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JOST SOLUTIONS AND THE DIRECT SCATTERING PROBLEM OF THE BENJAMIN–ONO EQUATION*

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Abstract. In this paper, we present a rigorous study of the direct scattering problem that arises from the complete integrability of the Benjamin–Ono (BO) equation. In particular, we establish existence, uniqueness, and asymptotic properties of the Jost solutions to the scattering operator in the Fokas–Ablowitz inverse scattering transform (IST). Formulas relating different scattering coefficients are proven, together with their asymptotic behavior with respect to the spectral parameter. This work is an initial step toward the construction of general solutions to the BO equation by IST.

Key words. Benjamin–Ono equation, completely integrable, inverse scattering transform, Jost solutions, scattering data

AMS subject classifications. 35A22, 35P25, 35Q53

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1. Introduction.

The Benjamin–Ono (BO) equation may be written as

$$u_t + 2uu_x - Hu_{xx} = 0. \tag{1.1}$$

Here we consider $u = u(x,t)$ a real-valued function of space and time, both one-dimensional, and H is the Hilbert transform defined by

$$Hf(x) = \text{P.V.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy. \tag{1.2}$$

Formulated by Benjamin [2] and Ono [16], the BO equation (1.1) is used to model long internal gravity waves in a two-layer fluid. Typical setup of the models requires the wave amplitudes to be much smaller than the depth of the upper layer, which in turn is small compared with the wavelengths, while the lower layer has infinite depth. See Davis and Acrivos [6], Choi and Camassa [4] and Xu [22] for more details on the derivation of (1.1). One can also observe (see [1]) that the BO equation (1.1) can be formally obtained from the intermediate long wave (ILW) equation by passing to the deep water limit, whereas the shallow water limit of the ILW equation gives the Korteweg–de Vries equation.

The BO equation (1.1) is known to be well-posed for initial data in Sobolev space $H^s(\mathbb{R})$. A large literature was devoted to this problem, and the following is a list of only the major results. Local well-posedness in $H^s(\mathbb{R})$ of (1.1) was obtained by Saut [18] for $s > 3$, Iório [8] for $s > \frac{3}{2}$, Ponce [17] for $s \geq \frac{3}{2}$, Koch and Tzvetkov [12] for $s > \frac{5}{4}$, Kenig and Koenig [11] for $s > \frac{9}{8}$, and Tao [20] for $s \geq 1$. Global well-posedness in $H^s(\mathbb{R})$ of (1.1) was obtained by Saut [18] for smooth solutions, Ponce [17] for $s \geq \frac{3}{2}$, and Tao [20] for $s \geq 1$.

The BO equation (1.1) was also found to be completely integrable. The Lax pair of (1.1) was discovered by Nakamura [15] and Bock and Kruskal [3]. An equivalent

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but formally different Lax pair was presented in Wu [21]. Fokas and Ablowitz [7] formulated the direct and inverse scattering problems for (1.1) and obtained soliton solutions. See also Kaup, Lakoba, and Matsuno [9], Kaup and Matsuno [10], and Xu [22]. As is the case for many other completely integrable equations, one expects to be able to construct solutions to the Cauchy problem of the BO equation using the Fokas–Ablowitz inverse scattering transform (IST). Even though the BO equation is known to be well-posed in $H^s(\mathbb{R})$, a solution by IST makes full use of the integrability structure of the equation and will provide key tools and insights for stability and asymptotic analysis. This plan was carried out by Coifman and Wickerhauser [5] for sufficiently small initial data. It turns out that the Fokas–Ablowitz IST does not behave well enough to be solved by iteration (contraction mapping principle) even under a small potential assumption, so Coifman and Wickerhauser actually used a more complicated regularized IST and solved it by iteration. Miller and Wetzel [13] studied the direct scattering problem of the Fokas–Ablowitz IST when the potential u is a rational function with simple poles and obtained explicit formulas for the scattering data. In [14], they further used these formulas to obtain small-dispersion limits for the scattering data. However, up to the present time, a rigorous analysis of the Fokas–Ablowitz or related IST for general, large potential u is still lacking, and as a result, no rigorous IST solution to the large data Cauchy problem of the BO equation has been proven.

As a first step toward this goal, the author [21] studied the L_u operator in the Lax pair of the BO equation and proved that its discrete spectrum is finite and simple. These are some key spectral assumptions made by Fokas and Ablowitz in their definition of the scattering data of the IST. A few other useful properties about the eigenfunctions were also established.

In this paper, we will examine the full spectrum of the L_u operator and provide a complete study of the direct scattering problem in the Fokas–Ablowitz IST. We will also investigate the asymptotic and regularity properties of the scattering data thus constructed. Such investigations may provide directions to the correct setup and future study of the inverse problem. The paper is organized as follows. We present the essential ingredients of the Fokas–Ablowitz IST in section 2. It will be evident that the central objects of study for the direct scattering problem are certain eigenfunctions of the L_u operator in the Lax pair. These are the so-called Jost solutions (or Jost functions). In section 3, we prove the existence and uniqueness of these Jost solutions. This will provide basis for the construction of the scattering data. As we will see in section 3, what we need to solve are certain Fredholm integral equations, and the main difficulty is to prove a vanishing lemma for the corresponding integral operator. In section 4, we construct the scattering coefficients in the Fokas–Ablowitz IST from the Jost solutions and prove certain important relations between them that are known only on the formal level in the literature. In section 5, we prove asymptotic formulas for the Jost solutions and scattering coefficients as the spectral parameter k approaches 0. These very useful asymptotic formulas obtained formally in [7] and [10] help clarify the global behavior of the scattering coefficients and may provide insight into the study of the inverse scattering problem. The key to proving these formulas is to solve a regularized Fredholm integral equation at $k = 0$, and the crucial difficulty is again to prove a vanishing lemma for a regularized integral operator. In section 6, we prove asymptotic formulas as the spectral parameter k approaches infinity. Finally, we discuss the time evolution of the scattering data in section 7. This point is worth discussing particularly because the operator that is used to define the Jost solutions is actually slightly different from the L_u operator in the Lax pair.

We now set up standard spaces and notation used throughout the paper. The following convention is employed for the Fourier transform and inverse Fourier transform:

$$F(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx, \tag{1.3}$$

$$F^{-1}(f)(x) = \check{f}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi} f(\xi)\, d\xi \tag{1.4}$$

with their usual extension to tempered distributions. The Cauchy projections $C_\pm$ are defined in terms of the Hilbert transform as

$$C_\pm f = \frac{\varphi \pm iHf}{2}. \tag{1.5}$$

In other words, $\widehat{C_\pm f} = \chi_{\mathbb{R}^\pm}\hat{f}$. We denote the L^p Hardy space of the upper half plane by $\mathbb{H}^{p,+}$. More specifically, $f(x) \in \mathbb{H}^{p,+}$ for $1 < p \leq \infty$ if it is the L^p (and almost everywhere) boundary value of an analytic function $F(x+iy)$ for $z = x+iy$ in the upper half plane $\{y > 0\}$, such that $\sup_{y>0}\|F(\cdot + iy)\|_p < \infty$. We denote $\mathbb{H}^{2,+}$ also by $\mathbb{H}^+$. Observe that $C_+(L^2(\mathbb{R})) = \mathbb{H}^+$. We fix the notation for weighted L^p spaces and weighted Sobolev spaces as follows. Let $w(x) = 1 + |x|$ be the weight function. We define for $1 \leq p \leq \infty$ and $s \in \mathbb{R}$

$$L^p_s(\mathbb{R}) = \{f \mid w^s f \in L^p(\mathbb{R})\} \tag{1.6}$$

and

$$H^s_s(\mathbb{R}) = \{f \mid f \in L^2_s, \text{ and } \hat{f} \in L^2_s\} \tag{1.7}$$

with norms $\|f\|_{L^p_s(\mathbb{R})} = \|w^s f\|_p$ and $\|f\|_{H^s_s(\mathbb{R})} = \|w^s f\|_2 + \|w^s \hat{f}\|_2$. We denote the $L^p(\mathbb{R})$ norm by $\|\cdot\|_p$. When doing estimates, we use C to mean a generic constant, whose value may be enlarged from step to step.

2. The Fokas–Ablowitz inverse scattering transform.

Throughout this section, we assume $u(x,t)$ is sufficiently smooth with sufficiently rapid decay in x for each t and present the Fokas–Ablowitz IST formulated in [7]. Since the current paper provides rigorous analysis of the direct scattering problem, we will freely quote results in the later sections when describing the direct problem and take note that the inverse problem calls for more analysis in future works. Since time is frozen when performing the IST, we drop the t dependence of u in the discussion.

We start by recalling the Lax pair of the BO equation (1.1) presented in [21]. There we see that when u is real, as is the case considered in this paper, we only need to take the Lax pair to be operators defined on $\mathbb{H}^+$:

$$L_u\varphi = \frac{1}{i}\varphi_x - C_+(uC_+\varphi), \tag{2.1}$$

$$B_u\varphi = \frac{1}{i}\varphi_{xx} + 2[(C_+u_x)(C_+\varphi) - C_+((uC_+\varphi)_x)]. \tag{2.2}$$

Since C_+ acts as the identity on $\mathbb{H}^+$, we simplify the Lax pair further by dropping the C_+ in $C_+\varphi$ and write

$$L_u\varphi = \frac{1}{i}\varphi_x - C_+(u\varphi), \tag{2.3}$$

$$B_u\varphi = \frac{1}{i}\varphi_{xx} + 2[(C_+u_x)\varphi - C_+((u\varphi)_x)]. \tag{2.4}$$

Notice that (2.3) and (2.4) may still make sense even if φ is not in $\mathbb{H}^+$. For instance, φ could be a function in a weighted L^p space. In fact, we will use (2.3) and (2.4) when defining the Jost solutions. On the other hand, the equivalence of the BO equation with the Lax equation does cling to the particular form (2.1) and (2.2). By dropping C_+ from the equations, we run a potential risk of destroying the equivalence of the BO equation with the Lax equation, when φ is not a function in $\mathbb{H}^+$. We will address this issue in section 7, since its effect comes into play only when time evolution is concerned.

According to [21], L_u given in (2.3), regarded as an operator on $\mathbb{H}^+$, is self-adjoint with finitely many negative simple eigenvalues λ_j, $j = 1, \dots, N$, and $[0, \infty)$ as the essential spectrum. We denote the resolvent set of L_u by $\rho(L_u) = \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_N\} \setminus [0, \infty)$. Now we use results established later in the current paper: by Lemma 3.1 and Theorem 3.5, for each $k \in \rho(L_u)$, there exists a unique Jost solution $m_1(x, k)$ in $L^\infty(\mathbb{R})$ to the equation

$$L_u m_1 = k(m_1 - 1) \tag{2.5}$$

such that $m_1(x, k) - 1 \to 0$ as $x \to \pm\infty$. $m_1(x, k)$ depends analytically on k. Furthermore, as k approaches a positive real λ from above or from below, $m_1(x, k)$ has limits $m_1(x, \lambda \pm 0i) \in L^\infty(\mathbb{R})$. We abbreviate $m_1(x, k)$ as $m_1(k)$ when convenient. Again by Proposition 2.1 and Corollary 2.2 in [21], for each negative simple eigenvalue λ_j, and normalized eigenfunction ϕ_j, there exists a number γ_j such that the Laurent expansion of $m_1(k)$ around λ_j is

$$m_1(k) = -\frac{i}{k - \lambda_j}\phi_j + (x + \gamma_j)\phi_j + (k - \lambda_j)h(k, \lambda_j), \tag{2.6}$$

where $h(k, \lambda_j)$ is analytic in k around λ_j. γ_j is called the phase constant in the literature.

The scattering data of the Fokas–Ablowitz IST consist of the eigenvalues $\{\lambda_j\}_{j=1}^N$, the phase constants $\{\gamma_j\}_{j=1}^N$, and the scattering coefficient

$$\beta(\lambda) = i \int_{\mathbb{R}} u(x) m_1(x, \lambda + 0i) e^{-i\lambda x}\, dx \tag{2.7}$$

for $\lambda > 0$.

The discussion above provides a minimal description of the direct scattering problem. However, to understand the connection to the inverse problem, we need to express the jump of $m_1(k)$ on the positive real line. To accomplish that we introduce another Jost function $m_e(x, \lambda - 0i) \in L^\infty(\mathbb{R})$ which for $\lambda > 0$ solves uniquely

$$L_u m_e = \lambda m_e \tag{2.8}$$

with asymptotic condition $m_e(x, \lambda - 0i) - e^{i\lambda x} \to 0$ as $x \to \infty$. The notation $\lambda - 0i$ in $m_e(x, \lambda - 0i)$ is natural in the integral equation it satisfies. The existence of m_e is established in Theorem 3.5. By Lemmas 4.2 and 4.4,

$$m_1(\lambda + 0i) - m_1(\lambda - 0i) = \beta(\lambda) m_e(\lambda - 0i) \tag{2.9}$$

and

$$\boldsymbol{e}(\lambda)\partial_\lambda(\bar{\boldsymbol{e}}(\lambda) m_e(\lambda - 0i)) = \frac{\overline{\beta(\lambda)}}{2\pi i \lambda} m_1(\lambda - 0i), \tag{2.10}$$

where $\boldsymbol{e}(\lambda) = \boldsymbol{e}(x,\lambda) = e^{i\lambda x}$. By Theorem 5.11,

$$\lim_{\lambda \searrow 0} m_1(\lambda - 0i) = \lim_{\lambda \searrow 0} m_e(\lambda - 0i). \tag{2.11}$$

Denoting the limit above by $m_1(0-0i) = m_e(0-0i)$, we obtain from (2.10)

$$\bar{\boldsymbol{e}}(\lambda) m_e(\lambda - 0i) = m_1(0-0i) + \int_0^\lambda \frac{\overline{\boldsymbol{e}(\mu)\beta(\mu)}}{2\pi i \mu} m_1(\mu - 0i)\, d\mu. \tag{2.12}$$

Note that by the study performed in section 5, for a large class of potential u called generic potentials, $m_1(0-0i)$ is actually equal to 0. Finally, by Theorem 6.5,

$$C_+ u = \lim_{k\to\infty} k(1 - m_1(k)), \tag{2.13}$$

where the limit holds in $L^\infty(\mathbb{R})$ in x.

Summarizing the above discussion, it is natural to cast the inverse scattering problem as follows. Given the negative eigenvalues $\{\lambda_j\}_{j=1}^N$, the phase constants $\{\gamma_j\}_{j=1}^N$ and suitable scattering coefficient $\beta(\lambda)$ for $\lambda > 0$, find an analytic mapping $k \mapsto m_1(k)$ from the resolvent set $\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_N\} \setminus [0,\infty)$ to a suitable function space in x such that

(a) around every λ_j, the Laurent expansion of $m_1(k)$ has the form (2.6) for some function ϕ_j and mapping $h(k,\lambda_j)$ analytic for k close to λ_j;

(b) $m_1(k)$ has limits $m_1(\lambda \pm 0i)$ in suitable function spaces as k approaches the positive real line from above and from below, such that

$$\begin{aligned} &m_1(\lambda + 0i) - m_1(\lambda - 0i) \\ &\quad = \beta(\lambda)\left(\boldsymbol{e}(\lambda) m_1(0-0i) + \int_0^\lambda \frac{\boldsymbol{e}(\lambda-\mu)\overline{\beta(\mu)}}{2\pi i \mu} m_1(\mu - 0i)\, d\mu\right); \end{aligned} \tag{2.14}$$

(c) $m_1(k) \to 1$ as $k \to \infty$.

Once $m_1(x,k)$ is obtained by solving the inverse problem, $u(x)$ may be recovered by

$$u = 2\ \mathrm{Re} \lim_{k\to\infty} k(1 - m_1(k)). \tag{2.15}$$

This completes the formulation of the inverse scattering problem.

The inverse problem is often called a nonlocal Riemann–Hilbert problem. Equation (2.14) is known as the nonlocal jump condition, in comparison with the usual jump condition appearing in a standard Riemann–Hilbert problem, where the integral in (2.14) is replaced by straightforward multiplication.

3. Existence and uniqueness of Jost solutions.

In this section, we solve certain modified eigenvalue equations for the operator $L_u = \frac{1}{i}\partial_x - C_+ u$, with specified asymptotic conditions at $\pm\infty$. These are the Jost solutions that play a central role in the Fokas–Ablowitz IST. They encode properties of the spectrum of L_u, which, according to [21], has the form $\{\lambda_1, \dots, \lambda_N\} \cup \{0\} \cup \mathbb{R}^+$, where $\mathbb{R}^+ = (0,\infty)$.

In the following, two Jost functions $m_1(x,k)$ and $m_e(x, \lambda \pm 0i)$ will be considered. These are solutions to the following equations, with suitable asymptotic conditions at infinity (stated in detail in Lemma 3.1):

$$\frac{1}{i}\partial_x m_1 - C_+(u m_1) = k(m_1 - 1), \tag{3.1}$$

$$\frac{1}{i}\partial_x m_e - C_+(u m_e) = \lambda m_e. \tag{3.2}$$

Here $\lambda \pm 0i \in \mathbb{R}^+ \pm 0i$, and

$$k \in \rho(L_u) \cup (\mathbb{R}^+ \pm 0i) = (\mathbb{C} \setminus \{\lambda_1, \dots \lambda_N\} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i), \tag{3.3}$$

which is the resolvent set glued with two copies of the positive real line. Later on, we will see that $m_1(x,k)$ has limits as k approaches the positive real line from above and below. The notation of $m_1(x,k)$ and $m_e(x,\lambda \pm 0i)$ is adapted to the asymptotic conditions at infinity and may be abbreviated as $m_1(k)$, $m_e(\lambda \pm 0i)$, $m_e(\lambda\pm)$, or simply m_1 and m_e. In [7], a different notation is used. We provide the translation of notation as follows:

$$M(x,\lambda) = m_1(x,\lambda+0i),\ \overline{M}(x,\lambda) = m_e(x,\lambda+0i), \tag{3.4}$$

$$N(x,\lambda) = m_e(x,\lambda-0i),\ \overline{N}(x,\lambda) = m_1(x,\lambda-0i). \tag{3.5}$$

The Jost functions can equivalently be described as solutions to certain Fredholm integral equations. To express these equations, we introduce the convolution kernels

$$G_k(x) = \frac{1}{2\pi}\int_0^\infty \frac{e^{ix\xi}}{\xi - k}\,d\xi \tag{3.6}$$

for $k \in \mathbb{C} \setminus [0,\infty)$ and

$$\widetilde{G}_k(x) = \frac{1}{2\pi}\int_{-\infty}^0 \frac{e^{ix\xi}}{\xi - k}\,d\xi \tag{3.7}$$

for $k \in \mathbb{C} \setminus (-\infty, 0]$. With $\epsilon > 0$, we have

$$\begin{aligned} G_{\lambda\pm i\epsilon}(x) &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{ix\xi}}{\xi - (\lambda \pm i\epsilon)}\,d\xi - \widetilde{G}_{\lambda\pm i\epsilon}(x) \\ &= \pm i e^{\mp\epsilon x} e^{i\lambda x}\chi_{\mathbb{R}^\pm}(x) - \widetilde{G}_{\lambda\pm i\epsilon}(x) \end{aligned} \tag{3.8}$$

with

$$G_{\lambda\pm 0i}(x) = \lim_{\epsilon\searrow 0} G_{\lambda\pm i\epsilon}(x) = \pm i e^{i\lambda x}\chi_{\mathbb{R}^\pm}(x) - \widetilde{G}_\lambda(x) \tag{3.9}$$

for $\lambda > 0$. The limit in (3.9) holds in the following sense: the first term in (3.8) converges pointwise, and the second term in (3.8) converges in $L^{p'}$ for every $p' \in [2,\infty)$. To see the latter, observe that $\widetilde{G}_{\lambda\pm i\epsilon}$ is the inverse Fourier transform of $\frac{\chi_{\mathbb{R}^-}(\xi)}{\xi-(\lambda\pm i\epsilon)}$, which converges to $\frac{\chi_{\mathbb{R}^-}(\xi)}{\xi-\lambda}$ in every L^p for $p \in (1,2]$, assuming $\lambda > 0$.

We are ready to describe the Fredholm integral equations satisfied by the Jost solutions. The following lemma may be considered as the definition for m_1 and m_e.

Lemma 3.1. *Let $p > 1$ and $s > s_1 > 1 - \frac{1}{p}$ be given, and let $u \in L^p_s(\mathbb{R})$. Suppose $m_1(x,k), m_e(x,\lambda\pm 0i) \in L^\infty_{-(s-s_1)}(\mathbb{R})$ for fixed $k \in (\mathbb{C}\setminus[0,\infty))\cup(\mathbb{R}^+ \pm 0i)$ and $\lambda \in \mathbb{R}^+$; then the following are equivalent:*

(a) *$m_1(x,k), m_e(x,\lambda \pm 0i)$ solve*

$$\frac{1}{i}\partial_x m_1 - C_+(um_1) = k(m_1 - 1), \tag{3.10}$$

$$\frac{1}{i}\partial_x m_e - C_+(um_e) = \lambda m_e, \tag{3.11}$$

together with the asymptotic conditions

$$m_1(x,k) - 1 \to 0 \begin{cases} \text{as } |x| \to \infty & \text{if } k \in \mathbb{C} \setminus [0,\infty), \\ \text{as } x \to \mp\infty & \text{if } k = \lambda \pm 0i \in \mathbb{R}^+ \pm 0i, \end{cases} \tag{3.12}$$

$$m_e(x, \lambda \pm 0i) - e^{i\lambda x} \to 0 \quad \text{as } x \to \mp\infty. \tag{3.13}$$

The above asymptotic conditions should be read with either the upper sign or the lower sign.

(b) $m_1(x,k), m_e(x,\lambda \pm 0i)$ *solve the following integral equations:*

$$m_1(x,k) = 1 + G_k * (um_1(\cdot,k))(x), \tag{3.14}$$

$$m_e(x,\lambda \pm 0i) = \boldsymbol{e}(x,\lambda) + G_{\lambda \pm 0i} * (um_e(\cdot, \lambda \pm 0i))(x), \tag{3.15}$$

where $\boldsymbol{e}(x,\lambda)$ *denotes* $e^{i\lambda x}$.

In addition, if either (a) *or* (b) *holds, we have the stronger bounds*

$$m_1(x,k) - 1 \in L^\infty(\mathbb{R}) \cap \mathbb{H}^{p,+} \tag{3.16}$$

for fixed $k \in \mathbb{C} \setminus [0,\infty)$ *and*

$$m_1(x,\lambda \pm 0i), m_e(x, \lambda \pm 0i) \in L^\infty(\mathbb{R}) \tag{3.17}$$

for fixed $\lambda \in \mathbb{R}^+$.

Proof. First of all, we notice from the conditions on u, m_1, and m_e that

$$um_1, um_e \in L^p_{s_1} \subset L^1 \cap L^p.$$

The terms $C_+(um_1)$, $C_+(um_e)$ in (3.10) and (3.11) are well-defined as C_+ is bounded on L^p. To see that the convolution in (3.14) and (3.15) are well-defined and belong to L^∞, we notice by (3.9) that $G_k \in L^{p'}$ if $k \in \mathbb{C} \setminus [0,\infty)$, and $G_{\lambda \pm 0i} \in L^\infty + L^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

We now study $m(x,k)$ for $k \in \mathbb{C} \setminus [0,\infty)$. In this case, we can actually show (3.10) is equivalent to (3.14) without using the asymptotic condition (3.12). To see this, we take the Fourier transform of (3.10) to get

$$\xi \widehat{m_1} - \chi_{\mathbb{R}^+} \widehat{um_1} = k\widehat{m_1} - k\hat{1} \tag{3.18}$$

or

$$\widehat{m_1} = \hat{1} + \frac{\chi_{\mathbb{R}^+}}{\xi - k} \widehat{um_1}. \tag{3.19}$$

Now take the inverse Fourier transform to get (3.14). The convolution formula for the inverse Fourier transform can be justified by the facts that $G_k \in L^2$ and $um_1 \in L^1$. The above calculation can be reversed. Hence (3.14) also implies (3.10). To obtain the limiting condition (3.12) when $k \in \mathbb{C} \setminus [0,\infty)$, we just observe that $\frac{\chi_{\mathbb{R}^+}}{\xi - k}\widehat{um_1} \in L^1$. Equation (3.19) also implies $m_1 - 1 \in \mathbb{H}^{p,+}$. To see this, we apply the Marcinkiewicz multiplier theorem to the multiplier $\mu(\xi) = \frac{\chi_{\mathbb{R}^+}(\xi) e^{-y\xi}}{\xi - k}$ for every $y > 0$. In fact

$$\begin{aligned} \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |\mu'(\xi)| \, d\xi &\le \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left(ye^{-y\xi} \frac{1}{|\xi - k|} + \frac{1}{|\xi - k|^2} \right) d\xi \\ &\le C_k \left(1 + \sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} ye^{-y\xi} \frac{1}{|\xi| + |k|} \, d\xi \right) \\ &\le C_k \left(1 + \sup_{j \in \mathbb{Z}} ye^{-y2^j} \log\left(\frac{2^{j+1} + |k|}{2^j + |k|} \right) \right) \end{aligned}$$

$$\leq C_k \left(1 + \sup_{j \in \mathbb{Z}} y e^{-y2^j} 2^j\right)$$

$$\leq C_k \left(1 + \sup_{y \geq 0} y e^{-y}\right) \leq C_k, \tag{3.20}$$

where C_k is a generic constant depending only on k. Estimate (3.20) implies that the L^p norm of

$$F(x+iy) = \frac{1}{2\pi} \int_0^\infty \frac{e^{i\xi(x+iy)}}{\xi - k} \widehat{um_1}(\xi)\, d\xi \tag{3.21}$$

is uniformly bounded for $y > 0$. On the other hand $F(x+iy)$ converges pointwise to $F(x+i0) = m_1(x,k)$ as $y \searrow 0$ by the dominated convergence theorem. Hence $m_1(x,k) - 1 \in \mathbb{H}^{p,+}$.

We now work on $m_1(x, \lambda \pm 0i)$ and $m_e(x, \lambda \pm 0i)$. To simplify notation, we suppress the x variable and $0i$ and only work on the case with the plus sign. The case with the minus sign can be treated similarly.

We first prove the passage from (3.10) and (3.12) to (3.14). In fact, the Fourier transform of (3.10) gives

$$\xi \widehat{m_1(\lambda+)} = \lambda \widehat{m_1(\lambda+)} - \lambda \widehat{1} + \chi_{\mathbb{R}^+} F(um_1(\lambda+)). \tag{3.22}$$

For every $\epsilon > 0$, we divide by $\xi - (\lambda + i\epsilon)$ to get

$$\widehat{m_1(\lambda+)} = -\frac{i\epsilon}{\xi - (\lambda+i\epsilon)} F(m_1(\lambda+) - 1) - \frac{\lambda + i\epsilon}{\xi - (\lambda + i\epsilon)} \widehat{1} + \frac{1}{\xi - (\lambda+i\epsilon)} \chi_{\mathbb{R}^+} F(um_1(\lambda+)). \tag{3.23}$$

Since $\widehat{1}$ is a multiple of δ, we have

$$-\frac{\lambda + i\epsilon}{\xi - (\lambda+i\epsilon)} \widehat{1} = \widehat{1}. \tag{3.24}$$

Now inverse Fourier transform (3.23) to get

$$m_1(\lambda+) = F^{-1}\left(-\frac{i\epsilon}{\xi - (\lambda+i\epsilon)} F(m_1(\lambda+) - 1)\right) + 1 + G_{\lambda+i\epsilon} * (um_1(\lambda+)). \tag{3.25}$$

By the decomposition (3.9), and the dominated convergence theorem,

$$\lim_{\epsilon \searrow 0} G_{\lambda+i\epsilon} * (um_1(\lambda+)) = G_{\lambda+0i} * (um_1(\lambda+)) \tag{3.26}$$

pointwise. Since

$$F^{-1}\left(-\frac{i\epsilon}{\xi - (\lambda+i\epsilon)}\right) = \epsilon \chi_{\mathbb{R}^+}(x) e^{i(\lambda+i\epsilon)x}, \tag{3.27}$$

we have

$$F^{-1}\left(-\frac{i\epsilon}{\xi - (\lambda+i\epsilon)} F(m_1(\lambda+) - 1)\right) = \epsilon \chi_{\mathbb{R}^+}(x) e^{i(\lambda+i\epsilon)x} * (m_1(\lambda+) - 1). \tag{3.28}$$

Equation (3.28) can be justified as follows. First of all, it is true if $m_1(\lambda+) - 1$ is replaced by a rapidly decaying function. By the conditions on $m_1(\lambda+)$, it is obvious that $w^{-2-(s-s_1)}(m_1(\lambda+) - 1) \in L^1$. Approximate $w^{-2-(s-s_1)}(m_1(\lambda+) - 1)$ in L^1 by a sequence of Schwartz class functions f_n, and take the limit as $n \to \infty$ of

$$F^{-1}\left(-\frac{i\epsilon}{\xi - (\lambda + i\epsilon)}\widehat{g_n}\right) = \epsilon\chi_{\mathbb{R}^+}(x)e^{i(\lambda+i\epsilon)x} * g_n(x), \tag{3.29}$$

where $g_n = w^{2+s-s_1}f_n$. The left-hand side of (3.29) converges as tempered distributions to the left-hand side of (3.28). To study convergence of the right-hand side, observe that

$$\begin{aligned} |\chi_{\mathbb{R}^+}(x)e^{i(\lambda+i\epsilon)x} * g_n| &= \left|\int_{-\infty}^{x} e^{-\epsilon(x-y)}e^{i\lambda(x-y)}g_n(y)\, dy\right| \\ &\le e^{-\epsilon x}(\sup_{y\le x} e^{\epsilon y}[w(y)]^{2+s-s_1})\|w^{-2-(s-s_1)}g_n\|_{L^1}. \end{aligned} \tag{3.30}$$

It follows that the right-hand side of (3.29) converges locally uniformly to the right-hand side of (3.28). Thus (3.28) holds. Now

$$\begin{aligned} &\epsilon\chi_{\mathbb{R}^+}(x)e^{i(\lambda+i\epsilon)x} * (m_1(\lambda+) - 1) \\ &\quad= \epsilon\int_0^\infty e^{i(\lambda+i\epsilon)y}(m_1(\lambda+) - 1)(x-y)\, dy \\ &\quad= \int_0^\infty e^{i\lambda y/\epsilon}e^{-y}(m_1(\lambda+) - 1)\left(x - \frac{y}{\epsilon}\right)\, dy. \end{aligned} \tag{3.31}$$

We take the limit as $\epsilon \searrow 0$ of (3.31). By (3.12), $(m_1(\lambda+) - 1)$ is bounded on $(-\infty, x]$ and approaches 0 as $x \to -\infty$, hence (3.31) tends to 0 for every x by the dominated convergence theorem. By (3.26), (3.28), and the above discussion about (3.31), the right-hand side of (3.25) tends to the right-hand side of (3.14) as $\epsilon \searrow 0$.

We can work similarly on $m_e(\lambda+)$. In this case, (3.25) is replaced by

$$m_e(\lambda+) = F^{-1}\left(-\frac{i\epsilon}{\xi - (\lambda + i\epsilon)}\widehat{m_e(\lambda+)}\right) + G_{\lambda+i\epsilon} * (um_e(\lambda+)). \tag{3.32}$$

Again,

$$\lim_{\epsilon\searrow 0} G_{\lambda+i\epsilon} * (um_e(\lambda+)) = G_{\lambda+i0} * (um_e(\lambda+)) \tag{3.33}$$

pointwise, and $F^{-1}(-\frac{i\epsilon}{\xi-(\lambda+i\epsilon)}\widehat{m_e(\lambda+)})$ equals

$$\begin{aligned} &\epsilon\chi_{\mathbb{R}^+}(x)e^{i(\lambda+i\epsilon)x} * m_e(\lambda+) \\ &\quad= \epsilon\int_{-\infty}^{x} e^{i(\lambda+i\epsilon)(x-y)}m_e(y, \lambda+)\, dy \\ &\quad= e^{i(\lambda+i\epsilon)x}\epsilon\int_{-\infty}^{x} e^{\epsilon y}e^{-i\lambda y}m_e(y, \lambda+)\, dy \\ &\quad= e^{i(\lambda+i\epsilon)x}\int_{-\infty}^{\epsilon x} e^{y}e^{-i\lambda y/\epsilon}m_e(y/\epsilon, \lambda+)\, dy. \end{aligned} \tag{3.34}$$

By (3.13), $m_e(x, \lambda+)$ is bounded on $(-\infty, |x|]$, and $e^{-i\lambda x}m_e(x, \lambda+) \to 1$ as $x \to -\infty$; therefore (3.34) tends to $e^{i\lambda x}$ for every x by the dominated convergence theorem. Equation (3.15) then follows as above.

We now prove that (3.14) implies (3.10) and (3.12). By the decomposition (3.9), we can write (3.14) as

$$m_1(\lambda+) = 1 + ie^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda y}u(y)m_1(y,\lambda+)\,dy - \widetilde{G}_\lambda * (um_1(\lambda+)). \tag{3.35}$$

Weakly differentiate (3.35) to get

$$\begin{aligned}\frac{1}{i}\partial_x m_1(\lambda+) &= i\lambda e^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda y}u(y)m_1(y,\lambda+)\,dy + um_1(\lambda+)\\ &\quad - \frac{1}{i}\partial_x\widetilde{G}_\lambda * (um_1(\lambda+)).\end{aligned} \tag{3.36}$$

To compute $\frac{1}{i}\partial_x\widetilde{G}_\lambda * (um_1(\lambda+))$, we take its Fourier transform

$$\begin{aligned}F\left(\frac{1}{i}\partial_x\widetilde{G}_\lambda * (um_1(\lambda+))\right) &= \xi\frac{\chi_{\mathbb{R}^-}}{\xi-\lambda}F(um_1(\lambda+))\\ &= \chi_{\mathbb{R}^-}F(um_1(\lambda+)) + \lambda\frac{\chi_{\mathbb{R}^-}}{\xi-\lambda}F(um_1(\lambda+))\\ &= F[C_-(um_1(\lambda+)) + \lambda\widetilde{G}_\lambda * (um_1(\lambda+))].\end{aligned} \tag{3.37}$$

All of the above steps can be justified using the fact that $um_1(\lambda+) \in L^p$. It follows that

$$\frac{1}{i}\partial_x\widetilde{G}_\lambda * (um_1(\lambda+)) = C_-(um_1(\lambda+)) + \lambda\widetilde{G}_\lambda * (um_1(\lambda+)). \tag{3.38}$$

Equation (3.36) thus gives

$$\begin{aligned}\frac{1}{i}\partial_x m_1(\lambda+) &= i\lambda e^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda y}um_1(y,\lambda+)\,dy + um_1(\lambda+) - C_-(um_1(\lambda+))\\ &\quad - \lambda\widetilde{G}_\lambda * (um_1(\lambda+))\\ &= C_+(um_1(\lambda+)) + \lambda G_{\lambda+0i} * (um_1(\lambda+))\\ &= C_+(um_1(\lambda+)) + \lambda(m_1(\lambda+) - 1).\end{aligned} \tag{3.39}$$

To get the last step, we have used (3.14) again. This proves (3.10). To get (3.12), we take the limit of (3.35) as $x \to -\infty$. It suffices to show

$$\lim_{x\to-\infty}\widetilde{G}_\lambda * (um_1(\lambda+))(x) = 0. \tag{3.40}$$

To see this, we write $\widetilde{G}_\lambda * (um_1(\lambda+))$ using the Fourier inversion formula as

$$\widetilde{G}_\lambda * (um_1(\lambda+)) = F^{-1}(F(\widetilde{G}_\lambda * (um_1(\lambda+)))) = F^{-1}\left(\frac{\chi_{\mathbb{R}^-}}{\xi-\lambda}F(um_1(\lambda+))\right). \tag{3.41}$$

Recall that $\frac{\chi_{\mathbb{R}^-}(\xi)}{\xi-\lambda}F(um_1(\lambda+)) \in L^1$, since $F(um_1(\lambda+)) \in L^{p'}$ by the Hausdorff–Young inequality, and $\frac{\chi_{\mathbb{R}^-}}{\xi-\lambda} \in L^p$. Thus (3.40) follows by the Riemann–Lebesgue lemma. That (3.15) implies (3.11) and (3.13) can be obtained in a similar way. ∎

To describe the Fredholm nature of the integral equations (3.14) and (3.15), define

$$T_k\varphi = G_k * (u\varphi) \tag{3.42}$$

for $k \in (\mathbb{C}\setminus[0,\infty))\cup(\mathbb{R}^+ \pm 0i)$. The integral equations (3.14) and (3.15) are of the form $(I - T_k)\varphi = g$ where $g \in L^\infty$. The existence and uniqueness of Jost solutions follow from the invertibility of $I - T_k$ on suitable spaces. In the following, we first prove that T_k are compact on certain weighted L^∞ spaces. Thus the Fredholm alternative theorem will reduce the question to a vanishing lemma.

LEMMA 3.2. *Let $p > 1$ and $s > s_1 > 1 - \frac{1}{p}$ be given, and let $u \in L^p_s(\mathbb{R})$. Then $T_k : L^\infty_{-(s-s_1)}(\mathbb{R}) \to L^\infty_{-(s-s_1)}(\mathbb{R})$ is compact for every $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$.*

Proof. We only provide an argument for $T_{\lambda+0i}$. The cases $T_{\lambda-0i}$ and T_k for $k \in \mathbb{C}\setminus[0,\infty)$ can be obtained analogously. Let $\{\varphi_n\}$ be a bounded sequence in $L^\infty_{-(s-s_1)}$. Recall from the proof of Lemma 3.1 that $G_{\lambda+0i} \in L^\infty + L^{p'}$, and $u\varphi_n \in L^1 \cap L^p$ with suitable estimates. Hence there exists $C_1 = C_1(u, \lambda, p, s_1)$ such that

$$\|T_{\lambda+0i}\varphi_n\|_\infty \le C_1\|\varphi_n\|_{L^\infty_{-(s-s_1)}}. \tag{3.43}$$

Also, one can compute the weak derivative of $T_{\lambda+0i}\varphi_n$ as in (3.39) to get

$$\frac{1}{i}\partial_x T_{\lambda+0i}\varphi_n = i\lambda e^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda y}u\varphi_n(y)\,dy + u\varphi_n^- - C_-(u\varphi_n) - \lambda\widetilde{G}_\lambda * (u\varphi_n). \tag{3.44}$$

The four terms above are in L^∞, L^p, L^p, and L^∞, respectively. As a consequence, for every natural number N, there exists $C_2 = C_2(u, \lambda, p, s_1, N)$ such that

$$\|\partial_x T_{\lambda+0i}\varphi_n\|_{L^p(-N,N)} \le C_2\|\varphi_n\|_{L^\infty_{-(s-s_1)}}. \tag{3.45}$$

From (3.43) and (3.45) we conclude that there exists $C = C(u, \lambda, p, s_1, N)$ such that

$$\|T_{\lambda+0i}\varphi_n\|_{W^{1,p}(-N,N)} \le C\|\varphi_n\|_{L^\infty_{-(s-s_1)}}. \tag{3.46}$$

By the Sobolev embedding thoerem, the sequence $\{T_{\lambda+0i}\varphi_n\}$ is uniformly bounded in every $C^{0,\frac{p-1}{p}}[-N,N]$, which is compactly embedded in $C^0[-N,N]$. By passing to a subsequence and a Cantor diagonal argument, we can assume that $\{T_{\lambda+0i}\varphi_n\}$ converges uniformly on any compact subset of $\mathbb{R}$ to a continuous function f. Obviously

$$\|f\|_\infty \le \sup_n \|T_{\lambda+0i}\varphi_n\|_\infty \le C_1 \sup_n \|\varphi_n\|_{L^\infty_{-(s-s_1)}}. \tag{3.47}$$

Hence $f \in L^\infty_{-(s-s_1)}$. For any $\epsilon > 0$, choose N large enough so that

$$w^{s_1-s}(x)C_1 \sup_n \|\varphi_n\|_{L^\infty_{-(s-s_1)}} < \frac{\epsilon}{2}$$

for $|x| > N$. We then have

$$w^{s_1-s}(x)|f(x) - T_{\lambda+0i}\varphi_n(x)| < \epsilon$$

for $|x| > N$. For $|x| \le N$, $T_{\lambda+0i}\varphi_n$ converges uniformly to f, and we obviously have

$$w^{s_1-s}(x)|f(x) - T_{\lambda+0i}\varphi_n(x)| < \epsilon$$

for n sufficiently large. Therefore $T_{\lambda+0i}\varphi_n$ converges to f in $L^\infty_{-(s-s_1)}$. ∎

By the Fredholm alternative theorem, what is left to show is that $(I - T_k)\varphi = 0$ and $\varphi \in L^\infty_{-(s-s_1)}$ imply $\varphi = 0$. We accomplish this in two steps. First we prove that, in suitable spaces, any such function φ must be in L^2. After that an L^2 vanishing lemma will close the argument. In fact, we can prove the following decay estimate for functions in the kernel of $I - T_k$.

LEMMA 3.3. *Suppose* $s > s_1 > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$, *and* $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$. *If* $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$ *and* $\varphi = T_k\varphi$, *then there exists* $C = C(u, k, s, s_1)$ *and* $r = r(s) > \frac{1}{2}$ *such that*

$$|\varphi(x)| \le C[w(x)]^{-r}. \tag{3.48}$$

In particular, $\varphi \in L^2(\mathbb{R})$.

Proof. We first assume $k \in \mathbb{C} \setminus [0,\infty)$. In this case we have

$$\varphi(x) = \int_{\mathbb{R}} G_k(x-y)u\varphi(y)\, dy, \tag{3.49}$$

where $G_k \in L^2$. As before, the conditions on u and φ imply $u\varphi \in L^1 \cap L^2$. Hence $\varphi = G_k * (u\varphi) \in L^\infty$. To prove the decay estimate for $|x|$ large, we split the integral in (3.49) into two pieces: one on $\{|y-x| \le \frac{|x|}{2}\}$ and the other on $\{|y-x| > \frac{|x|}{2}\}$. For the former, we have

$$\begin{aligned}
&\left|\int_{|y-x|\le\frac{|x|}{2}} G_k(x-y)u\varphi(y)\, dy\right| \\
&\quad\le C\|G_k\|_2\|\varphi\|_\infty \left(\int_{|y-x|\le\frac{|x|}{2}} w^{-2s}(y)w^{2s}(y)u^2(y)\, dy\right)^{\frac{1}{2}} \\
&\quad\le C\|G_k\|_2 w^{-s}\left(\frac{|x|}{2}\right)\|u\|_{L^2_s} \le C|x|^{-s}.
\end{aligned} \tag{3.50}$$

To estimate the other piece, we use the pointwise bound $|G_k(x)| \le \frac{C}{|x|}$ for $|x| > 1$, which is an easy consequence of (3.6) from integration by parts. Therefore

$$\left|\int_{|y-x|>\frac{|x|}{2}} G_k(x-y)u\varphi(y)\, dy\right| \le C\frac{1}{|x|}\|u\varphi\|_1 \le \frac{C}{|x|}\|u\|_{L^2_s}\|\varphi\|_\infty. \tag{3.51}$$

This completes the proof when $k \in \mathbb{C} \setminus [0,\infty)$. Next, we study the case when $k \in \mathbb{R}^+ \pm 0i$. Again, for simplicity, we work on $T_{\lambda+0i}$ only. Using (3.9), we have

$$\varphi(x) = T_{\lambda+0i}\varphi(x) = ie^{i\lambda x}\int_{-\infty}^{x} e^{-i\lambda y}u\varphi(y)\, dy - \int_{\mathbb{R}} \widetilde{G}_\lambda(x-y)u\varphi(y)\, dy. \tag{3.52}$$

By the same reason as above, $u\varphi \in L^1 \cap L^2$, so we get $\varphi \in L^\infty$. The decay estimate for the last term in (3.52) can be proved in a similar way as above, as $\widetilde{G}_\lambda \in L^2$. We now prove the decay estimate on the first integral in (3.52). To that end, we need the crucial identity (3.54), which follows from an integration calculation detailed in Lemma 4.3. Using that lemma, we have

$$\begin{aligned}
\langle\varphi, u\varphi\rangle &= \langle T_{\lambda+0i}\varphi, u\varphi\rangle \\
&= \langle G_{\lambda+0i} * u\varphi, u\varphi\rangle = i|\langle u\varphi, \boldsymbol{e}\rangle|^2 + \langle u\varphi, G_{\lambda+0i} * u\varphi\rangle \\
&= i|\langle u\varphi, \boldsymbol{e}\rangle|^2 + \langle u\varphi, T_{\lambda+0i}\varphi\rangle \\
&= i|\langle u\varphi, \boldsymbol{e}\rangle|^2 + \langle u\varphi, \varphi\rangle.
\end{aligned} \tag{3.53}$$

Therefore $\langle u\varphi, \boldsymbol{e}\rangle = 0$, or

$$\int_{\mathbb{R}} e^{-i\lambda x} u\varphi(x)\ dx = 0. \tag{3.54}$$

Using (3.54) on (3.52), we have

$$\varphi(x) = T_{\lambda+0i}\varphi(x) = i\int_{-\infty}^{x} e^{i\lambda(x-y)} u\varphi(y)\ dy - \widetilde{G}_\lambda * (u\varphi)(x) \tag{3.55}$$

$$= -i\int_{x}^{\infty} e^{i\lambda(x-y)} u\varphi(y)\ dy - \widetilde{G}_\lambda * (u\varphi)(x). \tag{3.56}$$

Denote

$$I(x) = i\int_{-\infty}^{x} e^{i\lambda(x-y)} u\varphi(y)\ dy = -i\int_{x}^{\infty} e^{i\lambda(x-y)} u\varphi(y)\ dy. \tag{3.57}$$

We want to show that $I(x)$ has $\frac{1}{|x|^r}$ decay at infinity for some $r > \frac{1}{2}$. Let us now use the first expression in (3.57) to study the decay of $I(x)$ as x tends to $-\infty$. Since $\varphi(x) = I(x) - \widetilde{G}_\lambda * (u\varphi)(x)$, we have

$$I(x) = i\int_{-\infty}^{x} e^{i\lambda(x-y)} u\varphi(y)\ dy = i\int_{-\infty}^{x} e^{i\lambda(x-y)} u(y)[I(y) - \widetilde{G}_\lambda * (u\varphi)(y)]\ dy. \tag{3.58}$$

Since $\widetilde{G}_\lambda * (u\varphi) \in L^2$ and $u \in L^2_s$, we get $u\widetilde{G}_\lambda * (u\varphi) \in L^1_s$. Thus

$$\left|\int_{-\infty}^{x} e^{i\lambda(x-y)} u(y)\widetilde{G}_\lambda * (u\varphi)(y)\ dy\right| \le C\|w^s u\widetilde{G}_\lambda * (u\varphi)\|_1 w^{-s}(x) \le Cw^{-s}(x) \tag{3.59}$$

for $x < 0$, as $w(y) > w(x)$ for $y < x < 0$. Since $s > \frac{1}{2}$, (3.59) already has the required decay as x tends to $-\infty$. We next use (3.58) to bootstrap decay estimates on $I(x)$. Recall that $I(x) \in L^\infty$. Suppose $I(x) \in L^\infty_r(-\infty, 0]$ for some $r \in [0, \frac{1}{2}]$. We have

$$\begin{aligned}
\int_{-\infty}^{x} |u(y)I(y)|\ dy &\le C\int_{-\infty}^{x} w^{-s-r}(y)|w^s u(y)|\ dy \\
&= C\int_{-\infty}^{x} w^{-r-\frac{1}{2}(s-\frac{1}{2})}(y) w^{-\frac{1}{2}(s+\frac{1}{2})}(y)|w^s u(y)|\ dy \\
&\le Cw^{-r-\frac{1}{2}(s-\frac{1}{2})}(x)\int_{\mathbb{R}} w^{-\frac{1}{2}(s+\frac{1}{2})}(y)|w^s u(y)|\ dy \\
&\le Cw^{-(r+\frac{1}{2}(s-\frac{1}{2}))}(x)\|w^{-\frac{1}{2}(s+\frac{1}{2})}\|_2\|u\|_{L^2_s}
\end{aligned} \tag{3.60}$$

for $x < 0$. By (3.58), (3.59), and (3.60), we get $I(x) \in L^\infty_{r+\frac{1}{2}(s-\frac{1}{2})}(-\infty, 0]$, which has a little more decay than what we started with. Finitely many iterations of this argument will bring the decay exponent r above $\frac{1}{2}$. A similar argument using the second expression in (3.57) shows that $I(x)$ has the required decay as x tends to ∞. ∎

The next result is the L^2 vanishing lemma alluded to in the previous discussion. It provides the key step for the proof of invertibility of $I - T_k$. It means, among other implications, that there is no embedded eigenvalues in the essential spectrum of L_u.

LEMMA 3.4. *Suppose* $s > s_1 > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$ *and* $k \in (\mathbb{C} \setminus [0, \infty)) \cup (\mathbb{R}^+ \pm 0i)$. *If* $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$ *and* $\varphi = T_k\varphi$, *then*

1. *If* $k \in \mathbb{C} \setminus \mathbb{R}$, $\varphi = 0$.
2. *If* $k \in (\mathbb{R} \setminus [0, \infty)) \cup (\mathbb{R}^+ \pm 0i)$,

$$\left| \int_{\mathbb{R}} u\varphi \; dx \right|^2 = -2\pi k \int_{\mathbb{R}} |\varphi|^2 \; dx. \tag{3.61}$$

In particular, $\varphi = 0$ *if* $k \in \mathbb{R}^+ \pm 0i$, *or if* k *is in the resolvent set of* $L_u = \frac{1}{i}\partial_x - C_+ u C_+$, *regarded as an operator on* $\mathbb{H}^+$.

Identity (3.61) is reminiscent of Lemma 2.5 in [21]. However, the proof of (3.61) is much more delicate when $k = \lambda \pm 0i$, as $\lambda > 0$ may introduce a singularity to $\hat{\varphi}$ and in particular make it nondifferentiable at λ.

Proof. Let us first assume $k \in \mathbb{C} \backslash \mathbb{R}$. By the same proof as in Lemma 3.1, $\varphi = T_k \varphi$ implies

$$\frac{1}{i}\partial_x \varphi - C_+(u\varphi) = k\varphi \tag{3.62}$$

and

$$\chi_{\mathbb{R}^+} \widehat{u\varphi} = (\xi - k)\hat{\varphi}. \tag{3.63}$$

Using (3.63) with the fact that $\varphi \in L^2$, proved in Lemma 3.3, we have $\varphi \in \mathbb{H}^+$. Thus $C_+\varphi = \varphi$. Multiply (3.62) by $\bar{\varphi}$ and take the imaginary part to get

$$-\frac{1}{2}|\varphi|_x^2 - \mathrm{Im}\; (C_+(u\varphi)\bar{\varphi}) = (\mathrm{Im}\; k)|\varphi|^2. \tag{3.64}$$

Integrating (3.64) and using the decay estimate (3.48) on φ, we get

$$0 = -\mathrm{Im} \int_{\mathbb{R}} u|\varphi|^2 \; dx = -\mathrm{Im} \int_{\mathbb{R}} C_+(u\varphi)\bar{\varphi} \; dx = (\mathrm{Im}\; k) \int_{\mathbb{R}} |\varphi|^2 \; dx. \tag{3.65}$$

To get the middle equality, we used the self-adjointness of C_+ and $C_+\varphi = \varphi$. Equation (3.65) implies $\varphi = 0$, as $k \in \mathbb{C} \setminus \mathbb{R}$.

Next, if $k \in \mathbb{R} \backslash [0, \infty)$, we obtain (3.62), (3.63), and $\varphi \in \mathbb{H}^+$ as before. In addition, Lemma 3.3 gives $\varphi \in L_r^\infty$ for some $r > \frac{1}{2}$. This together with the condition $u \in L_s^2$ for $s > \frac{1}{2}$ implies $xu\varphi \in L^2$. Therefore $\widehat{u\varphi} \in H^1(\mathbb{R})$. By the Sobolev embedding theorem, $\widehat{u\varphi}$ is continuous, and by (3.63) so is $\hat{\varphi}$ on $[0, \infty)$. Weakly differentiate (3.63) to get

$$\widehat{u\varphi}' = \hat{\varphi} + (\xi - k)\hat{\varphi}' \tag{3.66}$$

when $\xi > 0$. Multiply by $\overline{\hat{\varphi}}$ and take the real part to get

$$\mathrm{Re}\; (\widehat{u\varphi}'\overline{\hat{\varphi}}) = |\hat{\varphi}|^2 + (\xi - k)\mathrm{Re}\; (\varphi'\overline{\hat{\varphi}}) \tag{3.67}$$

or

$$\mathrm{Re}\; (\widehat{u\varphi}'\overline{\hat{\varphi}}) = |\hat{\varphi}|^2 + \frac{\xi - k}{2}(|\hat{\varphi}|^2)'. \tag{3.68}$$

Now integrate between 0 and ∞ to get

$$\mathrm{Re} \int_{\mathbb{R}} \widehat{u\varphi}'\overline{\hat{\varphi}} \; d\xi = \int_{\mathbb{R}} |\hat{\varphi}|^2 \; d\xi + \frac{k}{2}|\hat{\varphi}|^2(0+) - \frac{1}{2}\int_{\mathbb{R}} |\hat{\varphi}|^2 \; d\xi. \tag{3.69}$$

To obtain (3.69), we took the freedom to rewrite the integration domain as $\mathbb{R}$ whenever the integral involves $\hat{\varphi}$, a function supported on $[0,\infty)$, and have integrated the last term by part. To compute the boundary term for that step, we used the fact that $\lim_{\xi\to\infty}(\xi-k)|\hat{\varphi}|^2=0$, which is a consequence of (3.63) and the fact that $u\varphi\in L^1$. Now observe that the integral on the left-hand side of (3.69) is purely imaginary, by the Plancherel identity:

$$\int_{\mathbb{R}} \widehat{u\varphi}'\overline{\hat{\varphi}}\, d\xi = -2\pi i\int_{\mathbb{R}} xu|\varphi|^2\, dx. \tag{3.70}$$

Hence (3.69) gives

$$2\pi\int_{\mathbb{R}}|\varphi|^2\, dx = \int_{\mathbb{R}}|\hat{\varphi}|^2\, d\xi = -k|\hat{\varphi}|^2(0+). \tag{3.71}$$

Using (3.63) to write $-k\hat{\varphi}(0+)=\widehat{u\varphi}(0)$, (3.61) follows.

Finally, assume $k=\lambda\pm 0i$, where $\lambda\in\mathbb{R}^+$. Checking the signs of both sides of (3.61), one easily sees that $\varphi=0$ is the only way to avoid a contradiction. Therefore the key is to prove (3.61). By the same proof as in Lemma 3.1, $\varphi=T_{\lambda\pm 0i}\varphi$ implies

$$\frac{1}{i}\partial_x\varphi - C_+(u\varphi)=\lambda\varphi. \tag{3.72}$$

The Fourier transform of (3.72) gives

$$\widehat{u\varphi}\chi_{\mathbb{R}^+}=(\xi-\lambda)\hat{\varphi}. \tag{3.73}$$

This implies that φ has its frequencies supported on $\mathbb{R}^+$, and thus belongs to $\mathbb{H}^+$. Let ψ_n^2 be a smooth partition of unity on $(0,\infty)$:

$$\chi_{(0,\infty)}(\xi)=\sum_n\psi_n^2(\xi). \tag{3.74}$$

Here the ψ_n's are compactly supported smooth functions on $(0,\infty)$. An easy way to construct them is to make dyadic dilations of a fixed function. Let $P_n=F^{-1}\psi_nF$ be the Littlewood–Paley type projection associated with ψ_n. Letting P_n act on (3.72), we have

$$\frac{1}{i}(P_n\varphi)_x - P_n(u\varphi)=\lambda P_n\varphi. \tag{3.75}$$

Multiply by $ix\overline{P_n\varphi}$ and take the real part to get

$$\operatorname{Re}\, x(P_n\varphi)_x\overline{P_n\varphi}+\operatorname{Im}\, xP_n(u\varphi)\overline{P_n\varphi}=0 \tag{3.76}$$

or

$$\frac{1}{2}x\left(|P_n\varphi|^2\right)_x+\operatorname{Im}\, xP_n(u\varphi)\overline{P_n\varphi}=0. \tag{3.77}$$

We claim that $xP_n(u\varphi)\in L^2$. Indeed,

$$\begin{aligned} xP_n(u\varphi)(x) &= x\int_{\mathbb{R}}\check{\psi}_n(x-y)u(y)\varphi(y)\, dy\\ &= \int_{\mathbb{R}}(x-y)\check{\psi}_n(x-y)u(y)\varphi(y)\, dy+\int_{\mathbb{R}}\check{\psi}_n(x-y)yu(y)\varphi(y)\, dy\\ &= (x\check{\psi}_n)*(u\varphi)+\check{\psi}_n*(xu\varphi). \end{aligned} \tag{3.78}$$

Since $x\check{\psi}_n$ is in the Schwartz class, and the conditions on u and φ imply $u\varphi \in L^2$, we conclude that the first term in (3.78) is in L^2. The second term is also in L^2 because $xu\varphi$ is, as shown above. Integrate (3.77) on $\mathbb{R}$ and use the Plancherel identity on the last term to get

$$\frac{1}{2}x|P_n\varphi(x)|^2\Big|_{-\infty}^{\infty} - \frac{1}{2}\int_{\mathbb{R}} |P_n\varphi|^2 \, dx + \frac{1}{2\pi}\text{Re} \int_0^\infty (\psi_n \widehat{u\varphi})' \psi_n \bar{\hat{\varphi}} \, d\xi = 0. \tag{3.79}$$

We claim that the first term in (3.79) vanishes. In fact,

$$\begin{aligned}|P_n\varphi(x)| &= \left|\int_{\mathbb{R}} \check{\psi}_n(x-y)\varphi(y) \, dy\right| \\ &\le \int_{|y-x|\le \frac{|x|}{2}} |\check{\psi}_n(x-y)\varphi(y)| \, dy + \int_{|y-x|> \frac{|x|}{2}} |\check{\psi}_n(x-y)\varphi(y)| \, dy \\ &\le \sup_{\frac{|x|}{2}\le|y|\le\frac{3|x|}{2}} |\varphi(y)| \|\check{\psi}_n\|_1 + \|\varphi\|_\infty \int_{|y|>\frac{|x|}{2}} |\check{\psi}_n(y)| \, dy \\ &\le C|x|^{-r} + C|x|^{-1}\end{aligned}$$

when $|x|$ is large. The last inequality above follows from estimate (3.48) and the fact that $\check{\psi}_n$ is in the Schwartz class. We can now rewrite (3.79) as

$$-\frac{1}{2}\int_{\mathbb{R}} |P_n\varphi|^2 \, dx + \frac{1}{2\pi}\text{Re} \int_0^\infty \left(\left(\frac{\psi_n^2}{2}\right)' \widehat{u\varphi}\bar{\hat{\varphi}} + \psi_n^2 \widehat{u\varphi}' \bar{\hat{\varphi}}\right) \, d\xi = 0. \tag{3.80}$$

Take the sum over n to get

$$-\frac{1}{2}\int_{\mathbb{R}} |\varphi|^2 \, dx + \frac{1}{2\pi}\text{Re} \sum_n \int_0^\infty \left(\frac{\psi_n^2}{2}\right)' \widehat{u\varphi}\bar{\hat{\varphi}} \, d\xi + \frac{1}{2\pi}\text{Re} \int_{\mathbb{R}} \widehat{u\varphi}' \bar{\hat{\varphi}} \, d\xi = 0. \tag{3.81}$$

The frequency integration domain of the first and last terms in (3.81) was changed from $\mathbb{R}^+$ to $\mathbb{R}$. This is allowed because φ has frequencies supported on $\mathbb{R}^+$. The last integral in (3.81) is purely imaginary, as can be seen by the Plancherel identity, hence the real part vanishes. Since $\sum_n (\frac{\psi_n^2}{2})' = 0$, where the sum is locally finite, we may insert into the second integral in (3.81) a function χ that is compactly supported on $(0,\infty)$ and identically equal to 1 in a neighborhood of λ:

$$-\frac{1}{2}\int_{\mathbb{R}} |\varphi|^2 \, dx + \frac{1}{2\pi}\text{Re} \sum_n \int_0^\infty \left(\frac{\psi_n^2}{2}\right)' \widehat{u\varphi}\bar{\hat{\varphi}}(1-\chi) \, d\xi = 0. \tag{3.82}$$

Since $xu\varphi \in L^2$ as observed above, $\widehat{u\varphi} \in H^1$. Using (3.73) and the fact that $\chi = 1$ in a neighborhood of λ, we get $\hat{\varphi}(1-\chi) \in H^1(0,\infty)$. Therefore we can integrate the second term in (3.82) by parts and get

$$-\frac{1}{2}\int_{\mathbb{R}} |\varphi|^2 \, dx - \frac{1}{4\pi}\text{Re} \int_0^\infty \left(\widehat{u\varphi}\bar{\hat{\varphi}}(1-\chi)\right)' \, d\xi = 0 \tag{3.83}$$

or

$$-\frac{1}{2}\int_{\mathbb{R}} |\varphi|^2 \, dx + \frac{1}{4\pi}\text{Re}\, \widehat{u\varphi}(0)\bar{\hat{\varphi}}(0+) = 0. \tag{3.84}$$

The application of the fundamental theorem of calculus can be justified by the Sobolev embedding theorem. The fact that $\widehat{u\varphi}\bar{\hat{\varphi}}$ vanishes at infinity follows from $u\varphi \in L^1$ and (3.73). Equation (3.73) also implies $\widehat{u\varphi}(0) = -\lambda\hat{\varphi}(0+)$. Equation (3.61) then follows from (3.84). ◻

By [21], if $u \in L^\infty \cap L^2_s$ for some $s > \frac{1}{2}$, then $L_u = \frac{1}{i}\partial_x - C_+uC_+$ has finitely many negative simple eigenvalues and the resolvent set has the form $\rho(L_u) = \mathbb{C} \setminus \{\lambda_1, \dots \lambda_N\} \setminus [0,\infty)$. In fact, [21] required $u \in L^\infty \cap L^2_1$, but the same result is true with the slightly weaker decay assumption, if one uses the same kind of bootstrap argument in the proof of Lemma 3.3 to provide additional decay estimates on the eigenfunctions.

We are now ready to establish existence and uniqueness of the Jost solutions.

THEOREM 3.5. *Let $s > s_1 > \frac{1}{2}$ and $u \in L^2_s(\mathbb{R})$. Let $\rho(L_u)$ be the resolvent set of $L_u = \frac{1}{i}\partial_x - C_+uC_+$, regarded as an operator on $\mathbb{H}^+$. Then for every $k \in \rho(L_u) \cup (\mathbb{R}^+ \pm 0i)$, and every $\lambda > 0$, there exists unique $m_1(x,k)$ and $m_e(x,\lambda \pm 0i) \in L^\infty_{-(s-s_1)}(\mathbb{R})$ solving* (3.14) *and* (3.15)*, respectively, with improved bounds $m_1(x,k), m_e(x,\lambda\pm 0i) \in L^\infty(\mathbb{R})$. Furthermore, $k \mapsto m_1(k)$ is analytic from $\rho(L_u)$ to $L^\infty_{-(s-s_1)}(\mathbb{R})$, and $m_1(k) \in C^{0,\gamma}_{loc}((\rho(L_u) \cup (\mathbb{R}^+ \pm 0i)), L^\infty_{-(s-s_1)}(\mathbb{R}))$, while $m_e(\lambda \pm 0i) \in C^{0,\gamma}_{loc}((0,\infty), L^\infty_{-(s-s_1)}(\mathbb{R}))$. Here γ is some number between* 0 *and* 1.

Proof. Lemma 3.2, Lemma 3.4, and the Fredholm alternative theorem imply existence and uniqueness of $m_1(x,k)$ and $m_e(x,\lambda \pm 0i)$. The improved L^∞ bounds were proved in Lemma 3.1. The analytic dependence of $m_1(k)$ on k follows from the analytic dependence of T_k on k. That in turn follows from the fact that $\frac{1}{h}(G_{k+h} - G_k)$ converges in L^2 to $-\frac{1}{2\pi}\int_0^\infty \frac{1}{(\xi-k)^2}\, d\xi$, a result that is easy to see. What is left to show is the Hölder continuity of $m_1(k)$ as k approaches the positive half line from above and below, and of $m_e(\lambda \pm 0i)$. We write $m_1(k) = (I - T_k)^{-1}1$, and $m_e(\lambda \pm 0i) = (I - T_{\lambda\pm 0i})^{-1}\boldsymbol{e}(\lambda)$. Using the identity

$$(I - T_{k+h})^{-1} - (I - T_k)^{-1} = (I - T_k)^{-1}(T_{k+h} - T_k)(I - T_{k+h})^{-1}, \tag{3.85}$$

we reduce the problem to showing the following three points:

(a) $\boldsymbol{e}(\lambda) \in C^{0,\gamma}_{loc}((0,\infty), L^\infty_{-(s-s_1)}(\mathbb{R}))$.

(b) The $L^\infty_{-(s-s_1)}$ operator norm of $T_{k+h} - T_k$ is bounded by $C|h|^\gamma$ for fixed $k \in \mathbb{R}^+ \pm 0i$ and small h.

(c) The $L^\infty_{-(s-s_1)}$ operator norm of $(I - T_{k+h})^{-1}$ is uniformly bounded for fixed $k \in \mathbb{R}^+ \pm 0i$ and small h.

In the above, if $k = \lambda + 0i$, then Im $h \geq 0$, while if $k = \lambda - 0i$, then Im $h \leq 0$. To prove (a), we assume Im $h = 0$ and estimate

$$\begin{aligned} |w^{s_1-s}(x)(e^{i(\lambda+h)x} - e^{i\lambda x})| &= |w^{s_1-s}(x)(e^{ihx} - 1)| \\ &\leq w^{s_1-s}(x)\min\{|hx|, 2\}. \end{aligned} \tag{3.86}$$

If $|x| < 1/\sqrt{h}$, (3.86) is bounded by $|hx| \leq \sqrt{h}$. If $|x| \geq 1/\sqrt{h}$, (3.86) is bounded by $2w^{s_1-s}(1/\sqrt{h}) \leq Ch^{\frac{s-s_1}{2}}$. Hence $\|\boldsymbol{e}(\lambda + h) - \boldsymbol{e}(\lambda)\|_{L^\infty_{-(s-s_1)}} \leq Ch^{p_1}$ for $p_1 = \min(\frac{s-s_1}{2}, \frac{1}{2})$. This proves (a). Notice that (b) implies (c), as $I - T_k$ is already shown to be invertible. For simplicity, we only work on $T_{\lambda+0i+h} - T_{\lambda+0i}$ with Im $h \geq 0$. To prove (b), we estimate

$$\begin{aligned} &(T_{\lambda+0i+h} - T_{\lambda+0i})\psi(x) \\ &\quad = i\int_{-\infty}^x e^{i\lambda(x-y)}[e^{ih(x-y)} - 1]u(y)\psi(y)\, dy - (\widetilde{G}_{\lambda+h} - \widetilde{G}_\lambda) * (u\psi)(x). \end{aligned} \tag{3.87}$$

Since $u \in L^2_s$, and $\psi \in L^\infty_{-(s-s_1)}$, we can write $u = w^{-s}u_1$, with $\|u_1\|_2 = \|u\|_{L^2_s}$, and $\psi = w^{s-s_1}\psi_1$, with $\|\psi_1\|_\infty = \|\psi\|_{L^\infty_{-(s-s_1)}}$. Hence

$$u\psi = w^{-s_1}u_1\psi_1 = w^{-p_2}w^{-\frac{1}{2}-p_2}u_1\psi_1, \tag{3.88}$$

where $p_2 = \frac{1}{2}(s_1 - \frac{1}{2}) > 0$. Since $w^{-\frac{1}{2}-p_2} \in L^2$, we get $u_2 = w^{-\frac{1}{2}-p_2}u_1 \in L^1$. Notice that $|e^{ih(x-y)} - 1| \le C|h(x-y)|$ when $|h(x-y)| \le 1$, and $|e^{ih(x-y)} - 1| \le 2$, since $\operatorname{Im} h \ge 0$ and $x - y \ge 0$. Therefore

$$\begin{aligned}
&|w^{s_1-s}(x)(T_{\lambda+0i+h} - T_{\lambda+0i})\psi(x)| \\
&\quad\le C\Bigg(w^{s_1-s}(x)\int_{-\infty}^{x} \min\{|h(x-y)|, 2\}w^{-p_2}(y)|u_2(y)|\,dy \\
&\qquad\qquad + \|\widetilde{G}_{\lambda+h} - \widetilde{G}_\lambda\|_2\|u_1\|_2\Bigg)\|\psi\|_{L^\infty_{-(s-s_1)}} \\
&\quad\le C\left(\int_{-\infty}^{x} \min\{|h(x-y)|, 2\}w^{-p_3}(x-y)|u_2(y)|\,dy + |h|\right)\|\psi\|_{L^\infty_{-(s-s_1)}}
\end{aligned} \tag{3.89}$$

for $p_3 = \min(s - s_1, p_2)$. Here we have used the Plancherel identity to estimate $\|\widetilde{G}_{\lambda+h} - \widetilde{G}_\lambda\|_2$ and have used the elementary inequality $w(x - y) \le w(x)w(y)$. The term $(\min\{|h(x-y)|, 2\}w^{-p_3}(x-y))$ can be estimated as follows. If $|x-y| < 1/\sqrt{|h|}$,

$$\min\{|h(x-y)|, 2\}w^{-p_3}(x-y) \le |h(x-y)| \le \sqrt{|h|}.$$

On the other hand, if $|x - y| \ge 1/\sqrt{|h|}$,

$$\min\{|h(x-y)|, 2\}w^{-p_3}(x-y) \le 2w^{-p_3}(1/\sqrt{|h|}) \le C|h|^{\frac{p_3}{2}}$$

for h small. Therefore

$$|w^{s_1-s}(x)(T_{\lambda+0i+h} - T_{\lambda+0i})\psi(x)| \le C|h|^{p_4}\|\psi\|_{L^\infty_{-(s-s_1)}} \tag{3.90}$$

for $p_4 = \min(\frac{p_3}{2}, \frac{1}{2})$. This proves (b) with $\gamma = p_4$. ∎

4. Scattering coefficients between Jost solutions. Now that the Jost solutions are obtained, we may proceed to study relations between them that give rise to the scattering coefficients of the Fokas–Ablowitz IST. Such relations were obtained formally by Fokas and Ablowitz in [7]. In addition, there are also relations between different scattering coefficients, many of which are stated in [10]. However, the arguments used in [10] are formal as well and depend on certain identities involving the inverse scattering problem. Here we will prove these relations and construct the scattering data directly using the setup in section 3.

First, we want to establish differentiability with respect to λ of the function $\bar{\boldsymbol{e}}(\lambda)m_e(\lambda \pm 0i)$. The λ derivative of $\bar{\boldsymbol{e}}(\lambda)m_e(\lambda \pm 0i)$ will help produce an important scattering coefficient. In fact, we will show that $\bar{\boldsymbol{e}}(\lambda)m_e(\lambda \pm 0i)$ is differentiable as a map into the weighted L^∞ spaces used in section 3. It is curious that differentiability of the particular combination $\bar{\boldsymbol{e}}m_e$ can be proven under the same decay assumptions on u as in section 3, whereas any slightly different function, such as $m_e(\lambda \pm 0i)$ alone, $m_1(\lambda \pm 0i)$, or $\bar{\boldsymbol{e}}(\lambda)m_1(\lambda \pm 0i)$, will require significantly stronger decay conditions on u to be differentiable in the above sense. The basic reason is that the term $\partial_\lambda e^{i\lambda x} = xe^{i\lambda x}$ comes out and produces an extra factor of x when we differentiate (3.14) and

(3.15) with respect to λ. We would need $xe^{i\lambda x}$ to belong to $L^\infty_{-(s-s_1)}$, which forces $s - s_1 \geq 1$. Combined with the condition $s > s_1 > \frac{1}{2}$ in Theorem 3.5, this implies $s > \frac{3}{2}$. Certain other considerations seem to even require $s > \frac{5}{2}$. The special favor found only by $\bar{\boldsymbol{e}}(\lambda)m_e(\lambda \pm 0i)$ can be explained as follows. To simplify notation, we suppress the λ dependence when it is clear from the context. By (3.9) and (3.7), we formally have

$$\partial_\lambda G_{\lambda\pm 0i}(x) = -\frac{1}{2\pi\lambda} + ixG_{\lambda\pm 0i}(x). \tag{4.1}$$

Rewrite (3.15) as

$$\bar{\boldsymbol{e}} m_e(\lambda\pm) = 1 + \bar{\boldsymbol{e}} G_{\lambda\pm 0i} * (u\boldsymbol{e}\bar{\boldsymbol{e}} m_e(\lambda\pm)), \tag{4.2}$$

and differentiate with respect to λ formally:

$$\begin{aligned} &\partial_\lambda(\bar{\boldsymbol{e}} m_e(\lambda\pm)) \\ &= -ix\bar{\boldsymbol{e}} \left(G_{\lambda\pm 0i} * (u\boldsymbol{e}\bar{\boldsymbol{e}} m_e(\lambda\pm))\right) + \bar{\boldsymbol{e}}\left(-\frac{1}{2\pi\lambda} + ixG_{\lambda\pm 0i}\right) * (u\boldsymbol{e}\bar{\boldsymbol{e}} m_e(\lambda\pm)) \\ &\quad + \bar{\boldsymbol{e}} G_{\lambda\pm 0i} * (iyu\boldsymbol{e}\bar{\boldsymbol{e}} m_e(\lambda\pm)) + \bar{\boldsymbol{e}} G_{\lambda\pm 0i} * (u\boldsymbol{e}\partial_\lambda(\bar{\boldsymbol{e}} m_e(\lambda\pm))) \\ &= -\frac{\bar{\boldsymbol{e}}}{2\pi\lambda}\int_{\mathbb{R}} u(y)m_e(y,\lambda\pm)\, dy + \bar{\boldsymbol{e}} G_{\lambda\pm 0i} * (u\boldsymbol{e}\partial_\lambda(\bar{\boldsymbol{e}} m_e(\lambda\pm))). \end{aligned} \tag{4.3}$$

Multiply both sides of (4.3) by $\boldsymbol{e}$ to get

$$\boldsymbol{e}\partial_\lambda(\bar{\boldsymbol{e}} m_e(\lambda\pm)) = -\frac{1}{2\pi\lambda}\int_{\mathbb{R}} u(y)m_e(y,\lambda\pm)\, dy + G_{\lambda\pm 0i} * (u\boldsymbol{e}\partial_\lambda(\bar{\boldsymbol{e}} m_e(\lambda\pm))). \tag{4.4}$$

Notice (4.4) no longer involves any extra factor of x. In fact, the cancellation happening in (4.3) removed all extra factors of x. The proof of differentiability is to show that a similar, although no longer exact, cancellation happens on the level of difference quotients.

LEMMA 4.1. *Let $s > s_1 > \frac{1}{2}$, and $u \in L^2_s(\mathbb{R})$. Let $m_e(\lambda \pm 0i)$ be the Jost functions constructed in Theorem* 3.5. *Then $\bar{\boldsymbol{e}}(\lambda)m_e(\lambda \pm 0i) \in C^{1,\gamma}_{loc}((0,\infty), L^\infty_{-(s-s_1)}(\mathbb{R}))$ for some $0 < \gamma < 1$, and*

$$\partial_\lambda(\bar{\boldsymbol{e}}(\lambda)m_e(\lambda \pm 0i)) = -\frac{\bar{\boldsymbol{e}}}{2\pi\lambda}\left(\int_{\mathbb{R}} u(y)m_e(y,\lambda\pm 0i)\, dy\right) m_1(\lambda \pm 0i). \tag{4.5}$$

Proof. We denote the shift operator by $(\tau_h f)(\lambda) = f(\lambda + h)$ and the difference quotient operator by $D_h f = \frac{1}{h}(\tau_h f - f)$. One has the product rule:

$$D_h(fg) = (D_h f)g + (\tau_h f)(D_h g). \tag{4.6}$$

For simplicity, we only work on $m_e(\lambda - 0i)$ and write it simply as m_e. D_h acting on (4.2) gives

$$\begin{aligned} D_h(\bar{\boldsymbol{e}} m_e) &= (D_h\bar{\boldsymbol{e}})\left(G_{\lambda-0i} * (u\boldsymbol{e}\bar{\boldsymbol{e}} m_e)\right) + (\tau_h\bar{\boldsymbol{e}})[(D_h G_{\lambda-0i}) * (u\boldsymbol{e}\bar{\boldsymbol{e}} m_e)] \\ &\quad + (\tau_h\bar{\boldsymbol{e}})\left[(\tau_h G_{\lambda-0i}) * (u(D_h\boldsymbol{e})\bar{\boldsymbol{e}} m_e)\right] \\ &\quad + (\tau_h\bar{\boldsymbol{e}})\left[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})D_h(\bar{\boldsymbol{e}} m_e))\right]. \end{aligned} \tag{4.7}$$

We add up the first three terms in (4.7) as follows:

$$\begin{aligned}
&\frac{e^{-i(\lambda+h)x}-e^{-i\lambda x}}{h}\int_{\mathbb{R}} G_{\lambda-0i}(x-y)um_e(y)\ dy \\
&\quad + e^{-i(\lambda+h)x}\int_{\mathbb{R}} \frac{G_{\lambda+h-0i}(x-y)-G_{\lambda-0i}(x-y)}{h}um_e(y)\ dy \\
&\quad + e^{-i(\lambda+h)x}\int_{\mathbb{R}} G_{\lambda+h-0i}(x-y)\frac{e^{i(\lambda+h)y}-e^{i\lambda y}}{h}e^{-i\lambda y}um_e(y)\ dy \\
&= e^{-i(\lambda+h)x}\int_{\mathbb{R}} \frac{G_{\lambda+h-0i}(x-y)-G_{\lambda-0i}(x-y)e^{ih(x-y)}}{h}e^{ihy}um_e(y)\ dy \\
&= \frac{e^{-i(\lambda+h)x}}{2\pi}\int_{\mathbb{R}} \frac{1}{h}\left(\int_0^h \frac{e^{i(x-y)\xi}}{\xi-\lambda-h}\ d\xi\right)e^{ihy}um_e(y)\ dy. \qquad (4.8)
\end{aligned}$$

The last equality above follows from (3.9) and (3.7). Denote (4.8) by $S_h(um_e)$. We get from (4.7) that

$$\boldsymbol{e}D_h(\bar{\boldsymbol{e}}m_e) = \boldsymbol{e}S_h(um_e) + \boldsymbol{e}(\tau_h\bar{\boldsymbol{e}})\left[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}\boldsymbol{e}D_h(\bar{\boldsymbol{e}}m_e))\right]. \qquad (4.9)$$

Let φ be the solution to

$$\varphi = -\frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y)\ dy + G_{\lambda-0i} * (u\varphi), \qquad (4.10)$$

whose existence is guaranteed by Theorem 3.5. We want to show $\partial_\lambda(\bar{\boldsymbol{e}}m_e) = \bar{\boldsymbol{e}}\varphi$, which, as we will see in the following, is equivalent to (4.5). To that end, take the difference of (4.9) and (4.10) and rearrange terms to get

$$\begin{aligned}
&[\boldsymbol{e}D_h(\bar{\boldsymbol{e}}m_e)-\varphi] - \boldsymbol{e}(\tau_h\bar{\boldsymbol{e}})\left[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}[\boldsymbol{e}D_h(\bar{\boldsymbol{e}}m_e)-\varphi])\right] \\
&\quad = \boldsymbol{e}S_h(um_e) + \frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y)\ dy \\
&\qquad + \boldsymbol{e}(\tau_h\bar{\boldsymbol{e}})\left[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}\varphi)\right] - G_{\lambda-0i} * (u\varphi). \qquad (4.11)
\end{aligned}$$

Denote $T_{\lambda,h}\psi = \boldsymbol{e}(\tau_h\bar{\boldsymbol{e}})\left[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}\psi)\right]$, and recall the definition of $T_{\lambda-0i}$ by (3.42), (4.11) can be written as

$$\begin{aligned}
&(I-T_{\lambda,h})[\boldsymbol{e}D_h(\bar{\boldsymbol{e}}m_e)-\varphi] \\
&\quad = \boldsymbol{e}S_h(um_e) + \frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y)\ dy + (T_{\lambda,h}-T_{\lambda-0i})\varphi. \qquad (4.12)
\end{aligned}$$

In view of (4.12), it suffices to show the following three points:

(a) For $\lambda > 0$ fixed, $(I-T_{\lambda,h})^{-1}$ has uniformly bounded $L^\infty_{-(s-s_1)}$ operator norm for small h.

(b) $\|\boldsymbol{e}S_h(um_e) + \frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y)\ dy\|_{L^\infty_{-(s-s_1)}} \to 0$ as $h \to 0$.

(c) $\|(T_{\lambda,h}-T_{\lambda-0i})\varphi\|_{L^\infty_{-(s-s_1)}} \to 0$ as $h \to 0$.

In fact, we claim that the $L^\infty_{-(s-s_1)}$ operator norm of $T_{\lambda,h} - T_{\lambda-0i}$ tends to 0 as h tends to 0. This will imply both (a) and (c), as $(I - T_{\lambda-0i})$ is invertible by Theorem 3.5. We write

$$\begin{aligned}(T_{\lambda,h} &- T_{\lambda-0i})\psi \\ &= \boldsymbol{e}(\tau_h\bar{\boldsymbol{e}})\,[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}\psi)] - G_{\lambda-0i} * (u\psi) \\ &= [\boldsymbol{e}(\tau_h\bar{\boldsymbol{e}}) - 1]\,[(\tau_h G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}\psi)] \\ &\quad + (\tau_h G_{\lambda-0i} - G_{\lambda-0i}) * (u(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}}\psi) \\ &\quad + G_{\lambda-0i} * (u[(\tau_h\boldsymbol{e})\bar{\boldsymbol{e}} - 1]\psi) \\ &= I(x) + II(x) + III(x). \end{aligned} \tag{4.13}$$

We estimate the three terms separately. By (3.9) and (3.7), there exists $C = C(\lambda, u)$ such that

$$\begin{aligned}|w^{s_1-s}(x)I(x)| &\le Cw^{s_1-s}(x)|e^{-ihx} - 1|\left(\|u\psi\|_\infty + \|\widetilde{G}_{\lambda+h}\|_2\|u\psi\|_2\right) \\ &\le Cw^{s_1-s}(x)\min\{|hx|, 2\}\|\psi\|_{L^\infty_{-(s-s_1)}}.\end{aligned} \tag{4.14}$$

If $|x| < 1/\sqrt{h}$, $|hx| < \sqrt{h}$. On the other hand if $|x| > 1/\sqrt{h}$,

$$w^{s_1-s}(x) \le w^{s_1-s}(1/\sqrt{h}) \le Ch^{\frac{s-s_1}{2}}.$$

Therefore

$$|w^{s_1-s}(x)I(x)| \le Ch^{p_1}\|\psi\|_{L^\infty_{-(s-s_1)}}, \tag{4.15}$$

where $p_1 = \min(\frac{s-s_1}{2}, \frac{1}{2})$. Let $p_2 = \frac{1}{2}(s_1 - \frac{1}{2}) > 0$. By the conditions on u and ψ, we have

$$\begin{aligned}|u(y)\psi(y)| &\le w^{-s_1}(y)u_1(y)\|\psi\|_{L^\infty_{-(s-s_1)}} \\ &= w^{-p_2}(y)w^{-\frac{1}{2}-p_2}u_1(y)\|\psi\|_{L^\infty_{-(s-s_1)}} \\ &= w^{-p_2}(y)u_2(y)\|\psi\|_{L^\infty_{-(s-s_1)}},\end{aligned} \tag{4.16}$$

where $u_1 = w^s u \in L^2$, and $u_2 = w^{-\frac{1}{2}-p_2}u_1 \in L^1$. Letting $p_3 = \min(s - s_1, p_2)$, and using the relation $w(x - y) \le Cw(x)w(y)$, we have

$$\begin{aligned}&|w^{s_1-s}(x)II(x)| \\ &\quad\le Cw^{s_1-s}(x)\left(\int_x^\infty |e^{ih(x-y)} - 1||u(y)\psi(y)|\,dy + \|\widetilde{G}_{\lambda+h} - \widetilde{G}_\lambda\|_2\|u\psi\|_2\right) \\ &\quad\le Cw^{s_1-s}(x)\left(\int_x^\infty \min\{|h(x-y)|, 2\}w^{-p_2}(y)|u_2(y)|\,dy + h\right)\|\psi\|_{L^\infty_{-(s-s_1)}} \\ &\quad\le C\left(\int_x^\infty \min\{|h(x-y)|, 2\}w^{-p_3}(x-y)|u_2(y)|\,dy + h\right)\|\psi\|_{L^\infty_{-(s-s_1)}}.\end{aligned} \tag{4.17}$$

By the same argument as above, we get $\min\{|h(x-y)|, 2\}w^{-p_3}(x-y) \le Ch^{p_4}$ for $p_4 = \min(\frac{p_3}{2}, \frac{1}{2})$. Hence

$$|w^{s_1-s}(x)II(x)| \le Ch^{p_4}\|\psi\|_{L^\infty_{-(s-s_1)}}. \tag{4.18}$$

Similarly,

$$
\begin{aligned}
|w^{s_1-s}(x)III(x)| &\le |III(x)| \\
&\le C\left(\int_{\mathbb{R}} |u(y)\psi(y)||e^{ihy}-1|\ dy + \|\widetilde{G}_\lambda\|_2 \left(\int_{\mathbb{R}} |u(y)\psi(y)|^2|e^{ihy}-1|^2\right)^{1/2}\right) \\
&\le C\Bigg(\int_{\mathbb{R}} w^{-p_2}(y)u_2(y)\min\{|hy|,2\}\ dy \\
&\qquad + \left(\int_{\mathbb{R}} |w^{-s_1}(y)u_1(y)|^2(\min\{|hy|,2\})^2\ dy\right)^{1/2}\Bigg)\|\psi\|_{L^\infty_{-(s-s_1)}} \\
&\le Ch^{p_4}\|\psi\|_{L^\infty_{-(s-s_1)}}. \qquad (4.19)
\end{aligned}
$$

By (4.13), (4.15), (4.18), and (4.19), we have

$$
\|(T_{\lambda,h}-T_{\lambda-0i})\psi\|_{L^\infty_{-(s-s_1)}} \le Ch^{p_4}\|\psi\|_{L^\infty_{-(s-s_1)}}. \qquad (4.20)
$$

This proved points (a) and (c) mentioned above. To prove (b), we recall that $S_h(um_e)$ was defined by (4.8). So

$$
\begin{aligned}
&(\boldsymbol{e}S_h(um_e))(x) + \frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y)\ dy \\
&\quad = \frac{1}{2\pi}\int_{\mathbb{R}}\left[\frac{1}{h}\left(\int_0^h \frac{e^{i(x-y)(\xi-h)}}{\xi-\lambda-h}\ d\xi\right)+\frac{1}{\lambda}\right] um_e(y)\ dy \\
&\quad = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{h}\left(\int_0^h \frac{\lambda[e^{i(x-y)(\xi-h)}-1]+\xi-h}{\lambda(\xi-h-\lambda)}\ d\xi\right) um_e(y)\ dy. \qquad (4.21)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left|w^{s_1-s}(x)\left((\boldsymbol{e}S_h(um_e))(x) + \frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y)\ dy\right)\right| \\
&\quad \le Cw^{s_1-s}(x)\int_{\mathbb{R}} (\min\{|2h(x-y)|,2\}+h)\,|u(y)m_e(y)|\ dy \\
&\quad \le Cw^{s_1-s}(x)\int_{\mathbb{R}} (\min\{|2h(x-y)|,2\}+h)\,w^{-p_2}(y)|u_2(y)|\ dy \\
&\quad \le C\int_{\mathbb{R}} (\min\{|2h(x-y)|,2\}+h)\,w^{-p_3}(x-y)|u_2(y)|\ dy \\
&\quad \le Ch^{p_4}. \qquad (4.22)
\end{aligned}
$$

We have thus proved $\partial_\lambda(\bar{\boldsymbol{e}}m_e) = \bar{\boldsymbol{e}}\varphi$. Since φ satisfies (4.10), $\partial_\lambda(\bar{\boldsymbol{e}}m_e)$ satisfies (4.4). In other words,

$$
\begin{aligned}
\partial_\lambda(\bar{\boldsymbol{e}}m_e(\lambda-0i)) &= \bar{\boldsymbol{e}}(I-T_{\lambda-0i})^{-1}\left(-\frac{1}{2\pi\lambda}\int_{\mathbb{R}} um_e(y,\lambda-0i)\ dy\right) \\
&= -\frac{\bar{\boldsymbol{e}}}{2\pi\lambda}\left(\int_{\mathbb{R}} u(y)m_e(y,\lambda-0i)\ dy\right)m_1(\lambda-0i). \qquad (4.23)
\end{aligned}
$$

Its Hölder continuity follows from that of $\boldsymbol{e}(\lambda)$, $m_e(\lambda-0i)$, and $m_1(\lambda-0i)$, which was established in the proof of Theorem 3.5. ∎

We are now ready to describe the scattering coefficients for the Fokas–Ablowtiz IST.

LEMMA 4.2. *Let $m_1(x,k)$ and $m_e(x,\lambda\pm 0i)$ be the Jost solutions constructed in Theorem* 3.5. *Define*

$$\Gamma(\lambda) = 1 + i\int_{\mathbb{R}} u(x)m_e(x,\lambda+0i)e^{-i\lambda x}\,dx \tag{4.24}$$

$$= \frac{1}{1 - i\int_{\mathbb{R}} u(x)m_e(x,\lambda-0i)e^{-i\lambda x}\,dx}, \tag{4.25}$$

$$\beta(\lambda) = i\int_{\mathbb{R}} u(x)m_1(x,\lambda+0i)e^{-i\lambda x}\,dx, \tag{4.26}$$

and

$$f(\lambda) = -\frac{1}{2\pi\lambda}\int_{\mathbb{R}} u(x)m_e(x,\lambda-0i)\,dx. \tag{4.27}$$

Then the following relations between Jost solutions hold:

$$m_e(\lambda+0i) = \Gamma(\lambda)m_e(\lambda-0i), \tag{4.28}$$

$$m_1(\lambda+0i) - m_1(\lambda-0i) = \beta(\lambda)m_e(\lambda-0i), \tag{4.29}$$

and

$$\boldsymbol{e}\partial_\lambda(\bar{\boldsymbol{e}}m_e(\lambda-0i)) = f(\lambda)m_1(\lambda-0i). \tag{4.30}$$

Proof. By (3.9),

$$G_{\lambda+0i}(x) - G_{\lambda-0i}(x) = i\boldsymbol{e}(x,\lambda). \tag{4.31}$$

Therefore (3.15) implies

$$\begin{aligned} m_e(\lambda+0i) &= \boldsymbol{e}(\lambda) + (G_{\lambda-0i} + i\boldsymbol{e}(\lambda)) * (um_e(\lambda+0i)) \\ &= \boldsymbol{e}(\lambda) + G_{\lambda-0i} * (um_e(\lambda-0i)) + i\boldsymbol{e}(\lambda)\int_{\mathbb{R}} u(x)m_e(x,\lambda+0i)e^{-i\lambda x}\,dx \\ &= \left(1 + i\int_{\mathbb{R}} u(x)m_e(x,\lambda+0i)e^{-i\lambda x}\,dx\right)\boldsymbol{e}(\lambda) + G_{\lambda-0i} * (um_e(\lambda-0i)). \end{aligned} \tag{4.32}$$

By (3.15) and uniqueness of Jost solutions, we get

$$m_e(\lambda+0i) = \left(1 + i\int_{\mathbb{R}} u(x)m_e(x,\lambda+0i)e^{-i\lambda x}\,dx\right)m_e(\lambda-0i). \tag{4.33}$$

A similar calculation starting with the integral equation of $m_e(\lambda-0i)$ gives

$$m_e(\lambda-0i) = \left(1 - i\int_{\mathbb{R}} u(x)m_e(x,\lambda-0i)e^{-i\lambda x}\,dx\right)m_e(\lambda+0i). \tag{4.34}$$

This proves (4.28). Take the difference of the integral equations of $m_1(\lambda\pm 0i)$ given in (3.14) to get

$$\begin{aligned} m_1(\lambda+0i) - m_1(\lambda-0i) = {} & i\boldsymbol{e}(\lambda)\int_{\mathbb{R}} u(x)m_1(x,\lambda+0i)e^{-i\lambda x}\,dx \\ & + G_{\lambda-0i} * [u(m_1(\lambda+0i) - m(\lambda-0i))]. \end{aligned} \tag{4.35}$$

By (3.15) and uniqueness of Jost solutions, we get

$$m_1(\lambda+0i) - m_1(\lambda-0i) = m_e(\lambda-0i)\ i\int_{\mathbb{R}} u(x)m_1(x,\lambda+0i)e^{-i\lambda x}\ dx. \tag{4.36}$$

This proves (4.29). Finally, (4.30) is just the minus sign case of (4.5). ◻

Our next goal is to establish relations between different scattering coefficients. The following identity proves very useful in showing these relations.

LEMMA 4.3. *Denote* $\int_{\mathbb{R}} f(x)\overline{g(x)}\ dx$ *by* $\langle f,g\rangle$. *If* $f,g\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, $\lambda>0$, *then*

$$\langle G_{\lambda\pm 0i} * f, g\rangle = \pm i\langle f,\boldsymbol{e}\rangle\overline{\langle g,\boldsymbol{e}\rangle} + \langle f, G_{\lambda\pm 0i} * g\rangle. \tag{4.37}$$

Proof. We only present the calculation for $G_{\lambda+0i}$. Using (3.9), we see that

$$\overline{G_{\lambda+0i}(-x)} = -ie^{i\lambda x}\chi_{\mathbb{R}^-}(x) - \tilde{G}_\lambda(x) \tag{4.38}$$

or

$$G_{\lambda+0i}(x) = \overline{G_{\lambda+0i}(-x)} + ie^{i\lambda x}. \tag{4.39}$$

Thus

$$\begin{aligned}\langle G_{\lambda+0i} * f, g\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} G_{\lambda+0i}(x-y)f(y)\overline{g(x)}\ dy\ dx\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\overline{G_{\lambda+0i}(y-x)} + ie^{i\lambda(x-y)}\right)f(y)\overline{g(x)}\ dy\ dx\\ &= i\int_{\mathbb{R}} f(y)e^{-i\lambda y}\ dy\int_{\mathbb{R}}\overline{g(x)e^{-i\lambda x}}\ dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f(y)\overline{G_{\lambda+0i}(y-x)g(x)}\ dx\ dy\\ &= i\langle f,\boldsymbol{e}\rangle\overline{\langle g,\boldsymbol{e}\rangle} + \langle f, G_{\lambda+0i} * g\rangle.\end{aligned} \tag{4.40}$$

All of the above calculations are justified if $f,g\in L^1\cap L^2$. ◻

LEMMA 4.4. *Let* $s>s_1>\frac{1}{2}$ *and* $u\in L^2_s(\mathbb{R})$. *Define the Jost solutions as above, and let* $\Gamma(\lambda)$, $\beta(\lambda)$, *and* $f(\lambda)$ *be defined as in Lemma 4.2. Then* $\Gamma\in C^{1,\gamma}_{loc}(0,\infty)$, $\beta, f\in C^{0,\gamma}_{loc}(0,\infty)$ *for some* $0<\gamma<1$, *and the following relations hold:*

$$|\Gamma(\lambda)| = 1, \tag{4.41}$$

$$f(\lambda) = \frac{\overline{\beta(\lambda)}}{2\pi i\lambda}, \tag{4.42}$$

$$|\beta(\lambda)|^2 = 2\ \mathrm{Im}\int_{\mathbb{R}} u(x)m_1(x,\lambda+0i)\ dx, \tag{4.43}$$

and

$$\partial_\lambda\Gamma(\lambda) = \frac{|\beta(\lambda)|^2}{2\pi i\lambda}\Gamma(\lambda). \tag{4.44}$$

Proof. The regularity of Γ, β, and f follows easily from the corresponding regularity of the Jost solutions. What are left to show are the relations between them.

We start by multiplying the integral equation of $m_e(\lambda+)$ by $\overline{um_e(\lambda+)}$ and integrate on $\mathbb{R}$. Using Lemma 4.3, we have

$$\begin{aligned}
&\langle m_e(\lambda+), um_e(\lambda+)\rangle \\
&\quad = \langle \boldsymbol{e}, um_e(\lambda+)\rangle + \langle G_{\lambda+} * (um_e(\lambda+)), um_e(\lambda+)\rangle \\
&\quad = \langle \boldsymbol{e}, um_e(\lambda+)\rangle + i|\langle um_e(\lambda+, \boldsymbol{e})\rangle|^2 + \langle um_e(\lambda+), G_{\lambda+} * (um_e(\lambda+))\rangle \\
&\quad = \langle \boldsymbol{e}, um_e(\lambda+)\rangle + i|\langle um_e(\lambda+, \boldsymbol{e})\rangle|^2 + \langle um_e(\lambda+), m_e(\lambda+) - \boldsymbol{e}\rangle.
\end{aligned} \tag{4.45}$$

Since $\langle m_e(\lambda+), um_e(\lambda+)\rangle = \langle um_e(\lambda+), m_e(\lambda+)\rangle$, we get

$$i|\langle um_e(\lambda+), \boldsymbol{e}\rangle|^2 - \langle um_e(\lambda+), \boldsymbol{e}\rangle + \overline{\langle um_e(\lambda+), \boldsymbol{e}\rangle} = 0. \tag{4.46}$$

By the definition of $\Gamma(\lambda)$ given in (4.24), $\Gamma(\lambda) = 1 + i\langle um_e(\lambda+), \boldsymbol{e}\rangle$. Hence (4.46) means

$$i|\Gamma(\lambda) - 1|^2 + i(\Gamma(\lambda) - 1) + i\overline{\Gamma(\lambda) - 1} = 0, \tag{4.47}$$

from which it follows that $|\Gamma(\lambda)| = 1$.

Next we multiply the integral equation of $m_1(\lambda+)$ given in (3.14) by $\overline{um_e(\lambda+)}$ and integrate on $\mathbb{R}$. Use Lemma 4.3 again to get

$$\begin{aligned}
\langle m_1(\lambda+), um_e(\lambda+)\rangle &= \langle 1 + G_{\lambda+0i} * (um_1(\lambda+)), um_e(\lambda+)\rangle \\
&= \int_{\mathbb{R}} \overline{um_e(\lambda+)}\ dx + i\langle um_1(\lambda+), \boldsymbol{e}\rangle\overline{\langle um_e(\lambda+), \boldsymbol{e}\rangle} \\
&\quad + \langle um_1(\lambda+), G_{\lambda+0i} * (um_e(\lambda+)\rangle \\
&= \int_{\mathbb{R}} \overline{um_e(\lambda+)}\ dx + i\langle um_1(\lambda+), \boldsymbol{e}\rangle\overline{\langle um_e(\lambda+), \boldsymbol{e}\rangle} \\
&\quad + \langle um_1(\lambda+), m_e(\lambda+) - \boldsymbol{e}\rangle.
\end{aligned} \tag{4.48}$$

Since $\langle m_1(\lambda+), um_e(\lambda+)\rangle = \langle um_1(\lambda+), m_e(\lambda+)\rangle$, this implies

$$\int_{\mathbb{R}} \overline{um_e(\lambda+)}\ dx + \langle um_1(\lambda+), \boldsymbol{e}\rangle \left(i\overline{\langle um_e(\lambda+), \boldsymbol{e}\rangle} - 1\right) = 0. \tag{4.49}$$

By the definition of $\Gamma(\lambda)$, $\beta(\lambda)$, $f(\lambda)$, and the relation (4.28), we get

$$-2\pi\lambda\overline{f(\lambda)\Gamma(\lambda)} - \frac{1}{i}\beta(\lambda)\overline{\Gamma(\lambda)} = 0. \tag{4.50}$$

Divide both sides by $\overline{\Gamma(\lambda)}$ to obtain (4.42). This is allowed as $|\Gamma(\lambda)| = 1$.

Next we multiply the integral equation of $m_1(\lambda+)$ by $\overline{um_1(\lambda+)}$ and integrate on $\mathbb{R}$. Use Lemma 4.3 to get

$$\begin{aligned}
&\langle m_1(\lambda+), um_1(\lambda+)\rangle = \langle 1 + G_{\lambda+0i} * (um_1(\lambda+)), um_1(\lambda+)\rangle \\
&\quad = \int_{\mathbb{R}} \overline{um_1(\lambda+)}\ dx + i|\langle um_1(\lambda+), \boldsymbol{e}\rangle|^2 + \langle um_1(\lambda+), G_{\lambda+0i} * (um_1(\lambda+))\rangle \\
&\quad = \int_{\mathbb{R}} \overline{um_1(\lambda+)}\ dx + i|\langle um_1(\lambda+), \boldsymbol{e}\rangle|^2 + \langle um_1(\lambda+), m_1(\lambda+) - 1\rangle.
\end{aligned} \tag{4.51}$$

Since $\langle m_1(\lambda+), um_1(\lambda+)\rangle = \langle um_1(\lambda+), m_1(\lambda+)\rangle$, we have

$$-2i\ \mathrm{Im} \int_{\mathbb{R}} um_1(\lambda+)\ dx + i|\langle um_1(\lambda+), \boldsymbol{e}\rangle|^2 = 0, \tag{4.52}$$

from which (4.43) follows.

Finally, to get (4.44), we differentiate (4.24) using the plus sign case of (4.5) and apply (4.28) to get

$$\begin{aligned}\partial_\lambda \Gamma(\lambda) &= i\int_{\mathbb{R}} u(x)e^{-i\lambda x}\left(-\frac{1}{2\pi\lambda}\int_{\mathbb{R}} u(y)m_e(y,\lambda+)\ dy\right)m_1(x,\lambda+)\ dx\\ &= f(\lambda)\beta(\lambda)\Gamma(\lambda).\end{aligned} \tag{4.53}$$

Equation (4.44) now follows from (4.42). ∎

5. Asymptotic behavior near $k = 0$.

In this section, we discuss the asymptotic behavior of the Jost solutions and scattering coefficients as k approaches 0 within the set $\rho(L_u) \cup (\mathbb{R}^+ \pm 0i)$. It turns out that the convolution kernel G_k has a logarithmic singularity at $k = 0$, and so does the operator T_k. We employ the well-known method of subtracting a rank one operator from T_k so that the modified operator has a limit at $k = 0$. The limiting modified operator also has the form of identity plus a compact operator. We then obtain its invertibility through a vanishing lemma. The asymptotic behavior of the Jost functions can be recovered from the modified Jost functions. The asymptotics presented in this section was formally obtained in [7] and [10].

Let $\chi(\xi)$ be a smooth function on $[0,\infty)$, which is identically equal to 1 for $0 \le \xi \le 1$ and identically equal to 0 for $\xi \ge 2$. Later on we will see that it is crucial to allow the possibility of $\chi(\xi)$ being *complex* for $1 < \xi < 2$. For $k \in \rho(L_u) \cup (\mathbb{R}^+ \pm 0i)$, let

$$l(k) = \frac{1}{2\pi}\int_0^\infty \frac{\chi(\xi)}{\xi - k}\ d\xi, \tag{5.1}$$

and let

$$G_k^0(x) = G_k(x) - l(k) = \frac{1}{2\pi}\int_0^\infty \frac{e^{ix\xi} - \chi(\xi)}{\xi - k}\ d\xi, \tag{5.2}$$

$$T_k^0(\varphi) = G_k^0 * (u\varphi) = T_k(\varphi) - l(k)\langle \varphi, u\rangle. \tag{5.3}$$

We also define

$$G_0^0(x) = \frac{1}{2\pi}\int_0^\infty \frac{e^{ix\xi} - \chi(\xi)}{\xi}\ d\xi, \quad T_0^0(\varphi) = G_0^0 * (u\varphi). \tag{5.4}$$

We define the modified Jost functions $m_1^0(x,k)$ and $m_e^0(x,\lambda \pm 0i)$ to be solutions (if exist) to the integral equations

$$m_1^0(k) = 1 + T_k^0(m_1^0(k)) = 1 + G_k^0 * (um_1^0(k)), \tag{5.5}$$

$$m_e^0(\lambda \pm 0i) = \boldsymbol{e}(\lambda) + T_{\lambda\pm 0i}^0(m_e^0(\lambda \pm 0i)) = \boldsymbol{e}(\lambda) + G_{\lambda\pm 0i}^0 * (um_e^0(\lambda \pm 0i)). \tag{5.6}$$

Using (5.3), we obtain the relation between the original and the modified Jost functions:

$$m_1(k) = \frac{m_1^0(k)}{1 - l(k)\langle m_1^0(k), u\rangle}, \tag{5.7}$$

$$\begin{aligned}m_e(\lambda\pm) &= m_e^0(\lambda\pm) + l(\lambda\pm)\langle m_e^0(\lambda\pm), u\rangle m_1(\lambda\pm)\\ &= \frac{m_e^0(\lambda\pm) + l(\lambda\pm)\big(\langle m_e^0(\lambda\pm), u\rangle m_1^0(\lambda\pm) - \langle m_1^0(\lambda\pm), u\rangle m_e^0(\lambda\pm)\big)}{1 - l(\lambda\pm)\langle m_1^0(\lambda\pm), u\rangle}.\end{aligned} \tag{5.8}$$

To prove existence of the modified Jost functions when k is near 0, we carry out the plan introduced at the beginning of this section. The first step is to estimate the modified convolution kernel G_k^0.

Lemma 5.1. *There exists $k_0 > 0$ such that for every $\epsilon \in (0,1)$, there is $C > 0$ such that for all $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$ with $|k| < k_0$,*

$$|G_k^0(x) - G_0^0(x)| \le C|k|^\epsilon (1+|x|)^\epsilon. \tag{5.9}$$

Proof. By the definition of $\chi(\xi)$, we write

$$G_k^0(x) - G_0^0(x) = \frac{1}{2\pi}\int_0^1 \frac{e^{ix\xi}-1}{\xi}\frac{k}{\xi-k}\,d\xi + \frac{1}{2\pi}\int_1^\infty \frac{e^{ix\xi}-\chi(\xi)}{\xi}\frac{k}{\xi-k}\,d\xi. \tag{5.10}$$

We first estimate the second term in (5.10). Assuming $|k| < k_0 < \frac{1}{2}$, we obtain $|\xi - k| \ge \xi - |k| \ge \frac{1}{2}\xi$ for $\xi \ge 1$. Thus

$$\left|\int_1^\infty \frac{e^{ix\xi}-\chi(\xi)}{\xi}\frac{k}{\xi-k}\,d\xi\right| \le C|k|\int_1^\infty \frac{1}{\xi^2}\,d\xi \le C|k|. \tag{5.11}$$

We are left to estimate the first term in (5.10). Let us first consider the case $x > 0$. Make a change of variable to rewrite the integral as

$$kx\int_0^x \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi. \tag{5.12}$$

Notice that $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$ means kx can get arbitrarily close to the interval $(0,x)$. We deform the contour of integration when estimating (5.12). The work is split into two cases: when $|k|x < 1$ or when $|k|x \ge 1$. If $|k|x < 1$, we split the integral (5.12) as follows:

$$\int_{\Gamma_1} + \int_{2|k|x}^2 + \int_2^x. \tag{5.13}$$

Here Γ_1 is a semicircle centered at $|k|x$ with radius $|k|x$. Γ_1 is in the lower half plane if kx is in the upper half plane, and vice versa. With this choice, we have $|\xi - kx| \ge |k|x$ and $\left|\frac{e^{i\xi}-1}{\xi}\right| \le C$ when $\xi \in \Gamma_1$. Hence

$$|k|x\left|\int_{\Gamma_1} \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| \le C|k|x \le C|k|^\epsilon|x|^\epsilon. \tag{5.14}$$

We used $|k|x < 1$ to get the last inequality. For $2|k|x < \xi < 2$, we have $\left|\frac{e^{i\xi}-1}{\xi}\right| \le C$ and $|\xi - kx| \ge \xi - |k|x$. Hence

$$\begin{aligned} |k|x\left|\int_{2|k|x}^2 \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| &\le C|k|x\int_{2|k|x}^2 \frac{1}{\xi-|k|x}\,d\xi \\ &= C|k|x\log\left(\frac{2}{|k|x}-1\right) \\ &\le C|k|^\epsilon|x|^\epsilon. \end{aligned} \tag{5.15}$$

For ξ between 2 and x (either could be the larger of the two), $|\frac{e^{i\xi}-1}{\xi}| \le \frac{2}{\xi}$, and $|\xi - kx| \ge \xi - |k|x$. Thus

$$\begin{aligned}|k|x\left|\int_2^x \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| &\le C|k|x\left|\int_2^x \frac{1}{\xi(\xi-|k|x)}\,d\xi\right|\\ &= C\left|\log\left(\frac{2(1-|k|)}{2-|k|x}\right)\right|\\ &\le C\log[(1+|k|)(1+|k|x)]\\ &\le C|k|^\epsilon(1+|x|)^\epsilon. \end{aligned}\tag{5.16}$$

Now let's suppose $|k|x \ge 1$; we split the integral (5.12) into the following pieces:

$$\int_0^{\frac12} + \int_{\frac12}^{|k|x-\frac12} + \int_{\Gamma_2} + \int_{|k|x+\frac12}^x . \tag{5.17}$$

Here Γ_2 is a semicircle centered at $|k|x$ with radius $\frac12$. Again, Γ_2 is in the lower half plane if kx is in the upper half plane, and vice versa. For $0<\xi<\frac12$, $|\frac{e^{i\xi}-1}{\xi}| \le C$, and $|\xi - kx| \ge |k|x - \xi$. Hence

$$\begin{aligned}|k|x\left|\int_0^{\frac12} \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| &\le C|k|x\int_0^{\frac12}\frac{1}{|k|x-\xi}\,d\xi\\ &= C|k|x\log\left(\frac{|k|x}{|k|x-\frac12}\right)\\ &= C|k|x\log\left(1+\frac{1}{2|k|x-1}\right)\\ &\le C|k|x\log\left(1+\frac{1}{|k|x}\right) \le C|k|^\epsilon|x|^\epsilon. \end{aligned}\tag{5.18}$$

For $\frac12 < \xi < |k|x - \frac12$, $|\frac{e^{i\xi}-1}{\xi}| \le \frac{2}{\xi}$, and $|\xi - kx| \ge |k|x - \xi$. Hence

$$\begin{aligned}|k|x\left|\int_{\frac12}^{|k|x-\frac12} \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| &\le C|k|x\int_{\frac12}^{|k|x-\frac12}\frac{1}{\xi(|k|x-\xi)}\,d\xi\\ &\le C\log(2|k|x-1)\\ &\le C|k|^\epsilon|x|^\epsilon. \end{aligned}\tag{5.19}$$

For $\xi\in\Gamma_2$, $|\frac{e^{i\xi}-1}{\xi}| \le \frac{C}{|\xi|} \le \frac{C}{|k|x-\frac12}$, $|\xi-kx| \ge \frac12$. Thus

$$|k|x\left|\int_{\Gamma_2} \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| \le \frac{C|k|x}{|k|x-\frac12} \le C \le C|k|^\epsilon|x|^\epsilon. \tag{5.20}$$

Of course we used $|k|x\ge 1$. Finally, for $|k|x+\frac12 < \xi < x$, $|\frac{e^{i\xi}-1}{\xi}| \le \frac{2}{\xi}$, and $|\xi-kx| \ge \xi - |k|x$. Thus

$$\begin{aligned}|k|x\left|\int_{|k|x+\frac12}^x \frac{e^{i\xi}-1}{\xi}\frac{1}{\xi-kx}\,d\xi\right| &\le C|k|x\int_{|k|x+\frac12}^x\frac{1}{\xi(\xi-|k|x)}\,d\xi\\ &\le C\log[(1-|k|)(2|k|x+1)]\\ &\le C|k|^\epsilon(1+|x|)^\epsilon. \end{aligned}\tag{5.21}$$

This finishes the proof of (5.9) when $x > 0$. The proof for $x < 0$ is completely analogous. The case $x = 0$ is trivial. ∎

LEMMA 5.2.

$$G_0^0(x) = \lim_{N\to\infty} \frac{1}{2\pi}\int_0^N \frac{e^{ix\xi} - \chi(\xi)}{\xi}\, d\xi = -\frac{1}{2\pi}\log|x| + \frac{1}{2\pi}\begin{cases} c_1 & \text{if } x > 0,\\ c_2 & \text{if } x < 0,\end{cases} \tag{5.22}$$

and there is $C > 0$ *such that*

$$\left|\frac{1}{2\pi}\int_0^N \frac{e^{ix\xi} - \chi(\xi)}{\xi}\, d\xi\right| \le C + C|\log|x|| + \frac{C}{|x|^{\frac{1}{2}}}\chi_{\{|x|\le 1\}} \tag{5.23}$$

for all $N > 2$.

Proof. We write G_0^0 as

$$G_0^0(x) = \frac{1}{2\pi}\int_0^1 \frac{e^{ix\xi} - 1}{\xi}\, d\xi + \frac{1}{2\pi}\int_1^\infty \frac{e^{ix\xi}}{\xi}\, d\xi - \frac{1}{2\pi}\int_1^2 \frac{\chi(\xi)}{\xi}\, d\xi. \tag{5.24}$$

When $x > 0$, make a change of variable and recombine the integrals to get

$$\begin{aligned} G_0^0(x) &= -\frac{1}{2\pi}\int_1^x \frac{1}{\xi}\, d\xi + \frac{1}{2\pi}\int_0^1 \frac{e^{i\xi} - 1}{\xi}\, d\xi + \frac{1}{2\pi}\int_1^\infty \frac{e^{i\xi}}{\xi}\, d\xi \\ &\quad - \frac{1}{2\pi}\int_1^2 \frac{\chi(\xi)}{\xi}\, d\xi \\ &= -\frac{1}{2\pi}\log|x| + \frac{c_1}{2\pi}. \end{aligned} \tag{5.25}$$

When $x < 0$, a change of variable gives

$$\begin{aligned} G_0^0(x) &= -\frac{1}{2\pi}\int_{-1}^x \frac{1}{\xi}\, d\xi + \frac{1}{2\pi}\int_0^{-1} \frac{e^{i\xi} - 1}{\xi}\, d\xi + \frac{1}{2\pi}\int_{-1}^{-\infty} \frac{e^{i\xi}}{\xi}\, d\xi \\ &\quad - \frac{1}{2\pi}\int_1^2 \frac{\chi(\xi)}{\xi}\, d\xi \\ &= -\frac{1}{2\pi}\log|x| + \frac{c_2}{2\pi}. \end{aligned} \tag{5.26}$$

Next we assume $x > 0$, $N > 2$, and write the integral in (5.23) as

$$G_0^0(x) - \frac{1}{2\pi}\int_N^\infty \frac{e^{ix\xi}}{\xi}\, d\xi = G_0^0(x) - \frac{1}{2\pi}\int_{Nx}^\infty \frac{e^{i\xi}}{\xi}\, d\xi, \tag{5.27}$$

where $\int_{Nx}^\infty \frac{e^{i\xi}}{\xi}\, d\xi$ is easily seen to be bounded by

$$\begin{aligned} C + C\log\left(\frac{1}{Nx}\right)\chi_{\{Nx<1\}} &\le C + \frac{C}{|Nx|^{\frac{1}{2}}}\chi_{\{Nx<1\}} \\ &\le C + \frac{C}{|x|^{\frac{1}{2}}}\chi_{\{|x|\le 1\}} \end{aligned} \tag{5.28}$$

for $N > 2$. The proof for $x < 0$ is similar. ∎

The pointwise estimates established in the above lemmas imply estimates on the $L^\infty_{-(s-s_1)}$ operator norm of T_k^0.

LEMMA 5.3. *Let $s > s_1 > \frac{1}{2}$, and $u \in L^2_s(\mathbb{R})$. Let $k \in (\mathbb{C}\setminus[0,\infty))\cup(\mathbb{R}^+ \pm 0i)\cup\{0\}$. Then all T^0_k are compact on $L^\infty_{-(s-s_1)}(\mathbb{R})$, and there exist $\epsilon \in (0,1)$, $k_0 > 0$, and $C > 0$ such that for all $|k| < k_0$,*

$$\|T^0_k - T^0_0\|_{L^\infty_{-(s-s_1)} \to L^\infty_{-(s-s_1)}} \le C|k|^\epsilon. \tag{5.29}$$

Proof. When $k \neq 0$, T^0_k is compact since it is a rank one perturbation of T_k, which was shown to be compact in Lemma 3.2. The compactness of T^0_0 follows from (5.29), which we now show. In fact, by Lemma 5.1

$$\begin{aligned}
& w^{s_1-s}(x)|T^0_k(\varphi)(x) - T^0_0(\varphi)(x)| \\
&\quad \le w^{s_1-s}(x)\int_{\mathbb{R}} |G^0_k(x-y) - G^0_0(x-y)||u(y)\varphi(y)|\ dy \\
&\quad \le Cw^{s_1-s}(x)\int_{\mathbb{R}} |k|^\epsilon w^\epsilon(x-y)w^{-p_2}(y)|u_2(y)|\|\psi\|_{L^\infty_{-(s-s_1)}}\ dy \\
&\quad \le C|k|^\epsilon \|u_2\|_1 \|\psi\|_{L^\infty_{-(s-s_1)}},
\end{aligned} \tag{5.30}$$

where $p_2 = \frac{1}{2}(s_1 - \frac{1}{2}) > 0$ and $u_2 = uw^{s-s_1+p_2} \in L^1$. To get the last step above, we used $w(x-y) \le w(x)w(y)$, and $\epsilon < \min(s - s_1, p_2)$. ∎

The key to proving existence of modified Jost functions is to show invertibility of $I - T^0_0$, which by Lemma 5.3 reduces to showing triviality of its kernel. We accomplish this in several steps. First we show an identity that is crucial for later developments. It is for this identity that the complexity of $\chi(\xi)$ is needed. Recall that $\chi(\xi)$ is the cutoff function in the definition of G^0_0 and T^0_0 in (5.4).

LEMMA 5.4. *Suppose $s > s_1 > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$, $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$, and $\varphi = T^0_0\varphi$. If*

$$Im \int_1^2 \frac{\chi(\xi)}{\xi}\ d\xi \neq 0, \tag{5.31}$$

then

$$\int_{\mathbb{R}} u(y)\varphi(y)\ dy = 0. \tag{5.32}$$

Proof. By (5.4),

$$G^0_0(x) = \overline{G^0_0(-x)} - \frac{i}{\pi}\ \mathrm{Im}\int_1^2 \frac{\chi(\xi)}{\xi}\ d\xi. \tag{5.33}$$

Therefore by $\varphi = T^0_0\varphi = G^0_0 * (u\varphi)$,

$$\begin{aligned}
\langle\varphi, u\varphi\rangle = \langle G^0_0 * (u\varphi), u\varphi\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} G^0_0(x-y)u(y)\varphi(y)u(x)\overline{\varphi(x)}\ dy\ dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\overline{G^0_0(y-x)} - \frac{i}{\pi}\ \mathrm{Im}\int_1^2 \frac{\chi(\xi)}{\xi}\ d\xi\right) u(y)\varphi(y)u(x)\overline{\varphi(x)}\ dy\ dx \\
&= -\frac{i}{\pi}\left(\mathrm{Im}\int_1^2 \frac{\chi(\xi)}{\xi}\ d\xi\right)|\langle\varphi, u\rangle|^2 + \int_{\mathbb{R}}\int_{\mathbb{R}} u(y)\varphi(y)\overline{G^0_0(y-x)u(x)\varphi(x)}\ dx\ dy \\
&= -\frac{i}{\pi}\left(\mathrm{Im}\int_1^2 \frac{\chi(\xi)}{\xi}\ d\xi\right)|\langle\varphi, u\rangle|^2 + \langle u\varphi, G^0_0 * (u\varphi)\rangle \\
&= -\frac{i}{\pi}\left(\mathrm{Im}\int_1^2 \frac{\chi(\xi)}{\xi}\ d\xi\right)|\langle\varphi, u\rangle|^2 + \langle u\varphi, \varphi\rangle.
\end{aligned} \tag{5.34}$$

Since $\langle \varphi, u\varphi \rangle = \langle u\varphi, \varphi \rangle$, (5.34) and (5.31) imply $\langle \varphi, u \rangle = 0$, which is (5.32). We provide the estimates needed to apply Fubini's theorem in the calculation above. In fact, by (5.22),

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} |G_0^0(x-y)||u(y)\varphi(y)||u(x)\varphi(x)|\; dy\; dx\\ &\leq C\|u\varphi\|_1^2 + C\int_{\mathbb{R}}\int_{\mathbb{R}} |\log|x-y|||u(y)\varphi(y)||u(x)\varphi(x)|\; dy\; dx.\end{aligned} \tag{5.35}$$

We split the integral $\int_{\mathbb{R}} |\log|x-y|||u(y)\varphi(y)|\; dy$ at $|x-y|=1$ and estimate

$$\int_{|x-y|<1} |\log|x-y|||u(y)\varphi(y)|\; dy \leq \|\chi_{\{|x|<1\}} \log x\|_2 \|u\varphi\|_2 \tag{5.36}$$

and

$$\begin{aligned}\int_{|x-y|>1} \log|x-y||u(y)\varphi(y)|\; dy &\leq \int_{|x-y|>1} (1+|x-y|)^\epsilon |u(y)\varphi(y)|\; dy\\ &\leq (1+|x|)^\epsilon \int_{\mathbb{R}} (1+|y|)^\epsilon w^{-s_1}(y)|u_1(y)\varphi_1(y)|\; dy\\ &\leq C\|u_1\|_2(1+|x|)^\epsilon.\end{aligned} \tag{5.37}$$

Here $u_1 = w^s u \in L^2$, $\varphi_1 = w^{-(s-s_1)}\varphi \in L^\infty$, and $\epsilon > 0$ is chosen so that $s_1 - \epsilon > \frac{1}{2}$. The estimates above imply the finiteness of (5.35). ∎

The key vanishing integral (5.32) implies the following decay estimate for functions in the kernel of $I - T_0^0$.

Lemma 5.5. *Suppose $s > s_1 > \frac{1}{2}$, $u \in L_s^2(\mathbb{R})$, and that* (5.31) *holds. If $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$, and $\varphi = T_0^0\varphi$, then there exists $C = C(u, s, s_1)$ such that*

$$|\varphi(x)| \leq Cw^{-1}(x). \tag{5.38}$$

In particular $\varphi \in L^2(\mathbb{R})$.

Proof. By (5.22),

$$\begin{aligned}\varphi(x) = T_0^0\varphi(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} G_0^0(x-y)u(y)\varphi(y)\; dy\\ &= \frac{1}{2\pi}\int_{-\infty}^x (c_1 - \log(x-y))u(y)\varphi(y)\; dy\\ &\quad + \frac{1}{2\pi}\int_x^\infty (c_2 - \log(y-x))u(y)\varphi(y)\; dy\\ &= -\frac{1}{2\pi}\int_x^\infty \log(y-x)u(y)\varphi(y)\; dy\\ &\quad - \frac{1}{2\pi}\int_{-\infty}^x \log(x-y)u(y)\varphi(y)\; dy + R(x).\end{aligned} \tag{5.39}$$

By (5.32), $R(x)$ can be written in two ways:

$$R(x) = \frac{1}{2\pi}\int_x^\infty (c_2 - c_1)u(y)\varphi(y)\; dy = \frac{1}{2\pi}\int_{-\infty}^x (c_1 - c_2)u(y)\varphi(y)\; dy. \tag{5.40}$$

We now start a bootstrap argument, assuming $\varphi \in L^\infty_r$ for some $r \geq -(s-s_1)$. We have $u = w^{-s}u_1$ for some $u_1 \in L^2$ and $\varphi = w^{-r}\varphi_1$ for some $\varphi_1 \in L^\infty$. Hence $u\varphi = w^{-(r+s)}u_1\varphi_1$. Since $r \geq -(s-s_1)$, $r+s \geq s_1 > \frac{1}{2}$. Letting $p_2 = \frac{1}{2}(s_1 - \frac{1}{2}) > 0$, we get

$$u\varphi = w^{-(r+\Delta r)}w^{-\frac{1}{2}-p_2}u_1\varphi_1 = w^{-(r+\Delta r)}u_2\varphi_1. \tag{5.41}$$

Here

$$\Delta r = s - s_1 + p_2 > p_2 > 0, \tag{5.42}$$

$$r + \Delta r = r + s - s_1 + p_2 \geq p_2 > 0, \tag{5.43}$$

and $u_2 = w^{-\frac{1}{2}-p_2}u_1 \in L^1 \cap L^2$.

We assume $x > 0$. Use the first expression in (5.40) for $R(x)$ to get

$$|R(x)| \leq C\int_x^\infty w^{-(r+\Delta r)}(y)|u_2(y)\psi_1(y)|\, dy \leq Cw^{-(r+\Delta r)}(x). \tag{5.44}$$

Next we write the first integral in (5.39) as

$$\int_x^\infty \log(y-x)w^{-(r+\Delta r)}(y)u_2(y)\varphi_1(y)\, dy \tag{5.45}$$

and split the integral at $x+1$:

$$\begin{aligned}&\left|\int_x^{x+1} \log(y-x)w^{-(r+\Delta r)}(y)u_2(y)\varphi_1(y)\, dy\right| \\ &\quad \leq Cw^{-(r+\Delta r)}(x)\big\|\chi_{\{|y|\leq 1\}}\log y\big\|_2\|u_2\varphi_1\|_2.\end{aligned} \tag{5.46}$$

When $y - x > 1$, there exists $C = C(\epsilon)$ for every $\epsilon > 0$ such that $\log(y-x) \leq Cw^\epsilon(x-y) \leq Cw^\epsilon(x)w^\epsilon(y)$. Take $\epsilon = \frac{p_2}{4}$. We have $r + \Delta r - \epsilon \geq p_2 - \epsilon > 0$ and $\Delta r > p_2 > 4\epsilon$. Thus

$$\begin{aligned}&\left|\int_{x+1}^\infty \log(y-x)w^{-(r+\Delta r)}(y)u_2(y)\varphi_1(y)\, dy\right| \\ &\quad \leq Cw^\epsilon(x)\int_{x+1}^\infty w^{\epsilon-(r+\Delta r)}(y)|u_2(y)\varphi_1(y)|\, dy \\ &\quad \leq Cw^{2\epsilon-(r+\Delta r)}(x)\|u_2\varphi_1\|_1 \leq Cw^{-(r+\frac{\Delta r}{2})}(x).\end{aligned} \tag{5.47}$$

In summary, the first integral in (5.39) is bounded as follows:

$$\left|\int_x^\infty \log(y-x)u(y)\varphi(y)\, dy\right| \leq Cw^{-(r+\frac{\Delta r}{2})}(x). \tag{5.48}$$

We now focus on the second integral in (5.39). When $0 < x < 1$, we split the integral at -1 and estimate

$$\left|\int_{-\infty}^{-1} \log(x-y)u(y)\varphi(y)\, dy\right| \leq C\int_{-\infty}^{-1}\log(1+|y|)|u(y)\varphi(y)|\, dy \leq C, \tag{5.49}$$

$$\left|\int_{-1}^{x} \log(x-y)u(y)\varphi(y)\, dy\right| \leq C\big\|\chi_{\{|y|\leq 2\}}\log y\big\|_2\|u\varphi\|_2. \tag{5.50}$$

When $x > 1$, we use (5.32) again to rewrite the second integral in (5.39) as

$$\begin{aligned} &\int_{-\infty}^{x} (\log(x-y) - \log x) w^{-(r+\Delta r)}(y) u_2(y) \varphi_1(y)\ dy \\ &- \log x \int_{x}^{\infty} w^{-(r+\Delta r)}(y) u_2(y) \varphi_1(y)\ dy. \end{aligned} \tag{5.51}$$

The last term in (5.51) is easily seen to be bounded by

$$(\log x) w^{-(r+\Delta r)}(x) \|u_2 \varphi_1\|_1 \leq C w^{-(r+\frac{\Delta r}{2})}(x). \tag{5.52}$$

We split the first integral in (5.51) at $\frac{x}{2}$ and estimate as follows. When $y < \frac{x}{2}$, we have

$$\left|\log\left(1 - \frac{y}{x}\right)\right| \leq C \left|\frac{y}{x}\right|^p, \tag{5.53}$$

where $p = \min(1, r + \frac{\Delta r}{2})$. Thus

$$\begin{aligned} &\left| \int_{-\infty}^{\frac{x}{2}} \log\left(1 - \frac{y}{x}\right) w^{-(r+\Delta r)}(y) u_2(y) \varphi_1(y)\ dy \right| \\ &\quad \leq C|x|^{-p} \int_{-\infty}^{\frac{x}{2}} |y|^p w^{-(r+\Delta r)}(y) |u_2(y)\varphi_1(y)|\ dy \\ &\quad \leq C w^{-p}(x) \|u_2 \varphi_1\|_1. \end{aligned} \tag{5.54}$$

To estimate the $y > \frac{x}{2}$ piece of the first integral in (5.51), we use the first expression for $u\varphi$ in (5.41) and get

$$\begin{aligned} &\left| \int_{\frac{x}{2}}^{x} \log\left(1 - \frac{y}{x}\right) w^{-(r+\Delta r)-(\frac{1}{2}+p_2)}(y) u_1(y) \varphi_1(y)\ dy \right| \\ &\quad \leq C w^{-(r+\Delta r)-(\frac{1}{2}+p_2)}(x) \left(\int_{\frac{x}{2}}^{x} \left|\log\left(1 - \frac{y}{x}\right)\right|^2\ dy \right)^{\frac{1}{2}} \|u_1 \varphi_1\|_2 \\ &\quad \leq C w^{-(r+\Delta r)-(\frac{1}{2}+p_2)}(x) \left(\int_0^{\frac{1}{2}} |\log z|^2\ dz \right)^{\frac{1}{2}} x^{\frac{1}{2}} \\ &\quad \leq C w^{-(r+\Delta r)}(x). \end{aligned} \tag{5.55}$$

This completes the estimation of (5.39) when $x > 0$. The arguments for $x < 0$ are completely analogous, as long as one uses the second expression in (5.40) for $R(x)$. In summary, we get from the above estimates that $\varphi \in L^\infty_{r+\frac{\Delta r}{2}}$ if $r + \frac{\Delta r}{2} < 1$ and $\varphi \in L^\infty_1$ if $r + \frac{\Delta r}{2} \geq 1$. The result thus follows from finitely many iterations of the above estimates. ∎

Next we show that any function in the kernel of $I - T_0^0$ satisfies the same type of eigenvalue equation as do the Jost functions, regardless of the choice of $\chi(\xi)$.

LEMMA 5.6. *Let $s > s_1 > \frac{1}{2}$, and $u \in L^2_s(\mathbb{R})$. If $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$ satisfies $\varphi = T_0^0 \varphi$, then*

$$\frac{1}{i} \partial_x \varphi - C_+(u\varphi) = 0 \tag{5.56}$$

in the sense of tempered distributions.

Proof. Let $\psi(x)$ be any test function in $C_0^\infty(\mathbb{R})$. Let $M > 1$ be such that $[-M, M]$ contains the support of ψ. There exists $C = C(M, \psi)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left|1 + |\log|x-y|| + \frac{1}{|x-y|^{\frac{1}{2}}}\chi_{\{|x-y|\le 1\}}\right| |u(y)\varphi(y)\psi'(x)|\ dy\ dx \\
&\quad \le C\int_{\mathbb{R}}\int_{-M}^{M}\left(1 + |\log|x-y|| + \frac{1}{|x|^{\frac{1}{2}}}\right)\ dx\ |u(y)\varphi(y)|\ dy \\
&\quad \le C\int_{\mathbb{R}}(1+\log(|y|+M))|u(y)\varphi(y)|\ dy < \infty. \qquad (5.57)
\end{aligned}$$

Therefore, by (5.23), (5.57), and the dominated convergence theorem,

$$\begin{aligned}
\int_{\mathbb{R}}\varphi(x)\psi'(x)\ dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} G_0^0(x-y)u(y)\varphi(y)\ dy\ \psi'(x)\ dx \\
&= \lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^N \frac{e^{i(x-y)\xi}-\chi(\xi)}{\xi}\ d\xi\ u(y)\varphi(y)\psi'(x)\ dy\ dx. \qquad (5.58)
\end{aligned}$$

We want to use the Fubini theorem to change the order of integration. To that end, we observe that there is $C = C(M, \psi, N)$ such that

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^N \left|\frac{e^{i(x-y)\xi}-\chi(\xi)}{\xi}\right| |\psi'(x)||u(y)\varphi(y)|\ d\xi\ dx\ dy \\
&\quad \le C\int_{\mathbb{R}}\int_{-M}^{M}\left(\int_0^1\left|\frac{e^{i(x-y)\xi}-1}{\xi}\right|\ d\xi + C\right)|u(y)\varphi(y)|\ dx\ dy \\
&\quad \le C\int_{\mathbb{R}}\int_{-M}^{M}(1+|\log|x-y||)|u(y)\varphi(y)|\ dx\ dy \\
&\quad \le C\int_{\mathbb{R}}(1+\log(|y|+M))|u(y)\varphi(y)|\ dx\ dy < \infty. \qquad (5.59)
\end{aligned}$$

Hence (5.58) equals to

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}}\int_0^N\int_{\mathbb{R}}\left(\frac{e^{i(x-y)\xi}-\chi(\xi)}{\xi}\right)\psi'(x)\ dx\ d\xi\ u(y)\varphi(y)\ dy \\
&\quad = -i\lim_{N\to\infty}\int_{\mathbb{R}}\frac{1}{2\pi}\int_0^N\int_{\mathbb{R}} e^{i(x-y)\xi}\psi(x)\ dx\ d\xi\ u(y)\varphi(y)\ dy \\
&\quad = -i\lim_{N\to\infty}\int_{\mathbb{R}}\psi(x)\frac{1}{2\pi}\int_0^N e^{ix\xi}\int_{\mathbb{R}} e^{-iy\xi}u(y)\varphi(y)\ dy\ d\xi\ dx \\
&\quad = -i\int_{\mathbb{R}}\psi(x)C_+(u\varphi)(x)\ dx. \qquad (5.60)
\end{aligned}$$

The last step follows from the fact that $\frac{1}{2\pi}\int_0^N e^{ix\xi}\int_{\mathbb{R}} e^{-iy\xi}u(y)\varphi(y)\ dy\ d\xi$ converges to $C_+(u\varphi)(x)$ in L^2. The above calculation shows

$$\int_{\mathbb{R}}\varphi(x)\psi'(x)\ dx = -i\int_{\mathbb{R}}\psi(x)C_+(u\varphi)(x)\ dx, \qquad (5.61)$$

which gives (5.56). ∎

We are now ready to prove the key vanishing lemma.

LEMMA 5.7. *Suppose $s > s_1 > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$, and that* (5.31) *holds. If $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$ and $\varphi = T^0_0\varphi$, then $\varphi = 0$.*

Proof. At this point, we can basically repeat the proof of Lemma 3.4 for the case $k < 0$. Only in the present case, $k = 0$. All calculations can be justified now that we know the decay estimate (5.38). One has from Lemma 5.6 that

$$\chi_{\mathbb{R}^+}\widehat{u\varphi} = \xi\hat{\varphi}. \tag{5.62}$$

By (5.38) and the condition on u, we have $(1 + |x|)u\varphi \in L^1$, Hence $\widehat{u\varphi} \in C^1(\mathbb{R})$. Recall that $\widehat{u\varphi}(0) = 0$ by Lemma 5.4. Hence

$$\hat{\varphi}(0+) = \lim_{\xi\to 0^+} \frac{\widehat{u\varphi}(\xi)}{\xi} = \widehat{u\varphi}'(0). \tag{5.63}$$

We repeat the argument in Lemma 3.4 to get (3.71), which now becomes

$$2\pi \int_{\mathbb{R}} |\varphi|^2 \, dx = 0. \tag{5.64}$$ ∎

We can now prove existence of the modified Jost functions.

THEOREM 5.8. *Let $s > s_1 > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$, $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i) \cup \{0\}$, $\lambda \geq 0$, and $\chi(\xi)$ satisfy* (5.31). *Then there is $k_0 > 0$ such that for all $|k|, \lambda < k_0$, there exist unique solutions $m^0_1(x,k), m^0_e(x, \lambda \pm 0i) \in L^\infty_{-(s-s_1)}(\mathbb{R})$ to* (5.5) *and* (5.6). *Furthermore, there are $C > 0$ and $\epsilon \in (0,1)$ such that*

$$\|m^0_1(k) - m^0_1(0)\|_{L^\infty_{-(s-s_1)}} \leq C|k|^\epsilon, \tag{5.65}$$

$$\|m^0_e(\lambda \pm 0i) - m^0_1(0)\|_{L^\infty_{-(s-s_1)}} \leq C\lambda^\epsilon. \tag{5.66}$$

Proof. Lemma 5.3, Lemma 5.7, and the Fredholm alternative theorem imply the invertibility of $I - T^0_k$, from which we obtain existence and uniqueness of m^0_1 and m^0_e. The asymptotic bounds (5.65), (5.66) follow from Lemma 5.3 and the fact that $\|e^{i\lambda x} - 1\|_{L^\infty_{-(s-s_1)}} \leq C\lambda^\epsilon$. ∎

We can obtain asymptotic formulas for the original Jost functions and scattering coefficients as k approaches 0, since the original Jost functions can be expressed in terms of the modified Jost functions as in (5.7) and (5.8). At this point, it is useful to make a division between two distinct cases.

DEFINITION 5.9. *Let $u \in L^2_s(\mathbb{R})$, and let $m^0_1(x,0)$ be constructed as in Theorem* 5.8. *u is called a generic potential if $\int_{\mathbb{R}} u(x)m^0_1(x,0)\, dx \neq 0$, or a nongeneric potential if $\int_{\mathbb{R}} u(x)m^0_1(x,0)\, dx = 0$.*

Notice that $m^0_1(x,0)$ actually depends on the choice of the cutoff function $\chi(\xi)$ when we regularize T_k to T^0_k. However, the definition of genericity does not depend on the choice of $\chi(\xi)$, as is shown in the following lemma. To state the lemma, let $\chi^{(1)}(\xi)$ and $\chi^{(2)}(\xi)$ be smooth functions on $[0,\infty)$, which are identically equal to 1 on $[0,1]$, and identically equal to 0 on $[2,\infty)$. We use the notation $G^{0(1)}_0$, $G^{0(2)}_0$, etc., to denote the corresponding objects constructed using $\chi^{(1)}(\xi)$ and $\chi^{(2)}(\xi)$.

LEMMA 5.10. *Let $\chi^{(1)}(\xi)$ and $\chi^{(2)}(\xi)$ be given as above, and let u be given as in Theorem* 5.8. *Suppose $\chi^{(1)}(\xi)$ satisfy* (5.31), *and let $m^{0(1)}_1(x,0)$ be the Jost solution constructed in Theorem* 5.8. *If*

$$\int_{\mathbb{R}} u(x)m^{0(1)}_1(x,0)\, dx = 0, \tag{5.67}$$

then

(a) $I - T_0^{0(2)}$ *is invertible on* $L^\infty_{-(s-s_1)}(\mathbb{R})$,

(b) $m_1^{0(2)}(x,0) = (I - T_0^{0(2)})^{-1}1$ *is the same as* $m_1^{0(1)}(x,0)$.

Proof. Part (a) is of course already established in Theorem 5.8 if $\chi^{(2)}(\xi)$ satisfies (5.31). The interesting point, however, is that when u is nongeneric, $I - T_0^{0(2)}$ must still be invertible when $\int_1^2 \frac{\chi^{(2)}(\xi)}{\xi}\, d\xi$ is real. To prove that, we need to show that any $\varphi \in L^\infty_{-(s-s_1)}$ satisfying $\varphi = T_0^{0(2)}\varphi$ must be zero. Examining the sequence of lemmas before Theorem 5.8, we find that the only place (5.31) was used was to establish the key vanishing integral (5.32), which we now show by different means. In fact, we observe that

$$G_0^{0(2)}(x) = G_0^{0(1)}(x) + \frac{1}{2\pi}\int_1^2 \frac{\chi^{(1)}(\xi) - \chi^{(2)}(\xi)}{\xi}\, d\xi = G_0^{0(1)}(x) + c. \tag{5.68}$$

We have

$$\varphi = T_0^{0(2)}\varphi = G_0^{0(2)} * (u\varphi) = c\langle \varphi, u\rangle + G_0^{0(1)} * (u\varphi) = c\langle\varphi, u\rangle + T_0^{0(1)}\varphi. \tag{5.69}$$

Thus $\varphi = c\langle\varphi, u\rangle m_1^{0(1)}(0)$. Since $\langle m_1^{0(1)}(0), u\rangle = 0$ by (5.67), $\langle\varphi, u\rangle = 0$, which is the key vanishing integral (5.32). Part (a) can be proven by the same arguments following (5.32).

To show part (b), we observe that

$$\begin{aligned} G_0^{0(2)} * (u m_1^{0(1)}(0)) &= G_0^{0(1)} * (u m_1^{0(1)}(0)) + c\langle m_1^{0(1)}(0), u\rangle \\ &= G_0^{0(1)} * (u m_1^{0(1)}(0)) \\ &= m_1^{0(1)}(0) - 1. \end{aligned}$$

The result now follows by uniqueness. ∎

We are now ready to state and compute the asymptotics of the Jost functions and scattering coefficients as k approaches 0.

THEOREM 5.11. *Let* $s > s_1 > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$, $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$, *and* $\lambda > 0$. *Let* $m_1(x,k)$ *and* $m_e(x, \lambda \pm 0i)$ *be constructed as in Theorem* 3.5, *and let* $\Gamma(\lambda)$ *and* $\beta(\lambda)$ *be defined as in Lemma* 4.2. *Let* $m_0^0(x,0)$ *be constructed as in Theorem* 5.8. *Then there exists* $\epsilon \in (0,1)$ *such that as* k *approaches* 0 *and* λ *approaches* 0^+,

(a) *if* u *is a generic potential,*

$$m_1(k) = \frac{2\pi}{\langle m_1^0(0), u\rangle \log k} m_1^0(0) + O\left(\frac{1}{|\log^2 k|}\right), \tag{5.70}$$

$$m_e(\lambda \pm 0i) = \frac{2\pi}{\langle m_1^0(0), u\rangle \log(\lambda \pm 0i)} m_1^0(0) + O\left(\frac{1}{|\log^2 \lambda|}\right), \tag{5.71}$$

$$\Gamma(\lambda) = 1 + \frac{2\pi i}{\log(\lambda + 0i)} + O\left(\frac{1}{|\log^2 \lambda|}\right), \tag{5.72}$$

$$\beta(\lambda) = \frac{2\pi i}{\log(\lambda + 0i)} + O\left(\frac{1}{|\log^2 \lambda|}\right); \tag{5.73}$$

(b) *if u is a nongeneric potential,*

$$m_1(k) = m_1^0(0) + O(|k|^\epsilon |\log k|), \tag{5.74}$$

$$m_e(\lambda \pm 0i) = m_1^0(0) + O(\lambda^\epsilon |\log \lambda|), \tag{5.75}$$

$$\Gamma(\lambda) = 1 + O(\lambda^\epsilon |\log \lambda|), \tag{5.76}$$

$$\beta(\lambda) = O(\lambda^\epsilon |\log \lambda|). \tag{5.77}$$

Here the function log *takes the principle branch, with a branch cut on* $[0, \infty)$*. The big O notation has the usual meaning in equations involving Γ and β but holds in the sense of $L^\infty_{-(s-s_1)}(\mathbb{R})$ norm in equations involving m_1 and m_e.*

Proof. The proof is a straightforward calculation using (5.7), (5.8), (5.65), (5.66), and the definitions of $\Gamma(\lambda)$ and $\beta(\lambda)$. We only need to observe that

$$\begin{aligned} l(k) &= \frac{1}{2\pi} \int_0^\infty \frac{\chi(\xi)}{\xi - k}\, d\xi \\ &= \frac{1}{2\pi} \int_0^1 \frac{1}{\xi - k}\, d\xi + \frac{1}{2\pi} \int_1^2 \frac{\chi(\xi)}{\xi - k}\, d\xi \\ &= \frac{1}{2\pi} \log k + h(k), \end{aligned} \tag{5.78}$$

where log takes the principle branch with branch cut $[0, \infty)$, and $h(k)$ is analytic around $k = 0$. ∎

6. Asymptotic behavior near $k = \infty$. In this section, we obtain asymptotic formulas for the Jost functions and scattering coefficients as k approaches ∞ in the cut plane. The situation of large k limit is very different from that of small k limit discussed in section 5. As we will see in the following, the operator $I - T_k$ can be inverted explicitly when $|k|$ is sufficiently large. This allows explicit calculation and estimation of error. Similar to the situation of the Fourier transform, high regularity and decay of the potential u imply high regularity and decay of the scattering coefficients as k tends to ∞. The precise assumptions on u and the corresponding decay estimates on the scattering coefficients may vary according to the needs in application. As an example, we work in this section with the following three types of assumptions on u: $u \in L^2_s(\mathbb{R})$ with $s > \frac{1}{2}$, $u \in H^s_s(\mathbb{R})$ with $s > \frac{1}{2}$, and u in the Schwartz class $\mathcal{S}$. The first type of spaces keeps the same assumption on u as in the previous sections. The second type of spaces will provide the proper assumption to obtain a higher order term for $m_1(x, k)$. Finally, the choice of the Schwartz class will allow us to see how rapid decay of the scattering coefficients may be obtained, without having to formulate the regularity and decay assumptions on u too carefully.

To begin, let's use the weakest of the three types of assumptions, $u \in L^2_s(\mathbb{R})$, and show how $I - T_k$ can be inverted explicitly on $L^\infty_{-(s-s_1)}(\mathbb{R})$, when $s > s_1 > \frac{1}{2}$. First assume k is in a fixed Stolz angle away from the positive real line. In other words, there exists $\alpha \in (0, \frac{\pi}{2})$ such that

$$|\mathrm{Im}\ k| \geq (\tan \alpha) \mathrm{Re}\ k.$$

For any such k and any $\xi > 0$, $|\xi - k|$ and $\xi + |k|$ are comparable:

$$0 < \frac{1}{C_\alpha} \leq \frac{|\xi - k|}{\xi + |k|} \leq C_\alpha.$$

Therefore by the definition of T_k and G_k given in (3.42) and (3.6),

$$\|T_k\varphi\|_\infty \le \|G_k\|_2\|u\varphi\|_2 \le C\|G_k\|_2\|\varphi\|_{L^\infty_{-(s-s_1)}}, \tag{6.1}$$

where

$$\|G_k\|_2 \le C\left(\int_0^\infty \frac{1}{(\xi+|k|)^2}\, d\xi\right)^{\frac{1}{2}} \le \frac{C}{\sqrt{|k|}}. \tag{6.2}$$

It follows that $\|T_k\|_{L^\infty_{-(s-s_1)}\to L^\infty} \le \frac{C}{\sqrt{|k|}}$. Therefore $(I-T_k)^{-1} = \sum_{n=0}^\infty T_k^n$ when $|k|$ is large. To invert $I-T_k$ when k is close to the positive real line, we write $T_k = S_k - \widetilde{T}_k$ by (3.8), where for $k=\lambda\pm\mu i$, with $\lambda>0$, $\mu\ge 0$,

$$S_k\varphi(x) = i\int_{\mp\infty}^x e^{ik(x-y)}u(y)\varphi(y)\, dy \tag{6.3}$$

and

$$\widetilde{T}_k\varphi = \widetilde{G}_k * (u\varphi), \quad \widetilde{G}_k = \frac{1}{2\pi}\int_{-\infty}^0 \frac{e^{ix\xi}}{\xi-k}\, d\xi. \tag{6.4}$$

Now that $k=\lambda\pm\mu i$ with $\lambda>0$, k is in a fixed Stolz angle away from the negative real line. By the same argument as above, we have $\|\widetilde{T}_k\|_{L^\infty_{-(s-s_1)}\to L^\infty} \le \frac{C}{\sqrt{|k|}}$. On the other hand, $I-S_k$ can be inverted explicitly by solving an ODE. In fact, we can rewrite

$$\varphi = S_k\varphi + g = g + i\int_{\mp\infty}^x e^{ik(x-y)}u(y)\varphi(y)\, dy \tag{6.5}$$

as

$$(\varphi-g)e^{-ikx} = i\int_{\mp\infty}^x e^{-iky}u(y)\varphi(y)\, dy. \tag{6.6}$$

Differentiating with respect to x and rearranging terms using an integrating factor, we get

$$\left[e^{-i\int_{\mp\infty}^x u(t)\, dt}e^{-ikx}(\varphi-g)\right]_x = ie^{-i\int_{\mp\infty}^x u(t)\, dt}e^{-ikx}ug. \tag{6.7}$$

By (6.6), $\varphi(x)-g(x)\to 0$ as $x\to\mp\infty$. Hence we may integrate (6.7) from $\mp\infty$ and get

$$\varphi(x) = g(x) + i\int_{\mp\infty}^x e^{ik(x-y)}e^{i\int_y^x u(t)\, dt}u(y)g(y)\, dy. \tag{6.8}$$

The right-hand side of (6.8) is $(I-S_k)^{-1}g$. It is easy to see that the operator norm of $(I-S_k)^{-1}$ is bounded uniformly in k for $|k|$ large. Combining the calculation above, we may write

$$\begin{aligned}(I-T_k)^{-1} &= (I-S_k+\widetilde{T}_k)^{-1} = (I+(I-S_k)^{-1}\widetilde{T}_k)^{-1}(I-S_k)^{-1}\\ &= \sum_{n=0}^\infty(-(I-S_k)^{-1}\widetilde{T}_k)^n(I-S_k)^{-1}.\end{aligned} \tag{6.9}$$

We have thus proved the following lemma.

LEMMA 6.1. *Let $s > s_1 > \frac{1}{2}$ and $u \in L^2_s(\mathbb{R})$. Let $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$. There exists $k_0 > 0$ such that for $|k| > k_0$, $I - T_k$ is invertible on $L^\infty_{-(s-s_1)}(\mathbb{R})$, and*

(a) *if k is in a fixed Stolz angle away from the positive real line, i.e., there exists $\alpha \in (0, \frac{\pi}{2})$ such that*

$$|Im\ k| \geq (\tan\alpha) Re\ k,$$

then

$$\|T_k\|_{L^\infty_{-(s-s_1)} \to L^\infty} \leq \frac{C_\alpha}{\sqrt{|k|}} \tag{6.10}$$

and

$$(I - T_k)^{-1} = \sum_{n=0}^{\infty} T_k^n; \tag{6.11}$$

(b) *if $k = \lambda \pm i\mu$, with $\lambda > 0$, $\mu \geq 0$, then for $\widetilde{T}_k$ given in (6.4), and*

$$R_k\varphi(x) = i\int_{\mp\infty}^{x} e^{ik(x-y)} e^{i\int_y^x u(t)\ dt} u(y)\varphi(y)\ dy, \tag{6.12}$$

we have

$$\|\widetilde{T}_k\|_{L^\infty_{-(s-s_1)} \to L^\infty} \leq \frac{C}{\sqrt{|k|}}, \quad \|R_k\|_{L^\infty_{-(s-s_1)} \to L^\infty} \leq C \tag{6.13}$$

and

$$(I - T_k)^{-1} = \sum_{n=0}^{\infty} (-(I + R_k)\widetilde{T}_k)^n (I + R_k). \tag{6.14}$$

The calculation of the scattering coefficients will be simplified by the following lemma.

LEMMA 6.2. *Let u and $R_{\lambda+0i}$ be given as in Lemma 6.1. If $\varphi \in L^\infty_{-(s-s_1)}(\mathbb{R})$, then*

$$\int_{\mathbb{R}} u(x) e^{-i\lambda x} [(I + R_{\lambda+0i})\varphi](x)\ dx = \int_{\mathbb{R}} u(x) e^{-i\lambda x} e^{i\int_x^\infty u(t)\ dt} \varphi(x)\ dx. \tag{6.15}$$

Proof. Recall that $u \in L^1$ and $u\varphi \in L^1$ by the conditions on u and φ. Therefore by Fubini's theorem

$$\begin{aligned} &\int_{\mathbb{R}} u(x) e^{-i\lambda x} [R_{\lambda+0i}\varphi](x)\ dx \\ &= \int_{\mathbb{R}} u(x) e^{-i\lambda x} i \int_{-\infty}^{x} e^{i\lambda(x-y)} e^{i\int_y^x u(t)\ dt} u(y)\varphi(y)\ dy\ dx \\ &= \int_{\mathbb{R}} u(y)\varphi(y) e^{-i\int_{-\infty}^{y} u(t)\ dt} e^{-i\lambda y} \int_y^\infty \left(e^{i\int_{-\infty}^{x} u(t)\ dt}\right)_x dx\ dy \\ &= \int_{\mathbb{R}} u(y) e^{-i\lambda y} e^{i\int_y^\infty u(t)\ dt} \varphi(y)\ dy - \int_{\mathbb{R}} u(y) e^{-i\lambda y} \varphi(y)\ dy. \end{aligned} \tag{6.16}$$

Equation (6.15) thus follows. ∎

We want to use the inversion formulas in Lemma 6.1 to compute asymptotics of $m_1(x,k)$, $m_e(x,\lambda \pm 0i)$, $\Gamma(\lambda)$, $\beta(\lambda)$, and $f(\lambda)$. By relations (4.28) and (4.42), we only need to study $m_1(x,k)$, $m_e(x,\lambda+0i)$, $\Gamma(\lambda)$, and $\beta(\lambda)$.

THEOREM 6.3. *Let $s > \frac{1}{2}$, $u \in L^2_s(\mathbb{R})$, $k \in (\mathbb{C} \setminus [0,\infty)) \cup (\mathbb{R}^+ \pm 0i)$, and $\lambda > 0$. Then*

$$\lim_{k\to\infty} m_1(x,k) = 1, \tag{6.17}$$

$$\lim_{\lambda\to\infty} m_e(x,\lambda+0i) - e^{i\lambda x} e^{i\int_{-\infty}^x u(t)\ dt} = 0, \tag{6.18}$$

$$\Gamma(\lambda) - e^{i\int_{\mathbb{R}} u(t)\ dt} = O\left(\frac{1}{\lambda}\right) \text{ as } \lambda\to\infty, \tag{6.19}$$

and

$$\beta(\lambda) \in L^2(a,\infty), \quad \lim_{\lambda\to\infty} \beta(\lambda) = 0. \tag{6.20}$$

Here the limits for m_1 and m_e hold in $L^\infty(\mathbb{R})$ norm, and $a>0$ is any fixed number.

Proof. We first work on $m_1(k) = (I-T_k)^{-1}1$. If k is in the left half plane, we use (6.11) and the fact that $\|\sum_{n=1}^\infty T_k^n 1\|_\infty \le \frac{C}{\sqrt{|k|}}$ to conclude (6.17). If k is in the right half cut plane, we use (6.14) to write

$$m_1(k) = (I+R_k)1 + \sum_{n=1}^\infty (-(I+R_k)\widetilde{T}_k)^n (I+R_k)1 \tag{6.21}$$

and use (6.13) to conclude that the infinite sum in (6.21) has L^∞ norm bounded by $\frac{C}{\sqrt{|k|}}$. What is left to show is that $\|R_k 1\|_\infty \to 0$ as k approaches ∞ in the right half cut plane. For simplicity of presentation, let us work only with the case $k = \lambda + i\mu$ with $\lambda > 0$, $\mu \ge 0$. In the following proof, this is always assumed. Thus

$$R_k 1(x) = i\int_{-\infty}^x e^{ik(x-y)} e^{i\int_y^x u(t)\ dt} u(y)\ dy. \tag{6.22}$$

Recall that $u \in L^1$ if $u \in L^2_s$ with $s > \frac{1}{2}$, and $|e^{ik(x-y)}| \le 1$ when $x - y \ge 0$. So $\lim_{x\to-\infty} R_k 1(x) = 0$, and

$$\lim_{x\to\infty} R_k 1(x) = \begin{cases} 0 & \text{if } \mu > 0, \\ i\int_{\mathbb{R}} e^{i\lambda(x-y)} e^{i\int_y^x u(t)\ dt} u(y)\ dy & \text{if } \mu = 0, \end{cases} \tag{6.23}$$

by the dominated convergence theorem. By the Riemann–Lebesgue lemma,

$$\lim_{\lambda\to\infty} i\int_{\mathbb{R}} e^{i\lambda(x-y)} e^{i\int_y^x u(t)\ dt} u(y)\ dy = 0. \tag{6.24}$$

Therefore for every $\epsilon > 0$, there is $k_1 > 0$ such that if $|k| > k_1$, $|\lim_{x\to\infty} R_k 1(x)| < \epsilon$. Since $u \in L^1$, there exist finitely many points $\{x_n\}_{n=1}^N$ such that $|R_k 1(x) - R_k 1(y)| < \epsilon$ if none of the x_n's is between x and y. As we have already controlled $R_k 1(x)$ when x is at $\pm\infty$, it remains to control $R_k 1(x)$ if x is one of $\{x_n\}_{n=1}^N$. For each fixed x_n, we have

$$R_k 1(x_n) = i\int_{-\infty}^{x_n} e^{ik(x_n-y)} e^{i\int_y^{x_n} u(t)\ dt} u(y)\ dy. \tag{6.25}$$

We claim that $\lim_{k\to\infty} R_k 1(x_n) = 0$. In fact, one can mimic the proof of the Riemann–Lebesgue lemma, and approximate u in L^1 by a C_0^∞ function g, while integrating

$$i\int_{-\infty}^{x_n} e^{ik(x_n-y)} e^{i\int_y^{x_n} u(t)\ dt} g(y)\ dy \tag{6.26}$$

by parts to get

$$-\frac{1}{k}g(x_n) + \frac{1}{k}\int_{-\infty}^{x_n} e^{ik(x_n-y)}\left(e^{i\int_y^{x_n} u(t)\ dt} g(y)\right)_y\ dy, \tag{6.27}$$

which obviously tends to 0 as k tends to ∞. Thus by enlarging k_1 finitely many times, we get for $|k| > k_1$, $|R_k(x)| < 2\epsilon$ for all x. This completes the proof of (6.17). By a similar argument as above, the asymptotic behavior of $m_e(\lambda + 0i)$ is given by $(I + R_{\lambda+0i})\boldsymbol{e}$, which in this case can be computed explicitly as

$$\begin{aligned}
[R_{\lambda+0i}\boldsymbol{e}](x) &= i\int_{-\infty}^{x} e^{i\lambda(x-y)} e^{i\int_y^x u(t)\ dt} u(y) e^{i\lambda y}\ dy \\
&= -e^{i\lambda x} e^{i\int_{-\infty}^x u(t)\ dt} \int_{-\infty}^{x} \left(e^{-i\int_{-\infty}^y u(t)\ dt}\right)_y\ dy \\
&= -e^{i\lambda x} + e^{i\lambda x} e^{i\int_{-\infty}^x u(t)\ dt}.
\end{aligned} \tag{6.28}$$

Hence $[(I + R_{\lambda+0i})\boldsymbol{e}](x) = e^{i\lambda x} e^{i\int_{-\infty}^x u(t)\ dt}$. This finishes the proof of (6.18).

In order to obtain enough decay estimates of the scattering coefficients, we need to expand $m_1(\lambda + 0i)$ and $m_e(\lambda + 0i)$ by one more order. By (6.13), we have

$$\begin{aligned}
m_1(\lambda + 0i) &= (I - T_{\lambda+0i})^{-1} 1 \\
&= (I + R_{\lambda+0i})1 - (I + R_{\lambda+0i})\widetilde{T}_{\lambda+0i}(I + R_{\lambda+0i})1 + O\left(\frac{1}{\lambda}\right)
\end{aligned} \tag{6.29}$$

and

$$\begin{aligned}
m_e(\lambda + 0i) &= (I - T_{\lambda+0i})^{-1} \boldsymbol{e} \\
&= (I + R_{\lambda+0i})\boldsymbol{e} - (I + R_{\lambda+0i})\widetilde{T}_{\lambda+0i}(I + R_{\lambda+0i})\boldsymbol{e} + O\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{6.30}$$

By the definition of $\Gamma(\lambda)$ and $\beta(\lambda)$ given in (4.24) and (4.26), we have

$$\begin{aligned}
\Gamma(\lambda) = 1 &+ i\int_{\mathbb{R}} u(x) e^{-i\lambda x}[(I + R_{\lambda+0i})\boldsymbol{e}](x)\ dx \\
&- i\int_{\mathbb{R}} u(x) e^{-i\lambda x}[(I + R_{\lambda+0i})\widetilde{T}_{\lambda+0i}(I + R_{\lambda+0i})\boldsymbol{e}](x)\ dx + O\left(\frac{1}{\lambda}\right)
\end{aligned} \tag{6.31}$$

and

$$\begin{aligned}
\beta(\lambda) = i&\int_{\mathbb{R}} u(x) e^{-i\lambda x}[(I + R_{\lambda+0i})1](x)\ dx \\
&- i\int_{\mathbb{R}} u(x) e^{-i\lambda x}[(I + R_{\lambda+0i})\widetilde{T}_{\lambda+0i}(I + R_{\lambda+0i})1](x)\ dx + O\left(\frac{1}{\lambda}\right).
\end{aligned} \tag{6.32}$$

We first work on $\Gamma(\lambda)$. By Lemma 6.2,

$$\begin{aligned} i\int_{\mathbb{R}} u(x)e^{-i\lambda x}[(I+R_{\lambda+0i})\boldsymbol{e}](x)\ dx &= \int_{\mathbb{R}} iu(x)e^{i\int_x^\infty u(t)\ dt}\ dx \\ &= -\int_{\mathbb{R}}\left(e^{i\int_x^\infty u(t)\ dt}\right)_x\ dx \\ &= e^{i\int_{\mathbb{R}} u(t)\ dt}-1 \end{aligned} \tag{6.33}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} u(x)e^{-i\lambda x}[(I+R_{\lambda+0i})\widetilde{T}_{\lambda+0i}(I+R_{\lambda+0i})\boldsymbol{e}](x)\ dx \\ &\qquad = \int_{\mathbb{R}} u(x)e^{-i\lambda x}e^{i\int_x^\infty u(t)\ dt}[\widetilde{T}_{\lambda+0i}(I+R_{\lambda+0i})\boldsymbol{e}](x)\ dx, \end{aligned} \tag{6.34}$$

which is bounded by $\|u\|_2\|\widetilde{T}_{\lambda+0i}(I+R_{\lambda+0i})\boldsymbol{e}\|_2$. By the Plancherel identity,

$$\begin{aligned} \|\widetilde{T}_{\lambda+0i}(I+R_{\lambda+0i})\boldsymbol{e}\|_2 &\le C\left\|\frac{\chi_{\mathbb{R}^-}(\xi)}{\xi-\lambda}F(u(I+R_{\lambda+0i})\boldsymbol{e})\right\|_2 \\ &\le \frac{C}{\lambda}\|F(u(I+R_{\lambda+0i})\boldsymbol{e})\|_2 \\ &\le \frac{C}{\lambda}\|u(I+R_{\lambda+0i})\boldsymbol{e}\|_2 \le \frac{C}{\lambda}\|u\|_2. \end{aligned} \tag{6.35}$$

Hence

$$\Gamma(\lambda) = 1+e^{i\int_{\mathbb{R}} u(t)\ dt}-1+O\left(\frac{1}{\lambda}\right) = e^{i\int_{\mathbb{R}} u(t)\ dt}+O\left(\frac{1}{\lambda}\right). \tag{6.36}$$

This proves (6.19). The calculation of $\beta(\lambda)$ differs basically only in the main term

$$\int_{\mathbb{R}} u(x)e^{-i\lambda x}[(I+R_{\lambda+0i})1](x)\ dx = \int_{\mathbb{R}} u(x)e^{-i\lambda x}e^{i\int_x^\infty u(t)\ dt}\ dx. \tag{6.37}$$

The result (6.20) follows from the fact that $u \in L^1\cap L^2$. ∎

Our next result shows that a little more information on the asymptotic behavior of $m_1(x,k)$ may be obtained by imposing slightly stronger regularity assumptions on u. For this result, k is allowed to approach ∞ in a fixed Stolz angle away from the positive real line.

Theorem 6.4. *Let $s>\frac{1}{2}$ and $u\in H_s^s(\mathbb{R})$. Suppose there exists $\alpha\in(0,\frac{\pi}{2})$ such that $|Im\ k| > (\tan\alpha)Re\ k$. Then there exists $\epsilon>0$ such that*

$$m_1(x,k) = 1-\frac{C_+u(x)}{k}+O\left(\frac{1}{|k|^{1+\epsilon}}\right)\ \text{as } k\to\infty. \tag{6.38}$$

Here the big O notation holds in the sense of $L^\infty(\mathbb{R})$.

Proof. If k satisfies $|\mathrm{Im}\ k| > (\tan\alpha)\mathrm{Re}\ k$ and $|k|$ is sufficiently large, we use (6.11) to get

$$m_1(k) = (I-T_k)^{-1}1 = \sum_{n=0}^{\infty} T_k^n 1. \tag{6.39}$$

Since $u \in H^s_s(\mathbb{R})$, $w^s \hat{u} \in L^2$ and $\hat{u} \in L^1$. It follows that the L^∞ norm of

$$(T_k 1)(x) = \frac{1}{2\pi} \int_0^\infty \frac{e^{ix\xi}}{\xi - k} \hat{u}(\xi)\, d\xi \tag{6.40}$$

is bounded by $\frac{C}{|k|}$. Therefore by (6.10),

$$m_1(k) = 1 + T_k 1 + O\left(\frac{1}{|k|^{\frac{3}{2}}}\right). \tag{6.41}$$

We now write $T_k 1$ as

$$\begin{aligned}(T_k 1)(x) &= -\frac{1}{2\pi k}\int_0^\infty e^{ix\xi}\hat{u}(\xi)\, d\xi + \frac{1}{2\pi k}\int_0^\infty \frac{\xi e^{ix\xi}}{\xi - k}\hat{u}(\xi)\, d\xi \\ &= -\frac{C_+ u(x)}{k} + \frac{1}{2\pi k}\int_0^\infty \frac{\xi^{1-\epsilon} e^{ix\xi}}{\xi - k}[\xi^\epsilon \hat{u}(\xi)]\, d\xi.\end{aligned} \tag{6.42}$$

We estimate the last integral as follows. Since $w^s \hat{u} \in L^2$, by choosing $\epsilon > 0$ sufficiently small, we can make $\xi^\epsilon \hat{u}(\xi) \in L^1$. The integral is therefore bounded by

$$C \left\| \frac{\chi_{\mathbb{R}^+}(\xi)\xi^{1-\epsilon}}{\xi - k} \right\|_\infty \le C \left\| \frac{\chi_{\mathbb{R}^+}(\xi)\xi^{1-\epsilon}}{\xi + |k|} \right\|_\infty \le \frac{C}{|k|^\epsilon}. \tag{6.43}$$

Thus $T_k 1 = -\frac{C_+ u}{k} + O(\frac{1}{|k|^{1+\epsilon}})$, and the result follows. ∎

Our last result exemplifies how fast decay of the scattering coefficients can be obtained when u is assumed to be smooth with rapid decay.

Theorem 6.5. *Suppose $u \in \mathcal{S}$, the Schwartz class of rapidly decaying functions, and $k \in (\mathbb{C} \setminus [0, \infty)) \cup (\mathbb{R}^+ \pm 0i)$. Then there exists $k_0 > 0$ and $C > 0$ such that*

$$\left\| m_1(x, k) - 1 + \frac{C_+ u(x)}{k} \right\|_\infty \le \frac{C}{|k|^2} \tag{6.44}$$

for $|k| > k_0$, and for every positive integer N, there exists $C_N > 0$ such that

$$\|m_e(x, \lambda + 0i) - e^{i\lambda x} e^{i\int_{-\infty}^x u(t)\ dt}\|_\infty \le \frac{C_N}{\lambda^N}, \tag{6.45}$$

$$|\Gamma(\lambda) - e^{i\int_{\mathbb{R}} u(t)\ dt}| \le \frac{C_N}{\lambda^N},\ \text{and}\ |\beta(\lambda)| \le \frac{C_N}{\lambda^N} \tag{6.46}$$

for all $\lambda > k_0$.

Proof. The improvement from (6.38) to (6.44) is twofold: ϵ is improved to 1, and the restriction on the Stolz angle is removed. We first assume k is in the left half plane. The choice of ϵ in the proof of Theorem 6.4 is used only to make $\xi^\epsilon \hat{u}(\xi) \in L^1$. It is clear that we may choose $\epsilon = 1$ now that $u \in \mathcal{S}$.

To remove the restriction on the Stolz angle, let's assume k is in the right half plane with $|k|$ sufficiently large. This time we use (6.14) to write

$$m_1(k) = (I - T_k)^{-1} 1 = \sum_{n=0}^\infty (-(I + R_k)\widetilde{T}_k)^n (I + R_k) 1. \tag{6.47}$$

We again work only with the case $k = \lambda + i\mu$ with $\lambda > 0$, $\mu \geq 0$ and compute

$$\begin{aligned}
[(I + R_k)1](x) &= 1 + i \int_{-\infty}^{x} e^{ik(x-y)} e^{i\int_y^x u(t)\ dt} u(y)\ dy \\
&= 1 - \frac{u(x)}{k} + \frac{1}{k} \int_{-\infty}^{x} e^{ik(x-y)} \left(e^{i\int_y^x \int u(t)\ dt} u(y) \right)_y\ dy \\
&= 1 - \frac{u(x)}{k} + O\left(\frac{1}{|k|^2}\right). \qquad (6.48)
\end{aligned}$$

Here we have used integration by parts to compute the integral and used it one more time to estimate the remainder. It follows that

$$\begin{aligned}
[\widetilde{T}_k(I + R_k)1](x) &= \left[\widetilde{T}_k \left(1 - \frac{u}{k}\right)\right](x) + O\left(\frac{1}{|k|^2}\right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{0} \frac{e^{ix\xi}}{\xi - k} F\left(u\left(1 - \frac{u}{k}\right)\right)(\xi)\ d\xi + O\left(\frac{1}{|k|^2}\right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{0} \frac{e^{ix\xi}}{\xi - k} \hat{u}(\xi)\ d\xi + O\left(\frac{1}{|k|^2}\right) \\
&= -\frac{C_- u(x)}{k} + O\left(\frac{1}{|k|^2}\right). \qquad (6.49)
\end{aligned}$$

Therefore

$$m_1(k) = 1 - \frac{u}{k} + \frac{C_- u}{k} + O\left(\frac{1}{|k|^2}\right) = 1 - \frac{C_+ u}{k} + O\left(\frac{1}{|k|^2}\right). \qquad (6.50)$$

This completes the proof of (6.44).

Next, we use (6.14) to write

$$m_e(\lambda + 0i) = (I - T_{\lambda+0i})^{-1}\boldsymbol{e} = \sum_{n=0}^{\infty} (-(I + R_{\lambda+0i})\widetilde{T}_\lambda)^n (I + R_{\lambda+0i})\boldsymbol{e} \qquad (6.51)$$

and recall from the proof of Theorem 6.3 that $[(I + R_{\lambda+0i})\boldsymbol{e}](x) = e^{i\lambda x} e^{i\int_{-\infty}^{x} u(t)\ dt}$. Thus

$$\begin{aligned}
[\widetilde{T}_\lambda(I + R_{\lambda+0i})\boldsymbol{e}](x) &= \frac{1}{2\pi} \int_{-\infty}^{0} \frac{e^{ix\xi}}{\xi - \lambda} F\left(u(y) e^{i\lambda y} e^{i\int_{-\infty}^{y} u(t)\ dt}\right)(\xi)\ d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{0} \frac{e^{ix\xi}}{\xi - \lambda} F\left(u(y) e^{i\int_{-\infty}^{y} u(t)\ dt}\right)(\xi - \lambda)\ d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{-\lambda} \frac{e^{ix\xi}}{\xi} F\left(u(y) e^{i\int_{-\infty}^{y} u(t)\ dt}\right)(\xi)\ d\xi. \qquad (6.52)
\end{aligned}$$

Since $F(u(y)e^{i\int_{-\infty}^{y} u(t)\ dt})$ is also in the Schwartz class, we have $\|\widetilde{T}_\lambda(I + R_{\lambda+0i})\boldsymbol{e}\|_\infty \leq \frac{C_N}{\lambda^N}$, and (6.45) follows. The asymptotic bound on $\Gamma(\lambda)$ follows immediately from (6.45). Finally, to find the bound on $\beta(\lambda)$, we write

$$\beta(\lambda) = \sum_{n=0}^{\infty} \int_{\mathbb{R}} u(x) e^{-i\lambda x} [(-(I + R_{\lambda+0i})\widetilde{T}_\lambda)^n (I + R_{\lambda+0i})1](x)\ dx. \qquad (6.53)$$

By Lemma 6.2,

$$\begin{aligned}
&\int_{\mathbb{R}} u(x)e^{-i\lambda x}[((I+R_{\lambda+0i})\widetilde{T}_\lambda)\varphi](x)\ dx \\
&\quad= \int_{\mathbb{R}} u(x)e^{-i\lambda x}e^{i\int_x^\infty u(t)\ dt}[\widetilde{T}_\lambda\varphi](x)\ dx \\
&\quad= F\left(u(x)e^{i\int_x^\infty u(t)\ dt}[\widetilde{T}_\lambda\varphi](x)\right)(\lambda) \\
&\quad= \frac{1}{2\pi}\left[F\left(u(x)e^{i\int_x^\infty u(t)\ dt}\right)*F([\widetilde{T}_\lambda\varphi](x))\right](\lambda) \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^0 F\left(u(x)e^{i\int_x^\infty u(t)\ dt}\right)(\lambda-\xi)\frac{1}{\xi-\lambda}\widehat{u\varphi}(\xi)\ d\xi, \qquad (6.54)
\end{aligned}$$

whose L^∞ norm is bounded by

$$C\sup_{\xi\ge\lambda}\left|F\left(u(x)e^{i\int_x^\infty u(t)\ dt}\right)(\xi)\right|\|u\varphi\|_2 \le \frac{C_N}{\lambda^N}. \qquad (6.55)$$

Taking φ to be $((I+R_{\lambda+0i})\widetilde{T}_\lambda)^{n-1}(I+R_{\lambda+0i})1$, we easily obtain $|\beta(\lambda)|\le\frac{C_N}{\lambda^N}$ from (6.53). ∎

7. Time evolution of scattering data. In this section, we present a formal derivation of the time evolution of the Jost functions and scattering coefficients given in [7], assuming $u=u(x,t)$ is sufficiently smooth with sufficiently rapid decay and evolves with the BO equation (1.1). We spend no effort in justifying the change of order of derivatives with asymptotic notation. The reason that we don't try to make the steps rigorous is as follows. If our goal is to construct solutions to the Cauchy problem of the BO equation using IST, the shorter path is to evolve the scattering data by the formulas obtained formally in this section and prove that the solution constructed by the IST indeed solves the BO equation. Therefore, although it may be possible to prove the time evolution of scattering data using the H^s solution to the BO equation constructed in the PDE literature, we do not pursue that path here.

The derivation is done in two steps. In the first step, we will argue that the Jost functions $m_1(k)$, $m_e(\lambda\pm 0i)$ and the eigenfunctions ϕ_j defined in section 2 satisfy the following evolution equations:

$$\partial_t\phi_j = B_u\phi_j, \qquad (7.1)$$

$$\partial_t m_1(k) = B_u m_1(k), \qquad (7.2)$$

$$\partial_t m_e(\lambda\pm 0i) = B_u m_e(\lambda\pm 0i) - i\lambda^2 m_e(\lambda\pm 0i), \qquad (7.3)$$

where B_u is defined as (2.4). In the second step, we will use (7.1), (7.2), (7.3) to show the following time evolution for the eigenvalues $\{\lambda_j\}_{j=1}^N$, phase constants $\{\gamma_j\}_{j=1}^N$, and scattering coefficients $\Gamma(\lambda)$, $\beta(\lambda)$:

$$\partial_t\lambda_j = 0, \qquad (7.4)$$

$$\partial_t\gamma_j = 2\lambda_j, \qquad (7.5)$$

$$\partial_t\Gamma(\lambda) = 0, \qquad (7.6)$$

$$\partial_t\beta(\lambda) = i\lambda^2\beta(\lambda). \qquad (7.7)$$

We want to use the Lax equation $\partial_t L_u + [L_u, B_u] = 0$ to derive (7.1), (7.2), (7.3). However, as is pointed out in section 2, the equivalence of the Lax equation with the

BO equation has only been derived if L_u and B_u are regarded as operators on $\mathbb{H}^+$. We prove in the following lemma that the eigenfunctions and Jost functions are in fact boundary values of bounded analytic functions on the upper half plane. This is enough to justify the Lax equation.

LEMMA 7.1. *Let $\phi_j(x)$ be an eigenfunction of L_u corresponding to a negative eigenvalue λ_j. Let $m_1(x,k)$, $m_e(x,\lambda \pm 0i)$ be given as in Lemma* 3.1 *and satisfy either* (a) *or* (b) *in that lemma. Then $\phi_j, m_1(x,k), m_e(x,\lambda \pm 0i) \in \mathbb{H}^{\infty,+}$ for fixed k and λ.*

Proof. We first work on $m_1(x,k)$. If k is not on $\mathbb{R}^+ \pm 0i$, we can repeat the calculation in Lemma 3.1 to get (3.19), or

$$m_1(x) - 1 = \frac{1}{2\pi}\int_0^\infty \frac{e^{i\xi x}}{\xi - k}\widehat{um_1}(\xi)\, d\xi. \tag{7.8}$$

For $z = x + iy$ with $y > 0$, define

$$F(z) = \frac{1}{2\pi}\int_0^\infty \frac{e^{i\xi z}}{\xi - k}\widehat{um_1}(\xi)\, d\xi = \frac{1}{2\pi}\int_0^\infty \frac{e^{i\xi x}e^{-y\xi}}{\xi - k}\widehat{um_1}(\xi)\, d\xi. \tag{7.9}$$

Since $\widehat{um_1} \in L^{q'}$ for some $2 \le q' < \infty$, $F(z)$ is obviously bounded and analytic in the upper half plane. Furthermore, $F(x+iy)$ converges uniformly to $m_1(x) - 1$ as $y \searrow 0$. This shows $m_1(x,k) \in \mathbb{H}^{\infty,+}$. The eigenfunction $\phi_j(x)$ can be treated in a similar way.

Next we work on the cases $k = \lambda \pm 0i$. We provide arguments only for $m_1(x, \lambda + 0i)$. The other functions can be treated similarly. We abbreviated $m_1(x, \lambda + 0i)$ simply as m_1. Since

$$\frac{1}{i}\partial_x m_1 - C_+(um_1) = \lambda(m_1 - 1), \tag{7.10}$$

and $m_1(x) - 1 \to 0$ as $x \to -\infty$, we get

$$m_1(x) = 1 + i\int_{-\infty}^x e^{i\lambda(x-s)}C_+(um_1)(s)\, ds. \tag{7.11}$$

For $z = x + iy$ with $y > 0$, define

$$F(z) = m_1(0) + i\int_0^z e^{i\lambda(z-s)}C_+(um_1)(s)\, ds. \tag{7.12}$$

Here the integral is taken along any smooth contour in the upper half plane with end points at 0 and z. We have used the analytic extension of $C_+(um_1)$ into the upper half plane as $C_+(um_1) \in \mathbb{H}^{p,+}$. $F(z)$ is obviously analytic in the upper half plane. We now estimate $F(x + iy) - m_1(x)$. To do that, we take the contour of integration in (7.12) to be the straight line from 0 to x, followed by the straight line from x to $x + iy$. It follows that

$$F(x+iy) - m_1(x) = i\int_0^y e^{-\lambda(y-s)}C_+(um_1)(x+is)\, ds. \tag{7.13}$$

Using the elementary estimate on $\mathbb{H}^{p,+}$ functions (see Lemma 2.12 in [19])

$$|C_+(um_1)(x+is)| \le Cs^{-\frac{1}{p}}\|C_+(um_1)\|_{\mathbb{H}^{p,+}}, \tag{7.14}$$

we get

$$
\begin{aligned}
|F(x+iy)-m_1(x)| &\le C\|C_+(um_1)\|_{\mathbb{H}^{p,+}}\int_0^y e^{-\lambda(y-s)}s^{-\frac{1}{p}}\,ds\\
&= C\|C_+(um_1)\|_{\mathbb{H}^{p,+}}\int_0^1 y^{1-\frac{1}{p}}e^{-\lambda y(1-s)}s^{-\frac{1}{p}}\,ds. \qquad (7.15)
\end{aligned}
$$

For $\lambda > 0$, $y > 0$, $p > 1$, and $0 < s < 1$, we have the elementary estimate

$$
y^{1-\frac{1}{p}}e^{-\lambda y(1-s)} \le \min\left(y^{1-\frac{1}{p}}, \left[\frac{1-\frac{1}{p}}{\lambda(1-s)}\right]^{1-\frac{1}{p}} e^{-(1-\frac{1}{p})}\right). \qquad (7.16)
$$

Therefore for some constant $C = C(u, m_1, p, \lambda) > 0$

$$
|F(x+iy)-m_1(x)| \le C\min\left(1, y^{1-\frac{1}{p}}\right). \qquad (7.17)
$$

This implies that $F(z)$ is bounded and $F(x+iy)$ converges to $m_1(x)$ uniformly as $y \searrow 0$. In other words, $m_1 \in \mathbb{H}^{\infty,+}$. ∎

Next, we show that $(\partial_t L_u + [L_u, B_u])\varphi = 0$ if $\varphi \in \mathbb{H}^{\infty,+}$ and is suitably smooth. In fact, repeating the derivation of the Lax pair in [21] using the modified L_u and B_u given in (2.3) and (2.4) provides

$$
[L_u, B_u]\varphi = \frac{2}{i}(C_+u_{xx})\varphi - \frac{1}{i}C_+(u_{xx}\varphi) - 2C_+(u_xu\varphi). \qquad (7.18)
$$

Using the BO equation (1.1), we get

$$
(\partial_t L_u)\varphi = -C_+(u_t\varphi) = 2C_+(uu_x\varphi) + \frac{1}{i}C_+([(u_{xx} - 2(C_+u_{xx})]\varphi). \qquad (7.19)
$$

Hence

$$
\begin{aligned}
(\partial_t L_u + [L_u, B_u])\varphi &= \frac{2}{i}(C_+u_{xx})\varphi - \frac{2}{i}C_+((C_+u_{xx})\varphi)\\
&= \frac{2}{i}C_-\left[(C_+u_{xx})\varphi\right]. \qquad (7.20)
\end{aligned}
$$

Since $\varphi \in \mathbb{H}^{\infty,+}$, we get $(C_+u_{xx})\varphi \in \mathbb{H}^+$, and $C_-\left[(C_+u_{xx})\varphi\right] = 0$. Thus we may use the Lax equation on all eigenfunctions ϕ_j and Jost functions m_1 and m_e, by Lemma 7.1.

The standard argument of a Lax pair shows that all eigenvalues $\{\lambda_j\}_{j=1}^N$ do not change with time. We take the time derivative of $L_u\phi_j = \lambda_j\phi_j$ to get

$$
(\partial_t L_u)\phi_j + L_u(\partial_t\phi_j) = \lambda_j\partial_t\phi_j. \qquad (7.21)
$$

Using the Lax equation $(\partial_t L_u + [L_u, B_u])\phi_j = 0$, (7.21) becomes

$$
(L_u - \lambda_j)(\partial_t\phi_j - B_u\phi_j) = 0. \qquad (7.22)
$$

In other words, $\partial_t\phi_j - B_u\phi_j$ is an eigenfunction corresponding to λ_j. By the simplicity of the eigenvalues proven in [21], $\partial_t\phi_j - B_u\phi_j$ is a multiple of ϕ_j. To find out the multiplicity constant, we compare the asymptotics when $x \to \pm\infty$. By the normalization used in [21], $\phi_j(x) \sim \frac{1}{x}$ as $x \to \pm\infty$. On the other hand, we argue that

$$\partial_t \phi_j - B_u \phi_j = \partial_t \phi_j - \frac{1}{i}\partial_x^2 \phi_j - 2[(C_+ u_x)\phi_j - C_+((u\phi_j)_x)] = o\left(\frac{1}{x}\right), \tag{7.23}$$

which implies (7.1) as a consequence. In fact, $\partial_t \phi_j - \frac{1}{i}\partial_x^2 \phi_j = o\left(\frac{1}{x}\right)$ if we formally exchange derivatives with asymptotics. $xC_+ u_x \to 0$ as $x \to \pm\infty$ because

$$F(xC_+ u_x)(\xi) = -\partial_\xi(\chi_{\mathbb{R}^+}\xi\hat{u}) = -\chi_{\mathbb{R}^+}\hat{u} + i\chi_{\mathbb{R}^+}\xi\widehat{xu} \in L^1. \tag{7.24}$$

For a similar reason $xC_+((u\phi_j)_x) \to 0$ as $x \to \pm\infty$.

We can show (7.2) and (7.3) similarly. A few differences in the arguments are noted: the step where we used simplicity of eigenvalues is now replaced by uniqueness of Jost solutions; $\partial_t m_e(\lambda \pm 0i) - B_u m_e(\lambda \pm 0i) \sim -i\lambda^2 e^{i\lambda x}$ as $x \to \mp\infty$.

We now derive the time evolution of the scattering coefficients, starting with (7.5). Taking the time derivative of (2.6), we obtain

$$\partial_t m_1 = -\frac{i}{k-\lambda_j}\partial_t \phi_j + (\partial_t \gamma_j)\phi_j + (x+\gamma_j)\partial_t \phi_j + (k-\lambda_j)\partial_t h(k,\lambda_j). \tag{7.25}$$

Letting B_u act on (2.6), we obtain

$$B_u m_1 = -\frac{i}{k-\lambda_j} B_u \phi_j + B_u[(x+\gamma_j)\phi_j] + (k-\lambda_j)B_u h(k,\lambda_j). \tag{7.26}$$

Take the difference of (7.25) with (7.26), use (7.1), (7.2), and evaluate at $k = \lambda_j$ to get

$$(\partial_t \gamma_j)\phi_j + [x+\gamma_j, B_u]\phi_j = 0. \tag{7.27}$$

We compute the commutator term and get

$$\begin{aligned}
[x+\gamma_j, B_u]\phi_j &= -\frac{2}{i}\partial_x \phi_j + 2C_+(u\phi_j) - 2[xC_+(u\phi_j)_x - C_+(x(u\phi_j)_x)] \\
&= -\frac{2}{i}\partial_x \phi_j + 2C_+(u\phi_j) \\
&= -2L_u \phi_j = -2\lambda_j \phi_j.
\end{aligned} \tag{7.28}$$

The terms in the square brackets vanish as is easily seen by taking its Fourier transform. It follows that

$$(\partial_t \gamma_j - 2\lambda_j)\phi_j = 0, \tag{7.29}$$

from which we get (7.5). To obtain (7.6), we take the time derivative of (4.28) and also act on it by B_u. We get

$$\partial_t m_e(\lambda + 0i) = \Gamma \partial_t m_e(\lambda - 0i) + (\partial_t \Gamma) m_e(\lambda - 0i), \tag{7.30}$$

$$B_u m_e(\lambda + 0i) = \Gamma B_u m_e(\lambda - 0i). \tag{7.31}$$

Take the difference and use (7.3) to get

$$-i\lambda^2 m_e(\lambda + 0i) = (\partial_t \Gamma - i\lambda^2 \Gamma) m_e(\lambda - 0i). \tag{7.32}$$

Now use (4.28) again to get (7.6). Finally to obtain (7.7), we perform a similar calculation using (4.29). We first get

$$\partial_t m_1(\lambda + 0i) - \partial_t m_1(\lambda - 0i) = \beta \partial_t m_e(\lambda - 0i) + (\partial_t \beta) m_e(\lambda - 0i), \tag{7.33}$$

$$B_u m_1(\lambda + 0i) - B_u m_1(\lambda - 0i) = \beta B_u m_e(\lambda - 0i). \tag{7.34}$$

Take the difference and use (7.2) and (7.3) to get

$$(\partial_t \beta - i\lambda^2 \beta) m_e(\lambda - 0i) = 0, \tag{7.35}$$

from which (7.7) follows.

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