector of path departure rates h. We stipulate that for all $p \in \mathcal{P}$, the function $\Psi_p(\cdot, h)$ is measurable, almost everywhere positive, and square integrable. Furthermore, we use the notation

$$\Psi(t, h) \doteq \{\Psi_p(\cdot, h) : p \in \mathcal{P}\} \in \left(L^2_+[t_0, t_f]\right)^{|\mathcal{P}|}$$

to express the complete vector of effective path delays.

In order to develop an appropriate notion of minimum travel cost in the measure-theoretic context, we require the concept of *essential infimum*. In particular, for any measurable function $g : [t_0, t_f] \to \Re$, the essential infimum of $g(\cdot)$ on $[t_0, t_f]$ is given by

$$\text{essinf}\left\{g(s) : s \in [t_0, t_f]\right\} \doteq \sup\left\{x \in \Re : \ meas\{s \in [t_0, t_f] : g(s) < x\} = 0\right\} \tag{9}$$

where *meas* represents the (Lebesgue) measure. Note that for each $x > \text{essinf}\{g(s) : s \in [t_0, t_f]\}$ it must be true by definition that

$$meas\{s \in [t_0, t_f] : f(s) < x\} > 0$$

Let us define the essential infimum of the effective path delay, which depends on the path departure rate vector h, as follows:

$$v_p(h) = ess\inf\left\{\Psi_p(t, h) : t \in [t_0, t_f]\right\} > 0 \qquad \forall p \in \mathcal{P} \tag{10}$$

The minimum travel cost for a given OD pair (i, j) is, thus, defined as

$$v_{ij}(h) = \min\left\{v_p(h): p \in \mathcal{P}_{ij}\right\} > 0 \qquad \forall (i, j) \in \mathcal{W} \tag{11}$$

By definition, $v_{ij}(h)$ is the minimum travel cost for OD pair (i, j) among all associated route choices and departure time choices.

3. Definition of E-DUE

We introduce the trip matrix $\left(Q_{ij} : (i, j) \in \mathcal{W}\right)$, where each $Q_{ij} \in \Re_+$ is the (elastic) travel demand between the origin-destination (OD) pair $(i, j) \in \mathcal{W}$, and $\mathcal{W}$ is the set of all origin-destination pairs. The flow conservation constraints read

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij} \qquad \forall (i, j) \in \mathcal{W} \tag{12}$$

where (12) consists of Lebesgue integrals, and $\mathcal{P}_{ij} \subset \mathcal{P}$ is the set of paths connecting OD pair $(i, j) \in \mathcal{W}$. In the elastic demand case, the travel demand between OD pair (i, j) is assumed to be expressed as the following invertible function

$$Q_{ij} = G_{ij}[v] \qquad \forall (i, j) \in \mathcal{W}$$

where $v = \{v_{ij}(h): (i, j) \in \mathcal{W}\}$ is the vector of OD specific minimum travel costs $v_{ij}(h)$ defined in (10), (11). We note that Q_{ij} is the unknown travel demand between (i, j) that must ultimately be achieved by the end of the time horizon $t = t_f$. We will find it convenient to form the complete vector of travel demands by concatenating the OD-specific travel demands to obtain

$$Q = \left(Q_{ij}: (i, j) \in \mathcal{W}\right) = \left(G_{ij}[v]: (i, j) \in \mathcal{W}\right) \in \Re_+^{|\mathcal{W}|}$$

which defines a mapping from v to Q that, when invertible, gives rise to the *inverse demand function*:

$$\Theta : \Re_+^{|\mathcal{W}|} \to \Re_{++}^{|\mathcal{W}|}, \qquad Q \mapsto v$$

where

$$v = \left(v_{ij}: (i, j) \in \mathcal{W}\right) \quad \text{and} \quad v_{ij} = \Theta_{ij}[Q] \tag{13}$$

Notice that the inverse demand function defined in (13) is non-separable in the sense that each minimum OD travel cost v_{ij} is jointly determined by the entire vector of elastic demands $Q = \left(Q_{ij} : (i, j) \in \mathcal{W}\right)$.

Accordingly, we employ the following feasible set of departure flows when the travel demand between each origin-destination pair is endogenous.

$$\tilde{\Lambda} = \left\{(h, Q(t_f)) \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\, dt = Q_{ij}(t_f) \quad \forall (i, j) \in \mathcal{W}\right\} \subset \left(L^2[t_0, t_f]\right)^{|\mathcal{P}|} \times \Re^{|\mathcal{W}|} \tag{14}$$

where $\left(L^2[t_0, t_f]\right)^{|\mathcal{P}|} \times \Re^{|\mathcal{W}|}$ is the direct product of the $|\mathcal{P}|$-fold product of Hilbert spaces, and the $|\mathcal{W}|$-dimensional Euclidean space consisting of vectors of elastic travel demands.

With the preceding preparation, we are in a place where the simultaneous route-and-departure-time dynamic user equilibrium with elastic demand can be rigorously defined, as follows:

Definition 1. (Dynamic user equilibrium with elastic demand) A pair $(h^*, Q^*) \in \tilde{\Lambda}$ is said to be a dynamic user equilibrium with elastic demand if for all $(i, j) \in \mathcal{W}$,

$$h_p^*(t) > 0, \quad p \in \mathcal{P}_{ij} \implies \Psi_p(t, h^*) = \Theta_{ij}[Q^*(t_f)] \qquad \text{for almost every } t \in [t_0, t_f] \tag{15}$$

$$Q^*(t_f) > 0 \implies \Psi_p(t, h^*) = \Theta_{ij}[Q^*(t_f)] \qquad \text{for almost every } t \in [t_0, t_f], \quad \forall p \in \mathcal{P}_{ij} \tag{16}$$

4. The differential variational inequality formulation of E-DUE

To facilitate the differential variational inequality representation of E-DUE problems, it is convenient to re-state the constraints (14) as an initial value problem, as follows.

$$\Lambda = \left\{\left(h, Q(t_f)\right) \geq 0 : \frac{dQ_{ij}(t)}{dt} = \sum_{p \in P_{ij}} h_p(t), \ Q_{ij}(t_0) = 0 \quad \forall (i, j) \in \mathcal{W}\right\} \subseteq \left(L_+^2\left[t_0, t_f\right]\right)^{|\mathcal{P}|}, \tag{17}$$

Here, unlike in Section 3, $Q_{ij}(\cdot)$ is treated as a time-varying quantity representing the cumulative departures (number of vehicles) between OD pair (i, j) up to time $t \in [t_0, t_f]$. Each $Q_{ij}(t_f)$ is an unknown state value at the terminal time t_f to be found endogenously for all $(i, j) \in \mathcal{W}$.

Consider the subset Λ of the Hilbert space $\left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{P}|}$ and the subset $\Re_+^{|\mathcal{W}|}$ of the finite-dimensional space $\Re^{|\mathcal{W}|}$, with $h \in (L^2[t_0, t_f])^{|\mathcal{P}|}$ and $Q\left(t_f\right) \in \Re_+^{|\mathcal{W}|}$. We will investigate this differential variational inequality:

$$\begin{aligned} &\langle \Psi(t, h^*), h - h^* \rangle_{L^2} - \langle \Theta\left[Q^*\left(t_f\right)\right], Q\left(t_f\right) - Q^*\left(t_f\right) \rangle_2 \geq 0 \\ &\quad h, h^* \in \Lambda \subseteq \left(L^2\left[t_0, t_f\right]\right)^{|\mathcal{P}|}, \quad Q\left(t_f\right), Q^*\left(t_f\right) \in \Re_+^{|\mathcal{W}|} \end{aligned} \tag{18}$$

that arises in the study of dynamic user equilibrium with elastic travel demand, where

$$\langle \Psi(t, h^*), h - h^* \rangle_{L^2} = \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi(t, h^*)^T (h - h^*) dt$$

$$\langle \Theta\left[Q^*\left(t_f\right)\right], Q\left(t_f\right) - Q^*\left(t_f\right) \rangle_E = \Theta\left[Q^*\left(t_f\right)\right]^T \left[Q\left(t_f\right) - Q^*\left(t_f\right)\right]$$

The function $\Theta(Q)$ is a vector of inverse elastic travel demand functions. The desired differential variational inequality, in summation notation, is

$$\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h_p^*) dt - \sum_{(i,j) \in \mathcal{W}} \Theta_{ij}\left[Q^*\left(t_f\right)\right]\left[Q_{ij}\left(t_f\right) - Q_{ij}^*\left(t_f\right)\right] \geq 0 \tag{19}$$

where

$$\left(h, Q\left(t_f\right)\right), \left(h^*, Q^*\left(t_f\right)\right) \in \Lambda$$

We refer to (19) as $DVI(\Psi, \Theta, \Lambda)$. Note that when no confusion will occur we will sometimes use $Q\left(t_f\right)$ and Q interchangeably; the important thing to note is that the fundamental demand variables are $Q\left(t_f\right)$.

5. Monotonicity and lipshitz continuity

As we commented in the introduction of this paper, it is widely known that the effective travel delay operators needed in DUE modeling are potentially nonmonotonic. In this paper, we will assume that the vector effective path delay operator is *weakly monotone increasing*, and employ the following definition:

Definition 2. As presented in El Farouq (2001), a weakly monotone increasing operator $\Psi(t, h)$ is an operator that obeys the following inequality:

$$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_{L^2} \geq -1 \cdot K_\Psi \|h^1 - h^2\|^2 \qquad K_\Psi > 0 \tag{20}$$

for all feasible h^1 and h^2.

5.1. Numerical example of a weakly monotone function

It should be noted that weakly monotone increasing operators need not be monotone increasing; in fact they can take the form of sinusoidal curves and monotone decreasing curves, as well as undulating curves whose peaks and valleys are of arbitrary magnitudes. An example of a weakly monotone function that illustrates these characteristics now follows. Consider this function defined on the stipulated set:

$$F = 65 + x^2 - 2x^3 \cos 2x \qquad \text{such that} \qquad 0 \leq x \leq 4$$

Its associated graph is (Fig. 1).

In the context of this simple example, the condition that governs whether $F(x)$ is weakly monotone increasing is the following

$$\left[F\left(x^1\right) - F\left(x^2\right)\right]\left(x^1 - x^2\right) \geq -1 \cdot K_0 \left\|x^1 - x^2\right\|^2 \qquad K_0 > 0 \tag{21}$$

By inspection, $F(x)$ is monotone increasing from $x = 0$ to about $x = 2$, while $F(x)$ is monotone decreasing from about $x = 2$ to about $x = 3.3$. In fact, the constrained local minima and maxima of this problem, relative to $0 \leq x \leq 4$, belong to the set

$$S = \{0.00, 1.95, 3.33, 4.00\}$$

Let us test whether pairs of points from the set S obey the definition of some type of monotonicity. The relevant calculations are presented in Table 2.

The left-hand side (LHS) of the monotonicity definition, namely

$$LHS = \left[F\left(x^1\right) - F\left(x^2\right)\right]\left(x^1 - x^2\right),$$

makes it apparent that the function $F(x)$ is not strongly monotone. However, if we choose a constant K_0 such that

$$K_0 > \max\left\{\frac{191.90}{11.11}, \frac{99.70}{1.91}\right\} = 52.50,$$

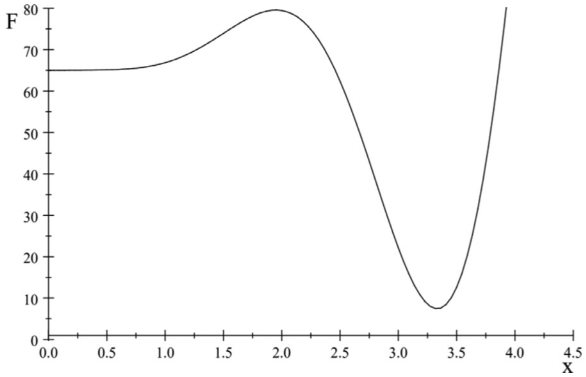

Fig. 1. Example function $F(x)$.

Table 2
Calculations related to weakly monotone example.

x^1	x^2	$F(x^1)$	$F(x^2)$	$F(x^1) - F(x^2)$	$x^1 - x^2$	$\|x^1 - x^2\|^2$	LHS
0.00	1.95	65.00	79.57	−14.57	−1.95	3.81	28.42
0.00	3.33	65.00	7.42	57.58	−3.35	11.11	−191.90
0.00	4.00	65.00	99.62	−34.62	−4.00	16.00	138.50
1.95	3.33	79.57	7.42	72.15	−1.38	1.91	−99.70
1.95	4.00	79.57	99.62	−20.06	−2.05	4.20	41.10
3.33	4.00	7.42	99.62	−92.21	−0.67	0.45	61.55

then $F(x)$ satisfies the definition of weakly monotone increasing. In particular, if we pick $K_0 = 55$, then we have

$$\left[F\left(x^1\right) - F\left(x^2\right)\right]\left(x^1 - x^2\right) \geq -K_0\left\|x^1 - x^2\right\|^2 = -55\left\|x^1 - x^2\right\|^2,$$

and all the feasible solution pairs from the table satisfy this inequality, thereby confirming $F(x)$ is merely weakly monotone and not monotone. In other words, weakly monotone operators constitute a very broad class of operators; that class appears to be much broader than any considered previously in the study of DUE algorithm convergence.

5.2. Weak and strong monotonicity

In this section, as well as subsequently, we will make use of the following shorthand:

$$\Psi\left(t, h^k\right) \equiv \Psi^k$$

$$\Theta\left(Q^k\right) \equiv \Theta^k$$

As we noted in the introduction, it is widely known that the effective travel delay operators used in DUE modeling are potentially nonmonotonic. In this paper, we will assume the effective path delay operator is *weakly monotone increasing*.

We will also assume that the inverse demand function is *strongly monotone decreasing*. That is

$$\langle\Theta^1 - \Theta^2, Q^1\left(t_f\right) - Q^2\left(t_f\right)\rangle_E \leq -1 \cdot K_\Theta\left\|Q^1\left(t_f\right) - Q^2\left(t_f\right)\right\|_E^2 \quad K_\Theta > 0 \tag{22}$$

for all feasible Q^1 and Q^2. Such an assumption about inverse demand functions is behaviorally sound, assuring, in principle, that inverse demands "fall rapidly."

It follows from (20) and (22) that

$$\langle\Psi^1 - \Psi^2, h^1 - h^2\rangle_{L^2} - \langle\Theta^1 - \Theta^2, Q^1\left(t_f\right) - Q^2\left(t_f\right)\rangle_E \geq -1 \cdot K_\Psi\left\|h^1 - h^2\right\|_{L_2}^2 + K_\Theta\left\|Q^1\left(t_f\right) - Q^2\left(t_f\right)\right\|_E^2 \tag{23}$$

$$\forall\left(h^1, Q^1\left(t_f\right)\right), \left(h^2, Q^2\left(t_f\right)\right) \in \Lambda \tag{24}$$

It will be important to recall that K_Ψ and K_Θ are strictly positive scalars.

5.3. Lipschitz continuity

We assume these forms of Lipschitz continuity for delay and inverse travel demand:

$$\left.\begin{aligned}\left\|\Psi^1 - \Psi^2\right\|_{L^2}^2 &\leq K_1\left\|h^1 - h^2\right\|_{L^2}^2 \\ \left\|\Theta^1 - \Theta^2\right\|_E^2 &\leq K_2\left\|Q^1\left(t_f\right) - Q^1\left(t_f\right)\right\|_E^2\end{aligned}\right\}\left(h^1, Q^1\left(t_f\right)\right), \left(h^2, Q^2\left(t_f\right)\right) \in \Lambda \tag{25}$$

with

$$K_1, K_2 > 0$$

We also assume the Lipschitz continuity of Q with respect to h:

$$\left\| Q^1\left(t_f\right) - Q^2\left(t_f\right) \right\|^2 \leq K_0 \left\| h^1 - h^2 \right\|^2 \qquad \left(h^1, Q^1\left(t_f\right)\right), \left(h^2, Q^2\left(t_f\right)\right) \in \Lambda \tag{26}$$

with

$$K_0 > 0 \tag{27}$$

6. Guidance for selecting K_0, K_1, K_2, K_Ψ and K_Θ

In this section, we show the following:

1. K_0 may be calculated explicitly.
2. If $\Psi(t, h)$ is Lipschitz continuous with constant K_1, then it is also weakly monotone with constant $K_\Psi = K_1$.
3. Working with a linear form of the inverse demand function leads to a lower bound for K_Θ.
4. For mild regularity conditions, a unique solution of $DVI(\Psi, \Theta, \Lambda)$ will exist and the set of feasible solutions Λ is compact.
5. Item 4 immediately above assures the following extremal problems have unambiguous formulations and solutions:
   (a) An optimal control problem in Mayer form whose solution is an upper bound for K_Θ.
   (b) An optimal control problem in Mayer form whose solution is a lower bound for K_2.

The above results assist in selecting numerical values for key monotonicity and Lipschitz constants presented in this paper.

Lipschitz continuity for Q with respect to h is the subject of the following result:

Theorem 1. *The functional*

$$Q(h) = \int_{t_0}^{t_f} \Gamma h dt,$$

where Γ is the path-OD incidence matrix, is Lipschitz continuous with constant K_0; that is,

$$\left\| Q^k - Q^* \right\|^2 \leq K_0 \left\| h^k - h^* \right\|^2 \tag{28}$$

$$K_0 > 0 \tag{29}$$

Moreover, K_0 may be calculated directly.

Proof. The Cauchy–Schwarz inequality states that for any vectors u and v of an inner product space it is true that

$$|\langle u, v\rangle| \leq ||u|| \cdot ||v||$$

Consider the following:

$$v = 1 \quad \forall t \in [t_0,\, t_f] \quad \forall p \in \mathcal{P}_{ij} \text{ and } u = h^2 - h^1$$

Then, by the Cauchy-Schwartz inequality, we have

$$\left| \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} (h_p^2(t) - h_p^1(t))\, dt \right| \leq \left\| h^2 - h^1 \right\| \cdot \parallel v \parallel$$

$$\left| \left(Q_{ij}^k - Q_{ij}^*\right) \right| \leq \left\| h^2 - h^1 \right\| \cdot \|v\|$$

$$\parallel v \parallel = \langle v, v\rangle^{\frac{1}{2}} = \sqrt{\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} (1)^2\, dt} = (|\mathcal{P}_{ij}| \cdot (t_f - t_0))^{\frac{1}{2}}$$

$$\left| Q_{ij}^k - Q_{ij}^* \right| \leq \left\| h^2 - h^1 \right\| \cdot (|\mathcal{P}_{ij}| \cdot (t_f - t_0))^{\frac{1}{2}}$$

So we select

$$\sqrt{K_{ij}} = (|\mathcal{P}_{ij}| \cdot (t_f - t_0))^{\frac{1}{2}} > 0 \tag{30}$$

$$\sqrt{K_0} \geq \max \sqrt{K_{ij}} : (i, j) \in \mathcal{W} \tag{31}$$

with the consequence that

$$\left\| Q^1 - Q^2 \right\|_E^2 \leq K_0 \left\| h^1 - h^2 \right\|_{L^2}^2 \qquad K_0 > 0, \tag{32}$$

which is recognized as a Lipschitz continuity condition. □

Clearly, (31) provides a means for determining K_0. Furthermore, the following result provides a sufficient condition for weak monotonicity:

Lemma 1. *Let $\Psi(t,h)$ be a Lipschitz continuous map defined on a subset Λ of the topological vector space V, with Lipschitz constant L. Then $\Psi(t,h)$ is weakly monotone increasing with constant L. That is,*

$$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_V \geq -1 \cdot K_1 \left\| h^1 - h^2 \right\|^2 \qquad K_1 > 0 \quad \forall h^1, h^2 \in \Lambda$$

holds for all $h^1, h^2 \in \Lambda$. Moreover, $K_\Psi = K_1$.

Proof. Clearly we have $\Lambda \subset V = (L^2_+[t_0, t_f])^{|\mathcal{P}|}$. By the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_V &\geq -1 \cdot \left\| \Psi^1 - \Psi^2 \right\| \cdot \left\| h^1 - h^2 \right\| \\ &\geq -1 \cdot K_1 \left\| h^1 - h^2 \right\|^2 \end{aligned} \tag{33}$$

Selecting the Lipschitz constant in inequality (25) will induce a value for K_Ψ. In particular, $K_\Psi = K_1$, as is apparent upon comparing (33) with (20). This proof originates with Han et al. (2015a). □

Lower Bound for K_Θ. We seek, in this example, guidance in selecting the Lipschitz constant K_Θ. Let us assume inverse demand is linear and separable:

$$\Theta_{ij}(Q) = A_{ij} - B_{ij} Q_{ij} \quad \forall (i,j) \in \mathcal{W}$$

It follows that Lipschitz continuity would require

$$\begin{aligned} \langle A - BQ^1 - A - BQ^2, A - BQ^1 - A - BQ^2 \rangle &\leq K_\Theta \left\| Q^1 - Q^2 \right\|^2 \\ \langle BQ^1 - BQ^2, BQ^1 - BQ^2 \rangle &\leq K_\Theta \left\| Q^1 - Q^2 \right\|^2 \\ \langle B\left(Q^1 - Q^2\right), B\left(Q^1 - Q^2\right) \rangle &\leq \\ \sum_{(i,j)} \left(B_{ij}\right)^2 \left(Q^1_{ij} - Q^2_{ij}\right)\left(Q^1_{ij} - Q^2_{ij}\right) &\leq \\ \sum_{(i,j)} \left(B_{ij}\right)^2 \left(Q^1_{ij} - Q^2_{ij}\right)^2 &\leq \\ b_0 \cdot \left\| Q^1 - Q^2 \right\|^2 &\leq K_\Theta \left\| Q^1 - Q^2 \right\|^2 \\ b_0 &\leq K_\Theta \end{aligned}$$

where

$$b_0 = \min\left\{ \left(B_{ij}\right)^2 : (i,j) \in \mathcal{W} \right\}$$

Lemma 2. *When (50) holds any solution of (19) is unique.*

Proof. We will see in Section 8 that

$$\begin{aligned} \langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_V - \langle \Theta^1 - \Theta^2, Q^1 - Q^2 \rangle_E \geq A \left\| h^1 - h^2 \right\|^2_{L^2} + B \left\| Q^1 - Q^2 \right\|^2_E \\ \forall \left(h^1, Q^1\right), \left(h^2, Q^2\right) \in \Lambda \end{aligned}$$

with A and B strictly positive. Therefore

$$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_{L^2} - \langle \Theta^1 - \Theta^2, Q^1 - Q^2 \rangle_E > 0 \tag{34}$$

provided

$$h^1 \neq h^2 \quad \text{and} \quad Q^1 \neq Q^2$$

In other words, the strongly monotone increasing operator

$$F \equiv \begin{pmatrix} \Psi \\ -1 \cdot \Theta \end{pmatrix} \tag{35}$$

is also strictly monotone increasing, thereby assuring

$$\begin{pmatrix} h^* \\ Q^* \end{pmatrix} \tag{36}$$

is an unique equilibrium solution of (19). □

We now introduce a result pertaining to compactness that is a prequisite to finding K_θ in finite time. That result is the following:

Lemma 3. *The set* Λ *is compact if the following conditions hold:*

1. The arrival penalty function $F(\cdot)$ appearing in (7) is continuous on $[t_0, t_f]$ and satisfies
$$F(t_2) - F(t_1) \geq \Delta(t_2 - t_1) \qquad \forall t_0 \leq t_1 \leq t_2 \leq t_f \tag{37}$$
for some $\Delta > -1$
2. The first-in-first-out (FIFO) queue discipline is obeyed at the path level. In addition, each link in the network has a finite exit flow capacity.
3. For any sequence of departure rate vectors $\{h^n(\cdot)\}_{n\geq 1}$ that are uniformly bounded point wise by a positive constant and converge weakly to $h^* \in (L^2[t_0, t_f])^{|P|}$, the corresponding effective path delays $\Psi_p(\cdot, h^n)$ converge to $\Psi_p(\cdot, h^*)$ uniformly for all $p \in \mathcal{P}$.

Proof. See Han et al. (2015a). □

Theorem 2. *Assuming the set* Λ *is compact and* Θ *is strictly deceasing, an upper bound for* K_Θ *may be found by solving the following optimal control problem in Mayer form:*

$$\max \left(- \frac{\langle \Theta(Q(t_f)) - \Theta[Q^*(t_f)], Q(t_f) - Q^*(t_f) \rangle}{\left\| Q(t_f) - Q^*(t_f) \right\|^2} \right) \tag{38}$$

$$s.t. \quad (h, Q(t_f)), (h^*, Q^*(t_f)) \in \Lambda \tag{39}$$

Proof. The constant K_Θ obeys (22). It follows that

$$- \frac{\langle \Theta(Q^1) - \Theta(Q^2), Q^1 - Q^2 \rangle}{\|Q^1 - Q^2\|^2} \geq K_\Theta$$

Therefore, the desired upper bound obeys

$$U_\Theta = \max \left(- \frac{\langle \Theta(Q(t_f)) - \Theta[Q^*(t_f)], Q(t_f) - Q^*(t_f) \rangle}{\left\| Q(t_f) - Q^*(t_f) \right\|^2} \right) \geq K_\Theta$$
$$s.t. \quad (h, Q(t_f)), (h^*, Q^*(t_f)) \in \Lambda$$

where

$$(h^*, Q^*(t_f)) \tag{40}$$

is the equilibrium solution. The upper bound K_Θ exists and is unambiguous because (1) an equilibrium solution exists and is unique, assuring the objective function (38) is well defined; and (2) a solution of (38), (39) is assured because Λ is compact and the objective is continuous on Λ. In the practical application of (39), one must provide an approximation to the equilibrium solution (40). □

Theorem 3. *Assuming the set* Λ *is compact, a lower bound for the Lipschitz constant* K_2 *may be determined by solving this optimal control problem of Mayer form:*

$$\min_{(h,Q)\in\Lambda} \frac{\left\| \Theta[Q(t_f)] - \Theta[Q^*(t_f)] \right\|_E^2}{\left\| Q(t_f) - Q^*(t_f) \right\|_E^2} \quad s.t. \quad (h, Q) \in \Lambda \tag{41}$$

Proof. Condition (25) may be stated as

$$\frac{\left\| \Theta(Q^k) - \Theta^* \right\|_E^2}{\left\| Q^k - Q^* \right\|_E^2} \leq K_2,$$

from which it is immediate that a lower bound L_2 for K_2 may be found by solving

$$L_2 = \min_{(h,Q)\in\Lambda} \frac{\left\| \Theta[Q(t_f)] - \Theta[Q^*(t_f)] \right\|_E^2}{\left\| Q(t_f) - Q^*(t_f) \right\|_E^2} \leq K_2 \tag{42}$$

where

$$\Lambda = \left\{ (h, Q) \geq 0 : \frac{dQ_{ij}(t)}{dt} = \sum_{p\in P_{ij}} h_p(t), \;\; Q_{ij}(t_0) = 0 \quad \forall (i, j) \in \mathcal{W} \right\}$$

The lower bound L_2 exists and is unambiguous because (h^*, Q^*) exists and is unique, while Λ is compact, assuring the objective function of (42) is well defined. In the practical application of (42), one must provide an approximate equilibrium demand vector $Q^*\left(t_f\right)$. □

7. Combined implications of weakly monotone delay, strongly monotone inverse demand and Lipschitz continuity

Let us now undertake to restate expression (23) in a fashion that will be helpful in proving convergence in the next section. We introduce the scalar η obeying

$$0 < \eta < 1 \tag{43}$$

Using η we may rewrite (20) as

$$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_{L^2} - \langle \Theta^1 - \Theta^2, Q^1 - Q^2 \rangle_E$$

$$\leq -K_\Psi \left\| h^k - h^* \right\|_{L^2}^2 + \eta K_\Theta \left\| Q^1 - Q^2 \right\|_E^2 + (1-\eta) K_\Theta \left\| Q^1 - Q^2 \right\|_E^2 \tag{44}$$

$$\leq -K_\Psi \left\| h^k - h^* \right\|_{L^2}^2 + \eta K_0 K_\Theta \left\| h^k - h^* \right\|_{L^2}^2 + (1-\eta) K_\Theta \left\| Q^1 - Q^2 \right\|_E^2 \tag{45}$$

$$= (\eta K_0 K_\Theta - K_\Psi) \left\| h^k - h^* \right\|_{L^2}^2 + (1-\eta) K_\Theta \left\| Q^1 - Q^2 \right\|_E^2 \tag{46}$$

$$= A \left\| h^k - h^* \right\|_{L^2}^2 + B \left\| Q^1 - Q^2 \right\|_E^2 \tag{47}$$

In (47), we have used these parameter definitions to simplify our notation:

$$A \equiv (\eta K_0 K_\Theta - K_\Psi) > 0 \tag{48}$$

$$B \equiv (1-\eta) K_\Theta > 0, \tag{49}$$

where the strict inequality in (48) is an assumption that is now being introduced, and will hold through the remainder of our presentation. The strict inequality in (49) follows from (43) and the strictly positive nature of K_0, K_Θ and K_Ψ. That is, we have this statement

$$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_V - \langle \Theta^1 - \Theta^2, Q^1 - Q^2 \rangle_E \geq A \left\| h^1 - h^2 \right\|_{L^2}^2 + B \left\| Q^1 - Q^2 \right\|_E^2$$
$$\forall \left(h^1, Q^1\right), \left(h^2, Q^2\right) \in \Lambda \tag{50}$$

assuring the operator

$$F \equiv \begin{pmatrix} \Psi \\ -1 \cdot \Theta \end{pmatrix} \tag{51}$$

is strongly monotone increasing.

8. Fixed-point contraction and convergence

It is well known that the differential variational inequality describing DUE may be stated as a fixed-point problem involving the minimum norm projection operator, giving rise to this algorithm:

$$\begin{pmatrix} h^{k+1} \\ Q^{k+1} \end{pmatrix} = P_\Lambda \begin{bmatrix} h^k - \alpha \Psi\left(t, h^k\right) \\ Q^k + \alpha \Theta\left(Q^k\right) \end{bmatrix} \tag{52}$$

where $P_\Lambda(.)$ is the minimum norm projection operator and α is a strictly positive scalar; that is,

$$\alpha > 0$$

In the context of DUE, this algorithm has been studied numerically by (Friesz and Mookherjee, 2006; Friesz et al., 2011; Friesz et al., 2013; Friesz and Meimand, 2014) and (Han et al., 2015a; Han et al., 2015b), and has proven very effective in numerical tests; see in particular the large-scale numerical studies found in Han et al. (2019). We next demonstrate that the assumptions of weakly monotone increasing path delay and strongly monotone decreasing inverse travel demand allow us to establish that algorithm (52) is, in fact, based on a contraction mapping. That demonstration will rely on the following strong convergence theorem:

Theorem 4. *Strong convergence of modified fixed point algorithm. Consider $M : V \longrightarrow V$, where V is a Hilbert space. The sequence generated by*

$$x^{k+1} = \beta_k z + (1-\beta_k) M(x^k) \tag{53}$$

converges strongly to a fixed point x^* of M; that is

$$\lim_{k\longrightarrow\infty} \{x^k\} \longrightarrow x^* \quad \text{and} \quad x^* = M(x^*),$$

when M is nonexpansive, $z \in V$ is an arbitrary point and

(i) $\beta_k \in [0, 1]$
(ii) $\lim_{k\to\infty} \beta_k = 0$
(iii) $\sum_{k=0}^{\infty} \beta_k = \infty$
(iv) $\lim_{k\to\infty} (\beta_k - \beta_{k-1})(\beta_k)^{-1} = 0$.

Proof. This result is generally attributed to Halpern (1967), although he used vastly different notation. A generalization, whose proof is much more readable is due to Bauschke (1996). Although concerned with a different problem class in Banach spaces, the paper by Xu (2003) gives an informative summary of Theorem 4 and related results stemming from Halpern (1967). □

Our main result is the following:

Theorem 5. *The modified fixed-point algorithm*

$$\begin{pmatrix} h^{k+1} \\ Q^{k+1} \end{pmatrix} = \beta_k \begin{pmatrix} z_h \\ z_Q \end{pmatrix} + (1-\beta_k) M(h^k, Q^k) \tag{54}$$

is strongly convergent, where

$$z = \begin{pmatrix} z_h \\ z_Q \end{pmatrix} \in \Lambda$$

is an arbitrary feasible point and

$$M(h, Q) = P_\Lambda \begin{bmatrix} h - \alpha\Psi(t, h) \\ Q + \alpha\Theta(Q) \end{bmatrix} \tag{55}$$

is nonexpansive for an appropriate choice of parameters associated with monotonicity, Lipschitz continuity, and the fixed-point formulation itself.

Proof. We note that

$$\begin{aligned}
Y^{k+1} &\equiv \left\| \left(h^{k+1}, Q^{k+1}\right) - (h^*, Q^*) \right\|^2 \\
&= \left\| \begin{pmatrix} h^k - \alpha\Psi^k \\ Q^k + \alpha\Theta^k \end{pmatrix} - \begin{pmatrix} h^* - \alpha\Psi^* \\ Q^* + \alpha\Theta^* \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} h^k - h^* - \alpha\left(\Psi^k - \Psi^*\right) \\ Q^k - Q^* + \alpha\left(\Theta^k - \Theta^*\right) \end{pmatrix} \right\|^2 \\
&= \left\| h^k - h^* \right\|_{L^2}^2 - 2\alpha \langle \Psi^k - \Psi^*, h^k - h^* \rangle_{L^2} + \alpha^2 \left\| \Psi^k - \Psi^* \right\|_{L^2}^2 \\
&\qquad + \left\| Q^k - Q^* \right\|_E^2 + 2\alpha \langle \Theta^k - \Theta^*, Q^k - Q^* \rangle_E + \alpha^2 \left\| \Theta^k - \Theta^* \right\|_E^2 \\
&= -2\alpha\left[\langle \Psi^k - \Psi^*, h^k - h^* \rangle_{L^2} - \langle \Theta^k - \Theta^*, Q^k - Q^* \rangle_E \right] \\
&\qquad + \left\| h^k - h^* \right\|_{L^2}^2 + \alpha^2 \left\| \Psi^k - \Psi^* \right\|_{L^2}^2 + \left\| Q^k - Q^* \right\|_E^2 + \alpha^2 \left\| \Theta^k - \Theta^* \right\|_E^2
\end{aligned} \tag{56}$$

Using the Lipschitz continuity assumptions (25), together with the monotonicity assumptions (20) and (22), we obtain the following from (56):

$$\begin{aligned}
Y^{k+1} &\le -2\alpha\left[A\left\| h^1 - h^2 \right\|^2 + B\left\| Q^1 - Q^2 \right\|_E^2 \right] \\
&\quad + \left\| h^k - h^* \right\|_{L^2}^2 + \alpha^2 K_1 \left\| h^k - h^* \right\|_{L^2}^2 + \left\| Q^k - Q^* \right\|_E^2 + \alpha^2 K_2 \left\| Q^k - Q^* \right\|_E^2 \\
&= \left(1 - 2\alpha A + \alpha^2 K_1\right) \left\| h^k - h^* \right\|_{L^2}^2 + \left(1 - 2\alpha B + \alpha^2 K_2\right) \left\| Q^k - Q^* \right\|_E^2
\end{aligned} \tag{57}$$

We assure that each iteration of the algorithm is a contraction by enforcing

$$Y^{k+1} < Y^k, \tag{58}$$

which is guaranteed by requiring

$$1 - 2\alpha A + \alpha^2 K_1 < 1$$

$$1 - 2\alpha B + \alpha^2 K_2 < 1$$

which may be restated as

$$\alpha^2 K_1 < 2\alpha A \tag{59}$$

$$\alpha^2 K_2 < 2\alpha B \tag{60}$$

We now use definitions (48) and (49) to restate (59) and (60) as

$$\alpha^2 K_1 < 2\alpha(\eta K_0 K_\Theta - K_\Psi) \tag{61}$$

$$\alpha^2 K_2 < 2\alpha(1-\eta)K_\Theta \tag{62}$$

Thus, a list of all parametric inequalities we have invoked is the following

$$K_\Psi < \eta K_0 K_\Theta \tag{63}$$

$$\frac{\alpha K_1 + 2K_\Psi}{2} < \eta K_0 K_\Theta \tag{64}$$

$$\frac{\alpha K_2}{2} < (1-\eta)K_\Theta \tag{65}$$

$$1 > \eta > 0 \tag{66}$$

Expression (63) is the previously introduced assumption (48). However, it should be noted (64) requires

$$K_\Psi < \frac{\alpha K_1}{2} + K_\Psi < \eta K_0 K_\Theta,$$

thereby making clear that (63) is redundant when (64) is enforced. As a consequence, our sufficiency conditions assuring convergence are (63), (64), (65) and (66). These conditions may be simplified by noting that Lemma 1 tells us that $K_\Psi = K_1$. It follows that

$$\frac{(2+\alpha)K_1}{2} < \eta K_0 K_\Theta \tag{67}$$

$$\frac{\alpha K_2}{2} < (1-\eta)K_\Theta \tag{68}$$

$$1 > \eta > 0 \tag{69}$$

□

9. Numerical example

Let us consider the network associated with Table 3. That network is depicted in Fig. 2 We employed the E-DUE fixed-point algorithm presented by Han et al. (2015a). The software instantiation of the E-DUE algorithm is based on an extension of the fixed-demand software employed by Han et al. (2019), when the latter is modified to treat elastic demand. The (Han et al., 2019) software is available to the public at https://github.com/DrKeHan/DTA. It includes a dynamic network loading (DNL) model based on LWR theory that determines effective path delay $\Psi(t, h)$ for use with a E-DUE solver.

Table 3
Network Topology.

arc	node pair	from node	to node
a_1	$(1, 2)$	1	2
a_2	$(2, 3)$	2	3
a_3	$(2, 4)$	2	4
a_4	$(3, 4)$	3	4
a_5	$(3, 5)$	3	5
a_6	$(4, 5)$	4	5
a_7	$(5, 6)$	5	6

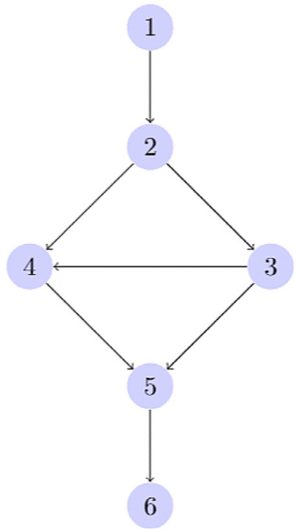

Fig. 2. Graph considered.

9.1. Detailed accounting of parameters

A single OD pair $(1, 6)$ is modeled with the linear inverse demand function

$$\Theta_{16} = a - bQ_{16} \tag{70}$$

where

$$a = 1.200 \tag{71}$$

$$b = \frac{1}{2000} \tag{72}$$

The functional form (70) and the parameters (71) and (72) are identical to that used in Han et al. (2015a). Other parameters employed are the following:

$$K_0 = 1.9 \times 10^7 \qquad K_\Psi = K_1 = 180 \qquad K_2 = 10^{-6} \tag{73}$$

$$K_\Theta = \frac{1}{2000} \tag{74}$$

$$\alpha = 50 \qquad \eta = \frac{1}{2} \tag{75}$$

Hereafter, we drop the subscripts for demand and inverse demand, since there is only one OD pair. Thus, we may write Lipschitz continuity condition (25) as

$$\left[(a - bQ^1) - \left(a - bQ^2\right)\right]^2 \leq K_2\left(Q^1 - Q^2\right)^2 \tag{76}$$

which is easily seen to yield

$$(b)^2\left(Q^1 - Q^2\right)^2 \leq K_2\left(Q^1 - Q^2\right)^2 \tag{77}$$

which tells us that

$$(b)^2 \leq K_2 \tag{78}$$

Since we have $K_2 = 10^{-6}$, we see that the choice

$$b \leq 10^{-3} \tag{79}$$

is permissible, and it is easy to show binding of the strongly monotone decreasing inequality for inverse demand will result. It should also be noted that the constants (73)–(75) satisfy inequalities (63), (64). In particular

$$\frac{\alpha K_1 + 2K_\Psi}{2} = \frac{50 \times 180 + 2 \times 180}{2} = 4680.0 \tag{80}$$

$$\eta K_0 K_\Theta = \frac{1}{2} \times 1.9 \times 10^7 \times \frac{1}{2000} = 4750.0 \tag{81}$$

$$4680.0 < 4750.0 \Longrightarrow \frac{\alpha K_1 + 2K_\Psi}{2} < \eta K_0 K_\Theta \tag{82}$$

$$\frac{\alpha K_2}{2} = \frac{50 \times 10^{-6}}{2} = 2.5 \times 10^{-5} \tag{83}$$

$$(1-\eta)K_\Theta = \frac{1}{2}\frac{1}{2000} = 25 \times 10^{-5} \tag{84}$$

$$2.5 \times 10^{-5} < 25 \times 10^{-5} \Longrightarrow \frac{\alpha K_2}{2} < (1-\eta)K_\Theta \tag{85}$$

Thus, we are certain that

$$\frac{\alpha K_1 + 2K_\Psi}{2} < \eta K_0 K_\Theta \tag{86}$$

$$\frac{\alpha K_2}{2} < (1-\eta)K_\Theta \tag{87}$$

and conclude that the parameters employed for this example satisfy the sufficiency conditions we have derived.

9.2. Numerical results

After 3000 iterations of the fixed-point algorithm, we obtain the information found in Tables 4–6. Naturally, these data support the weakly monotone increasing nature of the path delay operators and the strongly monotone decreasing nature of the inverse demand functions. That is

$$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_{L^2} \geq -1 \cdot K_\Psi \left\| h^1 - h^2 \right\|_{L^2}^2 \tag{88}$$

$$\langle \Theta^1 - \Theta^2, Q^1 - Q^2 \rangle_E \leq -1 \cdot K_\Theta \left\| Q^1 - Q^2 \right\|_E^2 \tag{89}$$

By inspection, it is easy to see that (88) and (89) are satisfied for each pair of iterates considered in Table 4 when appropriate $K_\Theta > 0$ and $K_\Psi > 0$ are selected, as explained above. It is especially important to observe that the path delay operators are weakly monotone increasing but not monotone over most of the 3000 iterations considered.

As such, Table 5 shows that use of the concept of weakly monotone increasing to describe path delays that are not monotone can in fact be accompanied by h-convergence and Q-convergence. Moreover, Table 5 demonstrates that inverse demand functions remain strongly monotone decreasing along the trajectory of computed demands, as must happen under the convergence theory we have presented in this manuscript.

It should likewise be noted that Table 6 shows Lipschitz continuity is satisfied at each iteration for Θ and Ψ. Lipschitz continuity is guarantied to hold for the demand functional $Q(h)$ by virtue of the calculations (76)–(79) appearing in Section 9.1.

Table 4
Weakly monotone increasing path delay.

iteration	$\left\| h^{k+1} - h^k \right\|_{L^2}^2$	$\langle \Psi^1 - \Psi^2, h^1 - h^2 \rangle_{L^2}$	$\geq$	$-1 \cdot K_\Psi \left\| h^{k+1} - h^k \right\|_{L^2}^2$
45	0.2247188923	−0.0127187380	>	−40.449400614
46	0.2202878088	−0.0114198099	>	−39.651805584
47	0.2168996439	−0.0121536923	>	−39.041935902
48	0.2134158305	−0.0180469126	>	−38.41484949
49	0.2110561700	−0.0135039437	>	−37.9901106
50	0.2083248243	−0.0092566072	>	−37.498468374
51	0.2017359652	−0.0104878449	>	−36.312473736
52	0.1968534081	−0.0028041714	>	−35.433613458
53	0.1939490297	0.0326045172	>	−34.910825346
54	0.1887019708	0.0179895322	>	−33.966354744
55	0.1828054340	0.0296159458	>	−32.90497812
⋮	⋮	⋮	⋮	⋮
2995	0.0003519890	0.0000000000	>	−0.06335802
2996	0.0003480617	−0.0000355291	>	−0.062651106
2997	0.0003494352	−0.0000162041	>	−0.062898336
2998	0.0003502586	−0.0000190425	>	−0.063046548
2999	0.0003512823	0.0001507855	>	−0.063230814
3000	0.0003479694	0.0000000000	>	−0.062634492

Table 5
Strongly monotone decreasing inverse demand.

iteration	$\left\|Q^{k+1}-Q^k\right\|_E^2$	$\langle\Theta^1-\Theta^2,Q^1-Q^2\rangle_E$	$\le$	$-1\cdot K_\Theta\left\|Q^{k+1}-Q^k\right\|_E^2$
45	10.6520894759	−0.0053260447	=	−0.0053260447
46	12.8269835301	−0.0064134918	=	−0.0064134918
47	15.1667854323	−0.0075833927	=	−0.0075833927
48	17.9056922848	−0.0089528461	=	−0.0089528461
49	20.2340244245	−0.0101170122	=	−0.0101170122
50	22.7588646616	−0.0113794323	=	−0.0113794323
51	26.1158431516	−0.0130579216	=	−0.0130579216
52	29.5682802229	−0.0147841401	=	−0.0147841401
53	31.3269735684	−0.0156634868	=	−0.0156634868
54	31.6595743201	−0.0158297872	=	−0.0158297872
55	31.8269224961	−0.0159134612	=	−0.0159134612
⋮	⋮	⋮	⋮	⋮
2995	0.0000011411	−0.0000000006	=	−0.0000000006
2996	0.0000243106	−0.0000000122	=	−0.0000000122
2997	0.0000456727	−0.0000000228	=	−0.0000000228
2998	0.0000283339	−0.0000000142	=	−0.0000000142
2999	0.0000460290	−0.0000000230	=	−0.0000000230
3000	0.0000145411	−0.0000000073	=	−0.0000000073

Table 6
Norms and Lipschitz continuity.

iteration	$\left\|\Psi^{k+1}-\Psi^k\right\|_{L^2}^2$	$\le$	$K_1\left\|h^{k+1}-h^k\right\|_{L^2}^2$	$\left\|\Theta^{k+1}-\Theta^k\right\|_E^2$	$\le$	$K_2\left\|Q^{k+1}-Q^k\right\|_E^2$
45	0.9054300600	≤	40.4494006061	0.0000026630	≤	0.0000106521
46	0.8692152192	≤	39.6518055850	0.0000032067	≤	0.0000128270
47	0.8646272586	≤	39.0419359009	0.0000037917	≤	0.0000151668
48	0.8988220638	≤	38.4148494817	0.0000044764	≤	0.0000179057
49	0.9154531800	≤	37.9901106034	0.0000050585	≤	0.0000202340
50	0.8463232350	≤	37.4984683777	0.0000056897	≤	0.0000227589
51	0.8044808940	≤	36.3124737285	0.0000065290	≤	0.0000261158
52	0.7637130018	≤	35.4336134583	0.0000073921	≤	0.0000295683
53	0.8793214308	≤	34.9108253479	0.0000078317	≤	0.0000313270
54	0.8514692964	≤	33.9663547387	0.0000079149	≤	0.0000316596
55	0.8262911124	≤	32.9049781256	0.0000079567	≤	0.0000318269
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2995	0.0000000000	≤	0.0633580197	0.0000000000	≤	0.0000000000
2996	0.0038003526	≤	0.0626511065	0.0000000000	≤	0.0000000000
2997	0.0032272614	≤	0.0628983384	0.0000000000	≤	0.0000000000
2998	0.0043042374	≤	0.0630465564	0.0000000000	≤	0.0000000000
2999	0.0075097260	≤	0.0632308130	0.0000000000	≤	0.0000000000
3000	0.0000000000	≤	0.0626344979	0.0000000000	≤	0.0000000000

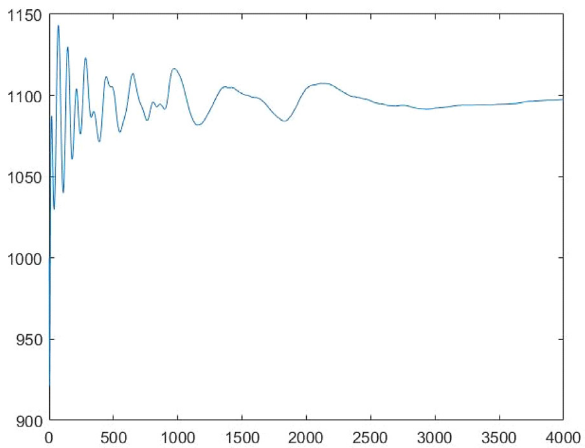

Fig. 3. Convergence in Q.

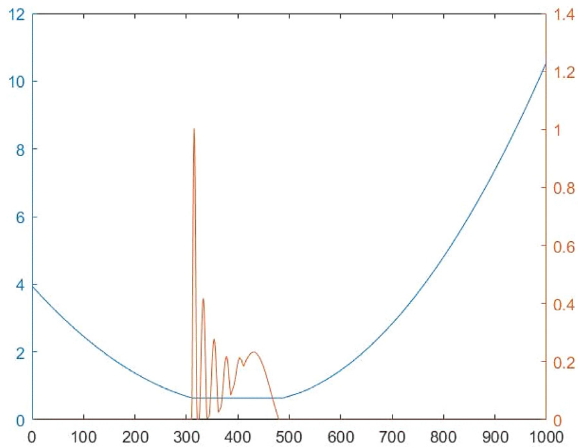

Fig. 4. DUE solution for a single path.

Note also that Fig. 3 presents the iterations converging to the equilibrium demand. Fig. 4 presents an illustrative DUE solution for one particular path. We see that departure rates increase when a local minimum of effective path delay emerges. Note also that in the approximate time interval [310, 475] the departure rate is never zero, although it is at times very small.

10. Conclusions

We have shown that convergence of an elementary class of algorithms (namely, fixed-point algorithms based on the minimum norm projection) for solving DVIs maybe assured when delay operators are weakly monotone increasing, demand fuctions are strongly monotone decreasing, and departure rates are bounded. These restrictions are met when we have appropriate Lipschitz continuity. weak monotonicity of the delay operator, and strongly monotone decreasing inverse demand functions. Other algorithms whose proofs of convergence depend on (strongly) monotone increasing path delay operators might be provably convergent for E-DUE using the ideas we have presented in this paper.

Declaration of Competing Interest

This is a revision of our original submission. The new title is “Convergence of Fixed-Point Algorithms for Elastic Demand Dynamic User Equilibrium.” It is a wholly original work never before published and not under review by any other journal.

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