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Rapidly rotating white dwarfs

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Abstract

A rotating star may be modeled as a continuous system of particles attracted to each other by gravity and with a given total mass and prescribed angular velocity. Mathematically this leads to the Euler–Poisson system. A white dwarf star is modeled by a very particular, and rather delicate, equation of state for the pressure as a function of the density. We prove an existence theorem for rapidly rotating white dwarfs that depend continuously on the speed of rotation. The key tool is global continuation theory, combined with a delicate limiting process. The solutions form a connected set \mathcal{K} in an appropriate function space. As the speed of rotation increases, we prove that either the supports of white dwarfs in \mathcal{K} become unbounded or their densities become unbounded. We also discuss the polytropic case with the critical exponent $\gamma = 4/3$.

Keywords: white dwarf star, Euler–Poisson equations, global continuation, rotating star, rapid rotation

Mathematics Subject Classification numbers: 85, 35.

1. Introduction

A white dwarf is a very dense remnant of a star that no longer undergoes fusion reactions [1]. They were discovered around 1910 by observations of the luminosity and temperature of stars at a known distance. White dwarfs have high effective surface temperature but low luminosity. It is now known that the evolution of a star can lead to three final states: white dwarfs, neutron stars or black holes. Most stars, especially those below about 6 solar masses eventually become white dwarfs. Their luminosities gradually fade away over a period of billions of years. At the end of this period, the star burns out its nuclear fuel and is no longer able to support itself against gravitational collapse by the thermal pressure generated by nuclear fusion. As a result, the size of the star shrinks dramatically, until it is supported again by a new mechanism known as the electron degeneracy pressure. A white dwarf is thereby formed. White dwarfs are extremely

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dense, of the order of 10^6 g cm^{-3} . The high densities imply that the electron velocities must be relativistic. Chandrasekhar [2] concluded in 1931 that there must be a limiting mass where the gravitational forces overwhelm electron degeneracy pressure so that a non-rotating white dwarf becomes unstable and collapses; this is the ‘Chandrasekhar limit’. The study of the electron degeneracy pressure also leads to the standard equation of state in the basic mathematical model for a white dwarf, sometimes called the relativistically degenerate model. In this paper we consider a white dwarf under such a model that rotates about a fixed axis and thereby loses its spherical shape. Fixing its mass, we construct a connected set of steady-state rotating solutions.

The equation of state relates the pressure p of a white dwarf to the density ρ by the formula

$$p(\rho) = A \int_0^{B\rho^{1/3}} \frac{\sigma^4}{\sqrt{m^2 c^2 + \sigma^2}} d\sigma, \quad (1.1)$$

where m is the mass of an electron, c is the speed of light, and A, B are constants. This equation is derived by considering the theory of quantum statistical mechanics on the electrons (see [appendix](#) for more details). Indeed, the number of electrons in phase space can be calculated by filling the momentum space according to the Pauli exclusion principle. One can then calculate the electron gas pressure by relating momentum to kinetic energy using the special relativistic energy–momentum law.

The density ρ evolves in time by the compressible Euler–Poisson equations (EP), subject to the internal forces of gravity due to the particles themselves.

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0, \\ (\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p = \rho \nabla U, \\ U(x, t) = \int_{\mathbb{R}^3} \frac{\rho(x', t)}{|x - x'|} dx'. \end{cases} \quad (1.2)$$

The first two equations hold where $\rho > 0$, and the last equation defines U on the entire \mathbb{R}^3 . The equation of state $p = p(\rho)$ given by (2.1) closes the system. To model a rotating star, one looks for a steady axisymmetric rotating solution to (1.2). That is, we assume ρ is symmetric about the x_3 -axis and $v = \kappa\omega(r)(-x_2, x_1, 0)$, where $r = r(x) = \sqrt{x_1^2 + x_2^2}$, with a prescribed function $\omega(r)$. κ is a constant measuring the intensity of rotation. (Note, however, that in sections 2 and 3, the letter r is used to mean $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.) With such specifications, the first equation in (1.2) concerning mass conservation is identically satisfied. The second equation in (1.2) concerning momentum conservation simplifies to

$$-\rho \kappa^2 r \omega^2(r) e_r + \nabla p = \rho \nabla \left(\frac{1}{|\cdot|} * \rho \right), \quad e_r = \frac{1}{r(x)}(x_1, x_2, 0). \quad (1.3)$$

The first term in (1.3) can be written as $-\rho \nabla \left(\int_0^r \omega^2(s) s ds \right)$.

As a consequence, in the region $\{x \in \mathbb{R}^3 \mid \rho(x) > 0\}$ occupied by the star, EP reduces to the equation

$$\frac{1}{|x|} * \rho + \kappa^2 \int_0^r s \omega^2(s) ds - h(\rho) + \alpha = 0, \quad (1.4)$$

where h is the enthalpy defined by $h'(\rho) = \frac{p'(\rho)}{\rho}$ with $h(0) = 0$, and α is a constant. We have normalized the physical constants. The density must vanish at the boundary of the star.

Non-rotating radial (spherically symmetric) white dwarfs were first analysed by Chandrasekhar [2] (see also chapter 9 of [3]). He proved that there is a maximum mass M_0 for a

white dwarf to exist. Auchmuty and Beals [4] proved that for any $M < M_0$, there exists a rotating white dwarf of mass M with compact support; it is obtained by minimizing the energy. Lieb and Yau [5] considered non-rotating white dwarfs as semi-classical limits of many quantum particles that are governed by a Schrödinger Hamiltonian.

Our goal in this article is to prove that there is a global connected set of rotating solutions, that is, it contains solutions which have arbitrarily large density somewhere or which have arbitrarily large support. They may rotate arbitrarily fast. The conclusion is stated somewhat informally in the following theorem.

Theorem 1.1. *Let M be the mass of the non-rotating solution. Assume the pressure $p(\cdot)$ is given by (1.1) and the angular velocity $\omega(\cdot)$ satisfies (4.1) and (4.2). By a ‘solution’ of the problem for a rotating white dwarf, we mean a triple (ρ, κ, α) , where ρ is an axisymmetric function with mass M that satisfies (1.4) and κ refers to the intensity of rotation speed. Then there exists a set \mathcal{K} of solutions satisfying the following three properties.*

- \mathcal{K} is a connected set in the function space $C_c^1(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$.
- \mathcal{K} contains the non-rotating solution.
- Either

$$\sup \{ \rho(x) \mid x \in \mathbb{R}^3, (\rho, \kappa, \alpha) \in \mathcal{K} \} = \infty$$

or

$$\sup \{ |x| \mid \rho(x) > 0, (\rho, \kappa, \alpha) \in \mathcal{K} \} = \infty.$$

The last statement means that either the densities become pointwise unbounded or the supports become unbounded along \mathcal{K} .

In [6] we constructed slowly rotating stars with fixed mass. In [7] we constructed a global connected set of slowly and rapidly rotating stars for a general class of equations of state. However, the white dwarf case does not fall into this class. Keeping the mass constant is a key to our methodology, so that there is no loss or gain of particles when the star changes its rotation speed. Moreover, we permit a non-uniform angular velocity.

A subtlety of the white dwarf case occurs in the proof that the total mass M is a strictly monotone function both of the central density $\rho(0)$ and of the radius R of the star in the non-rotating radial case. We give a self-contained proof of this fact in section 3. It is based on a fundamental lemma given in section 2. The monotonicity is ultimately a consequence of the virial identity and the minimization of the energy. In a different context a weaker form of the monotonicity was proven in [5]. This monotonicity property of the mass is used in two crucial places in our proof in section 4.

In section 4 we use the same basic method as in [7]. That means we force the total mass M to be fixed and introduce the constant α as a variable. We get the support to be compact by artificially forcing the parameter α to be sufficiently negative (see lemma 4.1). Then we begin the construction of rotating star solutions in the standard way by continuation from a non-rotating solution ($\kappa = 0$). Letting κ increase, we continue the construction by applying a global implicit function theorem, which is based on the Leray–Schauder degree. Later on, in theorem 4.1, we obtain the whole global connected set \mathcal{K} of solutions by allowing α to increase.

The equation of state (1.1) for the white dwarf satisfies $p(\rho) = O(\rho^{4/3})$ as $\rho \rightarrow \infty$. However, the exact polytropic case $p = \rho^\gamma$ with $\gamma = \frac{4}{3}$ was also excluded from [7] because in that case the constant mass condition introduces a non-trivial nullspace of the linearized operator, which prevents the employment of the implicit function theorem. Here we supplement our discussion of the white dwarf stars with a discussion of the polytropic case $p = \rho^\gamma$ with $\gamma = \frac{4}{3}$. In that

case we prove in section 5 that there is no slowly uniformly rotating solution at all with the given mass M . In particular, one has

Theorem 1.2. *Assuming the equation of state $p = \rho^{4/3}$ and the uniform rotation profile $\omega \equiv \kappa$, there are no solutions close to ρ_0 with the same total mass as ρ_0 that are slowly rotating.*

2. Preliminaries

With the physical constants set to be 1, the equation of state is

$$p(\rho) = \int_0^{\rho^{1/3}} \frac{\sigma^4}{\sqrt{1+\sigma^2}} d\sigma. \quad (2.1)$$

We write $s = \rho$ for simplicity. Note that explicit calculations yield

$$p'(s) = \frac{s^{2/3}}{3\sqrt{1+s^{2/3}}}, \quad (2.2)$$

$$h(s) = \sqrt{1+s^{2/3}} - 1, \quad (2.3)$$

$$h'(s) = \frac{p'(s)}{s} = \frac{1}{3s^{1/3}\sqrt{1+s^{2/3}}}. \quad (2.4)$$

Writing $t = h(s) \geq 0$, we have

$$h^{-1}(t) = (2t + t^2)^{3/2}, \quad (2.5)$$

$$(h^{-1})'(t) = 3(1+t)\sqrt{2t+t^2}. \quad (2.6)$$

The key properties (2.9) and (2.10) of our paper [6] are

$$\lim_{s \rightarrow 0^+} s^{3-\gamma} p'''(s) = \text{constant} < 0 \quad (2.7)$$

and

$$\lim_{s \rightarrow \infty} s^{1-\gamma^*} p'(s) = \text{constant} > 0. \quad (2.8)$$

They are true for $\gamma = 5/3$ and $\gamma^* = 4/3$, respectively.

In section 3 we will have to study the equation

$$w_{rr} + g(w, r) = 0, \quad (2.9)$$

where

$$g(w, r) = 4\pi r h^{-1}\left(\frac{w_+}{r}\right) = 4\pi r \left(2\frac{w_+}{r} + \frac{w_+^2}{r^2}\right)^{3/2} \quad (2.10)$$

for $r > 0$. Here

$$w_+ = \max(w, 0). \quad (2.11)$$

We need to understand how the solution depends on its data at $r = 0$. This is described in the following lemma.

Lemma 2.1. *For $a > 0$, denote by $w(r, a)$ the solution of $w_{rr} + g(w, r) = 0$ with $w(0, a) = 0$ and $w_r(0, a) = a$. Assume that for some $a_0 > 0$ and some $R > 0$ we have $w(R, a_0) = 0$ and $w(r, a_0) > 0$ for all $0 < r < R$. Then there exists $r_0 \in (0, R)$ such that*

$$w_a(r, a_0) > 0 \quad \text{in } (0, r_0), \quad w_a(r, a_0) < 0 \quad \text{in } (r_0, R]. \quad (2.12)$$

Proof. The proof is closely related to lemma 4.9 in [6]. We calculate

$$\frac{1}{4\pi}(g - wg_w) = \left(2\frac{w}{r} + \frac{w^2}{r^2}\right)^{1/2} \left\{-w - \frac{2}{r}w^2\right\} < 0, \quad (2.13)$$

which is (4.52) in [6], and

$$\frac{1}{4\pi}g_r = \left(2\frac{w}{r} + \frac{w^2}{r^2}\right)^{1/2} \left\{-\frac{w}{r} - \frac{2}{r^2}w^2\right\} < 0, \quad (2.14)$$

which is (4.53) in [6]. Furthermore, calculate

$$\frac{1}{4\pi}(rg_r + 2g) = \left(2\frac{w}{r} + \frac{w^2}{r^2}\right)^{1/2} \{3w\} > 0, \quad (2.15)$$

which is weaker than (4.54) in [6].

Where convenient, we write $\frac{\partial}{\partial r}$ as $'$. We define the three auxiliary functions

$$x(r; a) = rw'(r; a), \quad y(r; a) = w'(r; a), \quad z(r; a) = w_a(r; a). \quad (2.16)$$

Their values at $r = 0$ are

$$x(0^+; a) = 0, \quad x'(0^+; a) = w'(0; a) - \lim_{r \rightarrow 0^+} rg(w, r) = a. \quad (2.17)$$

$$y(0^+; a) = a, \quad y'(0^+; a) = -\lim_{r \rightarrow 0^+} g(w, r) = 0. \quad (2.18)$$

$$z(0^+; a) = 0, \quad z'(0^+; a) = 1. \quad (2.19)$$

Now

$$x'' = (rw')'' = rw''' + 2w'' = r(-g_r - g_w w') - 2g = -rg_r - g_w x - 2g. \quad (2.20)$$

So x satisfies the equation

$$x'' + g_w x + rg_r + 2g = 0. \quad (2.21)$$

Similarly,

$$y'' + g_w y + g_r = 0, \quad z'' + g_w z = 0. \quad (2.22)$$

The derivatives of various Wronskians are

$$W(x, z)' = \begin{vmatrix} x & z \\ x' & z' \end{vmatrix}' = z(r g_r + 2g). \quad (2.23)$$

$$W(y, z)' = \begin{vmatrix} y & z \\ y' & z' \end{vmatrix}' = z g_r. \quad (2.24)$$

$$W(w, z)' = \begin{vmatrix} w & z \\ w' & z' \end{vmatrix}' = z(g - w g_w). \quad (2.25)$$

In the rest of the proof we set a equal to a_0 in all functions. Because $w > 0$ and $w'' = -g < 0$ for $r \in (0, R)$, we see that w is a positive concave function with a unique maximum and zero boundary values on $[0, R]$. By (2.19), $z(r) > 0$ for r close to 0.

We claim that z vanishes somewhere in $(0, R)$. On the contrary, suppose that $z(r) > 0$ for all $r \in (0, R)$. Integrating (2.25) on $(0, R)$ and using the boundary conditions of w and z , we have

$$-w'(R)z(R) = \int_0^R z(g - w g_w) dr < 0. \quad (2.26)$$

The inequality is a consequence of (2.13). However, since $w'(R) < 0$ and $z(R) = z(R-) \geq 0$, the left side of (2.26) is non-negative. This contradiction shows that z vanishes somewhere in the open interval.

Let r_0 be the smallest value in $(0, R)$ for which $z(r_0) = 0$. Integrating (2.23) on $(0, r_0)$, we find

$$x(r_0)z'(r_0) = \int_0^{r_0} z(r g_r + 2g) dr > 0 \quad (2.27)$$

by (2.15) and the fact that $z(r) > 0$ for $r \in (0, r_0)$. Since $z'(r_0) < 0$, we deduce that $x(r_0) < 0$, and hence $w'(r_0) < 0$.

Thus it suffices to show that $z(r) < 0$ for all $r_0 < r \leq R$. Again supposing the contrary, let $r_1 \in (r_0, R]$ be the first zero of z strictly bigger than r_0 . Integrating (2.24) on (r_0, r_1) and recalling the definition $y = w'$, we obtain

$$w'(r_1)z'(r_1) - w'(r_0)z'(r_0) = y(r_1)z'(r_1) - y(r_0)z'(r_0) = \int_{r_0}^{r_1} z g_r dr \geq 0. \quad (2.28)$$

The last inequality follows from (2.14) and the fact that $z(r) < 0$ for $r \in (r_0, r_1)$. However, since w is concave and $w'(r_0) < 0$, it must also be the case that $w'(r_1) < 0$. We also have $z'(r_0) < 0$, and $z'(r_1) > 0$. These conditions together imply that the left side of (2.28) is negative. This contradiction implies $z(r) < 0$ for all $r_0 < r \leq R$. \square

3. Monotonicity of the mass

For a non-rotating (spherical) star, $\rho(0)$ is the density at its centre. Let $a = h(\rho(0))$. Denote the density of this star at any radius $r = |x|$ by $\rho(r; a)$ and denote the radius of the star by $R(a)$. Defining $u = h(\rho)$, it turns out that $\Delta u + 4\pi h^{-1}(u) = 0$ for $r < R(a)$. The star's radius $R(a)$ is

finite for all $a > 0$, as is seen by applying the criterion $\int_0^1 h^{-1}(t)t^{-4} dt = \infty$ of theorem 1 in [8]. The total mass of the star is defined as

$$M(a) = \int_{\mathbb{R}^3} \rho dx = 4\pi \int_0^{R(a)} \rho(r; a) r^2 dr. \quad (3.1)$$

So $M'(a) = \int_{B(R(a))} \rho_a(x; a) dx$. Our goal is to prove the following lemma.

Lemma 3.1. $M'(a) > 0$ for all $a > 0$.

As already hinted in the introduction, a proof that $M(a)$ is strictly increasing appears within a different context in [5]. However, the comparison method in their proof cannot show that $M'(a) > 0$. In section 4 we will require the strict positivity of the derivative. The present section is devoted to providing a completely self-contained proof of this stronger statement.

To this end we define the total energy as

$$\mathcal{E}(\rho) = \int H(\rho) dx - D(\rho, \rho), \quad D(\rho, \rho) = \frac{1}{2} \iint \frac{\rho(x)\rho(y)}{|x-y|} dx dy, \quad (3.2)$$

where

$$H(\rho) = \int_0^\rho h(s) ds. \quad (3.3)$$

Lemma 3.2. Any radial solution satisfies the virial identity

$$\mathcal{E}(\rho) = \int [4H(\rho) - 3\rho h(\rho)] dx. \quad (3.4)$$

Proof. We have $u = h(\rho)$ in $\Omega =: \{\rho > 0\}$ and $\Delta u = \frac{1}{r^2}(r^2 u_r)_r = -4\pi\rho$ in \mathbb{R}^3 . We consider ρ to vanish outside Ω . From the latter equation, we have

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx = 4\pi \int_{\mathbb{R}^3} \rho u dx = 4\pi \int_{\mathbb{R}^3} \rho \left(\frac{1}{|\cdot|} * \rho \right) dx = 2\pi D(\rho, \rho). \quad (3.5)$$

We therefore have

$$\int_0^\infty r^3 \rho h'(\rho) \rho_r dr = \int_0^\infty r^3 \rho u_r dr = -\frac{1}{4\pi} \int_0^\infty r^2 u_r (r^2 u_r)_r \frac{1}{r} dr. \quad (3.6)$$

Integrating by parts, the right side equals

$$\frac{1}{8\pi} \int_0^\infty (r^2 u_r)^2 \frac{1}{r^2} dr = \frac{1}{8\pi} \int_0^\infty u_r^2 r^2 dr = \frac{1}{32\pi^2} \int_{\mathbb{R}^3} |\nabla u|^2 dx = \frac{1}{8\pi} D(\rho, \rho). \quad (3.7)$$

On the other hand, the left side of (3.6) equals

$$\int_0^\infty r^3 [\rho h(\rho) - H(\rho)]_r dr = -\int_0^\infty 3r^2 [\rho h(\rho) - H(\rho)] dr = -\frac{3}{4\pi} \int_{\mathbb{R}^3} [\rho h(\rho) - H(\rho)] dx. \quad (3.8)$$

Combining the last three equations, we have

$$3 \int_{\mathbb{R}^3} [\rho(h(\rho) - H(\rho))] dx = D(\rho, \rho). \quad (3.9)$$

This proves (3.2). \square

Proof of lemma 3.1. The function $u(r; a) = h(\rho(r; a))$ defined for $r \leq R(a)$ satisfies $\Delta u + 4\pi h^{-1}(u) = 0$, $u(r; a) > 0$ and $u_r(r; a) < 0$ for $0 < r < R(a)$, as well as the boundary conditions $u(R(a); a) = 0$, $u(0, a) = a$. This function u is extended to all of \mathbb{R}^3 by solving $\Delta u = -4\pi h^{-1}(u_+)$ in \mathbb{R}^3 . Thus u is harmonic outside the star.

Now we define $w = ru$. This change of variables gives us $w_{rr} + g(w; r) = 0$ for $0 \leq r \leq R(a)$, where g is defined in (2.10). Also $w(0, a) = 0$, $w_r(0, a) = a$, $w(R(a); a) = 0$ and $w(r, a) > 0$ for $0 < r < R(a)$. Therefore lemma 2.1 is applicable, so that w_a strictly changes sign in the interval $(0, R(a))$. Now $w_a = ru_a = rh'(\rho)\rho_a$ and $h' > 0$, so that ρ_a also strictly changes sign in the interval $(0, R(a))$.

From the definition of the energy \mathcal{E} , we have

$$\begin{aligned} \frac{d}{da} \mathcal{E}(\rho(\cdot; a)) &= \int_{B(R(a))} h(\rho(x; a)) \rho_a(x; a) - \iint_{B(R(a))^2} \frac{\rho_a(x; a) \rho(x; a)}{|x - y|} dx dy \\ &= \int_{B(R(a))} \left\{ h(\rho(x; a)) - \left(\frac{1}{|\cdot|} * \rho(\cdot; a) \right)(x) \right\} \rho_a(x; a) dx \end{aligned} \quad (3.10)$$

since $\rho(R(a); a) = 0$ and $H(0) = 0$. By (1.4) in section 1, the factor in curly brackets is a constant $\alpha < 0$, so that

$$\frac{d}{da} \mathcal{E}(\rho(\cdot, a)) = \alpha M'(a). \quad (3.11)$$

We will prove by contradiction that $M'(a) \neq 0$.

Now suppose that $M'(a) = 0$ for some a . Then $\frac{d}{da} \mathcal{E}(\rho(\cdot, a)) = 0$. Using the virial identity, we therefore have

$$0 = \int [h(\rho) - 3\rho h'(\rho)] \rho_a dx = \int \left[\left(1 + \rho^{2/3}\right)^{-1/2} - 1 \right] \rho_a dx. \quad (3.12)$$

The function $k(s) = 1 - (1 + s^{2/3})^{-1/2}$ is positive and increasing for $s > 0$, so that the radial function $r \rightarrow \tilde{k}(r) =: k(\rho(r))$ is positive and decreasing as a function of $r = |x|$ and it vanishes at $r = R(a)$. Now we have both $\int \rho_a dx = 0$ and $\int \tilde{k} \rho_a dx = 0$. This is impossible, due to the facts that ρ_a strictly changes sign from positive to negative, while \tilde{k} is positive and decreasing. This contradiction means that $M'(a) \neq 0$.

Thus we have shown that $M(a)$ is either strictly increasing or strictly decreasing. We claim that $M(a) \leq Ca^{3/4}$ for sufficiently small a . To prove this claim, we let $v(x; a)$ be the unique solution of

$$\Delta v + (2v_+ + av_+^2)^{3/2} = 0, \quad v(0) = 1, \quad v'(0) = 0 \quad (3.13)$$

for $a \geq 0$ and $|x| \geq 0$. For $a > 0$, a simple rescaling, using the formula for h^{-1} given in (2.5) and the definition of u , shows that $v(x; a) = \frac{1}{a} u(a^{-1/4}x; a)$, now by [8] the solution $v(x; 0)$ has a unique zero R_0 . We obviously have $v'(R_0; 0) < 0$. By the continuous dependence of solution of the ODE on the parameter a , for arbitrarily small $\epsilon > 0$ we have $v(R_0; a) < \epsilon$ and $v'(R_0; a) < v'(R_0; 0) + \epsilon < 0$, provided that a is sufficiently small. Furthermore, $|v''(x; a)|$

is uniformly bounded for $|x| < R_0 + 1$ and a small. Thus $v'(x; a) < v'(R_0; 0) + 2\epsilon$ for $R_0 < |x| < R_0 + \delta$ and some constant δ . If $v(R_0; a) < 0$, the zero of $v(x; a)$ occurs before $|x|$ reaches R_0 . Otherwise $v(x; a)$ must cross zero before $|x|$ reaches $R_0 + \delta$. Thus we have the following estimate on the radius of the star, which is the zero of $u(x; a)$: $R(a) \leq a^{-1/4}(R_0 + \delta)$ for small a . Because $u(x; a)$ and $\rho(r; a)$ are radially decreasing, we have $\rho(r; a) \leq h^{-1}(a)$ and $M(a) \leq Ch^{-1}(a)[R(a)]^3 \leq Ca^{3/2}a^{-3/4} = Ca^{3/4}$ for sufficiently small a . This proves the claim. Now if we assume by contradiction that $M(\cdot)$ is decreasing, then let $0 < \epsilon < a$. It follows that $0 \leq M(a) \leq M(\epsilon) \leq C\epsilon^{3/4}$ for small ϵ . Hence $M(\cdot)$ cannot be decreasing. Therefore $M' > 0$. \square

4. Existence of rotating white dwarf solutions

With the key difficulty about the mass function $M(a)$ having been resolved in section 3, we will be able to prove theorem 1.1. In order to formulate the result precisely, let us put the following conditions on the rotation profile $\omega(s)$:

$$s\omega^2(s) \in L^1(0, \infty), \quad \omega^2(s) \text{ is not compactly supported,} \quad (4.1)$$

$$\lim_{r(x) \rightarrow \infty} r(x) \left(\sup_x j - j(x) \right) = 0, \quad (4.2)$$

where

$$j(x) = \int_0^{r(x)} s\omega^2(s) ds. \quad (4.3)$$

Let $\rho_0(x)$ be the unique non-rotating ($\kappa = 0$) solution with mass $M = \int \rho_0(x) dx$. In view of (1.4) and the mass constraint, we define the pair $\mathcal{F}(\rho, \kappa, \alpha) = (\mathcal{F}_1(\rho, \kappa, \alpha), \mathcal{F}_2(\rho))$, where

$$\mathcal{F}_1(\rho, \kappa, \alpha) = \rho(\cdot) - h^{-1} \left(\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+ \right), \quad (4.4)$$

and

$$\mathcal{F}_2(\rho) = \int_{\mathbb{R}^3} \rho(x) dx - M. \quad (4.5)$$

As usual, a solution to $\mathcal{F}(\rho, \kappa, \alpha) = 0$ with $\rho \in C_{loc}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ will give rise to a steady solution of the EP with rotation profile $\kappa\omega(s)$, and mass M . In particular, $\mathcal{F}(\rho_0, 0, \alpha_0) = 0$. We have the following main theorem, more precise than theorem 1.1.

Theorem 4.1. *For given $\omega(s)$ satisfying the above assumptions, and given non-rotating white dwarf solution ρ_0 , there exists a connected set \mathcal{K} in $C_c^1(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ such that*

- (a) $\mathcal{F}(\rho, \kappa, \alpha) = 0$ for all $(\rho, \kappa, \alpha) \in \mathcal{K}$. In other words, \mathcal{K} is a set of rotating white dwarf solutions.
- (b) $(\rho_0, 0, \alpha_0) \in \mathcal{K}$.
- (c) Either

$$\sup \{ \|\rho\|_\infty \mid (\rho, \kappa, \alpha) \in \mathcal{K} \} = \infty$$

or

$$\sup \{ |x| \mid \rho(x) > 0, (\rho, \kappa, \alpha) \in \mathcal{K} \} = \infty.$$

This means that either the densities become unbounded or the supports of the stars become unbounded. The proof of theorem 4.1 is basically parallel to the argument in [7] now that we have proven $M'(a) \neq 0$ in section 3. For completeness, we provide a sketch of the complete argument below. We refer to [7] for more details.

For fixed constants $s > 3$, let us define the weighted space

$$C_s = \{f : \mathbb{R}^3 \rightarrow \mathbb{R} \mid f \text{ is continuous, axisymmetric, even in } x_3, \text{ and } \|f\|_s < \infty\},$$

where

$$\|f\|_s = \sup_{x \in \mathbb{R}^3} \langle x \rangle^s |f(x)| < \infty.$$

Also define for $N > 0$,

$$\mathcal{O}_N = \left\{ (\rho, \kappa, \alpha) \in C_s \times \mathbb{R}^2 \mid \alpha + \kappa^2 \sup_x j(x) < -\frac{1}{N} \right\}. \quad (4.6)$$

We begin by showing an elementary support estimate for the nonlinear part of \mathcal{F}_1 on \mathcal{O}_N .

Lemma 4.1. *There exists a constant C_0 such that for all $(\rho, \kappa, \alpha) \in \mathcal{O}_N$ the expression $\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+$ is supported in the ball $\{x \in \mathbb{R}^3 : |x| \leq C_0 N \|\rho\|_s\}$.*

Proof. First we note that $\left| \frac{1}{|\cdot|} * \rho(\cdot)(x) \right| \leq C_0 \|\rho\|_s \frac{1}{\langle x \rangle}$ because $s > 3$. Hence for $|x| > C_0 N \|\rho\|_s$,

$$\left[\frac{1}{|\cdot|} * \rho(\cdot)(x) + \kappa^2 j(x) + \alpha \right] \leq C_0 \|\rho\|_s \frac{1}{\langle x \rangle} - \frac{1}{N} < 0$$

since $(\rho, \kappa, \alpha) \in \mathcal{O}_N$. Therefore its positive part vanishes for such x . \square

We see from this lemma that \mathcal{F}_1 differs from ρ only by a perturbation on a compact set. Using this observation and the smoothing effect of Δ^{-1} , it is easy to obtain

Lemma 4.2. *\mathcal{F} maps \mathcal{O}_N to $C_s \times \mathbb{R}$. It is C^1 Fréchet differentiable, where $\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho, \kappa, \alpha)$ is Fredholm of index zero. The nonlinear part of \mathcal{F}_1 (i.e. $\mathcal{F}_1 - \rho$) is compact from \mathcal{O}_N to C_s .*

Proof. By lemma 4.1, if (ρ, κ, α) is bounded, the support of $\left[\frac{1}{|\cdot|} * \rho(\cdot) + \kappa^2 j(\cdot) + \alpha \right]_+$ is contained in some ball B_R . The map is obviously compact from \mathcal{O}_N to $C^0(\overline{B_R})$. Using again the trivial bound $\|u\|_{C_s} \leq \langle R \rangle^s \|u\|_{C^0(\overline{B_R})}$ for $u \in C_s$ supported in B_R , we obtain the compactness of this mapping into C_s . \square

Lemma 4.3. *$\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho_0, 0, \alpha_0) : C_s \times \mathbb{R} \rightarrow C_s \times \mathbb{R}$ is an isomorphism.*

Proof. This lemma is the first place where the crucial condition $M'(a) \neq 0$ proven in section 3 will be used. Let $(\delta\rho, \delta\kappa)$ belong to the nullspace of $\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho_0, 0, \alpha_0)$. Let $w = \frac{1}{|\cdot|} * \delta\rho + \alpha_0$. As shown in lemma 4.3 of [7], w is radial. Indeed, that argument shows that w must be a radial solution of the boundary value problem

$$\Delta w + 4\pi \left[(h^{-1})'(u_0) \right] w = 0, \quad w'(0) = w'(R_0) = 0 \quad (4.7)$$

in the ball B_{R_0} , where B_{R_0} is the support of ρ_0 , and $u_0 = h(\rho_0)$. Being an ODE, (4.7) can have at most a one-dimensional solution space. On the other hand, we recall the definition for any $a > 0$ that $u(r; a)$ solves

$$\Delta u + 4\pi h^{-1}(u) = 0, \quad u'(0) = 0, \quad u(0; a) = a. \quad (4.8)$$

Denoting $u_a = \partial_a u(r; u_0(0))$, we obviously have

$$\Delta u_a + 4\pi \left[(h^{-1})'(u_0) \right] u_a = 0, \quad u'_a(0) = 0. \quad (4.9)$$

Comparing (4.7) with (4.9), we see that $w = Cu_a$ for some constant C . Integrating (4.7), we also have

$$\int_{B_{R_0}} \left[(h^{-1})'(u_0) \right] Cu_a \, dx = 0. \quad (4.10)$$

On the other hand, taking account of $\rho = h^{-1}(u)$ and (3.1), we see that

$$\int_{B_{R_0}} \left[(h^{-1})'(u_0) \right] u_a \, dx = \frac{d}{da} \bigg|_{a=u_0(0)} \int_{u(x;a)>0} h^{-1}(u(x;a)) \, dx = M'(u_0(0)) \neq 0. \quad (4.11)$$

There is no boundary term because $h^{-1}(0) = 0$. The last two equations imply that $C = 0$, so that $w = 0$. This implies that the kernel of $\frac{\partial \mathcal{F}}{\partial(\rho, \alpha)}(\rho_0, 0, \alpha_0)$ is trivial, which is the key ingredient of the operator being an isomorphism. \square

Proof of theorem 4.1. With the suitable compactness properties given by lemma 4.2 and local solvability given by lemma 4.3, one is in a position to apply a global implicit function theorem of Rabinowitz (see theorem 3.2 in [10], theorem 2.6.1 of [9], or [11]). The result is a connected set $\mathcal{K}_N \subset \mathcal{O}_N$ of solutions to $\mathcal{F} = 0$ for which at least one of the following three properties holds:

- (a) $\mathcal{K}_N \setminus \{(\rho_0, 0, \alpha_0)\}$ is connected.
- (b) \mathcal{K}_N is unbounded, i.e.

$$\sup_{\mathcal{K}_N} (\|\rho\|_s + |\kappa| + |\alpha|) = \infty.$$

- (c) \mathcal{K}_N approaches the boundary of \mathcal{O}_N , i.e.

$$\inf_{\mathcal{K}_N} \left| \kappa^2 \sup_x j(x) + \alpha + \frac{1}{N} \right| = 0.$$

The first alternative (the ‘loop’) can be eliminated by observing that, since \mathcal{K}_N is even in κ , if $\mathcal{K}_N \setminus \{(\rho_0, 0, \alpha_0)\}$ were connected, it must contain a different non-rotating solution $(\rho_1, 0, \alpha_1) \neq (\rho_0, 0, \alpha_0)$. As in lemma 5.1 of [7], it must be a radial non-rotating white-dwarf solution with a different centre density $\rho_1(0) \neq \rho_0(0)$ but with the same total mass $\int_{\mathbb{R}^3} \rho_1(x) \, dx = \int_{\mathbb{R}^3} \rho_0(x) \, dx$. This contradicts the strict monotonicity of $M(a)$ established in section 3.

The sets \mathcal{K}_N are nested, so their union $\mathcal{K} = \bigcup_{N=1}^{\infty} \mathcal{K}_N$ is also connected. Therefore one of the following statements is true:

- (a) $\sup_{\mathcal{K}} (\|\rho\|_s + |\kappa| + |\alpha|) = \infty$.

(b) $\inf_{\mathcal{K}} |\kappa^2 \sup_x j(x) + \alpha| = 0$.

We suppose that both $\sup_{\mathcal{K}} \sup_{x \in \mathbb{R}^3} \rho(x) < \infty$ and $R_* =: \sup_{\mathcal{K}} \sup_{\rho(x) > 0} |x| < \infty$, and will derive a contradiction.

We will first prove that (a) is true. On the contrary, suppose that (a) is false. Then (b) must be true and $\sup_{\mathcal{K}} (\|\rho\|_s + |\kappa| + |\alpha|) < \infty$. Since $|x - y| \leq |x| + R_*$ for all y in the support of ρ , we have

$$\left(\frac{1}{|\cdot|} * \rho \right)(x) = \int \frac{1}{|x - y|} \rho(y) dy \geq \frac{M}{|x| + R_*}.$$

We may now write

$$\frac{1}{|\cdot|} * \rho(x) + \kappa^2 j(x) + \alpha \geq \frac{M}{|x| + R_*} - \kappa^2 (\sup j - j(x)) + (\kappa^2 \sup j + \alpha). \quad (4.12)$$

Let $\kappa_0 = \sup_{\mathcal{K}} |\kappa|$. Considering a point x in the plane $\{x_3 = 0\}$, we have $|x| = r(x)$. By (4.2), $\sup j - j(x) = o\left(\frac{1}{|x|}\right)$ as $|x| \rightarrow \infty$. Thus by (4.12),

$$\frac{1}{|\cdot|} * \rho(x) + \kappa^2 j(x) + \alpha \geq \frac{M}{|x| + R_*} - o\left(\frac{\kappa_0^2}{|x|}\right) + \left(\kappa^2 \sup_x j(x) + \alpha\right). \quad (4.13)$$

Choosing $|x| > R_*$ sufficiently large, we can make the sum of the first two terms on the right side of (4.13) positive. Then because of (b), there exists a solution $(\rho, \kappa, \alpha) \in \mathcal{K}$ such that the right side of (4.13) is positive. Hence, due to $\mathcal{F}_1(\rho, \kappa, \alpha) = 0$, we have $\rho(x) > 0$. This contradicts the assumption that the support of ρ is bounded by R_* . Thus (a) must be true.

Since we have assumed that ρ is pointwise bounded and its support is also bounded all along \mathcal{K} , it follows that ρ is also bounded in the weighted space C_s . Because of (a), it must be the case that $|\kappa| + |\alpha|$ is unbounded. From the definition of \mathcal{O}_N , we know that $\alpha < 0$. In case κ were bounded, it would have to be the case that $\alpha \rightarrow -\infty$ along a sequence. Then the equation $\mathcal{F}_1 = 0$ would imply that $\rho \equiv 0$, which contradicts the mass constraint.

So it follows that $\kappa_n \rightarrow \infty$ for some sequence $(\rho_n, \kappa_n, \alpha_n) \in \mathcal{K}$ with $\alpha_n < 0$. For each n , let us choose any point x_n such that $\rho_n(x_n) > 0$. By (4.1), we may also choose a point y_0 such that $r(y_0) > R_*$ and $j(y_0) > j(R_*)$. Since $\rho_n(y_0) = 0$ and $\rho_n(x_n) > 0$, we have

$$0 \geq \left[\frac{1}{|\cdot|} * \rho_n(\cdot) + \kappa_n^2 j(\cdot) + \alpha_n \right](y_0) \geq \left[\frac{1}{|\cdot|} * \rho_n(\cdot) + \kappa_n^2 j(\cdot) + \alpha_n \right] \Big|_{x_n}^{y_0}.$$

On the right side, the α_n cancels. Due to our assumption that the values of ρ_n and the supports of ρ_n are uniformly bounded, we deduce that

$$0 \geq \kappa_n^2 [j(r(y_0)) - j(r(x_n))] - C,$$

where C is a fixed constant. Thus $j(r(x_n)) \rightarrow j(r(y_0))$ since $\kappa_n \rightarrow \infty$. But $r(x_n) \leq R_* < r(y_0)$ and j is an increasing function of r , so that $j(r(x_n)) \leq j(R_*) < j(r(y_0))$. This is the desired contradiction.

Finally, we remark on why \mathcal{K} is also connected in $C_c^1(\mathbb{R}^3) \times \mathbb{R}^2$. In fact, we know that for each N the set \mathcal{K}_N is connected in $C_s \times \mathbb{R}^2$. We also know from lemma 4.1 that all the solutions in $\mathcal{K}_N \subset \mathcal{O}_N$ have a uniform bound on their supports. This bound may depend on N . The regularizing effect of Δ^{-1} then implies that \mathcal{K}_N is connected in $C^1(F_N) \times \mathbb{R}^2$ for a

suitable compact set $F_N \subset \mathbb{R}^3$. Thus \mathcal{K}_N is connected in $C_c^1(\mathbb{R}^3) \times \mathbb{R}^2$ under the usual direct limit topology. Because \mathcal{K} is a nested union of \mathcal{K}_N , it too is connected in $C_c^1(\mathbb{R}^3) \times \mathbb{R}^2$. \square

5. Pure 4/3 power under uniform rotation

In this section, we briefly study the Euler–Poisson equation under the pure power equation of state $p = \rho^{4/3}$ and constant angular velocity profile. Analogously to the white dwarf case, we define $\mathcal{F} = \mathcal{F}(\rho, \kappa, \alpha) = (\mathcal{F}_1, \mathcal{F}_2)$ by

$$\mathcal{F}_1(\rho, \kappa, \alpha) = \rho(x) - \left[\frac{1}{|\cdot|} * \rho(x) - \frac{1}{|\cdot|} * \rho(0) + \kappa r^2(x) + \alpha \right]_+^3, \quad (5.1)$$

$$\mathcal{F}_2(\rho, \kappa, \alpha) = \int_{B_1} \rho(x) dx - M, \quad (5.2)$$

and solve for $\mathcal{F}(\rho, \kappa, \alpha) = (0, 0)$. The cubic function in (5.1) corresponds to the pure $\frac{4}{3}$ power in the equation of state. As before, the radial non-rotating solution $\rho_0 = u_0^3$ satisfies the equivalent equation

$$\Delta u_0 + 4\pi u_0^3 = 0 \quad (5.3)$$

on its support, which we may take to be the unit ball B_1 without loss of generality. Let $\alpha_0 = u_0(0)$, and $M = \int_{B_1} \rho_0(x) dx$. We readily check that $\mathcal{F}(\rho_0, 0, \alpha_0) = (0, 0)$. By the scaling symmetry of (5.3), we easily see that for any $\alpha > 0$ and

$$\rho^\alpha(x) = \left(\frac{\alpha}{\alpha_0} \right)^3 \rho_0 \left(\frac{\alpha}{\alpha_0} x \right),$$

we have $\mathcal{F}(\rho^\alpha, 0, \alpha) = (0, 0)$. This ρ^α has the same mass M for all α .

Let $X = C_{\text{sym}}(\overline{B_2})$ be defined to have the same symmetry properties as C_s but only defined on B_2 . We will show that the linear operator $\frac{\partial \mathcal{F}}{\partial(\rho, \kappa)}(\rho_0, 0, \alpha_0) : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$ is isomorphic. Once this is proven, the implicit function theorem implies that (ρ, κ) is locally uniquely determined by α . Therefore the trivial solutions $(\rho^\alpha, 0, \alpha)$ defined above are the unique local solutions and they are non-rotating. We thus obtain the curious conclusion given in theorem 1.2.

Proof of theorem 1.2. We just need to prove the bijectivity. We compute the derivative of \mathcal{F} as follows, recalling that $u_0 = \rho_0^{1/3} = \left[\frac{1}{|\cdot|} * \rho_0(x) - \frac{1}{|\cdot|} * \rho_0(0) + \alpha_0 \right]_+$.

$$\left. \frac{\partial \mathcal{F}_1}{\partial(\rho, \kappa)} \right|_{(\rho, \kappa, \alpha) = (\rho_0, 0, \alpha_0)} (\delta \rho, \delta \kappa) = \delta \rho - 3u_0^2 \left[\frac{1}{|\cdot|} * \delta \rho(x) - \frac{1}{|\cdot|} * \delta \rho(0) + \delta \kappa r^2(x) \right], \quad (5.4)$$

$$\left. \frac{\partial \mathcal{F}_2}{\partial(\rho, \kappa)} \right|_{(\rho, \kappa, \alpha)} (\delta \rho, \delta \kappa) = \int_{B_2} \delta \rho(x) dx. \quad (5.5)$$

This derivative is a compact perturbation of the identity and thus is Fredholm of index zero. Hence we merely need to show it is injective. To that end, let us assume that (5.4) and (5.5)

both vanish. Denoting

$$\varphi(x) = \frac{1}{|\cdot|} * \delta\rho(x) - \frac{1}{|\cdot|} * \delta\rho(0) + \delta\kappa r^2(x), \quad (5.6)$$

we then have

$$\Delta\varphi = -4\pi\delta\rho + 4\delta\kappa = -12\pi u_0^2\varphi + 4\delta\kappa, \quad (5.7)$$

and

$$\int_{B_1} u_0^2 \varphi \, dx = 0. \quad (5.8)$$

We project (5.7) onto the radial component (integrating against 1 on \mathbb{S}^2), where φ_{00} denotes the radial component of φ , to obtain

$$\Delta\varphi_{00} = -12\pi u_0^2 \varphi_{00} + 4\delta\kappa, \quad (5.9)$$

while (5.8) naturally selects the radial component so that

$$\int_{B_1} u_0^2 \varphi_{00} \, dx = 0. \quad (5.10)$$

If $\delta\kappa \neq 0$, we can divide (5.9) by it, and without loss of generality, we may assume $\delta\kappa = 1$. Integrating (5.9) on B_1 and using (5.10), we get

$$4\pi\varphi'_{00}(1) = 4\frac{4\pi}{3}, \quad (5.11)$$

$$\varphi'_{00}(1) = \frac{4}{3}. \quad (5.12)$$

Then the function $u(|x|) = \varphi_{00}(|x|) - \frac{2}{3}|x|^2$ satisfies

$$\Delta u + 12\pi u_0^2 u = -8\pi u_0^2 |x|^2 \quad (5.13)$$

and

$$u'(1) = 0. \quad (5.14)$$

Referring to the proofs of lemmas 4.3 and 4.7 in [6] in the case that $\gamma = \frac{4}{3}$, the radial function

$$v(|x|) = \frac{\partial}{\partial\alpha}(\rho^\alpha(|x|))^{1/3} \Big|_{\alpha=\alpha_0} = u_0(|x|) + ru'_0(|x|) \quad (5.15)$$

satisfies on B_1

$$\Delta v + 12\pi u_0^2 v = 0, \quad (5.16)$$

and

$$v'(1) = 0, \quad (5.17)$$

$$\int_{B_1} u_0^2 v \, dx = 0. \quad (5.18)$$

In fact, (5.16) is a special case of (4.27) in [6] (where $h^{-1}(s) = s^3$). (4.28) in [6] shows the left-hand side of (5.17) and that of (5.18) are the same. (4.45) in [6] implies (5.17), and finally (5.15) follows from (4.44) in [6] (ignoring an irrelevant constant multiple). We multiply (5.13) by v , multiply (5.16) by u , and take the difference, obtaining

$$v\Delta u - u\Delta v = -8\pi u_0^2 |x|^2 v. \quad (5.19)$$

Integrating (5.19) over B_1 , using Green's identity and the boundary conditions (5.14) and (5.17), we get

$$\int_{B_1} u_0^2 v |x|^2 dx = 0. \quad (5.20)$$

But notice that (5.18) and (5.20) contradict each other! Indeed, $v' = 2u_0' + ru_0'' = -4\pi |x| u_0^3 < 0$ for $|x| < 1$. It follows from (5.18) that $u_0^2 v$ is positive near 0 and negative near ∂B_1 , and only switches sign once. Therefore (5.20) cannot hold. This contradiction implies that $\delta\kappa = 0$. Then the same argument as in the proof of lemma 4.3 in [6] shows that $\delta\rho = 0$.

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Appendix

For completeness, we give a heuristic derivation of (1.1) from quantum statistics. See [3] and [12] for more discussions.

A star is considered to be a completely degenerate electron gas, which means that all the lowest quantum states are occupied. The Pauli exclusion principle states that no two electrons can occupy the same quantum state. Therefore there is an upper limit of the number of electrons in a cube of side h in momentum space, where h is Planck's constant. The size of the momenta must be less than a certain maximum value $P_0 > 0$, called the Fermi momentum. Since every momentum state produces two quantum states corresponding to different spins, the distribution function of momenta is $f(P) = \frac{2}{h^3} \chi_{|P| \leq P_0}$. Here $P \in \mathbb{R}^3$ is the momentum variable.

The density of the electron gas at a point in space is proportional to the total number of electrons:

$$\rho = \int_{\mathbb{R}^3} f(P) dP = \frac{8\pi}{3h^3} P_0^3. \quad (6.1)$$

The pressure p at a point in space is proportional to the momentum transfer on an imaginary surface of unit size per unit time. Without loss of generality, assume the surface is normal to the x_3 -axis. Denote by $S(|P|) > 0$ the speed of an electron with momentum P . Thus the velocity of such an electron is $V(P) = S(|P|) \frac{P}{|P|}$. Supposing the electrons hit the surface and bounce back elastically, the momentum transfer caused by electrons with momentum P is given by $2P_3$. The number of electrons with momentum P that hit the surface per unit time is $f(P)V_3(P) = f(P)S(P) \frac{P_3}{|P|}$. Thus the pressure is given by

$$p = \int_{P_3 < 0} 2P_3 f(P) V_3(P) dP = \int_{P_3 < 0} 2f(P) S(|P|) \frac{P_3^2}{|P|} dP = \frac{8\pi}{3h^3} \int_0^{P_0} \sigma^3 S(\sigma) d\sigma. \quad (6.2)$$

The speed $S(|P|) = E'(|P|)$, where the energy $E(|P|)$ is given by the special relativistic energy–momentum relation:

$$E(|P|) = c\sqrt{m^2c^2 + |P|^2} - mc^2. \quad (6.3)$$

It follows that

$$p = \frac{8\pi c}{3h^3} \int_0^{P_0} \frac{\sigma^4}{\sqrt{m^2c^2 + \sigma^2}} d\sigma. \quad (6.4)$$

One readily observes that (6.1) and (6.4) imply (1.1).

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