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Stability and robustness analysis of minmax solutions for differential graphical games[☆]

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ABSTRACT

Recent studies have shown that, in general, Nash equilibrium cannot be achieved by the players of a differential graphical game by using distributed control policies. Alternative solution concepts that do not necessarily lead to Nash equilibrium can be proposed to allow the players in the game determine distributed optimal strategies. This paper analyzes the performance properties of the solution concept regarded as minmax strategies. The minmax formulation is shown to provide distributed control policies for linear systems under mild assumptions. The stability and robustness characteristics of the proposed solution are studied in terms of gain and phase margins, and related to the robustness properties of the single-agent LQR controller. The results of our analysis are finally tested by means of a simulation example.

1. Introduction

Analyzing the performance of the decision-making processes of multiagent systems has become indispensable with the increasing use of autonomous systems in industrial and urban areas. In many applications, a single system, regarded as an *agent*, must complete a task while observing the state information of only a subset of other systems, regarded as its neighbors. The multiple connections among the agents form a communication network.

The literature available for cooperative control in networked systems is extensive (see Hong, Hu, & Gao, 2006; Kamalapurkar, Dinh, Walters, & Dixon, 2013; Lewis, Zhang, Hengster-Movric, & Das, 2013; Li, Wang, & Chen, 2004; Olfati-Saber, Fax, & Murray, 2007; Qu, 2009; Ren, Beard, & Atkins, 2005; Ren, Moore, & Chen, 2007; Zhang, Feng, Yang & Liang, 2015 and references therein). The standard study of consensus and synchronization in multiagent systems do not consider optimization procedures. If the agents in a network have the goal of minimizing cost functions, then they must take into account not only their own behavior, but also the behavior of their neighbors. This property leads to the formulation of a *game* (Basar & Olsder, 1999; Isaacs, 1965; Shoham & Leyton-Brown, 2008). Game-theoretic approaches have been recently proposed to provide optimality to the cooperative (Vamvoudakis, Lewis, & Hudas, 2012; Yaghmaie, Lewis, & Su, 2016) and noncooperative (Cao, Ertin, & Arora, 2008; Qu & Simaan, 2009) interactions of networked agents. The most important solution concept in game theory, Nash equilibrium, is obtained when all agents use their best strategies simultaneously. Every admissible solution for a graphical game, however, requires the use of distributed control policies. This means that the agents are allowed to use only local information received through the communication graph. The distributed-policy requirement makes Nash equilibrium generally unattainable among the agents. As we show in this paper, the information restriction imposed by the graph topology prevents the multiagent system from reaching an equilibrium.

The unattainability of Nash equilibrium in graphical games can be addressed by proposing alternative solution concepts for the agents. The solutions regarded as *minmax strategies* have been used to describe the behavior of players in strictly adversarial games (Basar & Bernhard, 1995; Shoham & Leyton-Brown,

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2008). In this paper, we propose the use of minmax strategies to solve non-adversarial and cooperative games and analyze the resulting behavior of the agents connected in a graph. Using minmax strategies, each agent prepares its best response against the worst-case behavior of its neighbors. From the perspective of an individual agent, the resulting formulation of this graphical game is the same as an H_∞ control problem (Zames, 1979). In turn, the H_∞ formulation can be solved as the zero-sum game between a system and a disturbance term (Basar & Bernhard, 1995; Lewis, Vrabie, & Syrmos, 2012). The H_∞ controller has been thoroughly studied due to its attractive robust characteristics (Doyle, Glover, & Khargonekar, 1998; Kwakernaak, 1993; Li, Duan, & Chen, 2011; Modares, Lewis, & Jiang, 2015). Nash equilibrium and minmax strategies are known to be equivalent for zero-sum games (Shoham & Leyton-Brown, 2008). In the nonzero-sum games studied in this paper, minmax strategies do not necessarily lead to Nash equilibrium.

The main contributions of this paper are summarized as follows. Minmax strategies are proposed to solve non-adversarial differential graphical games for synchronization, i.e., a leader–followers structure. This differs from the current minmax formulations in the literature that are only relevant to strictly adversarial games (Basar & Bernhard, 1995; Cao et al., 2008). Different from the Nash equilibrium solution, minmax strategies are proven to provide distributed control policies under mild conditions in the system dynamics and the performance functions. The conditions for stability of the global multiagent system when all agents use their minmax policies are studied. Robustness of the control policy of each agent is also analyzed. The gain and phase margins of the minmax policies are obtained; as per the authors' knowledge, analysis of phase and gain margins for minmax or H_∞ controllers had not been yet performed. Comparing our results with those of Safonov and Athans (1977), the robustness properties of minmax strategies are shown to improve the corresponding characteristics of the linear quadratic regulator (LQR).

The remainder of this paper is structured as follows. Section 2 presents the preliminaries of the paper. In Section 3, minmax strategies are proposed to solve the graphical games and their corresponding distributed control policies are obtained. Sections 4 and 5 study the stability and robustness properties of minmax strategies, respectively. Simulation studies are presented in Section 6.

1.1. Notation and preliminary definitions

The following notations are used in the paper (Khalil, 1996; Safonov & Athans, 1977).

The space $\mathcal{L}_2^n$ is defined as the set of all piecewise continuous functions $x : [0, \infty) \to \mathbb{R}^n$ such that

$$\|x\|_{\mathcal{L}_2} = \left(\int_0^\infty x^T(t)x(t) \right)^{1/2} dt < \infty,$$

i.e., the space $\mathcal{L}_2^n$ defines the set of all square-integrable functions $x(t)$.

The extended space $\mathcal{L}_{2e}^n$ is defined by

$$\mathcal{L}_{2e}^n = \left\{ x | x_\tau \in \mathcal{L}_2^n, \forall \tau \geq 0 \right\}$$

where x_τ is a truncation of x defined by $x_\tau(t) = x(t)$ if $0 \leq t \leq \tau$, and $x_\tau(t) = 0$ otherwise.

Define the inner-product in space $\mathcal{L}_2^n[0, \infty)$ as

$$\langle x, y \rangle = \int_0^\infty x^T(t)y(t)dt \tag{1}$$

where $x,\ y \in \mathcal{L}_2^n[0, \infty)$.

A mapping $H : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^n$ is finite-gain $\mathcal{L}_2$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \beta \tag{2}$$

for all $u \in \mathcal{L}_{2e}^m$ and $\tau \in [0, \infty)$.

When the inequality (2) is satisfied for some $\gamma \geq 0$, the system is said to have $\mathcal{L}_2$ gain less than or equal to γ.

2. Differential graphical games

Differential graphical games study the interactions of multiagent systems connected in a communication graph, such that every player is able to interact only with its neighbors. Formally, consider a set of N agents connected by a communication graph $\mathcal{G}_r = (V, E)$. Each player of the game is represented by a node $v_i \in V$ of $\mathcal{G}_r$, and their interconnections are described by the set of edges $E \subseteq V \times V$. The edge weights of the graph are represented as a_{ij}, with $a_{ij} > 0$ if $(v_j, v_i) \in E$ and $a_{ij} = 0$ otherwise. The set of neighbors of node v_i is $\mathcal{N}_i = \{v_j : a_{ij} > 0\}$. The graph is assumed to have no self-loops, i.e., $a_{ii} = 0$ for all agents i; this means that an agent obtains its own state information from internal sensors and not from the communication graph. Define the graph adjacency matrix as $\mathcal{A} = [a_{ij}]$. The weighted in-degree of node i is $d_i = \sum_{j=1}^N a_{ij}$, and the in-degree matrix of the graph is $D = \text{diag}_i\{d_i\}$. The Laplacian matrix is defined as $L = D - \mathcal{A}$ (Lewis et al., 2013).

2.1. Agent dynamics

The mathematical model of each agent i, $i = 1, \ldots, N$, is given by the linear dynamics

$$\dot{x}_i = Ax_i + Bu_i \tag{3}$$

where $x_i(t) \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ are the state variable vector and the control input vector of agent i, respectively. The pair (A, B) is assumed to be controllable. Define an additional agent, regarded as the leader or target node, with dynamics

$$\dot{x}_0 = Ax_0 \tag{4}$$

where the eigenvalues of A have non-positive real parts. The communication links between the leader and the other agents are represented by the pinning gains $g_i \geq 0$. In this paper, the general objective of all agents is to achieve synchronization with the leader state.

Let each agent be able to observe the full state vector of its neighbors in the graph. The local synchronization error of agent i is thus defined as

$$\delta_i = \sum_{j=1}^N a_{ij} \left(x_i - x_j \right) + g_i \left(x_i - x_0 \right) \tag{5}$$

and the local error dynamics are

$$\begin{aligned} \dot{\delta}_i &= \sum_{j=1}^N a_{ij} \left(\dot{x}_i - \dot{x}_j \right) + g_i \left(\dot{x}_i - \dot{x}_0 \right) \\ &= A\delta_i + (d_i + g_i) Bu_i - \sum_{j=1}^N a_{ij} Bu_j \end{aligned} \tag{6}$$

where the dynamics (3)–(4) have been used.

Each agent i expresses its individual objectives in the game by means of a cost function

$$J_i := J_i \left(\delta_i, \delta_{-i}, u_i, u_{-i} \right)$$

where $J_i \left(\delta_i, \delta_{-i}, u_i, u_{-i} \right)$ is a positive definite scalar function. δ_{-i} and u_{-i} represent the local errors and control inputs of the neighbors of agent i, respectively. For synchronization games, the function

$$J_i = \int_0^\infty \left(\delta_i^T Q_i \delta_i + u_i^T R_i u_i + \sum_{j=1}^N a_{ij} u_j^T R_j u_j \right) dt \tag{7}$$

with $Q_i \geq 0$, $R_i > 0$ and $R_{ij} \geq 0$, is commonly employed (Lewis et al., 2012; Vamvoudakis et al., 2012).

2.2. Nash equilibrium

The best response of agent i for fixed neighbor policies u_{-i} is defined as the control policy u_i^* such that the inequality $J_i(\delta, u_i^*, u_{-i}) \leq J_i(\delta, u_i, u_{-i})$ holds for all policies u_i. Nash equilibrium is achieved if every agent plays its best response with respect to all its neighbors, i.e.,

$$J_i\left(\delta, u_i^*, u_{-i}^*\right) \leq J_i\left(\delta, u_i, u_{-i}^*\right) \tag{8}$$

for all agents $i = 1, \ldots, N$.

It is proven in Vamvoudakis et al. (2012) that the best response of agent i with cost function (7) is given by

$$u_i^* = -\frac{1}{2}(d_i + g_i) R_i^{-1} B^T \nabla V_i(\delta) \tag{9}$$

where the functions $V_i(\delta)$ solve the Hamilton–Jacobi (HJ) equations

$$\begin{aligned} &\delta_i^T Q_i \delta_i + \nabla V_i^T A \delta_i - \frac{(d_i + g_i)^2}{4} \nabla V_i^T B R_i^{-1} B^T \nabla V_i \\ &+ \frac{1}{4} \sum_{j=1}^N a_{ij} (d_j + g_j)^2 \nabla V_j^T B R_j^{-1} R_{ij} R_j^{-1} B^T \nabla V_j \\ &+ \frac{1}{2} \sum_{j=1}^N a_{ij} (d_j + g_j) \nabla V_i^T B R_j^{-1} B^T \nabla V_j = 0. \end{aligned} \tag{10}$$

When each agent i uses its best policy (9), Nash equilibrium is achieved in the game.

The Nash equilibrium solution for differential graphical games presents, however, a drawback. Consider the following. A valid distributed control policy (9) requires that the value function for agent i employs only local information, i.e., $V_i(\delta) = V_i(\delta_i)$. Make this assumption and rearrange the HJ equation (10) as

$$\begin{aligned} &\frac{1}{4} \sum_{j=1}^N a_{ij} \left[\nabla V_i + \nabla \bar{V}_j\right]^T B R_j^{-1} B^T \left[\nabla V_i + \nabla \bar{V}_j\right] \\ &= \frac{1}{4} \nabla V_i^T B \left((d_i + g_i)^2 R_i^{-1} + \sum_{j=1}^N a_{ij} R_j^{-1} \right) B^T \nabla V_i \\ &\quad - \delta_i^T Q_i \delta_i - \nabla V_i^T A \delta_i \end{aligned} \tag{11}$$

where $\nabla \bar{V}_j = (d_j + g_j) \nabla V_j$. This equation has the form $f_1(\delta_i, \delta_{-i}) = f_2(\delta_i)$ and, in most cases, it will not hold true for all possible neighbor trajectories δ_{-i}.

In general, there may not exist a set of functions $V_i(\delta_i)$ that solve the HJ equations (10) and provide distributed control policies for the agents. This is an expected result due to the limited knowledge of the agents connected in the communication graph. If agent i does not know the local information of his neighbors, δ_j, then it cannot determine their best responses in the game and prepare its best strategy accordingly.

In the following section, *minmax strategies* are proposed as a practical alternative to the Nash equilibrium solution of graphical games.

3. Minmax strategies for graphical games

In this section we remedy the inconveniences presented in the previous section by defining the minmax strategies for differential graphical games. Intuitively, minmax strategies are obtained when each agent prepares itself for the worst-case behavior of its neighbors. As it is shown below, the corresponding Hamilton–Jacobi–Isaacs (HJI) equations for minmax strategies are generally solvable for linear systems and distributed control policies are obtained accordingly.

3.1. Formulation of minmax strategies

Let agent i prepare its minmax strategy by making the conservative assumption that the goal of its neighbors is to maximize its own performance index, J_i. The following definition formalizes the concept of minmax strategy employed in this paper.

Definition 1 (*Minmax Strategies*). In a differential graphical game, the minmax strategy of agent i is given by

$$u_i^* = \arg\min_{u_i} \max_{u_{-i}} J_i(\delta_i, u_i, u_{-i}). \tag{12}$$

The performance index (7) requires to be modified to formulate a zero-sum game between agent i and its neighbors. To this end, define the function

$$\begin{aligned} J_i = \int_0^\infty \Bigg(&\delta_i^T Q_i \delta_i + (d_i + g_i) u_i^T R_i u_i \\ &- \gamma^2 \sum_{j=1}^N a_{ij} u_j^T R_j u_j \Bigg) dt \end{aligned} \tag{13}$$

where $Q_i \geq 0$, $R_i, R_j > 0$ and γ is a positive scalar. To determine its minmax strategy, agent i assumes that the goal of its neigbors is to maximize its cost function (13).

Define the Hamiltonian function associated with the cost index (13) as

$$\begin{aligned} H_i = &\ \delta_i^T Q_i \delta_i + (d_i + g_i) u_i^T R_i u_i \\ &- \gamma^2 \sum_{j=1}^N a_{ij} u_j^T R_j u_j + \nabla V_i^T(\delta_i) \dot{\delta}_i \end{aligned} \tag{14}$$

with $\dot{\delta}_i$ as in (6). If the value function V_i has a quadratic form, i.e.,

$$V_i(\delta_i) = \delta_i^T P_i \delta_i, \tag{15}$$

then (14) can be expressed as

$$\begin{aligned} H_i = &\ \delta_i^T Q_i \delta_i + (d_i + g_i) u_i^T R_i u_i - \gamma^2 \sum_{j=1}^N a_{ij} u_j^T R_j u_j \\ &+ 2\delta_i^T P_i \left(A\delta_i + (d_i + g_i) B u_i - \sum_{j=1}^N a_{ij} B u_j \right) \end{aligned} \tag{16}$$

The optimal control policy for agent i is now obtained by means of the stationary condition $\frac{\partial H_i}{\partial u_i} = 0$, which yields

$$u_i^* = -R_i^{-1} B^T P_i \delta_i. \tag{17}$$

Similarly, the worst-case policy of the neighbors of agent i can be obtained as

$$v_j^* = -\frac{1}{\gamma^2} R_j^{-1} B^T P_i \delta_i. \tag{18}$$

Notice that v_j^* is not necessarily the actual control policy employed by agent j, u_j.

The HJ equation to be solved for matrix P_i is finally obtained by substituting the policies (17) and (18) in (16), and equating it to zero. This procedure yields the algebraic Riccati equations (ARE)

$$Q_i + P_i A + A^T P_i - (d_i + g_i)\, P_i B R_i^{-1} B^T P_i + \frac{1}{\gamma^2} \sum_{j=1}^{N} a_{ij} P_i B R_j^{-1} B^T P_i = 0. \tag{19}$$

Remark 1. Control policies (17) are distributed, in contrast to the Nash solution policies given by (9).

Remark 2. If the ARE (19) has a solution $P_i > 0$, then substituting the function (15) and the control policies (17)–(18) in the Hamiltonian (14) satisfies the HJ equation $H_i(u_i^*, \upsilon_i^*, \nabla V_i) = 0$, verifying that (15) is indeed the value function of the game.

Remark 3. Equations in the form of (19) are known to have solutions for P_i if $\left(A, \sqrt{Q_i}\right)$ is observable, (A, B) is stabilizable, and $(d_i + g_i)R_i^{-1} - \frac{1}{\gamma^2}\sum_{j=1}^{N} R_j^{-1} > 0$ (Lewis et al., 2012). Notice the influence of the parameter γ to make this inequality hold.

The following theorem shows that control policy (17) with P_i being the solution of (19) provides the minmax strategy for agent i. The proof of this theorem assumes stability of the error dynamics (6); such stability will be analyzed in Section 4.

Theorem 1. *Let the agents of a differential graphical game with dynamics (3) and a leader with dynamics (4) use the control policies (17) where matrices P_i are the solutions of the AREs (19). Moreover, assume these control policies achieve asymptotical stability of the local synchronization errors (6) for all agents i. Then, all agents follow their minmax strategies as defined in Definition 1 and the minmax value of the game is $V_i(\delta_i(0))$.*

Proof. Consider the value function (15) and express the performance index (13) as

$$J_i = \int_0^\infty \left(\delta_i^T Q_i \delta_i + (d_i + g_i)\, u_i^T R_i u_i - \gamma^2 \sum_{j=1}^{N} a_{ij} u_j^T R_j u_j \right) dt - \int_0^\infty \dot{V}_i(\delta_i) dt + \int_0^\infty 2\delta_i^T P_i \left(A\delta_i + (d_i + g_i)\, B u_i - \sum_{j=1}^{N} a_{ij} B u_j \right) dt.$$

Using the inner-product notation (1), express J_i as

$$J_i = \langle \delta_i, Q_i \delta_i \rangle + (d_i + g_i) \langle u_i, R_i u_i \rangle - \gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle u_j, R_j u_j \right\rangle + V_i(\delta(0)) + 2 \langle \delta_i, P_i A \delta_i \rangle + 2(d_i + g_i) \langle \delta_i, P_i B u_i \rangle - 2 \sum_{j=1}^{N} a_{ij} \left\langle \delta_i, P_i B u_j \right\rangle$$

where we have used the fact that $\int_0^\tau \dot{V}_i(\delta_i)dt = V_i(\delta_i(\tau)) - V_i(\delta_i(0))$, and that, because the global closed-loop system is asymptotically stable, $V_i(\delta_i(\tau)) = 0$ as $\tau \to \infty$. As P_i makes the ARE (19) hold, we get

$$\begin{aligned} J_i &= (d_i + g_i) \left\langle u_i^*, R_i u_i^* \right\rangle - \gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle \upsilon_i^*, R_j \upsilon_i^* \right\rangle + (d_i + g_i) \langle u_i, R_i u_i \rangle - \gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle u_j, R_j u_j \right\rangle \\ &\quad - 2(d_i + g_i) \left\langle u_i^*, R_i u_i \right\rangle + 2\gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle \upsilon_i^*, R_j u_j \right\rangle + V_i(\delta(0)) \\ &= (d_i + g_i) \left\langle u_i - u_i^*, R_i(u_i - u_i^*) \right\rangle - \gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle u_j - \upsilon_j^*, R_j(u_j - \upsilon_j^*) \right\rangle + V_i(\delta(0)). \end{aligned} \tag{20}$$

Therefore, u_i^* in (17) is the minmax strategy of agent i and the value of the game is given by $V_i(\delta_i(0))$. □

The following two sections analyze the stability and robustness properties of the proposed minmax solutions. In Section 4, the conditions under which the control policies (17) stabilize the system are studied.

4. Stability of minmax strategies

Conditions for the stability of minmax strategies are well known for adversarial games (Basar & Bernhard, 1995; Cao et al., 2008). However, in non-adversarial games the worst-case behavior assumption made by the agents is not the true strategy followed by its neighbors. Here, we determine the conditions on which the minmax assumption leads to stability of the global system.

Two stability concepts are analyzed in this section for minmax strategies. First, it is proven that the system (6) with control policies u_i^* in (17) is finite-gain $\mathcal{L}_2$ stable. Then, conditions for asymptotic stability of the global multiagent system are provided.

4.1. $\mathcal{L}_2$ Stability

When using minmax strategies, an agent with error dynamics (6) considers the effect of its neighbors policies, $\sum_{j=1}^{N} a_{ij} B u_j$, as a disturbance term to be rejected. The nominal system for agent i can then be defined as

$$\dot{\bar{\delta}}_i = A\bar{\delta}_i + (d_i + g_i)\, B u_i. \tag{21}$$

This idea provides the foundation of the following analysis. Define the performance output of agent i as

$$\|z_i(t)\|^2 = \delta_i^T Q_i \delta_i + (d_i + g_i) u_i^T R_i u_i. \tag{22}$$

Similarly, the *disturbance* input produced by the neighbors of agent i is defined as

$$\|\zeta_i(t)\|^2 = \sum_{j=1}^{N} a_{ij} u_j^T R_j u_j. \tag{23}$$

According to (2), the output (22) is $\mathcal{L}_2$ stable if

$$\|z_i(t)\|_{\mathcal{L}_2} \leq \gamma\, \|\zeta_i(t)\|_{\mathcal{L}_2} + \beta \tag{24}$$

for some $\gamma,\ \beta \geq 0$. The following theorem shows the $\mathcal{L}_2$ stability properties of the minmax policies (17).

Theorem 2. *The system (6) with policy u_i^* as in (17) and P_i as the solution of (19) is $\mathcal{L}_2$ stable with $\mathcal{L}_2$-gain bounded by γ according to (24).*

Proof. In the proof of Theorem 1, the final step (20) showed that

$$\begin{aligned} J_i &= \langle \delta_i, Q_i \delta_i \rangle + (d_i + g_i) \langle u_i, R_i u_i \rangle - \gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle u_j, R_j u_j \right\rangle \\ &= (d_i + g_i) \left\langle u_i - u_i^*, R_i(u_i - u_i^*) \right\rangle - \gamma^2 \sum_{j=1}^{N} a_{ij} \left\langle u_j - \upsilon_j^*, R_j(u_j - \upsilon_j^*) \right\rangle + V_i(\delta(0)). \end{aligned}$$

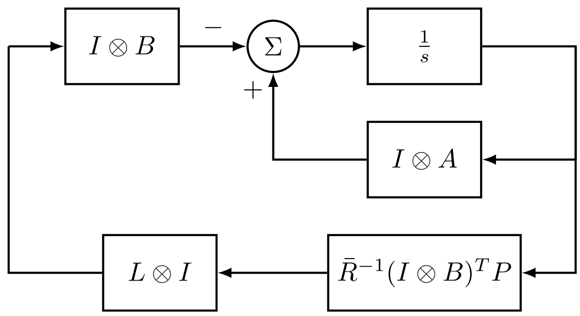

Fig. 1. Closed-loop multiagent system.

Because $\mathcal{L}_2$ stability must hold for all initial conditions, select $\delta_i(0) = 0$. This implies $V_i(\delta_i(0)) = 0$. Let $u_i = u_i^*$ to obtain

$$\langle \delta_i, Q_i\delta_i\rangle + (d_i + g_i)\langle u_i, R_i u_i\rangle - \gamma^2 \sum_{j=1}^{N} a_{ij}\langle u_j, R_j u_j\rangle$$

$$= -\gamma^2 \sum_{j=1}^{N} a_{ij}\left\langle u_j - v_j^*, R_j(u_j - v_j^*)\right\rangle \le 0$$

which implies

$$\langle \delta_i, Q_i\delta_i\rangle + (d_i + g_i)\langle u_i, R_i u_i\rangle \le \gamma^2 \sum_{j=1}^{N} a_{ij}\langle u_j, R_j u_j\rangle$$

Taking the square root to both sides of the inequality shows that (24) holds. □

The asymptotic stability of minmax strategies is studied in the following subsection.

4.2. Asymptotic stability

The conditions for asymptotic stability of the dynamics (6) for all $i = 1, \ldots, N$, are now studied. Substitute the control policies (17) in the error dynamics (6) to get

$$\dot{\delta}_i = \left(A - (d_i + g_i)BR_i^{-1}B^TP_i\right)\delta_i + \sum_{j=1}^{N} a_{ij}BR_j^{-1}B^TP_j\delta_j. \tag{25}$$

System (25) can be expressed in global form by defining the variable $\delta = [\delta_1^T, \ldots, \delta_N^T]^T$, such that

$$\dot{\delta} = \left[(I \otimes A) - ((L+G) \otimes B)\bar{R}^{-1}(I \otimes B^T)P\right]\delta \tag{26}$$

where I is an identity matrix of appropriate dimensions, $\otimes$ represents the Kronecker product, $G = \text{diag}_i\{g_i\}$, $\bar{R} = \text{diag}_i\{R_i\}$ and $P = \text{diag}_i\{P_i\}$. Fig. 1 shows the block diagram of the feedback global system.

Two additional assumptions will be considered to complete the stability analysis of the global system (26).

Assumption 1. The matrices R_i in the performance indices (13) are selected such that $R_i = R_j = R$ for all agents i and j.

Assumption 2. The graph weights a_{ij} and g_i have sufficiently small magnitudes for all pairs i, j, such that

$$\lambda_{\min}(Q_i) \ge \left[\left(1 - \frac{1}{\gamma^2}\right)d_i + g_i\right]\lambda_{\max}(P_iBR^{-1}B^TP_i) \tag{27}$$

where $d_i = \sum_{j=1}^{N} a_{ij}$, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the minimum and maximum eigenvalues of a matrix, respectively, and all matrices are defined in the ARE (19).

Assumption 2 is a restriction on the graph weights that will be used to guarantee asymptotic stability of the global system. Assumption 1 allows us to write $\bar{R}^{-1} = I \otimes R^{-1}$, such that (26) can be expressed as

$$\dot{\delta} = \left[(I \otimes A) - ((L+G) \otimes BR^{-1}B^T)P\right]\delta. \tag{28}$$

Similarly, the ARE (19) can be written as

$$0 = Q_i + P_iA + A^TP_i - \left[\left(1 - \frac{1}{\gamma^2}\right)d_i + g_i\right]P_iBR^{-1}B^TP_i. \tag{29}$$

The following lemmas are used in our main proof of stability below. Recall that if the graph $\mathcal{G}_r$ is strongly connected, then $\lambda = 0$ is a simple eigenvalue of the Laplacian matrix L (Lewis et al., 2013).

Lemma 1. *Let L be the Laplacian matrix of a strongly connected graph, let $G = diag_i\{g_i\}$ be the pinning gain and let R be a symmetric, positive definite matrix. Then, there exists a matrix $W = diag_i\{w_i\}$ such that*

$$S_1 := \left((L+G)W^{-1} + W^{-1}(L+G)^T\right) \otimes BR^{-1}B^T \ge 0. \tag{30}$$

Proof. It is proven in Zhang, Li, Qu and Lewis (2015) that there exists a diagonal matrix W, such that $W(L+G) + (L+G)^TW > 0$ for strongly connected graphs. By properties of the Kronecker product (Brewer, 1978) and the fact that $BR^{-1}B^T$ is a symmetric, positive semidefinite matrix, we get $\left(W(L+G) + (L+G)^TW\right) \otimes BR^{-1}B^T \ge 0$. Premultiplying and postmultiplying both sides of this expression by $W^{-1} \otimes I$, we obtain (30). □

Lemma 2. *Let matrix A have all eigenvalues with non-positive real parts, let matrix P_i solve the ARE (29) and let Assumption 2 hold. Define the matrix $P_w = (W \otimes I)P$ with W defined in Lemma 1 and P is as in (26). Then,*

$$S_2 := -P_w^{-1}(I \otimes A^T) - (I \otimes A)P_w^{-1} \ge 0. \tag{31}$$

Proof. Express the ARE (29) as

$$-P_iA - A^TP_i = Q_i - \left[\left(1 - \frac{1}{\gamma^2}\right)d_i + g_i\right]P_iBR^{-1}B^TP_i. \tag{32}$$

From (32), if (27) holds it is clear that $-P_iA - A^TP_i \ge 0$. Because the graph is strongly connected, $w_i > 0$ for all elements of the vector w (Zhang, Li et al., 2015), and the inequality $-w_i^{-1}P_iA - w_i^{-1}A^TP_i \ge 0$ holds. Premultiplying and postmultiplying both sides of this expression by P_i^{-1}, we get $-w_i^{-1}AP_i^{-1} - w_i^{-1}P_i^{-1}A^T \ge 0$, which leads directly to (31). □

The result in Lemma 3 is used in the proof of the following theorem.

Lemma 3. *Consider the matrix S_1 defined in Lemma 1. Then, $\delta^TS_1\delta = 0$ if and only if for all the subvectors δ_i of δ, we have $BR^{-1}B^T\delta_i = 0$.*

Proof. The null space of any symmetric, positive semidefinite matrix S is given by $Null(S) = \{\delta | \delta^TS\delta = 0\}$ (Bernstein, 2009, Fact 8.15.2). This implies that $\delta^TS_1\delta = 0$ if and only if $S_1\delta = 0$; thus, δ must be an eigenvector of S_1, associated with an eigenvalue equal to zero. From the properties of the Kronecker product (Brewer, 1978), δ is such eigenvector if it can be expressed as $\delta = z_1 \otimes z_2$, where either $(((L+G)W^{-1} + W^{-1}(L+G)^T))z_1 = 0$ or $BR^{-1}B^Tz_2 = 0$. As the matrix $(L+G)W^{-1}+W^{-1}(L+G)^T$ is nonsingular (Zhang, Li et al., 2015), we get that each component δ_i can be expressed as $\delta_i = z_2$, where $BR^{-1}B^Tz_2 = 0$. □

We are ready to prove that the minmax policies (17) make the global system (28) asymptotically stable. The following theorem is our main result in this section.

Theorem 3. *Let the conditions in Theorem 1 hold, and make Assumptions 1 and 2. Furthermore, let the graph $\mathcal{G}_r$ have a spanning tree with the leader node as the root. Then, control policies (17) make the system (28) asymptotically stable.*

Proof. The proof of this theorem is divided in two parts. First, we prove that asymptotic stability is achieved for strongly connected graphs. Then, we generalize the result for graphs with a spanning tree.

Assume a strongly connected graph $\mathcal{G}_r$ and notice that Lemmas 1 and 2 hold from Assumptions 1 and 2. Now,

$$\begin{aligned}
&P_w^{-1}\left[(I\otimes A) - ((L+G)\otimes BR^{-1}B^T)P\right]^T \\
&\quad + \left[(I\otimes A) - ((L+G)\otimes BR^{-1}B^T)P\right]P_w^{-1} \\
&= P_w^{-1}(I\otimes A^T) - (W^{-1}\otimes I)((L+G)^T\otimes BR^{-1}B^T) \\
&\quad + (I\otimes A)P_w^{-1} - ((L+G)\otimes BR^{-1}B^T)(W^{-1}\otimes I) \\
&= P_w^{-1}(I\otimes A^T) - (W^{-1}(L+G)^T\otimes BR^{-1}B^T) \\
&\quad + (I\otimes A)P_w^{-1} - ((L+G)W^{-1}\otimes BR^{-1}B^T) \\
&= -S_1 - S_2 \le 0.
\end{aligned}$$

By Lyapunov theory (Khalil, 1996), the system (26) is stable. Moreover, using LaSalle's extension (Khalil, 1996), the system trajectories δ converge to the largest invariant set such that $\delta^T(S_1+S_2)\delta = 0$. We now prove that this happens only when $\delta = 0$. Consider the ARE (19) and notice that $P_iA + A^TP_i - (d_i+g_i)P_iBR_i^{-1}B^TP_i < 0$. Premultiplying and postmultiplying this inequality by P_i^{-1}, we get

$$AP_i^{-1} + P_i^{-1}A^T - (d_i+g_i)BR_i^{-1}B^T < 0 \tag{33}$$

By Lemma 3, $\delta^TS_1\delta = 0$ if and only if $BR^{-1}B^T\delta_i = 0$. It is also clear that $\delta^TS_2\delta = 0$ if and only if $\delta_i^T(P_i^{-1}A^T + AP_i^{-1})\delta_i = 0$. Assume now that there exists a vector $\delta \neq 0$ such that $\delta^TS_1\delta = 0$ and $\delta^TS_2\delta = 0$. This would imply $\delta_i^T(AP_i^{-1} + P_i^{-1}A^T - (d_i+g_i)BR_i^{-1}B^T)\delta_i = 0$, which contradicts the negative definiteness of (33). Therefore, $\delta = 0$ and the system is asymptotically stable.

Consider now the case when the graph has a spanning tree with the leader as a root. If $\mathcal{G}_r$ has a spanning tree but is not strongly connected, then the Laplacian matrix is reducible and can be expressed by means of a permutation transformation as the Frobenius form Lewis et al. (2013)

$$\begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & & & \\ L_{M1} & L_{M2} & \cdots & L_{MM} \end{bmatrix}$$

where each submatrix L_{kk} is irreducible. Irreducibility of matrix L_{kk} implies that the subgraph connecting only the agents in the kth block row of L is strongly connected. This implies that the dynamics of the agents in the first block row are asymptotically stable.

We can now prove stability of the global system by induction. Assume all agents in the block rows 1 to $k-1$ have stable dynamics. Thus, as $t \to \infty$, the influence of their local errors δ_j over the dynamics (25) of the agents in the kth block row vanishes. This leaves only the strongly connected interaction of the agents in the kth block row, which is proven to be stable. □

Theorem 3 provides sufficient conditions for the asymptotic stability of the minmax policies. In the following section, we show that minmax strategies also provide strong robustness properties to the closed-loop system.

5. Robustness analysis for minmax strategies

In this section, we are particularly interested in determining the gain and phase margins of the agents provided by the minmax policies (17). Our approach to perform this analysis is to consider each individual agent using its minmax input (17) and determine how the neighbor policies, seen as a disturbance, affect its stability.

Let the perturbed version of the nominal system (21) be given by the dynamics

$$\dot{\hat{\delta}}_i = A\hat{\delta}_i + (d_i+g_i)\,(B\mathscr{P}u_i) \tag{34}$$

where the disturbance $\mathscr{P}$ is assumed to be a finite-gain operator with $\mathscr{P}0 = 0$, and $\hat{\delta}_i$ represents the state trajectories of the perturbed system. We let $\hat{\delta}_i(0) = \delta_i(0)$.

The subsequent robustness analysis follows a similar procedure as in Safonov and Athans (1977). The following lemma shows a sufficient condition on the disturbance $\mathscr{P}$ that guarantees the stability of $\hat{\delta}_i$. The gain and phase margins of the system are then shown to be a particular case of such condition. Notice also that guaranteeing the stability of $\hat{\delta}_i$ implies the stability of δ_i.

Lemma 4. *If the perturbation $\mathscr{P}$ of the system (34) is such that*

$$\langle u, \left[(d_i+g_i)(2\mathscr{P}-I)R_i^{-1} + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}R_j^{-1}\right] u\rangle \ge 0 \tag{35}$$

for all $u \in L_2^m[0,\infty)$, then

$$\delta_i^T(0)P_i\delta_i(0) \ge \langle \hat{\delta}_i, Q_i\hat{\delta}_i\rangle. \tag{36}$$

If, additionally, $\left[\sqrt{Q}, A\right]$ is detectable, then $\hat{\delta}_i$ is asymptotically stable.

Proof. Using the definition of the perturbed system (34) and the ARE (29), we have for every τ,

$$\begin{aligned}
&\delta_i^T(0)P_i\delta_i(0) \\
&= \hat{\delta}_i^T(\tau)P_i\hat{\delta}_i(\tau) - \int_0^\tau \frac{d}{dt}\left(\hat{\delta}_i^T(t)P_i\hat{\delta}_i(t)\right)dt \\
&\ge -2\langle\hat{\delta}_{i\tau}, P_i\left(A - (d_i+g_i)B\mathscr{P}R_i^{-1}B^TP_i\right)\hat{\delta}_{i\tau}\rangle \\
&= \langle\hat{\delta}_{i\tau}, (-P_iA - A^TP_i)\hat{\delta}_{i\tau}\rangle \\
&\quad + \langle\hat{\delta}_{i\tau}, (2(d_i+g_i)P_iB\mathscr{P}R_i^{-1}B^TP_i)\hat{\delta}_{i\tau}\rangle \\
&= \langle\hat{\delta}_{i\tau}, (d_i+g_i)P_iB(2\mathscr{P}-I)R_i^{-1}B^TP_i\hat{\delta}_{i\tau}\rangle \\
&\quad + \langle\hat{\delta}_{i\tau}, \left(\frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}P_iBR_j^{-1}B^TP_i + Q_i\right)\hat{\delta}_{i\tau}\rangle
\end{aligned}$$

where $\delta_{i\tau}$ is the truncation of δ_i as defined in Section 1.1. Let $\Pi_i = (d_i+g_i)(2\mathscr{P}-I)R_i^{-1} + (1/\gamma^2)\sum_{j=1}^{N} a_{ij}R_j^{-1}$ and write

$$\begin{aligned}
\delta_i^T(0)P_i\delta_i(0) - \langle\hat{\delta}_{i\tau}, Q_i\hat{\delta}_{i\tau}\rangle &\ge \langle\hat{\delta}_{i\tau}, P_iB\Pi_iB^TP_i\hat{\delta}_{i\tau}\rangle \\
&= \langle B^TP_i\hat{\delta}_{i\tau}, \Pi_iB^TP_i\hat{\delta}_{i\tau}\rangle \\
&= \langle\bar{u}, \Pi_i\bar{u}\rangle \ge 0
\end{aligned}$$

where $\bar{u} = B^T P_i \hat{\delta}_{i\tau}$ and the last inequality holds by assumption. In the limit $\tau \to \infty$, it follows that $\delta_i^T(0)P_i\delta_i(0) \geq \langle \hat{\delta}_i, Q_i\hat{\delta}_i \rangle$ which implies that $\langle \hat{\delta}_i, Q_i\hat{\delta}_i \rangle$ is bounded. As $[\sqrt{Q}, A]$ is detectable, $\hat{\delta}_i$ is square-integrable. Because $\mathscr{P}$ has finite gain, so does the matrix $A - (d_i + g_i)B\mathscr{P}R_i^{-1}B^T P_i$. From (34), we notice that $\dot{\hat{\delta}}_i$ is also square integrable. Since both $\hat{\delta}_i$ and $\dot{\hat{\delta}}_i$ are square integrable, $\hat{\delta}_i$ is asymptotically stable (Safonov & Athans, 1977). □

The following theorem sets the result in Lemma 4 for the case when the disturbance $\mathscr{P}$ is a linear operator.

Theorem 4. *Let $\mathscr{P}$ be a finite-gain, linear time-invariant operator $\mathscr{H}$ with transfer matrix $H(j\omega)$. If for all frequencies $\omega \in \mathbb{R}$*

$$(d_i + g_i)\left(H(j\omega)R_i^{-1} + R_i^{-1}H^*(j\omega) - R_1^{-1}\right) + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}R_j^{-1} \geq 0 \tag{37}$$

and if $\left[Q^{1/2}, A\right]$ is detectable, then the system (34) is asymptotically stable.

Proof. Expressing $\mathscr{P}$ as a linear operator $\mathscr{H}$ and using Parseval's theorem (Khalil, 1996), we get

$$\begin{aligned}
&\langle u_i, \left[(d_i + g_i)(2\mathscr{P} - I)R_i^{-1} + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}R_j^{-1}\right] u_i\rangle \\
&= \langle u_i, \left[(d_i + g_i)(2\mathscr{H} - I)R_i^{-1} + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}R_j^{-1}\right] u_i\rangle \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{U}_i^*(j\omega)\Bigg[(d_i + g_i)\Big(H(j\omega)R_i^{-1} \\
&\quad + R_i^{-1}H^*(j\omega) - R_1^{-1}\Big)\frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}R_j^{-1}\Bigg]\mathcal{U}(j\omega)d\omega \\
&\geq 0
\end{aligned}$$

where the last inequality holds by the assumption in (37). The proof is completed by Lemma 4. □

From Theorem 4 we derive our main robustness results. The following corollary holds for a particular selection of matrices R_i and the disturbance $\mathscr{P}$.

Corollary 1. *Let $R_i = diag_k\left\{r_{i,k}\right\}$ and let the disturbance $\mathscr{P}$ be such that*

$$\mathscr{P}u_i = \begin{bmatrix} \mathscr{P}_1 u_{i,1} \\ \vdots \\ \mathscr{P}_m u_{i,m} \end{bmatrix}. \tag{38}$$

If each of the perturbations $\mathscr{P}_k$ is linear time-invariant with proper transfer function $H_k(s)$, $Re\{s_j\} < 0$ for all poles s_j of $H_k(s)$, and

$$Re\{H_k(j\omega)\} \geq \frac{1}{2}\left[1 - \frac{1}{(d_i + g_i)\gamma^2}\sum_{j=1}^{N} a_{ij}\frac{r_{i,k}}{r_{j,k}}\right] \tag{39}$$

for all ω, then the system (34) is asymptotically stable.

Proof. $Re\{s_j\} < 0$ assures that $\mathscr{P}$ has finite gain. Take $H(s) = \mathrm{diag}_k\{H_k(s)\}$. Now,

$$\begin{aligned}
&(d_i + g_i)r_{i,k}^{-1}\left(H_k(j\omega) + H_k^*(j\omega) - 1\right) + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}r_{j,k}^{-1} \\
&= 2(d_i + g_i)\left(r_{i,k}^{-1}Re\{H_k(j\omega)\} - r_{i,k}^{-1}\right) + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}r_{j,k}^{-1} \\
&\geq 2(d_i + g_i)r_{i,k}^{-1}\left(\frac{1}{2} - \frac{1}{2\gamma^2}\frac{1}{d_i + g_i}\sum_{j=1}^{N} a_{ij}\frac{r_{i,k}}{r_{j,k}}\right) \\
&\quad - (d_i + g_i)r_{i,k}^{-1} + \frac{1}{\gamma^2}\sum_{j=1}^{N} a_{ij}r_{j,k}^{-1} \\
&= 0
\end{aligned}$$

As the conditions of Theorem 4 are satisfied, (34) is asymptotically stable. □

We can finally determine the phase and gain margins of minmax strategies by means of the following result, which follows from Corollary 1.

Corollary 2. *Let the conditions of Corollary 1 hold. A phase shift ϕ_i with*

$$|\phi_i| \leq 60^\circ + \theta \tag{40}$$

where $\theta = arccos\left(0.25(c + \sqrt{12 - 3c^2})\right)$ and $c = 1 - (d_i + g_i)^{-1}\gamma^{-2}\sum a_{ij}r_{i,k}r_{j,k}^{-1}$, in the respective feedback loops of each of the controls u_i will leave an asymptotically stable system. Moreover, inserting a gain of α_k such that

$$\alpha_k \geq \frac{1}{2}\left[1 - \frac{1}{(d_i + g_i)\gamma^2}\sum_{j=1}^{N} a_{ij}\frac{r_{i,k}}{r_{j,k}}\right] \tag{41}$$

in the feedback loops of the controllers $u_{i,k}$, leaves the system asymptotically stable.

Proof. Expressing the complex number $H_k(j\omega)$ in polar form, it is clear from Corollary 1 that the condition for stability is $cos\phi_k(\omega) \geq 0.5 - 0.5(d_i + g_i)^{-1}\gamma^{-2}\sum a_{ij}r_{i,k}r_{j,k}^{-1}$, or $|\phi_k(\omega)| \leq arccos(0.5 - 0.5(d_i + g_i)^{-1}\gamma^{-2}\sum a_{ij}r_{i,k}r_{j,k}^{-1})$. Using trigonometric identities, we can express this result as

$$|\phi_i(\omega)| \leq \arccos\left(\frac{1}{2}\right) + \arccos\left(\frac{c + \sqrt{12 - 3c^2}}{4}\right)$$

with the constant c defined in the corollary statement. This proves the phase margin of minmax policies.

From Corollary 1, if $H_k(j\omega)$ represents a scalar gain α_k, then (41) guarantees stability of the system. □

Remark 4. Corollary 2 shows that minmax strategies have infinite gain margin, gain reduction tolerance of more than 50% and phase margin of more than 60°. The amount of additional phase delay and of additional gain reduction tolerance depend on the selection of matrices R_i, parameter γ and the graph topology. This is an improved result from the LQR controller, which is known to have infinite gain margin, 50% gain reduction tolerance and 60° of phase margin (Safonov & Athans, 1977).

Remark 5. Minmax strategies are not a best response against the actual behavior of the neighbors. Thus, the improved minmax robustness properties are obtained at the cost of paying a higher payoff when compared to the Nash, non-distributed policies.

6. Simulation results

Two numerical examples are presented to test the validity of our theoretical results. First, we show the applicability of minmax strategies to achieve synchronization. Then, the robustness properties of the minmax policies are compared with those of the LQR solution.

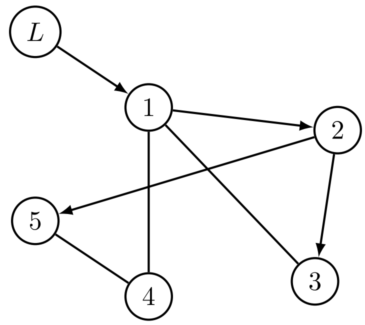

Fig. 2. Graph topology for simulation.

6.1. Leader–followers synchronization

Consider a set of 5 agents and one leader connected as shown in Fig. 2. If $j \in \mathcal{N}_i$, let $a_{ij} = 0.3$. Each agent is taken with linear dynamics given by (3), where

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}. \tag{42}$$

We first show that the agents in this system do not have distributed control policies of the form (17) that achieve Nash equilibrium. Consider, then the performance indices (7) and quadratic value functions as in (15) and notice that the HJ equations (10) are now expressed as

$$\begin{aligned} &\delta_i^{\mathrm{T}} \left(Q_i + P_i A + A^{\mathrm{T}} P_i - (d_i + g_i)^2 P_i B R_i^{-1} B^{\mathrm{T}} P_i \right) \delta_i \\ &\quad + \sum_{j=1}^{N} a_{ij} (d_j + g_j)^2 \delta_j P_j B R_j^{-1} R_{ij} R_j^{-1} B^{\mathrm{T}} P_j \delta_j \\ &\quad + 2 \sum_{j=1}^{N} a_{ij} (d_j + g_j) \delta_i P_i B R_j^{-1} B^{\mathrm{T}} P_j \delta_j = 0. \end{aligned} \tag{43}$$

which must now be solved for matrices P_i for all agents i. Because (43) must hold for all values of δ_i and δ_j, then the matrices P_i and P_j, $j \in \mathcal{N}_i$, must solve simultaneously the matrix equations

$$\begin{aligned} &Q_i + P_i A + A^{\mathrm{T}} P_i - (d_i + g_i)^2 P_i B R_i^{-1} B^{\mathrm{T}} P_i = 0, \\ &P_j B R_j^{-1} R_{ij} R_j^{-1} B^{\mathrm{T}} P_j = 0, \\ &P_i B R_j^{-1} B^{\mathrm{T}} P_j = 0. \end{aligned} \tag{44}$$

Note that there are N sets of equations of the form (44) that need to be solved simultaneously. It is clear that these equations do not have positive definite solutions.

Minmax strategies are now designed as proposed in this paper. The minmax performance indices of the agents are defined by (13) with $Q_1 = Q_3 = 2I$, $Q_2 = Q_5 = 3I$ and $Q_4 = I$, where I is the identity matrix. Let all agents use the same values for $R = 2I$ and $\gamma = 2$.

The matrices P_i that solve the minmax strategies problem are obtained by solving the algebraic Riccati equations (19). The resulting matrices are shown below.

$$P_1 = \begin{bmatrix} 0.5055 & 0.0314 \\ 0.0314 & 0.3537 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.1928 & 0.1314 \\ 0.1314 & 0.7597 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.6756 & 0.0612 \\ 0.0612 & 0.4441 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0.4991 & 0.0707 \\ 0.0707 & 0.3068 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 0.8049 & 0.0542 \\ 0.0542 & 0.5558 \end{bmatrix}.$$

The minmax control policies are now given by (17) with the appropriate matrix P_i for each agent. Using these policies, the agents successfully achieve synchronization with trajectories shown in Fig. 3. Fig. 4 shows the trajectories of the agents also along a time axis.

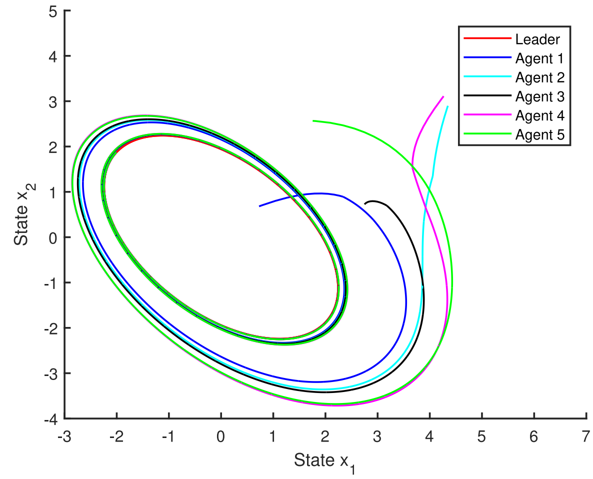

Fig. 3. State trajectories with minmax policies.

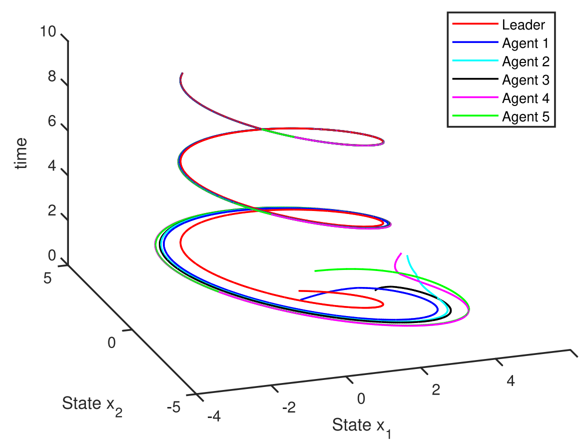

Fig. 4. Synchronization in time with minmax policies.

6.2. Robustness comparison

To test the results presented in Section 5, we present a simulation comparison when the system (21) presents parametric uncertainties. The LQR and the minmax policies used here differ only in the selection of matrix P_i. For minmax, the ARE (19) is solved, while for the single-agent LQR P_i is the solution of the ARE

$$Q_i + P_i A + A^T P_i - P_i B R_i^{-1} B^T P_i = 0. \tag{45}$$

These equations are solved using the matrices in (42). However, let the actual system be given by the matrices

$$\bar{A} = \begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1.7 & 0.2 \\ 0.2 & 1.2 \end{bmatrix}.$$

For clarity, we show the following results only for agent 1. Similar results are obtained for all other agents. Using the simulation parameters in (40) and (41), we get

$$|\phi_1| \le 60^\circ + \arccos \frac{1}{4} \left(\frac{5}{6} + \sqrt{12 - \frac{25}{12}} \right) = 65.37^\circ,$$

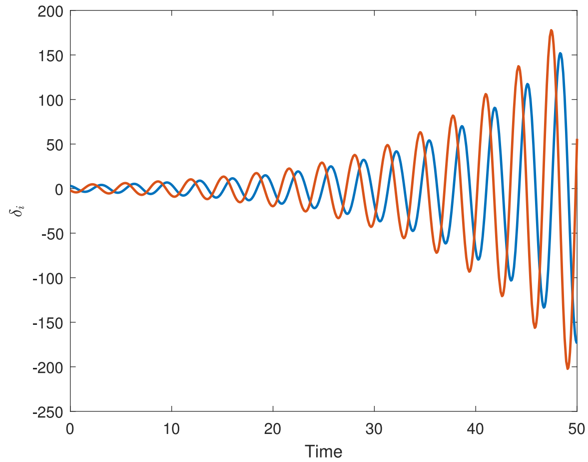

Fig. 5. The LQR policy leaves the uncertain system unstable.

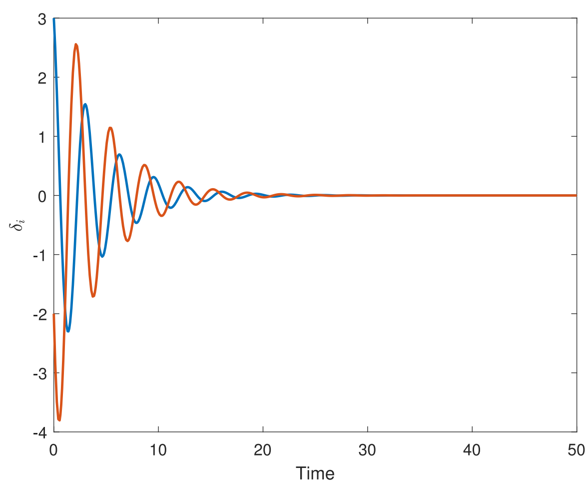

Fig. 6. Minmax policies stabilize the uncertain system.

$$\alpha_1 \geq \frac{1}{2}\left(1 - \frac{1}{6}\right) = \frac{5}{12}.$$

These results improve the 60° phase margin and the 0.5 gain reduction tolerance of the LQR controller.

The simulation results are shown in Figs. 5 and 6. Fig. 5, shows that the system with parametric uncertainties remains unstable when the LQR controller is applied. On the other hand, the minmax policy is shown in Fig. 6 to stabilize the system in spite of the incorrect model used to design it.

7. Conclusion

Minmax strategies were designed and analyzed as an alternative solution concept for differential graphical games. The assumption made by each agent about the worst intentions of its neighbors yields robust control policies, as analyzed in Section 5. Such policies are always distributed in the sense that the agents use only local information obtained from the graph topology. Although the agents prepare their strategies against neighbor policies that are not being used, the global system still reaches asymptotic stability and synchronization with the leader node.

Despite its attractive features, the robustness properties of minmax may be too conservative for certain applications. For this reason, research about differential graphical games is being continued by the authors considering different solution concepts that still allow solutions using distributed input policies.

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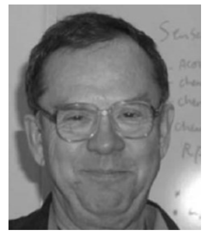

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