



Existence of rotating magnetic stars

Juhi Jang^{a,b}, Walter A. Strauss^c, Yilun Wu^{d,*}

^a Department of Mathematics, University of Southern California, Los Angeles, CA 90089, USA

^b Korea Institute for Advanced Study, Seoul, Republic of Korea

^c Department of Mathematics and Lefschetz Center for Dynamical Systems, Brown University, Providence, RI 02912, USA

^d Department of Mathematics, University of Oklahoma, Norman, OK 73069, USA

HIGHLIGHTS

- Rotating magnetic star modeled by the Euler–Poisson and MHD equations.
- First existence proof with both rotation and magnetic field.
- The star has an oblate shape for small rotation and magnetic field.

ARTICLE INFO

Article history:

Received 22 October 2017

Received in revised form 11 March 2019

Accepted 16 March 2019

Available online 25 March 2019

Communicated by S. Wang

ABSTRACT

We consider a star as a compressible fluid subject to gravitational and magnetic forces. This leads to an Euler–Poisson system coupled to a magnetic field, which may be regarded as an MHD model together with gravity. The star executes steadily rotating motion about a fixed axis. We prove, for the first time, the existence of such stars provided that the rotation speed and the magnetic field are sufficiently small.

© 2019 Published by Elsevier B.V.

1. Introduction

There have been extensive mathematical studies of stars subject only to gravitational forces but very few that incorporate magnetic forces. The only study that we are aware of is [1], in which the star does not rotate. It is well known that magnetic forces have major physical effects, for instance in the reconnection phenomenon of solar flares. Stellar magnetism is a very active area of physical theory [2,3], typically modeled by MHD, as well as of observation [4]. Because it is rare for stars to have a net charge, it is frequently assumed that the electric field vanishes.

Our model consists of the steady compressible Euler equations together with gravity and magnetic terms. It is as follows.

$$\nabla \cdot (\rho v) = 0 \quad (1)$$

$$\rho(v \cdot \nabla)v + \nabla p = \rho \nabla U + (\nabla \times B) \times B \quad (2)$$

$$\nabla \times (v \times B) = 0 \quad (3)$$

$$\nabla \cdot B = 0 \quad (4)$$

$$\Delta U = -4\pi \rho \quad (5)$$

Eqs. (1)–(3) should hold in the fluid domain $\{\rho > 0\}$, while (4) and (5) should hold in all of $\mathbb{R}^3$. For simplicity, the magnetic permeability is set equal to 1 throughout $\mathbb{R}^3$, although more realistically it could differ inside and outside the star. We further impose the boundary conditions $\lim_{|x| \rightarrow \infty} U(x) = 0$, an equation of state $p = p(\rho)$, and

$$p = 0 \text{ on the set } \partial\{\rho > 0\}. \quad (6)$$

The first two equations express mass and momentum conservation. The magnetic force is $J \times B$, where $J = \nabla \times B$ (from Ampère's Law in Maxwell's equations without E) is the magnetic current, omitting the usual 4π factor. The third equation comes from Faraday's Law in Maxwell's equations, where the electric field and the conductivity have been neglected due to the large length scale in astrophysics. The fourth equation is one of Maxwell's equations and the fifth is gravity.

We assume a steady rotation $v = \omega e_\theta$ around the x_3 axis, where ω is a constant rotation speed and $e_\theta = (-\sin \theta, \cos \theta, 0)$ in cylindrical coordinates. Then (1) is satisfied and $(v \cdot \nabla)v = -\omega^2 \nabla(r^2)/2$ where $r^2 = x_1^2 + x_2^2$. Furthermore an equation of state is assumed: p is a function of ρ . For instance, we allow $p(\rho) = \rho^\gamma$ with $\frac{6}{5} < \gamma < 2$, $\gamma \neq \frac{4}{3}$. The specific enthalpy is defined as

$$h(\rho) = \int_0^\rho \frac{p'(s)}{s} ds. \quad (7)$$

We will show in Section 2 that, due to the cylindrical symmetry, there is a scalar function ψ such that $rB^r = \partial_3 \psi$, $rB^3 = -\partial_r \psi$.

* Corresponding author.

E-mail address: allenwu@ou.edu (Y. Wu).

Under these conditions we will show in Section 2 that the system reduces to the three equations

$$-\frac{1}{2}\omega^2 r^2 + h(\rho) - U + \epsilon K(\psi) = \text{constant} \quad \text{in } \{\rho > 0\}, \quad (8)$$

$$L\psi = \epsilon k(\psi)\rho \quad \text{in } \mathbb{R}^3, \quad (9)$$

$$U = |\chi|^{-1} * \rho \quad \text{in } \mathbb{R}^3, \quad (10)$$

where $L = \nabla \cdot r^{-2} \nabla$ and where $k = K'$ is an arbitrary function. We call k the magnetic current function, because it takes the magnetic potential ψ to a multiple of the magnetic current $J = \nabla \times B$ (see Section 2).

We will prove the existence of solutions by a perturbation analysis starting from a spherically symmetric stationary solution. The tool we employ is the standard implicit function theorem in Banach space.

For any radius $R > 0$, it is well-known [5] that there exists a unique spherical solution $\rho_0(|x|) \geq 0$, called the Lane–Emden solution, with $\omega = \epsilon = 0$ and $\psi \equiv 0$ such that $\rho_0 > 0$ in $B_R = \{|x| < R\}$ and $\rho_0 \in C^2(B_R) \cap C^{1,\alpha}(\mathbb{R}^3)$, where $\alpha = \min\left(\frac{2-\gamma}{\gamma-1}, 1\right)$. Our main theorem is as follows.

Theorem 1.1. *Let $R > 0$. Let ρ_0 be the unique solution mentioned above. Let $k \in C^2(\mathbb{R})$. Let $p(\rho) = \rho^\gamma$ where $6/5 < \gamma < 2$ and $\gamma \neq 4/3$. Then there exist $\bar{\omega} > 0$ and $\bar{\epsilon} > 0$ and solutions $(\rho = \rho_{\omega,\epsilon}, \psi = \psi_{\omega,\epsilon})$ for all $|\omega| < \bar{\omega}$ and $|\epsilon| < \bar{\epsilon}$, with the following properties.*

- $\rho \in C_c^{1,\alpha}(\mathbb{R}^3)$, $\psi \in C^{3,\alpha}(\mathbb{R}^3)$, where $\alpha = \min\left(\frac{2-\gamma}{\gamma-1}, 1\right)$.
- Both functions are axisymmetric and even in x_3 .
- $\rho \geq 0$ has compact support (near B_R).
- $\int \rho \, dx = M(\rho_0)$ (independently of ω, ϵ).
- The mapping $(\omega, \epsilon) \rightarrow (\rho, \psi)$ is continuous from $(-\bar{\omega}, \bar{\omega}) \times (-\bar{\epsilon}, \bar{\epsilon})$ into $C^1(B_{2R}) \times C_0^2(\mathbb{R}^3)$.

More generally, we permit $p(\rho)$ to be any function that satisfies (11) and (12) and we assume that $M'(\rho_0(0)) \neq 0$, where $M(\rho(0)) = \int_{\mathbb{R}^3} \rho \, dx$ is the total mass of the unique radial nonrotating star solution with center density $\rho(0)$ (more details explained in Theorem 2.1 in [5]).

$$p(s) \in C^3(0, \infty), p' > 0, p(0) = 0 \quad (11)$$

$$\exists \gamma \in (1, 2), \lim_{s \rightarrow 0^+} s^{3-\gamma} p'''(s) < 0, \text{ and} \quad (12)$$

$$\exists \gamma^* \in \left(\frac{6}{5}, 2\right), \lim_{s \rightarrow \infty} s^{1-\gamma^*} p'(s) > 0.$$

Our construction shows that the solutions are modified from the Lane–Emden solution by a simple radial stretching or contraction. The support of $\rho_{\omega,\epsilon}$ takes an oblate shape, as we remark at the end of Section 2. The shape is only affected by the magnetic field at higher orders in ω^2 and ϵ . We remark that the case $\gamma = \frac{4}{3}$ is excluded from Theorem 1.1, because in that case the key linearized operator in our construction has a non-trivial kernel. This corresponds to the fact that there exists a family of non-rotating radial solutions with zero magnetic field, obtained by simple rescaling of an unperturbed one. The solutions in this family all have the same total mass, due to the special scaling symmetry in this case. With the mass constraint in Theorem 1.1, the nearby solutions must come from this trivial class. However, if we were to remove the mass constraint, non-trivial solutions also could arise at the $\frac{4}{3}$ power.

There have been many studies, including by the giants MacLaurin, Jacobi, Poincaré, Liapunov and Chandrasekhar, of stationary and steadily rotating stars subject to gravitational forces but without any magnetic field. There are two modern methods of

analysis of rotating stars, the variational method introduced by Auchmuty and Beals [6] and Li [7] and the perturbation method introduced by Lichtenstein [8]. The perturbation method was recently revived and further developed in [5] and [9], where further references and discussions may be found. Furthermore, the two papers [10] and [11] appeared after this paper was originally submitted. There are a number of excellent general expositions, notably the treatises [12,13] and [14].

However, the only mathematical reference of which we are aware that deals with a magnetic effect is [1], which considers a stationary ($v = 0$) magnetic star. The authors of [1] find solutions by a variational method and permit $\gamma \geq 2$, but they require k to be a constant function of ψ . Our paper is very different from [1] with regard to its methodology and most importantly with regard to the rotation of the star. Besides permitting rotation, we use a perturbation method and we permit the magnetic current function $k(\psi)$ to be completely arbitrary rather than a constant. To the best of our knowledge, ours is the first mathematically rigorous result that establishes the existence of rotating magnetic stars.

In Section 2 we state the assumptions in detail, specialize the model to our situation, and outline the proof of the theorem. Section 3 is devoted to studying the detailed properties of the inverse operator L^{-1} . Section 4 is devoted to the proof of Fréchet differentiability, which is a key requirement of the implicit function theorem.

2. Setup and outline

We are looking for axisymmetric steady rotating solutions to the magnetic star equations (1)–(5). To that end, let $v = r\omega e_\theta$, $B = B^r e_r + B^3 e_3$, and assume that all the functions ρ, B, U depend only on the cylindrical coordinates r and x_3 . The magnetic star equations simplify considerably under the aforementioned assumptions. First of all, by these assumptions we have $\nabla \cdot v = 0$, $\nabla \rho \perp e_\theta$, and $v \parallel e_\theta$. As a consequence, the mass conservation equation (1) is automatically satisfied. Eq. (3) is also satisfied as is seen from the following calculation:

$$\nabla \times (v \times B) = -\nabla_v B + \nabla_B v = -\omega B^r e_\theta + \omega B^r e_\theta = 0. \quad (13)$$

Because there is no θ -dependence, Eq. (4) gives us

$$0 = r \nabla \cdot B = \partial_r(rB^r) + \partial_3(rB^3), \quad (14)$$

which is satisfied if the components of the magnetic field are induced by a scalar axisymmetric ‘magnetic potential’ ψ in the following way:

$$rB^r = \partial_3 \psi, \quad rB^3 = -\partial_r \psi. \quad (15)$$

This is equivalent to assuming that the vector magnetic potential A , given by $B = \nabla \times A$, is $A = -\frac{\psi}{r} e_\theta$.

Our next step is to express the term $(\nabla \times B) \times B$ in (2) by ψ . A short computation shows that

$$\nabla \times B = (\partial_3 B^r - \partial_r B^3) e_\theta = r(L\psi) e_\theta, \quad (16)$$

where

$$L\psi = \frac{1}{r} \partial_r \left(\frac{\partial_r \psi}{r} \right) + \frac{\partial_3^2 \psi}{r^2} = \nabla \cdot \left(\frac{1}{r^2} \nabla \psi \right). \quad (17)$$

So

$$(\nabla \times B) \times B = (L\psi) e_\theta \times (\partial_3 \psi e_r - \partial_r \psi e_3) = -(L\psi) \nabla \psi. \quad (18)$$

Because $v \cdot \nabla v = -\nabla(\frac{1}{2}\omega^2 r^2)$ and $\frac{\nabla p}{\rho} = \nabla(h(\rho))$ from (7), the momentum equation (2) becomes

$$\nabla(-\frac{1}{2}\omega^2 r^2 + h(\rho)) = \nabla U - \frac{1}{\rho}(L\psi) \nabla \psi \quad (19)$$

Notice that every term but the last one in (19) is a gradient. It follows that the last term must be curl free, namely,

$$\nabla \left(\frac{L\psi}{\rho} \right) \times \nabla \psi = 0. \quad (20)$$

Thus the gradients of $\frac{L\psi}{\rho}$ and ψ are parallel. A natural sufficient condition for this is that $\frac{L\psi}{\rho}$ is a function of ψ . Motivated by this condition, we look for a special but quite wide class of solutions to (19) for which $L\psi = \epsilon \rho k(\psi)$ with an arbitrarily prescribed function k . The constant ϵ is conveniently included here as a small parameter. For these solutions, the momentum equation (19) can now be written as

$$h(\rho) - \frac{1}{2} \omega^2 r^2 - U + \epsilon K(\psi) = \text{constant}, \quad (21)$$

where $K(s) = \int_0^s k(t) dt$.

To summarize, the magnetic star equations have now been simplified to the following problem:

$$h(\rho) - \frac{1}{2} \omega^2 r^2 - U + \epsilon K(\psi) = \text{constant} \quad \text{in the region } \{\rho > 0\}, \quad (22)$$

$$L\psi = \epsilon \rho k(\psi) \quad \text{in } \mathbb{R}^3, \quad (23)$$

$$U = \rho * \frac{1}{|x|} \quad \text{in } \mathbb{R}^3. \quad (24)$$

The last equation comes from (5), together with the assumption that $\rho(x)$ vanishes appropriately at infinity. Thus we have (8), (9), (10). We also assume the boundary condition $\psi(\infty) = 0$. Solutions of (22)–(24) together with the boundary condition (6) satisfy our original system (1)–(6). The rest of the paper is devoted to the existence of these solutions, thereby proving Theorem 1.1.

We will construct solutions to (22), (23), (24) which are close to the nonrotating, magnetic-free Lane–Emden solutions. We thus begin by considering a Lane–Emden solution ρ_0 supported on B_1 , as is explained in [5], and the deformation

$$g_\zeta(x) = x \left(1 + \frac{\zeta(x)}{|x|^2} \right) \quad (25)$$

used in [5]. Here $\zeta : B_1 \rightarrow \mathbb{R}$ is an axisymmetric function that is even in x_3 . If ζ is small in a suitable norm, g_ζ is invertible, ζ can be extended to $\mathbb{R}^3$ preserving the symmetry requirements. The deformation g_ζ can then be extended to a homeomorphism on $\mathbb{R}^3$ (as well as diffeomorphic on $\mathbb{R}^3 - \{0\}$) accordingly. See [5] for detailed estimates of these facts. We look for a solution of the form

$$\rho_\zeta(z) = \mathcal{M}(\zeta) \rho_0(g_\zeta^{-1}(z)), \quad (26)$$

where $\mathcal{M}(\zeta)$ is chosen such that $\rho_\zeta(x)$ has the same mass as $\rho_0(x)$.

Our model (22), (23), (24) may thus be recast as the pair of equations

$$\begin{aligned} & \left(\rho_\zeta * \frac{1}{|\cdot|} \right)(z) - \left(\rho_\zeta * \frac{1}{|\cdot|} \right)(0) + \frac{1}{2} \omega^2 (z_1^2 + z_2^2) \\ & - h(\rho_\zeta(z)) + h(\rho_\zeta(0)) - \epsilon K(\psi(z)) + \epsilon K(\psi(0)) = 0 \\ & \text{for } z \in g_\zeta(\overline{B_1}), \end{aligned} \quad (27)$$

$$\psi(z) - \epsilon L^{-1}(\rho_\zeta k(\psi))(z) = 0 \quad \text{for all } z \in \mathbb{R}^3. \quad (28)$$

The precise definition and properties of L^{-1} are given in Section 3.

We reduce the problem further by observing that (28) only needs to be solved for $z \in g_\zeta(\overline{B_1})$. Indeed, as ρ_ζ is supported on $g_\zeta(\overline{B_1})$, if we can find a smooth enough function $\psi : g_\zeta(\overline{B_1}) \rightarrow \mathbb{R}$ for which (28) holds for all $z \in g_\zeta(\overline{B_1})$, then we can extend $\rho_\zeta k(\psi)$ to $\mathbb{R}^3$ by setting it to be zero outside $g_\zeta(\overline{B_1})$. Now we extend

ψ to $\mathbb{R}^3$ by (28) and observe that (28) holds for all $z \in \mathbb{R}^3$. In summary, when solving (27) and (28), we only need them to hold for $z \in g_\zeta(\overline{B_1})$.

In order that the functions are defined on a fixed domain, we make the change of variable $z = g_\zeta(x)$, and replace ψ by the function

$$\varphi(x) = \psi(g_\zeta(x)). \quad (29)$$

It follows that (27), (28) can be written as

$$\begin{aligned} & \left(\rho_\zeta * \frac{1}{|\cdot|} \right)(g_\zeta(x)) - \left(\rho_\zeta * \frac{1}{|\cdot|} \right)(0) + \frac{1}{2} \omega^2 (x_1^2 + x_2^2) \left(1 + \frac{\zeta(x)}{|x|^2} \right)^2 \\ & - h(\mathcal{M}(\zeta) \rho_0(x)) + h(\mathcal{M}(\zeta) \rho_0(0)) - \epsilon K(\varphi(x)) \\ & + \epsilon K(\varphi(0)) = 0, \end{aligned} \quad (30)$$

$$\varphi(x) - L^{-1}(\rho_\zeta \epsilon k(\varphi(g_\zeta^{-1}))) (g_\zeta(x)) = 0, \quad (31)$$

where both equations are now required to be valid only in $\overline{B_1}$.

Now we begin to set up the scenario for the implicit function theorem. We define the operators

$$\begin{aligned} \mathcal{F}_1(\zeta, \varphi, \omega, \epsilon)(x) &= \left(\rho_\zeta * \frac{1}{|\cdot|} \right)(g_\zeta(x)) - \left(\rho_\zeta * \frac{1}{|\cdot|} \right)(0) \\ &+ \frac{1}{2} \omega^2 (x_1^2 + x_2^2) \left(1 + \frac{\zeta(x)}{|x|^2} \right)^2 \\ &- h(\mathcal{M}(\zeta) \rho_0(x)) + h(\mathcal{M}(\zeta) \rho_0(0)) \\ &- \epsilon K(\varphi(x)) + \epsilon K(\varphi(0)), \end{aligned} \quad (32)$$

$$\begin{aligned} \mathcal{F}_2(\zeta, \varphi, \omega, \epsilon)(x) &= \varphi(x) \\ &- \epsilon L^{-1}(\rho_\zeta \epsilon k(\varphi(g_\zeta^{-1}))) (g_\zeta(x)) \end{aligned} \quad (33)$$

for $x \in \overline{B_1}$. We define $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)^t$. Note that $\mathcal{F}(0, 0, 0, 0) = 0$ because ρ_0 is the Lane–Emden solution in B_1 .

Let $\overline{B_1} = \overline{B_1} \setminus \{0\}$. We consider the space

$$\begin{aligned} X &= C^1(\overline{B_1}) \cap \{f \mid f \text{ is axisymmetric and even in } x_3, f(0) = 0, \\ &\|f\|_X < \infty\} \end{aligned} \quad (34)$$

where the norm is

$$\|f\|_X = \sup_{x \in \overline{B_1}} \frac{|\nabla f(x)|}{|x|}.$$

It is easy to see that X is a Banach space. In fact, a Cauchy sequence $\{u_n\}$ in X is also Cauchy and thus converges to u in $C^1(\overline{B_1})$. Now for any $x \in \overline{B_1}$, $|\nabla u_n(x) - \nabla u_m(x)| < \epsilon|x|$ for n, m sufficiently large. Taking m to infinity, we get $|\nabla u_n(x) - \nabla u(x)| < \epsilon|x|$, which means $\{u_n\}$ converges to u in X . Finally, we define $X_\delta = \{f \in X \mid \|f\|_X \leq \delta\}$.

Lemma 2.1. *The operator $\mathcal{F}$ is continuously Fréchet differentiable from $X_\delta \times X_\delta \times \mathbb{R} \times \mathbb{R}$ into $X \times X$ provided δ is sufficiently small.*

Lemma 2.1 will be proven in Section 4.

Lemma 2.2. *$\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(0, 0, 0, 0) : X^2 \rightarrow X^2$ is an isomorphism.*

Proof. We write $\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(0, 0, 0, 0) : X^2 \rightarrow X^2$ as a matrix:

$$\begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial \zeta} & \frac{\partial \mathcal{F}_1}{\partial \varphi} \\ \frac{\partial \mathcal{F}_2}{\partial \zeta} & \frac{\partial \mathcal{F}_2}{\partial \varphi} \end{pmatrix} (0, 0, 0, 0) = \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial \zeta}(0, 0, 0, 0) & 0 \\ 0 & I \end{pmatrix}. \quad (35)$$

By Theorem 4.1 in [5], and the fact that $\mathcal{L}$ in [5] is the same as $\frac{\partial \mathcal{F}_1}{\partial \zeta}(0, 0, 0, 0)$, we immediately see that $\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(0, 0, 0, 0)$ is an isomorphism. $\square$

Given these two lemmas and the fact that $\mathcal{F}(0, 0, 0, 0) = 0$, the implicit function theorem provides a solution for every small enough ϵ and ω .

In addition, with the linearized operators, we may study the first order approximation of $\zeta(\omega, \epsilon)$ and $\varphi(\omega, \epsilon)$ as functions of ω^2 and ϵ . In fact, one easily obtains

$$\begin{pmatrix} \zeta(\omega, \epsilon) \\ \varphi(\omega, \epsilon) \end{pmatrix} = \begin{pmatrix} \frac{\partial \zeta}{\partial \omega^2}(0, 0)\omega^2 + \frac{\partial \zeta}{\partial \epsilon}(0, 0)\epsilon \\ \frac{\partial \varphi}{\partial \omega^2}(0, 0)\omega^2 + \frac{\partial \varphi}{\partial \epsilon}(0, 0)\epsilon \end{pmatrix} + o(|\omega|^2 + |\epsilon|), \quad (36)$$

where

$$\begin{pmatrix} \frac{\partial \zeta}{\partial \omega^2}(0, 0) \\ \frac{\partial \varphi}{\partial \omega^2}(0, 0) \end{pmatrix} = - \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial \zeta}(0, 0, 0, 0) & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2}r^2 \\ 0 \end{pmatrix} \\ = \begin{pmatrix} -\frac{1}{2}[\frac{\partial \mathcal{F}_1}{\partial \zeta}(0, 0, 0, 0)]^{-1}r^2 \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \frac{\partial \zeta}{\partial \epsilon}(0, 0) \\ \frac{\partial \varphi}{\partial \epsilon}(0, 0) \end{pmatrix} = - \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial \zeta}(0, 0, 0, 0) & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -L^{-1}(\rho_0 k(0)) \end{pmatrix} \\ = \begin{pmatrix} 0 \\ k(0)L^{-1}(\rho_0) \end{pmatrix}.$$

In other words, the first order approximation of $\zeta(\omega, \epsilon)$ and $\varphi(\omega, \epsilon)$ are given by $-\frac{\omega^2}{2}[\frac{\partial \mathcal{F}_1}{\partial \zeta}(0, 0, 0, 0)]^{-1}r^2$ and $\epsilon k(0)L^{-1}(\rho_0)$ respectively.

Notice that the magnetic field does not affect the lowest order change in the shape of the star, because ϵ does not appear in the approximation for $\zeta(\omega, \epsilon)$. By the discussion in Section 4.6 of [5], we see that the first order approximation of $\zeta(\omega, \epsilon)$ gives rise to an oblate star, which is wider at the equator than at the poles. On the other hand, if $k(0) \neq 0$, the first order approximation of $\varphi(\omega, \epsilon)$ is necessarily nonzero, giving rise to a genuine magnetic star.

3. The operator L^{-1}

Before we prove the Fréchet differentiability of $\mathcal{F}$, we carefully define L^{-1} and compile a few useful estimates of it. Consider the equation $Lu = f$ in $\mathbb{R}^3$, where both u and f are axisymmetric and even in x_3 , where f is compactly supported, and where $u(\infty) = 0$. If we define $v = u/r^2$, then a simple calculation gives

$$v_{rr} + \frac{3}{r}v_r + v_{zz} = f. \quad (37)$$

We write $z = x_3$, $r = \sqrt{x_1^2 + x_2^2}$, $\hat{v}(r, z) = v(x_1, x_2, x_3)$. Following [1], we use the five-dimensional extension of the equation to obtain a simple explicit formula for the solution. Let $\tilde{v}$ be the 5D extensions of v defined by

$$\tilde{v}(x_1, x_2, x_3, x_4, x_5) = v \left(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \cos \theta, \right. \\ \left. \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \sin \theta, x_5 \right) \\ = \hat{v} \left(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, x_5 \right), \quad (38)$$

with $\tilde{f}$ defined in the same way. Thus (37) can be written as $\Delta_5 \tilde{v} = \tilde{f}$. Moreover, the condition $u(\infty) = 0$ implies $\tilde{v}(\infty) = 0$.

It follows that

$$v(x_1, x_2, x_3) = v(\sqrt{x_1^2 + x_2^2}, 0, x_3) = \tilde{v}(x_1, x_2, 0, 0, x_3) \\ = C_5 \int_{\mathbb{R}^5} \frac{1}{|(x_1, x_2, 0, 0, x_3) - y|^3} \tilde{f}(y) dy. \quad (39)$$

Thus

$$(L^{-1}f)(x) = u(x_1, x_2, x_3) = C_5(x_1^2 + x_2^2)(L_1^{-1}f)(x), \quad (40)$$

where we define the integral operator

$$(L_1^{-1}f)(x) = \int_{\mathbb{R}^5} \frac{1}{|(x_1, x_2, 0, 0, x_3) - y|^3} \tilde{f}(y) dy. \quad (41)$$

Since

$$\left| \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} - \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2} \right| \leq |x - y|$$

for $x, y \in \mathbb{R}^5$, there exists a constant $C = C(\beta, R)$ such that

$$\|\tilde{f}\|_{C^{0,\beta}(B_R(\mathbb{R}^5))} \leq C\|f\|_{C^{0,\beta}(B_R(\mathbb{R}^3))} \quad (42)$$

for any $0 < \beta \leq 1$ and $R > 0$. In other words, $\tilde{f}$ is as regular as f . It is worthwhile to keep in mind that, by (40), L^{-1} is basically a quadratic function vanishing at the origin multiplied by the inverse Laplacian in 5D. Therefore, in addition to enjoying all the regularization properties of the inverse Laplacian, it also automatically vanishes to quadratic degree at the origin. This observation gives rise to the next lemma.

Lemma 3.1. *Let $f \in L^\infty(B_2)$. Then $L^{-1}f \in C^{1,\beta}(\overline{B_2})$ for every $0 < \beta < 1$. There is a constant $C > 0$ such that*

$$|L^{-1}f(z)| + |\nabla L^{-1}f(z)| \leq C\|f\|_{L^\infty(B_2)}|z| \quad (43)$$

for any $z \in B_2$.

Proof. Using (40) and its gradient, we have

$$(\nabla L^{-1}f)(z) = C_5 \nabla(z_1^2 + z_2^2)(L_1^{-1}f)(z) + C_5(z_1^2 + z_2^2)(\nabla L_1^{-1}f)(z), \quad (44)$$

the inequality follows from the boundedness of $\int_{B_2 \subset \mathbb{R}^5} \frac{1}{|x-y|^3} dy$ and $\int_{B_2 \subset \mathbb{R}^5} \frac{1}{|x-y|^4} dy$. $\square$

Combining the $C^{1,\beta}$ regularity of $L_1^{-1}f$ with the properties of g_ζ , we get the next lemma.

Lemma 3.2. *Let $f \in L^\infty(B_2)$, $\zeta_1, \zeta_2 \in X_\delta$ where $\delta > 0$ is sufficiently small. For any $0 < \beta < 1$, there exists $C_\beta > 0$ such that*

$$|[\nabla L^{-1}f](g_{\zeta_1}(x)) - [\nabla L^{-1}f](g_{\zeta_2}(x))| \leq C_\beta \|f\|_{L^\infty(B_2)} \|\zeta_1 - \zeta_2\|_X^\beta |x| \quad (45)$$

for $x \in \overline{B_1}$.

Proof. By (44), $\nabla L^{-1}f$ is of the form $f_1 + (z_1^2 + z_2^2)f_2$, where $\|f_1\|_{C_1(B_2)} \leq C\|f\|_{L^\infty}$, and $\|f_2\|_{C^{0,\beta}(B_2)} \leq C_\beta\|f\|_{L^\infty}$. It follows that

$$|[\nabla L^{-1}f](g_{\zeta_1}(x)) - [\nabla L^{-1}f](g_{\zeta_2}(x))| \\ \leq C\|f\|_{L^\infty} |g_{\zeta_1}(x) - g_{\zeta_2}(x)| + C_\beta\|f\|_{L^\infty} |g_{\zeta_1}(x)| |g_{\zeta_1}(x) - g_{\zeta_2}(x)|^\beta \\ \leq C_\beta\|f\|_{L^\infty(B_2)} \|\zeta_1 - \zeta_2\|_X^\beta |x|. \quad (46)$$

In the last step, we used $|g_{\zeta_1}(x) - g_{\zeta_2}(x)| \leq \|\zeta_1 - \zeta_2\|_X |x|$, and $|g_{\zeta_1}(x)| \leq (1 + \|\zeta\|_X)|x|$ (cf. Lemma 3.4 of [5]). $\square$

The standard elliptic Schauder estimates immediately imply the following lemma.

Lemma 3.3. Let $f \in C^{0,\beta}(\overline{B_2})$ for some $0 < \beta < 1$. Then $L^{-1}f \in C^{2,\beta}(\overline{B_2})$, and there is a constant $C_\beta > 0$ such that

$$\|\nabla^2 L^{-1}f\|_{C^{0,\beta}(\overline{B_2})} \leq C_\beta \|f\|_{C^{0,\beta}(\overline{B_2})}. \quad (47)$$

Next we discuss a few more delicate estimates involving interior composition with the deformation g_ζ^{-1} before the action of L^{-1} . To that end, we first write an alternative expression for $L^{-1}(u(g_\zeta^{-1}))(z)$:

$$\begin{aligned} L^{-1}(u(g_\zeta^{-1}))(z) &= C_5(z_1^2 + z_2^2) \int_{\mathbb{R}^5} \frac{1}{|(z_1, z_2, 0, 0, z_3) - y|^3} \widetilde{u(g_\zeta^{-1})}(y) dy \\ &= C_5(z_1^2 + z_2^2) \int_{\mathbb{R}^5} \frac{1}{|(z_1, z_2, 0, 0, z_3) - y|^3} \widetilde{u}(\widetilde{g_\zeta^{-1}})(y) dy \\ &= C_5(z_1^2 + z_2^2) \int_{\text{supp } \widetilde{u}} \frac{1}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_\zeta}(y)|^3} \\ &\quad \times \widetilde{u}(y) \det D\widetilde{g_\zeta}(y) dy \end{aligned} \quad (48)$$

where we define $\widetilde{g_\zeta} : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ as

$$\widetilde{g_\zeta}(x) = x \left(1 + \frac{\widetilde{\zeta}(x)}{|x|^2} \right). \quad (49)$$

In the preceding identity we have used the nontrivial but straightforward fact that

$$u(g_\zeta^{-1})(y) = \widetilde{u}(\widetilde{g_\zeta^{-1}})(y). \quad (50)$$

Similarly, for a row-vector-valued function u_a and a column-vector-valued function u_b , we obtain the alternative expression

$$\begin{aligned} L^{-1}(u_a(g_\zeta^{-1})D(g_\zeta^{-1})u_b(g_\zeta^{-1}))(z) &= C_5(z_1^2 + z_2^2) \int_{\mathbb{R}^5} \frac{[u_a(\nabla g_\zeta)^{-1}u_b](y)}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_\zeta}(y)|^3} \det D\widetilde{g_\zeta}(y) dy. \end{aligned} \quad (51)$$

The reason we write such alternative expressions is that we want to estimate the difference $L^{-1}(u(g_\zeta^{-1})) - L^{-1}(u(g_{\zeta_2}^{-1}))$ assuming only an L^∞ control on u . In order to get an estimate depending on $\|\zeta_1 - \zeta_2\|_X$, we must use (48) to move the interior composition of g_ζ^{-1} out of u . The following lemma will be useful in deriving the difference estimate of $L^{-1}(u(g_\zeta^{-1})) - L^{-1}(u(g_{\zeta_2}^{-1}))$:

Lemma 3.4. Suppose $\zeta_1, \zeta_2 \in X_\delta$ where $\delta > 0$ is sufficiently small and $u \in L^\infty(B_1)$. Then

$$\begin{aligned} &\left| \nabla_z \int_{B_1 \subset \mathbb{R}^5} \left(\frac{1}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_1}}(y)|^3} - \frac{1}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_2}}(y)|^3} \right) \widetilde{u}(y) dy \right| \\ &\leq Cd(1 + |\log d|) \|u\|_{L^\infty}, \end{aligned} \quad (52)$$

where $d = \|\zeta_1 - \zeta_2\|_X$, for $z \in B_2 \subset \mathbb{R}^3$.

Proof. The singularities of the integral are at $y_{(1)} = \widetilde{g_{\zeta_1}^{-1}}(z_1, z_2, 0, 0, z_3)$ and $y_{(2)} = \widetilde{g_{\zeta_2}^{-1}}(z_1, z_2, 0, 0, z_3)$. For any $y \in B_2$, we claim that $|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_1}}(y)|$ is comparable to $|y_{(1)} - y|$, and that $|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_2}}(y)|$ is comparable to $|y_{(2)} - y|$. In fact,

$$|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_1}}(y)| = |\widetilde{g_{\zeta_1}}(y_{(1)}) - \widetilde{g_{\zeta_1}}(y)| \leq \|D\widetilde{g_{\zeta_1}}\|_{L^\infty} |y_{(1)} - y|.$$

Similarly

$$|y_{(i)} - y| \leq \|D\widetilde{g_{\zeta_i}}\|_{L^\infty} |(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_i}}(y)|.$$

The L^∞ bound on $D\widetilde{g_{\zeta_i}}$ and $D\widetilde{g_{\zeta_i}^{-1}}$ follows in a similar fashion to Lemma 3.4 of [5]. Using similar estimates involving differences of

ζ_1 and ζ_2 , we obtain that $|\widetilde{g_{\zeta_1}}(y) - \widetilde{g_{\zeta_2}}(y)|$ and $|y_{(1)} - y_{(2)}|$ are both bounded by a constant multiple of $\|\zeta_1 - \zeta_2\|_X$. By the preceding distance estimates, we may choose a ball B_d to be centered at the midpoint of $y_{(1)}$ and $y_{(2)}$ with radius comparable to $d = \|\zeta_1 - \zeta_2\|_X$, such that the following facts hold whenever y is outside B_d :

- (a) $|y - y_{(i)}| \geq 2|y_{(1)} - y_{(2)}|, i = 1, 2.$
- (b) $|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_i}}(y)| \geq 2|\widetilde{g_{\zeta_1}}(y) - \widetilde{g_{\zeta_2}}(y)|, i = 1, 2.$

We split the integral into one piece on B_d and another piece off B_d . On B_d , we use the fact that $|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_i}}(y)|$ is comparable to $|y_{(i)} - y|$, so the integral is bounded by

$$C \int_0^d \frac{2}{r^4} r^4 dr \|u\|_{L^\infty} = Cd \|u\|_{L^\infty}. \quad (53)$$

Off B_d , we use the distance estimates above to conclude that for all $0 \leq t \leq 1$, $|(z_1, z_2, 0, 0, z_3) - t\widetilde{g_{\zeta_1}}(y) - (1-t)\widetilde{g_{\zeta_2}}(y)|$ is comparable to the distance between y and the center of B_d , so the integral is bounded by

$$C \int_d^1 \frac{1}{r^5} r^4 dr \|\widetilde{g_{\zeta_1}} - \widetilde{g_{\zeta_2}}\|_{L^\infty} \|u\|_{L^\infty} \leq Cd |\log d| \|u\|_{L^\infty}. \quad \square \quad (54)$$

Now we use the preceding lemma to prove the relevant estimate on L^{-1} .

Lemma 3.5. Let u, u_a, u_b be respectively scalar, row-vector-valued and column-vector-valued bounded functions supported on $\overline{B_1}$. Let $\zeta_1, \zeta_2 \in X_\delta$ where $\delta > 0$ is sufficiently small. Then there is a constant $C > 0$ such that

$$\begin{aligned} &|\{\nabla L^{-1}[u(g_{\zeta_1}^{-1})]\}(z) - \{\nabla L^{-1}[u(g_{\zeta_2}^{-1})]\}(z)| \\ &\leq Cd(1 + |\log d|) \|u\|_{L^\infty} |z|, \end{aligned} \quad (55)$$

and

$$\begin{aligned} &|\{\nabla L^{-1}[u_a(g_{\zeta_1}^{-1})\nabla(g_{\zeta_1}^{-1})u_b(g_{\zeta_1}^{-1})]\}(z) \\ &\quad - \{\nabla L^{-1}[u_a(g_{\zeta_2}^{-1})\nabla(g_{\zeta_2}^{-1})u_b(g_{\zeta_2}^{-1})]\}(z)| \\ &\leq Cd(1 + |\log d|) \|u_a\|_{L^\infty} \|u_b\|_{L^\infty} |z|. \end{aligned} \quad (56)$$

Here $d = \|\zeta_1 - \zeta_2\|_X$.

Proof. We write $\{\nabla L^{-1}[u(g_{\zeta_1}^{-1})]\}(z) - \{\nabla L^{-1}[u(g_{\zeta_2}^{-1})]\}(z)$ as

$$\begin{aligned} &C_5 \nabla(z_1^2 + z_2^2) \left(L^{-1}[u(g_{\zeta_1}^{-1})](z) - L^{-1}[u(g_{\zeta_2}^{-1})](z) \right) \\ &\quad + C_5(z_1^2 + z_2^2) \left(\{\nabla L^{-1}[u(g_{\zeta_1}^{-1})]\}(z) - \{\nabla L^{-1}[u(g_{\zeta_2}^{-1})]\}(z) \right). \end{aligned} \quad (57)$$

Using (48), $L^{-1}[u(g_{\zeta_1}^{-1})](z) - L^{-1}[u(g_{\zeta_2}^{-1})](z)$ can be written as

$$\begin{aligned} &\int_{B_1} \frac{\widetilde{u}(y) D\widetilde{g_{\zeta_1}}(y)}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_1}}(y)|^3} - \frac{\widetilde{u}(y) D\widetilde{g_{\zeta_2}}(y)}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_2}}(y)|^3} dy \\ &= \int_{B_1} \frac{\widetilde{u}(y) [D\widetilde{g_{\zeta_1}}(y) - D\widetilde{g_{\zeta_2}}(y)]}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_1}}(y)|^3} dy \\ &\quad + \int_{B_1} \widetilde{u}(y) D\widetilde{g_{\zeta_2}}(y) \left(\frac{1}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_1}}(y)|^3} - \frac{1}{|(z_1, z_2, 0, 0, z_3) - \widetilde{g_{\zeta_2}}(y)|^3} \right) dy \end{aligned}$$

which is easily seen to be bounded by $Cd\|u\|_{L^\infty}$ in view of the estimates $|D\widetilde{g_{\zeta_1}}(y) - D\widetilde{g_{\zeta_2}}(y)| \leq Cd$ and $|\widetilde{g_{\zeta_1}}(y) - \widetilde{g_{\zeta_2}}(y)| \leq Cd$. We can treat $\{\nabla L^{-1}[u(g_{\zeta_1}^{-1})]\}(z) - \{\nabla L^{-1}[u(g_{\zeta_2}^{-1})]\}(z)$ in a similar way, this time using $|D\widetilde{g_{\zeta_1}}(y) - D\widetilde{g_{\zeta_2}}(y)| \leq Cd$ and Lemma 3.4 to draw the conclusion. The difference expression (56) can be treated analogously using (51). $\square$

Finally, we will also need a lemma concerning the Hölder estimate of $u(g_{\zeta_1}^{-1}) - u(g_{\zeta_2}^{-1})$.

Lemma 3.6. *If $u \in C^{0,\beta}(\mathbb{R}^3)$ for some $0 < \beta \leq 1$, and u is supported in B_1 , and $\zeta_1, \zeta_2 \in X_\delta$ where $\delta > 0$ is sufficiently small, then $\|u(g_{\zeta_1}^{-1}) - u(g_{\zeta_2}^{-1})\|_{C^{0,\alpha}(\mathbb{R}^3)} \leq C_\alpha \|\zeta_1 - \zeta_2\|_X^{\beta-\alpha}$ for every $0 < \alpha < \beta$.*

Remark 1. Notice that $\|u(g_{\zeta_1}^{-1}) - u(g_{\zeta_2}^{-1})\|_{C^{0,\alpha}(\mathbb{R}^3)}$ might not tend to zero as $\|\zeta_1 - \zeta_2\|_X \rightarrow 0$ for $\alpha = \beta$, as is suggested by the simple example $f_\epsilon(x) = |x + \epsilon|^\beta - |x|^\beta$. The $C^{0,\beta}$ norm of f_ϵ does not decrease as $\epsilon \rightarrow 0$.

Proof. Let $v = u(g_{\zeta_1}^{-1}) - u(g_{\zeta_2}^{-1})$, we have $\|v\|_{C^{0,\beta}} \leq C$, and $\|v\|_\infty \leq C \|\zeta_1 - \zeta_2\|_X^\beta$. In fact, $v(x) = u(g_{\zeta_1}^{-1}(x)) - u(g_{\zeta_2}^{-1}(x))$ is bounded by a constant multiple of $|g_{\zeta_1}^{-1}(x) - g_{\zeta_2}^{-1}(x)|^\beta$, due to the Hölder continuity of u . The latter is bounded by $\|\zeta_1 - \zeta_2\|_X^\beta$ by the property of g_ζ given in Lemma 3.4 of [5]. For any $x, y, x \neq y$, note that

$$\begin{aligned} \frac{|v(x) - v(y)|}{|x - y|^\alpha} &= \left(\frac{|v(x) - v(y)|}{|x - y|^\beta} \right)^{\frac{\alpha}{\beta}} |v(x) - v(y)|^{1-\frac{\alpha}{\beta}} \\ &\leq C \|v\|_{C^{0,\beta}}^{\frac{\alpha}{\beta}} \|v\|_\infty^{1-\frac{\alpha}{\beta}} \\ &\leq C \|\zeta_1 - \zeta_2\|_X^{\beta-\alpha}, \end{aligned}$$

from which we deduce the result. $\square$

4. Fréchet differentiability

In this section, we prove the Fréchet differentiability of $\mathcal{F}(\zeta, \varphi, \omega, \epsilon)$.

Theorem 4.1. *The operator $\mathcal{F} : X_\delta^2 \times \mathbb{R}^2 \rightarrow X^2$ is continuously Fréchet differentiable if $\delta > 0$ is sufficiently small.*

4.1. $\mathcal{F}$ Maps into X^2

Lemma 4.1. *There exists a constant $C > 0$ depending on ρ_0, k , and δ such that*

$$\|\mathcal{F}_1(\zeta, \varphi, \omega, \epsilon)\|_X \leq C(1 + \omega^2 + \epsilon) \text{ and } \|\mathcal{F}_2(\zeta, \varphi, \epsilon)\|_X \leq C(1 + \epsilon) \quad (58)$$

if $\zeta \in X_\delta$ and $\varphi \in X_\delta$ for sufficiently small $\delta > 0$.

Proof. We start with $\mathcal{F}_1$. The terms except $-\epsilon K(\varphi(x)) + \epsilon K(\varphi(0))$ in $\mathcal{F}_1$ have been shown to map into X in [5] (cf. Lemma 5.1 in [5]). In order for $-\epsilon K(\varphi(x)) + \epsilon K(\varphi(0))$ to also map into X , it suffices to show that $|K'(\varphi(x))\nabla\varphi(x)|$ is bounded by $C|x|$. But this immediately follows from the fact that $\varphi \in X$ and $|K'(\varphi(x))|$ is bounded.

We next move on to $\mathcal{F}_2$. Let us rewrite $\mathcal{F}_2$ as

$$\mathcal{F}_2(\zeta, \varphi, \omega, \epsilon) = \varphi(x) - \epsilon \mathcal{M}(\zeta) \mathcal{F}_3(\zeta, \varphi) \quad (59)$$

where

$$\mathcal{F}_3(\zeta, \varphi) := L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))) (g_\zeta(x)). \quad (60)$$

Since $\varphi \in X$, we only need to show that $\mathcal{F}_3$ maps into X . To this end, we compute the spatial derivative of (60):

$$\partial_i \mathcal{F}_3(\zeta, \varphi) = [\nabla L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1})))](g_\zeta(x)) \cdot \partial_i g_\zeta(x). \quad (61)$$

Since $|g_\zeta(x) - x| \leq \|\zeta\|_X |x|$ and $|\partial_i g_\zeta(x)| \leq C(1 + \|\zeta\|_X)$ (cf. Lemma 3.4 in [5]), in order to bound $|\partial_i \mathcal{F}_3|$ by $C|x|$, it is sufficient to show that $[\nabla L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1})))](z)$ is bounded by $C|z|$. This, in turn, is a consequence of Lemma 3.1. $\square$

4.2. Formal derivative of $\mathcal{F}$

To simplify notation, we suppress the ω, ϵ dependence in $\mathcal{F}$. Another reason for doing so is that the differentiability with respect to ω and ϵ is simpler. Therefore, for the moment, we think of them as being fixed. Let $\zeta, \varphi \in X_\delta$ and $\xi, \eta \in X$ be given. Consider $s \in \mathbb{R}$ in a sufficiently small neighborhood of 0 so that $\zeta + s\xi, \varphi + s\eta \in X_\delta$. We define the formal derivatives of $\mathcal{F}$ with respect to ζ , and φ respectively as the pointwise limits

$$\begin{aligned} \left[\frac{\partial \mathcal{F}}{\partial \zeta}(\zeta, \varphi) \xi \right](x) &= \partial_s \Big|_{s=0} \mathcal{F}(\zeta + s\xi, \varphi)(x), \\ \left[\frac{\partial \mathcal{F}}{\partial \varphi}(\zeta, \varphi) \eta \right](x) &= \partial_s \Big|_{s=0} \mathcal{F}(\zeta, \varphi + s\eta)(x), \end{aligned} \quad (62)$$

for every fixed x . We do not yet claim that this is a Gateaux derivative, which would require a specific condition making use of the norm, while for the time being we are only defining it pointwise. The formal derivative of $\mathcal{F}_1$ with respect to ζ was computed in Lemma 5.2 in [5]. It is easy to see that the formal derivative of $\mathcal{F}_1$ with respect to φ is

$$\left[\frac{\partial \mathcal{F}_1}{\partial \varphi}(\zeta, \varphi) \eta \right](x) = -\epsilon K'(\varphi(x)) \eta(x) \quad (63)$$

As for $\mathcal{F}_2$ it is clear from (59) that

$$\left[\frac{\partial \mathcal{F}_2}{\partial \zeta}(\zeta, \varphi) \xi \right] = -\epsilon [\mathcal{M}'(\zeta) \xi] \mathcal{F}_3(\zeta, \varphi) - \epsilon \mathcal{M}(\zeta) \left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta, \varphi) \xi \right] \quad (64)$$

and

$$\left[\frac{\partial \mathcal{F}_2}{\partial \varphi}(\zeta, \varphi) \eta \right] = \eta - \epsilon \mathcal{M}(\zeta) \left[\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta, \varphi) \eta \right] \quad (65)$$

where $\mathcal{F}_3$ is given in (60). According to (5.18) in [5], $\mathcal{M}'(\zeta) \xi$ is given by

$$\begin{aligned} \mathcal{M}'(\zeta) \xi &= \frac{-M}{\left(\int_{B_1} \rho_0(x) \det Dg_\zeta(x) dx \right)^2} \\ &\quad \times \int_{B_1} \rho_0(x) \det Dg_\zeta(x) \operatorname{tr} \left[(Dg_\zeta)^{-1}(x) D \left(\xi(x) \frac{x}{|x|^2} \right) \right] dx \end{aligned} \quad (66)$$

The computation of $\left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta, \varphi) \xi \right]$ is similar to the one for the first term in $\mathcal{F}_1$, both of which are integral operators involving $\rho_0(g_\zeta^{-1})$. Using the formula (see (5.30) of [5]),

$$\partial_s g_{\zeta+s\xi}^{-1}(y) = -Dg_{\zeta+s\xi}^{-1}(y) \xi(g_{\zeta+s\xi}^{-1}(y)) \frac{g_{\zeta+s\xi}^{-1}(y)}{|g_{\zeta+s\xi}^{-1}(y)|^2}, \quad (67)$$

we deduce that

$$\begin{aligned} &\left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta, \varphi) \xi \right](x) \\ &= -L^{-1} \left[(\nabla \rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))) + \rho_0(g_\zeta^{-1}) k'(\varphi(g_\zeta^{-1})) \nabla \varphi(g_\zeta^{-1}) \right] \cdot \\ &\quad D(g_\zeta^{-1}) \cdot \frac{\xi(g_\zeta^{-1}) g_\zeta^{-1}}{|g_\zeta^{-1}|^2} (g_\zeta(x)) \\ &\quad + [\nabla L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1})))](g_\zeta(x)) \cdot \frac{\xi(x)}{|x|^2} x. \end{aligned} \quad (68)$$

Lastly, the formal derivative with respect to φ is given by

$$\left[\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta, \varphi) \eta \right](x) = L^{-1} [\rho_0(g_\zeta^{-1}) k'(\varphi(g_\zeta^{-1})) \eta(g_\zeta^{-1})] (g_\zeta(x)). \quad (69)$$

A rigorous justification of the above formulas involving the cut-off function method may be found in Lemma 5.2. of [5]. The details are omitted.

Next we will show that the formal derivative $\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}$, just computed, is a bounded linear map on $X \times X$.

Lemma 4.2. *If $\zeta, \varphi \in X_\delta$ and δ is sufficiently small, there exists a constant $C > 0$ such that*

$$\left\| \frac{\partial \mathcal{F}_1}{\partial \zeta}(\zeta, \varphi) \xi \right\|_X \leq C(1 + \omega^2) \|\xi\|_X, \quad \left\| \frac{\partial \mathcal{F}_1}{\partial \varphi}(\zeta, \varphi) \eta \right\|_X \leq C\epsilon \|\eta\|_X \quad (70)$$

and

$$\left\| \frac{\partial \mathcal{F}_2}{\partial \zeta}(\zeta, \varphi) \xi \right\|_X \leq C\epsilon \|\xi\|_X, \quad \left\| \frac{\partial \mathcal{F}_2}{\partial \varphi}(\zeta, \varphi) \eta \right\|_X \leq C(1 + \epsilon) \|\eta\|_X. \quad (71)$$

Proof. The first inequality in (70) was derived in Lemma 5.5 in [5]. The second inequality in (70) directly follows from (63) since $|K'(\varphi(x))|$ is bounded. The rest of the proof is devoted to (71). It suffices to prove the boundedness of $\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta, \varphi)$ and $\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta, \varphi)$ on X .

We start with $\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta, \varphi)$. To estimate its X norm, we take the spatial derivatives

$$\begin{aligned} & \partial_i \left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta, \varphi) \xi \right](x) \\ &= -\nabla L^{-1} \left[\left(\nabla \rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1})) + \rho_0(g_\zeta^{-1}) k'(\varphi(g_\zeta^{-1})) \nabla \varphi(g_\zeta^{-1}) \right) \cdot \right. \\ & \quad \left. Dg_\zeta^{-1} \cdot \frac{\xi(g_\zeta^{-1}) g_\zeta^{-1}}{|g_\zeta^{-1}|^2} \right] (g_\zeta(x)) \cdot \partial_i g_\zeta(x) \quad (72) \end{aligned}$$

$$+ \left[\nabla^2 L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))) \right] (g_\zeta(x)) \partial_i g_\zeta(x) \cdot \frac{\xi(x)}{|x|^2} x \quad (73)$$

$$+ \left[\nabla L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))) \right] (g_\zeta(x)) \cdot \partial_i \left(\frac{\xi(x)}{|x|^2} x \right). \quad (74)$$

We claim, starting with (72), that each term on the right-hand side is bounded by $C \|\xi\|_X |x|$. Due to Lemma 3.1 and the estimates $|g_\zeta(x) - x| \leq \|\zeta\|_X |x|$ and $|\partial_i g_\zeta(x)| \leq C(1 + \|\zeta\|_X)$ (cf. Lemma 3.4 in [5]), we can bound (72) by

$$\begin{aligned} |(72)| &\leq C \left\| \left[\nabla \rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1})) + \rho_0(g_\zeta^{-1}) k'(\varphi(g_\zeta^{-1})) \nabla \varphi(g_\zeta^{-1}) \right] \cdot \right. \\ & \quad \left. \nabla g_\zeta^{-1} \cdot \frac{\xi(g_\zeta^{-1}) g_\zeta^{-1}}{|g_\zeta^{-1}|^2} \right\|_{L^\infty} |x| \end{aligned}$$

Note that the first factor is bounded in the L^∞ norm due to the assumptions on k and φ . As for the second factor, since ∇g_ζ^{-1} is bounded and since $|g_\zeta^{-1}(y)|$ is comparable to $|y|$, and $|\xi(y)y|/|y|^2$ is dominated by $C \|\xi\|_X$, we have $\|\nabla g_\zeta^{-1} \cdot \xi(g_\zeta^{-1}) g_\zeta^{-1} / |g_\zeta^{-1}|^2\|_{L^\infty} \leq C \|\xi\|_X$.

For (73), since $|\partial_i g_\zeta(x) \cdot (\xi(x)/|x|^2)x| \leq C \|\xi\|_X |x|$, it is enough to have $\nabla^2 L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))) (z)$ be bounded and continuous. With our assumptions on ζ and φ , $\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))$ is in C^1 and therefore (73) satisfies the desired estimate due to Lemma 3.3. As for (74), since $|\partial_i (\frac{\xi(x)}{|x|^2} x)| \leq C \|\xi\|_X$, it is enough to have $|\nabla L^{-1}(\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))) (z)| \leq C|z|$, which is indeed the case by Lemma 3.1, using the fact that $\rho_0(g_\zeta^{-1}) k(\varphi(g_\zeta^{-1}))$ is in L^∞ .

Finally we examine the spatial derivative $\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta, \varphi)$:

$$\begin{aligned} & \partial_i \left[\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta, \varphi) \eta \right](x) \\ &= \nabla L^{-1} \left[\rho_0(g_\zeta^{-1}) k'(\varphi(g_\zeta^{-1})) \eta(g_\zeta^{-1}) \right] (g_\zeta(x)) \cdot \partial_i g_\zeta(x). \quad (75) \end{aligned}$$

Since $\|\rho_0(g_\zeta^{-1}) k'(\varphi(g_\zeta^{-1})) \eta(g_\zeta^{-1})\|_{L^\infty} \leq C \|\eta\|_X$, from Lemma 3.1 we deduce that $\|\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta, \varphi) \eta\|_X \leq C \|\eta\|_X |x|$. This completes the proof of the lemma. $\square$

4.3. Continuity of $\mathcal{F}'$

Now we establish the continuity of the formal derivative $\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi, \omega, \epsilon)}$ from $X_\delta \times X_\delta \times (-\delta, \delta) \times (-\delta, \delta)$ into $X \times X$.

Lemma 4.3. *If $\zeta, \varphi \in X_\delta$ and δ is sufficiently small, there exist constants $C > 0$, $0 < \alpha < 1$, and a constant C_β for each $0 < \beta < 1$ such that*

$$\left\| \left(\frac{\partial \mathcal{F}_1}{\partial \zeta}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_1}{\partial \zeta}(\zeta_2, \varphi_2) \right) \xi \right\|_X \leq C \|\zeta_1 - \zeta_2\|_X^\alpha \|\xi\|_X, \quad (76)$$

$$\left\| \left(\frac{\partial \mathcal{F}_1}{\partial \varphi}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_1}{\partial \varphi}(\zeta_2, \varphi_2) \right) \eta \right\|_X \leq C\epsilon \|\varphi_1 - \varphi_2\|_X \|\eta\|_X, \quad (77)$$

$$\begin{aligned} & \left\| \left(\frac{\partial \mathcal{F}_2}{\partial \zeta}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_2}{\partial \zeta}(\zeta_2, \varphi_2) \right) \xi \right\|_X \leq C_\beta \epsilon (\|\zeta_1 - \zeta_2\|_X^\beta \\ & \quad + \|\varphi_1 - \varphi_2\|_X^\beta) \|\xi\|_X, \quad (78) \end{aligned}$$

$$\begin{aligned} & \left\| \left(\frac{\partial \mathcal{F}_2}{\partial \varphi}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_2}{\partial \varphi}(\zeta_2, \varphi_2) \right) \eta \right\|_X \leq C_\beta \epsilon (\|\zeta_1 - \zeta_2\|_X^\beta \\ & \quad + \|\varphi_1 - \varphi_2\|_X) \|\eta\|_X, \quad (79) \end{aligned}$$

Proof. The inequality (76) was derived in Lemma 5.6 of [5]. We focus on the rest of the estimates, beginning with (77). Recalling (63), we have

$$\begin{aligned} & \partial_i \left[\left(\frac{\partial \mathcal{F}_1}{\partial \varphi}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_1}{\partial \varphi}(\zeta_2, \varphi_2) \right) \eta \right](x) \\ &= \epsilon (K''(\varphi_2(x)) \partial_i \varphi_2(x) - K''(\varphi_1(x)) \partial_i \varphi_1(x)) \eta(x) \\ & \quad + \epsilon (K'(\varphi_2(x)) - K'(\varphi_1(x))) \partial_i \eta(x) \end{aligned}$$

The second term on the right-hand side is bounded by $C\epsilon \|\varphi_1 - \varphi_2\|_X \|\eta\|_X |x|$ since $|K'(\varphi_2(x)) - K'(\varphi_1(x))|$ is bounded by $C \|\varphi_1 - \varphi_2\|_X$. Rewriting the first term as

$$\epsilon (K''(\varphi_2(x)) (\partial_i \varphi_2(x) - \partial_i \varphi_1(x)) + (K''(\varphi_2(x)) - K''(\varphi_1(x))) \partial_i \varphi_1(x)) \eta(x),$$

we see that it is also bounded by $C\epsilon \|\varphi_1 - \varphi_2\|_X \|\eta\|_X |x|$. Thus (77) holds.

As for (78), we write

$$\left(\frac{\partial \mathcal{F}_2}{\partial \zeta}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_2}{\partial \zeta}(\zeta_2, \varphi_2) \right) \xi = J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \epsilon [\mathcal{M}'(\zeta_2) \xi - \mathcal{M}'(\zeta_1) \xi] \mathcal{F}_3(\zeta_1, \varphi_1)$$

$$J_2 = \epsilon [\mathcal{M}'(\zeta_2) \xi] (\mathcal{F}_3(\zeta_2, \varphi_2) - \mathcal{F}_3(\zeta_1, \varphi_1))$$

$$J_3 = \epsilon (\mathcal{M}(\zeta_2) - \mathcal{M}(\zeta_1)) \frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_1, \varphi_1) \xi$$

$$J_4 = \epsilon \mathcal{M}(\zeta_2) \left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_2, \varphi_2) \xi - \frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_1, \varphi_1) \xi \right]$$

We will estimate the X norm of each J_i , $i = 1, 2, 3, 4$. We start with J_1 . The X norm of $\mathcal{F}_3$ was shown to be bounded in Lemma 4.1. The estimate of I_1 in Lemma 5.6 of [5] shows

$\|\mathcal{M}'(\zeta_2)\xi - \mathcal{M}'(\zeta_1)\xi\|_{L^\infty} \leq C\|\zeta_1 - \zeta_2\|_X \|\xi\|_X$, and therefore we obtain

$$\|J_1\|_X \leq C\epsilon\|\zeta_1 - \zeta_2\|_X \|\xi\|_X.$$

We next estimate J_2 . By Lemma 5.5 of [5], we have $\|\mathcal{M}'(\zeta_2)\xi\|_X \leq C\|\xi\|_X$. In order to estimate the X norm of $\mathcal{F}_3(\zeta_2, \varphi_2) - \mathcal{F}_3(\zeta_1, \varphi_1)$, we rewrite its spatial derivative as

$$\begin{aligned} \partial_i(\mathcal{F}_3(\zeta_2, \varphi_2) - \mathcal{F}_3(\zeta_1, \varphi_1)) &= \\ &\{ \nabla L^{-1}[\rho_0(g_{\zeta_2}^{-1})k(\varphi_2(g_{\zeta_2}^{-1}))](g_{\zeta_2}(x)) - \nabla L^{-1}[\rho_0(g_{\zeta_1}^{-1})k(\varphi_2(g_{\zeta_1}^{-1}))] \\ &\quad \times (g_{\zeta_2}(x)) \} \partial_i g_{\zeta_2}(x) \\ &+ \{ \nabla L^{-1}[\rho_0(g_{\zeta_1}^{-1})k(\varphi_2(g_{\zeta_1}^{-1}))](g_{\zeta_2}(x)) - \nabla L^{-1}[\rho_0(g_{\zeta_1}^{-1})k(\varphi_2(g_{\zeta_1}^{-1}))] \\ &\quad \times (g_{\zeta_1}(x)) \} \partial_i g_{\zeta_2}(x) \\ &+ \{ \nabla L^{-1}[\rho_0(g_{\zeta_1}^{-1})k(\varphi_2(g_{\zeta_1}^{-1}))](g_{\zeta_1}(x)) - \nabla L^{-1}[\rho_0(g_{\zeta_1}^{-1})k(\varphi_1(g_{\zeta_1}^{-1}))] \\ &\quad \times (g_{\zeta_1}(x)) \} \partial_i g_{\zeta_2}(x) \\ &+ \nabla L^{-1}[\rho_0(g_{\zeta_1}^{-1})k(\varphi_1(g_{\zeta_1}^{-1}))](g_{\zeta_1}(x)) \partial_i(g_{\zeta_2}(x) - g_{\zeta_1}(x)) \\ &=: J_{21} + J_{22} + J_{23} + J_{24} \end{aligned}$$

J_{21} can be estimated by the representation (48) and Lemma 3.5 with $u = \rho_0 k(\varphi_2)$. Since $\rho_0 k(\varphi_2)$ is in L^∞ , we deduce that $|J_{21}| \leq C_\beta \|\zeta_1 - \zeta_2\|_X^\beta |x|$ for any $0 < \beta < 1$. For J_{22} , we apply Lemma 3.2 to deduce that $|J_{22}| \leq C_\beta \|\zeta_1 - \zeta_2\|_X^\beta |x|$ for any $0 < \beta < 1$, as $\rho_0(g_{\zeta_1}^{-1})k(\varphi_2(g_{\zeta_1}^{-1})) \in C^1$. To estimate J_{23} , we note that $|k(\varphi_2(z)) - k(\varphi_1(z))| \leq \|k'\|_{L^\infty} |\varphi_2(z) - \varphi_1(z)| \leq C\|\varphi_1 - \varphi_2\|_X$, which results in $|J_{23}| \leq C\|\varphi_1 - \varphi_2\|_X |x|$ by Lemma 3.2. Moreover, we have $|J_{24}| \leq C\|\zeta_1 - \zeta_2\|_X |x|$, since $|\partial_i(g_{\zeta_2}(x) - g_{\zeta_1}(x))| \leq C\|\zeta_1 - \zeta_2\|_X$. To sum up, we deduce that

$$\|J_2\|_X \leq C\epsilon(\|\zeta_1 - \zeta_2\|_X^\beta + \|\varphi_1 - \varphi_2\|_X) \|\xi\|_X$$

for any $0 < \beta < 1$.

The estimation of J_3 is similar to that of J_1 . From Lemma 4.2 we deduce that

$$\|J_3\|_X \leq C\epsilon\|\zeta_1 - \zeta_2\|_X \|\xi\|_X.$$

We now turn into J_4 . Introducing some notation to denote the functions appearing in (72), (73), (74), we define

$$\begin{aligned} v(\zeta, \varphi) &:= (\nabla \rho_0(g_\zeta^{-1})k(\varphi(g_\zeta^{-1})) + \rho_0(g_\zeta^{-1})k'(\varphi(g_\zeta^{-1}))\nabla \varphi(g_\zeta^{-1})) \\ &\quad \cdot Dg_\zeta^{-1} \cdot \frac{\xi(g_\zeta^{-1})g_\zeta^{-1}}{|g_\zeta^{-1}|^2} \\ w(\zeta, \varphi) &:= \rho_0(g_\zeta^{-1})k(\varphi(g_\zeta^{-1})) \end{aligned}$$

Then we may write $\partial_i \left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_2, \varphi_2)\xi - \frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_1, \varphi_1)\xi \right]$ as

$$\begin{aligned} &\partial_i \left[\frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_2, \varphi_2)\xi - \frac{\partial \mathcal{F}_3}{\partial \zeta}(\zeta_1, \varphi_1)\xi \right] \\ &= \nabla L^{-1}(v(\zeta_1, \varphi_1))(g_{\zeta_1}(x)) \cdot \partial_i g_{\zeta_1}(x) \\ &\quad - \nabla L^{-1}(v(\zeta_2, \varphi_2))(g_{\zeta_2}(x)) \cdot \partial_i g_{\zeta_2}(x) \\ &+ \left[\nabla^2 L^{-1}(w(\zeta_2, \varphi_2))(g_{\zeta_2}(x)) \partial_i g_{\zeta_2}(x) \right. \\ &\quad \left. - \nabla^2 L^{-1}(w(\zeta_1, \varphi_1))(g_{\zeta_1}(x)) \partial_i g_{\zeta_1}(x) \right] \cdot \frac{\xi(x)}{|x|^2} x \\ &+ \left[\nabla L^{-1}(w(\zeta_2, \varphi_2))(g_{\zeta_2}(x)) - \nabla L^{-1}(w(\zeta_1, \varphi_1))(g_{\zeta_1}(x)) \right] \\ &\quad \cdot \partial_i \left(\frac{\xi(x)}{|x|^2} x \right) \\ &=: J_{41} + J_{42} + J_{43}. \end{aligned}$$

We will estimate each J_{4i} , $i = 1, 2, 3$ separately. We start with J_{41} following the same steps as for J_2 . Namely, we rewrite it as $J_{411} + J_{412} + J_{413} + J_{414}$ (see $J_{21} + J_{22} + J_{23} + J_{24}$) and estimate each term. The only novelty is that we have $v(\zeta, \varphi)$ instead of $w(\zeta, \varphi) = \rho_0(g_\zeta^{-1})k(\varphi(g_\zeta^{-1}))$. Consequently, we just use estimate (56) instead of (55). We deduce that

$$|J_{41}| \leq C_\beta(\|\zeta_1 - \zeta_2\|_X^\beta + \|\varphi_1 - \varphi_2\|_X) \|\xi\|_X |x|$$

for any $0 < \beta < 1$.

For J_{42} , we rewrite the square bracket term as

$$\begin{aligned} [\cdot] &= \nabla^2 L^{-1}[w(\zeta_2, \varphi_2) - w(\zeta_1, \varphi_2)](g_{\zeta_2}(x)) \partial_i g_{\zeta_2}(x) \\ &\quad + \nabla^2 L^{-1}[w(\zeta_1, \varphi_2) - w(\zeta_1, \varphi_1)](g_{\zeta_2}(x)) \partial_i g_{\zeta_2}(x) \\ &\quad + \{ \nabla^2 L^{-1}[w(\zeta_1, \varphi_1)](g_{\zeta_2}(x)) - \nabla^2 L^{-1}[w(\zeta_1, \varphi_1)](g_{\zeta_1}(x)) \} \\ &\quad \times \partial_i g_{\zeta_2}(x) \\ &\quad + \nabla^2 L^{-1}[w(\zeta_1, \varphi_1)](g_{\zeta_1}(x)) \partial_i(g_{\zeta_2}(x) - g_{\zeta_1}(x)) \\ &=: J_{421} + J_{422} + J_{423} + J_{424} \end{aligned}$$

We focus first on $\nabla^2 L^{-1}[w(\zeta_2, \varphi_2) - w(\zeta_1, \varphi_2)](z)$ in J_{421} . Since $\rho_0 k(\varphi_2) \in C^{0,1}$, Lemma 3.5 yields $\|w(\zeta_2, \varphi_2) - w(\zeta_1, \varphi_2)\|_{C^{0,\alpha}} \leq C_\alpha \|\zeta_1 - \zeta_2\|_X^{1-\alpha}$ for any $0 < \alpha < 1$. Lemma 3.3 then gives the bound

$$\|\nabla^2 L^{-1}(w(\zeta_2, \varphi_2) - w(\zeta_1, \varphi_2))\|_{L^\infty} \leq C_\beta \|\zeta_1 - \zeta_2\|_X^\beta$$

for any $0 < \beta < 1$. Similarly, in J_{422} we also have

$$\|\nabla^2 L^{-1}[w(\zeta_1, \varphi_2) - w(\zeta_1, \varphi_1)]\|_{L^\infty} \leq C_\beta \|\varphi_1 - \varphi_2\|_X^\beta$$

for any $0 < \beta < 1$. The most serious term in J_{423} is as follows. Since $\rho_0 k(\varphi_1) \in C^{0,1}$, Lemma 3.3 again gives

$$\begin{aligned} &|\nabla^2 L^{-1}[w(\zeta_1, \varphi_1)](g_{\zeta_2}(x)) - \nabla^2 L^{-1}[w(\zeta_1, \varphi_1)](g_{\zeta_1}(x))| \\ &\leq C_\beta |g_{\zeta_1}(x) - g_{\zeta_2}(x)|^\beta \leq C_\beta \|\zeta_1 - \zeta_2\|_X^\beta \end{aligned}$$

for any $0 < \beta < 1$. Lastly, $|J_{424}| \leq C\|\zeta_1 - \zeta_2\|_X$ since $\nabla^2 L^{-1}[w(\zeta_1, \varphi_1)](g_{\zeta_1}(x))$ is bounded and $|\partial_i(g_{\zeta_2}(x) - g_{\zeta_1}(x))| \leq C\|\zeta_1 - \zeta_2\|_X$. Together with $|\frac{\xi(x)}{|x|^2} x| \leq C\|\xi\|_X |x|$, we arrive at

$$|J_{42}| \leq C_\beta(\|\zeta_1 - \zeta_2\|_X^\beta + \|\varphi_1 - \varphi_2\|_X) \|\xi\|_X |x|$$

for any $0 < \beta < 1$.

The square bracket in J_{43} can be estimated in the same way as was done for J_{21} , J_{22} and J_{23} . Together with $|\partial_i(\frac{\xi(x)}{|x|^2} x)| \leq C\|\xi\|_X$, we have

$$|J_{43}| \leq C_\beta(\|\zeta_1 - \zeta_2\|_X^\beta + \|\varphi_1 - \varphi_2\|_X) \|\xi\|_X |x|$$

for any $0 < \beta < 1$. Combining the estimates of J_{41} , J_{42} and J_{43} , we find that

$$\|J_4\|_X \leq C_\beta(\|\zeta_1 - \zeta_2\|_X^\beta + \|\varphi_1 - \varphi_2\|_X) \|\xi\|_X$$

for any $0 < \beta < 1$. This completes the proof of (78).

It remains to show (79). Note that

$$\begin{aligned} \left(\frac{\partial \mathcal{F}_2}{\partial \varphi}(\zeta_1, \varphi_1) - \frac{\partial \mathcal{F}_2}{\partial \varphi}(\zeta_2, \varphi_2) \right) \eta &= \epsilon(\mathcal{M}(\zeta_2) - \mathcal{M}(\zeta_1)) \\ &\quad \times \frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta_1, \varphi_1) \eta \\ &+ \epsilon \mathcal{M}(\zeta_2) \left[\frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta_2, \varphi_2) \eta - \frac{\partial \mathcal{F}_3}{\partial \varphi}(\zeta_1, \varphi_1) \eta \right] \\ &=: K_1 + K_2 \end{aligned}$$

Each factor of K_1 has been previously estimated and it is easy to see that

$$\|K_1\|_X \leq C\epsilon\|\zeta_1 - \zeta_2\|_X \|\eta\|_X.$$

The other term K_2 can be estimated in the same spirit as for J_2 , for instance. We obtain

$$\|K_2\|_X \leq C_\beta (\|\zeta_1 - \zeta_2\|_X^\beta + \|\varphi_1 - \varphi_2\|_X) \|\eta\|_X$$

for any $0 < \beta < 1$. We omit the details. This completes the proof of the proposition. $\square$

Before proving that the formal derivatives are genuine Fréchet derivatives, we need a technical lemma that establishes the equality of the mixed partial derivatives of the functions

$$G_1(x, s) = \mathcal{F}(\zeta + s\xi, \varphi)(x) = (\mathcal{F}_1(\zeta + s\xi, \varphi)(x), \mathcal{F}_2(\zeta + s\xi, \varphi)(x))^t$$

and

$$G_2(x, s) = \mathcal{F}(\zeta, \varphi + s\eta)(x) = (\mathcal{F}_1(\zeta, \varphi + s\eta)(x), \mathcal{F}_2(\zeta, \varphi + s\eta)(x))^t$$

This will of course be true if the G_1, G_2 are C^2 . However, rather than showing such regularity, we make a direct computation.

Lemma 4.4. *Let G_1, G_2 be defined as above, where $\zeta, \varphi \in X_\delta$ for $\delta > 0$ sufficiently small, and $\xi, \eta \in X$ are such that $\zeta + s\xi, \varphi + s\eta \in X_\delta$ for all $|s| \leq 1$. Then*

$$\partial_i \partial_s G_j(x, s) = \partial_s \partial_i G_j(x, s) \quad (80)$$

for all $x \in \overline{B_1}$, all $|s| \leq 1$ and $j = 1, 2$.

Proof. We only show the proof for G_1 , as G_2 is similar. For the $\mathcal{F}_1$ part of G_1 , (80) is an immediate consequence of Lemma 5.4 of [5]. For $\mathcal{F}_2$, we only focus on the part

$$G_3(x, s) = \mathcal{F}_3(\zeta + s\xi, \varphi)(x) = L^{-1}[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))](g_{\zeta+s\xi}(x)). \quad (81)$$

In the following, to simplify notation, we plug in $s = 0$ after taking the derivative ∂_s . The general case will be similar but has more clumsy notation. $\partial_i \partial_s G_3(x, 0)$ is given by (72), (73) and (74). To find the other mixed partial derivative, we compute

$$\partial_i G_3(x, s) = \nabla L^{-1}[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))](g_{\zeta+s\xi}(x)) \cdot \partial_i g_{\zeta+s\xi}(x). \quad (82)$$

If we let the ∂_s derivative fall on the various terms as follows

$$\nabla L^{-1}\{\partial_s[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))]\}(g_{\zeta+s\xi}(x)) \cdot \partial_i g_{\zeta+s\xi}(x) \quad (83)$$

$$+ \nabla^2 L^{-1}[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))](g_{\zeta+s\xi}(x)) \cdot \partial_s g_{\zeta+s\xi}(x) \cdot \partial_i g_{\zeta+s\xi}(x) \quad (84)$$

$$+ \nabla L^{-1}[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))](g_{\zeta+s\xi}(x)) \cdot \partial_s \partial_i g_{\zeta+s\xi}(x), \quad (85)$$

we will recover (72), (73) and (74), and the proof will be complete. To justify the calculation above, we define

$$G_4(z, s) = \nabla L^{-1}[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))](z) \quad (86)$$

and want to show that

$$(a) \quad \partial_s G_4(z, s) = \nabla L^{-1}\{\partial_s[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))]\}(z).$$

$$(b) \quad \partial_z G_4(z, s) = \nabla^2 L^{-1}[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))](z).$$

$$(c) \quad \text{Both } \partial_s G_4(z, s) \text{ and } \partial_z G_4(z, s) \text{ are continuous.}$$

The calculation will therefore be justified by the chain rule.

Since $\partial_s[\rho_0(g_{\zeta+s\xi}^{-1})k(\varphi(g_{\zeta+s\xi}^{-1}))]$ is bounded, (a) follows directly from the dominated convergence theorem. (b) is obvious. The continuity of $\partial_s G_4(z, s)$ and $\partial_z G_4(z, s)$ can be proven by the same kind of estimates employed in the proof of Lemma 4.3. We omit the straightforward details. $\square$

We finally show that the formal derivatives are genuine Fréchet derivatives.

Lemma 4.5. *Let $\zeta, \varphi \in X_\delta$ where $\delta > 0$ is sufficiently small. There exist $\delta_1 > 0$, $0 < \alpha < 1$ and $C > 0$ such that if $\|\xi\|_X, \|\eta\|_X < \delta_1$, then*

$$\left\| \mathcal{F}(\zeta + \xi, \varphi + \eta) - \mathcal{F}(\zeta, \varphi) - \frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(\xi, \eta) \right\|_{X^2} \leq C(\|\xi\|_X + \|\eta\|_X)^{1+\alpha}, \quad (87)$$

where $\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}$ denotes the formal derivative defined in (62).

Proof. In order to estimate the X norm on the left hand side of (87), we compute the spatial derivatives by

$$\begin{aligned} & \partial_i [\mathcal{F}(\zeta + \xi, \varphi + \eta) - \mathcal{F}(\zeta, \varphi)](x) \\ &= \partial_i \mathcal{F}(\zeta + s\xi, \varphi + s\eta)(x) \Big|_{s=0}^{s=1} \\ &= \partial_s \partial_i \mathcal{F}(\zeta + s\xi, \varphi + s\eta)(x) \Big|_{s=\theta(x)}^{s=\theta(x)} \\ &= \partial_i \partial_s \mathcal{F}(\zeta + s\xi, \varphi + s\eta)(x) \Big|_{s=\theta(x)}^{s=\theta(x)} \\ &= \partial_i \left[\left(\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(\zeta + s\xi, \varphi + s\eta) \right) (\xi, \eta) \right] (x) \Big|_{s=\theta(x)}, \end{aligned} \quad (88)$$

where $0 < \theta(x) < 1$. We have used Lemma 4.4 to exchange the order of mixed partial derivatives. It follows that

$$\begin{aligned} & \partial_i \left[\mathcal{F}(\zeta + \xi, \varphi + \eta) - \mathcal{F}(\zeta, \varphi) - \frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(\xi, \eta) \right] (x) \\ &= \partial_i \left[\left(\frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(\zeta + s\xi, \varphi + s\eta) - \frac{\partial \mathcal{F}}{\partial(\zeta, \varphi)}(\zeta, \varphi) \right) (\xi, \eta) \right] (x) \Big|_{s=\theta(x)}. \end{aligned} \quad (89)$$

Using Lemma 4.3, the components of (89) are bounded by

$$\begin{aligned} & C(\|\xi\|_X + \|\eta\|_X)^\alpha (\|\xi\|_X + \|\eta\|_X) |x| \Big|_{s=\theta(x)} \\ & \leq C(\|\xi\|_X + \|\eta\|_X)^{1+\alpha} |x|. \end{aligned}$$

The estimate (87) thus follows. $\square$

Since the dependence of $\mathcal{F}$ on ω and ϵ is very simple, we easily see that $\mathcal{F}$ is continuously Fréchet differentiable with respect to these two variables as well. Thus Theorem 4.1 immediately follows from Lemma 4.5.

Acknowledgments

JJ is supported in part by NSF grant DMS-1608494. YW is supported in part by NSF grant DMS-1714343. We acknowledge the support of the spring 2017 semester program at ICERM (Brown U.) where this work began. We would also like to thank the anonymous referee for valuable comments which have improved the presentation of the paper.

References

- [1] P. Federbush, T. Luo, J. Smoller, Existence of magnetic compressible fluid stars, *Arch. Ration. Mech. Anal.* 215 (2015) 611–631.
- [2] L. Mestel, *Stellar Magnetism*, Oxford U. Press, 2012.
- [3] S.H. Shapiro, S.A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars*, Wiley-VCH, 2014.
- [4] L. Gizon, et al., Shape of a slowly rotating star measured by asteroseismology, *Sci. Adv.* 2 (2016) e1601777.
- [5] W.A. Strauss, Y. Wu, Steady states of rotating stars and galaxies, *SIAM J. Math. Anal.* 49 (6) (2017) 4865–4914.

- [6] J. Auchmuty, R. Beals, Variational solutions of some nonlinear free boundary problems, *Arch. Ration. Mech. Anal.* 43 (4) (1971) 255–271.
- [7] Y.-Y. Li, On uniformly rotating stars, *Arch. Ration. Mech. Anal.* 115 (1991) 367–393.
- [8] L. Lichtenstein, Untersuchungen über die Gleichgewichtsfiguren rotierender Flüssigkeiten, deren Teilchen einander nach dem Newtonschen Gesetze anziehen, *Math. Z.* 36 (1) (1933) 481–562.
- [9] J. Jang, T. Makino, On slowly rotating axisymmetric solutions of the Euler–Poisson equations, *Arch. Ration. Mech. Anal.* 225 (2) (2017) 873–900.
- [10] J. Jang, T. Makino, On rotating axisymmetric solutions of the Euler–Poisson equations, *J. Differential Equations* 266 (7) (2019) 3942–3972.
- [11] W.A. Strauss, Y. Wu, Rapidly rotating stars, *Comm. Math. Phys.* (2019) (in press).
- [12] S. Chandrasekhar, *Introduction to the Stellar Structure*, U. of Chicago Press, 1939.
- [13] J.-L. Tassoul, *Stellar Rotation*, Cambridge U. Press, 2000.
- [14] W.S. Jardetzky, *Theories of Figures of Celestial Bodies*, Courier Corporation, 2013.