Reduced-Order Estimation of the Uniform Completely Connected Homogeneous Influence Model (UCC-HIM)

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***Abstract*—The influence model (IM) is a discrete-time stochastic automaton that captures spatiotemporal network dynamics. It constitutes a reduced-order representation of networked Markov chains and has found broad stochastic network applications. Parameter estimation from observation data is critical for utilizing IM in real applications. The master Markov chain approach used in the literature incurs significant computational cost. In this letter, we develop an efficient estimation algorithm for a special class of IM, named the uniform completely connected homogeneous influence model (UCC-HIM), through exploiting its special network topology. Specially, we introduce a reduced-order Markov chain representation for the UCC-HIM, analyze its relationship with the master Markov chain, based on which an efficient estimation algorithm is developed. Two simulation studies verify the accuracy and computation reduction of the proposed estimation approach.**

***Index Terms*—Stochastic automaton, spatiotemporal processes, influence model, parameter estimation, reduced-order analysis.**

I. INTRODUCTION

THE INFLUENCE model (IM) is a discrete-time stochastic model that captures spatiotemporal network dynamics [1], [2]. It constitutes a reduced-order representation of networked Markov chains through abstracting network-level interactions and local-level update rules. IM has been used in diverse stochastic network applications, such as power networks, social networks, virus spreads, and weather evolution [3]–[6].

In order to use IM in stochastic network applications, one critical step is to estimate underlying model parameters from observation data. Identifiability deals with the uniqueness of IM estimates. In [7], the identifiability conditions for homogeneous IM were recently provided. Through exploiting the mapping structure between IM and its equivalent master Markov chain, the identifiability analysis led to a linear algebra-based estimator. This letter also developed a baseline maximum likelihood estimator (MLE) for comparison. Several other MLE based estimators have been developed in the literature [4], [8]. Of our interest, [9] developed an IM estimation algorithm based on its corresponding first-order representation, i.e., the influence matrix. All of these existing estimation algorithms have limitations of some sorts in their performance. The computation of the linear algebra-based approach grows exponentially with the increase of network size. The performance of the MLE is sensitive to initial guesses, and local optima are difficult to avoid. Furthermore, obtaining the influence matrix from data is complex and prone to errors due to the coupling effect of network- and local- level interactions. To overcome these challenges, we here develop an IM estimation algorithm that is both accurate and computationally effective, including for large networks.

In this letter, we take a structural approach to study reduced-computation IM estimation methods for IM. Network topology plays an important role in network dynamics, and topology-based approaches have been widely used in studies such as network identification, state estimation, network design and control [10]–[12]. As a first step, we here focus on a canonical class of IM, named the uniform completely connected homogeneous influence model (UCC-HIM). In UCC-HIM, all sites are fully connected with common mutual influence and local status update rules. Stochastic networks of such a topology capture agent interactions in close proximity and has been used in studies in a wide range of applications, including, e.g., banking systems, the emergence of social norms, wireless sensor networks and protein interaction networks [13]–[16].

For this UCC-HIM, we develop an efficient estimation algorithm that exploits its symmetric topological property. Compared to the MLE and linear algebra-based estimators developed for general IMs, the proposed algorithm significantly reduces the computational complexity while maintains accuracy.

The rest of this letter is structured as follows. The fundamentals of the IM and the UCC-HIM problem formulation are introduced in Section II. In Section III, a reduced-order

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Markov chain is introduced to facilitate the analysis for UCC-HIM based on its special network topology. Then an efficient parameter estimation algorithm is developed through exploiting structures of the reduced-order Markov chain. In Section IV, two simulation studies are conducted to verify effectiveness of the proposed estimation algorithm. Section V concludes this letter.

II. PRELIMINARIES OF THE INFLUENCE MODEL

A. The influence Model (IM)

An IM is composed of N interacting sites. Each site i has M_i possible statuses, where $i \in \{1, 2, \ldots, N\}$. A scalar $s_i[k] \in \{1, 2, \ldots, M_i\}$ indexes the status of site i at time k. $S_i[k]$, a row vector of length M_i, also denotes site i's status at time k, where all entries are filled with '0's except a '1' at the location corresponding to the status index $s_i[k]$. For example, $S_i[k] = [1\ 0\ \cdots\ 0]$ when $s_i[k] = 1$.

At each time step, site i updates its status based on the network influence matrix $D \in \mathbb{R}^{N\times N}$ and local Markov chain transition matrix $A_{ji} \in \mathbb{R}^{M_j\times M_i}$, where $j \in \{1, 2, \ldots, N\}$. D and A_{ji} are row stochastic matrices. The 4-step update rule is summarized as follows.

1) Choose site j as site i's determining site with probability $d_{i,j}$, where $d_{i,j}$ is the element of D denoting the probability that i is influenced by j.
2) Calculate $p_{ij}[k+1] \in \mathbb{R}^{1\times M_i}$, the probability of site i's next status, based on the current status of site j as $p_{ij}[k+1] = S_j[k]A_{ji}$, where each element $a_{m,n}$ of A_{ji} is the conditional probability for site i's next status to be n given that site j's current status is m.
3) Determine $P_i[k+1] \in \mathbb{R}^{1\times M_i}$, the probability mass function of site i's next status by considering the influence of all sites on site i as
$$P_i[k+1] = \sum_{j=1}^{N} d_{i,j}p_{ij}[k+1] = \sum_{j=1}^{N} d_{i,j}S_j[k]A_{ji}. \quad (1)$$
4) $S_i[k+1]$ is then obtained by realizing $P_i[k+1]$, i.e., sampling random numbers according to the distribution $P_i[k+1]$.

Cascading $S_i[k]$ and $P_i[k+1]$ into row vectors $S^H[k]$ and $P^H[k+1]$ of length $\sum_{i=1}^{N} M_i$, we have
$$S^H[k] = \begin{bmatrix} S_1[k] & S_2[k] & \cdots & S_N[k] \end{bmatrix}, \quad (2)$$
$$P^H[k+1] = \begin{bmatrix} P_1[k+1] & P_2[k+1] & \cdots & P_N[k+1] \end{bmatrix}. \quad (3)$$

Then the above 4-step update rule leads to the IM dynamics succinctly captured by the two following iterative equations,
$$P^H[k+1] = S^H[k]H, \quad (4)$$
$$S^H[k+1] = MultiRealize(P^H[k+1]), \quad (5)$$
where $S^H[k+1]$ is obtained by realizing each $P_i[k+1]$ respectively, and $H \in \mathbb{R}^{\sum_{i=1}^{N} M_i\times\sum_{i=1}^{N} M_i}$ is the influence matrix defined as:
$$H \triangleq \begin{bmatrix} d_{1,1}A_{11} & \ldots & d_{N,1}A_{1N} \\ \vdots & \vdots & \vdots \\ d_{1,N}A_{N1} & \ldots & d_{N,N}A_{NN} \end{bmatrix}. \quad (6)$$

If all sites have the same number of statuses, M, and $A_{ji} = A$ for all i and j, the IM is referred to as the homogeneous influence model (HIM), with the state vector $S^H[k]$ of length MN. The dimension of state grows linearly with network size. A state matrix $S^G[k] \in \mathbb{R}^{N\times M}$ is further introduced to capture the HIM state in its matrix form
$$S^G[k] = \begin{bmatrix} S_1[k]' & S_2[k]' & \cdots & S_N[k]' \end{bmatrix}'. \quad (7)$$

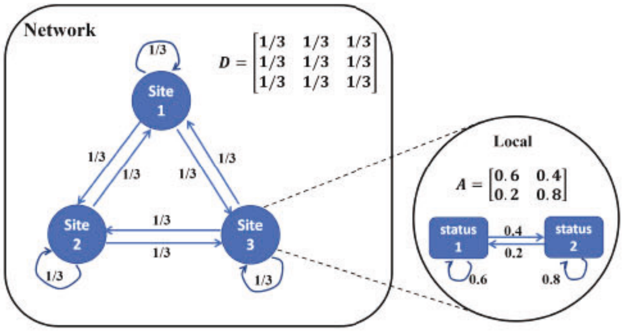

Fig. 1. An example of UCC-HIM of 3 sites and 2 statuses at each site.

The corresponding influence matrix in (6) is then simplified to $H \triangleq D' \otimes A$, where $\otimes$ is the Kronecker product, and the superscript $'$ denotes the transpose operation.

In this letter, we focus on the HIM with a canonical network topology, referred to as the uniform completely connected homogeneous influence model (UCC-HIM). In UCC-HIM, all sites are fully connected with the same mutual influence, i.e., $d_{i,j} = \frac{1}{N}\ \forall\ i, j \in \{1, 2, \ldots, N\}$. An example of UCC-HIM is shown in Figure 1. The UCC-HIM has practical values, e.g., it captures the voting behaviors in leaderless social networks and other types of network interactions in close proximity.

B. The Master Markov Chain Representation of IM

The dynamics of IM can also be captured by its equivalent master Markov chain [1]. As the name suggests, the master Markov chain representation uses the Markov properties of IM and constructs a big Markov chain with states as the combination of all site statuses. There are a total of M^N states in the master Markov chain representation. A scalar $s^g[k] \in \{1, 2, \ldots, M^N\}$ is adopted to index the states based on the statuses of all sites, i.e., $s_i[k]\ \forall\ i \in \{1, 2, \ldots, N\}$ as
$$s^g[k] = \sum_{i=1}^{N}(s_i[k] - 1)M^{N-i} + 1. \quad (8)$$

The event matrix $B \in \mathbb{R}^{M^N\times MN}$ captures all states of the master Markov chain [1]. The qth row of B is the state vector S^H corresponding to $s^g = q$. For example, for the UCC-HIM network in Figure 1 with 3 sites and 2 statuses for each site, i.e., $N = 3$ and $M = 2$, there are 8 states in the Markov chain in total. The state $s^g = 4$ corresponds to $s_1 = 1$, $s_2 = 2$ and $s_3 = 2$, and hence $S_1 = [1\ 0]$, $S_2 = [0\ 1]$, $S_3 = [0\ 1]$, to form $S^H = [1\ 0\ 0\ 1\ 0\ 1]$ in the 4th row of B,
$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The master Markov chain is characterized by its state transition matrix $G \in \mathbb{R}^{M^N \times M^N}$, which gives the conditional probability of its next state $s^g[k+1]$ given its current state $s^g[k]$. Let $S_p^G[k]$ and $S_q^G[k+1]$ denote the state matrices corresponding to $s^g[k] = p$ and $s^g[k+1] = q$, respectively. The elements of G can be obtained in a succinct form as:

$$g_{p,q} = P(s^g[k+1] = q | s^g[k] = p) = \prod_{n=1}^{N} \prod_{m=1}^{M} z_{n,m}, \quad (9)$$

where $z_{n,m}$ is the nth row and mth column element of matrix $Z_{pq} \in \mathbb{R}^{N \times M}$.

$$Z_{pq} = \left(D S_p^G[k] A \right)^{\circ S_q^G[k+1]}. \quad (10)$$

The superscript $\circ$ denotes the element-wise exponential operator. In particular, for two matrices X and Y with the same dimension, the mth row and nth column element of the outcome of $X^{\circ Y}$, $(X^{\circ Y})_{m,n}$, is calculated as ${X_{m,n}}^{Y_{m,n}}$. Let $P^G[k+1] = D S_p^G[k] A$, the ith row of $P^G[k+1]$ indicates $P_i[k+1]$ with $i \in \{1, \ldots, N\}$ according to (1). Z_{pq} is constructed to obtain the probabilities of all sites at the local statuses captured in $S_q^G[k+1]$. $g_{p,q}$ is obtained by multiplying these probabilities since the statuses of all sites evolve independently. More details can be found in [7].

The comparison between the influence model dynamics (4) and its Master Markov chain representation (9) clearly shows the effectiveness of the influence model. The HIM of MN states captures the dynamics of its equivalent master Markov chain of M^N states.

C. Problem Formulation

Despite the tractability of the influence model due to its reduced-order formulation, model estimation cannot easily be achieved in an effective way. The master Markov chain G has been used as a step for the estimation of the IM [7]. However, as expected, the computation involved in the master Markov chain-based estimation approach increases exponentially with the network size. The large computational cost of G limits its practical use, especially for large networks.

In this letter, we take a structural approach to study a class of IM, namely, the UCC-HIM, and provide an efficient parameter estimation algorithm. The problem is formulated as follows.

Problem: Consider a UCC-HIM of N sites with M statuses for each site. Given L independent observation sequences $O = \{O_1, O_2, \ldots, O_L\}$ with $O_i = [S^H[1]_i, S^H[2]_i, \ldots, S^H[K]_i]$, where the initial network state $S^H[1]_i$ can be arbitrary, estimate the underlying local transition matrix A with $L, K \to \infty$.

III. PARAMETER ESTIMATION FOR THE UNIFORM COMPLETELY CONNECTED HOMOGENEOUS INFLUENCE MODEL (UCC-HIM)

In this section, we develop an efficient parameter estimation algorithm for the UCC-HIM. We first study the identifiability. Then we construct a reduced-order Markov chain through exploiting the symmetric network topology of the UCC-HIM. The mapping relationship between the reduced-order Markov chain and the master Markov chain is illustrated next. The analysis of the reduced-order Markov chain leads to an efficient and accurate estimation algorithm.

A. The Identifiability of the UCC-HIM

Lemma 1 [7]: The influence model is identifiable from the observation sequences O with $L, K \to \infty$, if and only if the underlying parameters A and D can be uniquely determined from the master Markov chain G.

Lemma 2 [7]: A can be uniquely determined from G as

$$a_{m,n} = \sqrt[N]{g_{\sum_{i=1}^{N}(m-1)M^{N-i}+1, \sum_{j=1}^{N}(n-1)M^{N-j}+1}}, \quad (11)$$

where $a_{m,n}$ is the mth row and nth column entry of A and $\sum_{i=1}^{N}(m-1)M^{N-i}+1$ is the master Markov chain's state with all sites in the same local status m.

Based on the above lemmas, we prove the identifiability of the UCC-HIM in Theorem 1.

Theorem 1: Any UCC-HIM is identifiable.

Proof: For a UCC-HIM of N sites, D is determined and takes the form of $d_{i,j} = \frac{1}{N}$ $\forall\, i, j \in \{1, 2, \ldots, N\}$. Because A can be determined uniquely from G according to Lemma 2, the theorem is proved naturally according to Lemma 1. ■

B. The Reduced-Order Markov Chain R

From (11), we see that the transition matrix of the master Markov chain G is needed to obtain A. The dimension of G is M^N, making the estimation computation grow exponentially with network size. To efficiently estimate A, we introduce a reduced-order Markov chain R by first showing its states and then the transition matrix.

The reduced-order Markov chain records the number of sites in each status. We adopt a length-M vector $S^R[k] = [r_1 \ \cdots \ r_m \ \cdots \ r_M]$ to denote the state of the reduced-order Markov chain R at time k, where r_m is the number of sites whose local statuses are m. Hence $0 \le r_m \le N$ and $\sum_{m=1}^{M} r_m = N$. In other words, $S^R[k] = \sum_{i=1}^{N} S_i[k]$. By counting all the possible $S^R[k]$, the reduced-order Markov chain R has r states, where

$$r = \left(\!\!\binom{M}{N}\!\!\right) = \binom{M+N-1}{N} = \frac{(M+N-1)!}{(M-1)!N!}. \quad (12)$$

The notation $((\cdot))$ and $(\cdot)$ denote the multiset and combination operations, respectively. r is the number of ways to assign M statuses to the N sites, with repetitions allowed and ordering disregarded. See [17, p. 71], for the details of this multiset operation. To index the r states $S^R[k]$, we introduce a scalar $s^r[k] \in \{1, 2, \ldots, r\}$. Given $S^R[k]$, $s^r[k]$ is calculated as

$$s^r[k] = \sum_{i=1}^{M-1} \sum_{j=0}^{N-1-\sum_{m=1}^{i} r_m} \binom{M-i+j-1}{j} + 1. \quad (13)$$

Note that in the summation, if a term's upper bound is less than the lower bound, the term is zero [18].

For the example of $N = 3$ and $M = 2$ in Figure 1, the states of R, $S^R[k]$, are [3 0], [2 1], [1 2] and [0 3]. $r = \frac{(2+3-1)!}{(2-1)!3!} = 4$ in this case. The state $S^R[k] = [1 \ 2]$ is indexed with $s^r[k] = 3$.

The transition matrix $R \in \mathbb{R}^{r \times r}$ of the reduced-order Markov chain indicates the conditional probability mass functions (PMFs) of its next state given its current state. Let $S_p^R[k]$ and $S_q^R[k+1]$ denote the state vectors corresponding to $s^r[k] = p$ and $s^r[k+1] = q$, respectively. R can be obtained according to the following theorem.

Theorem 2: For a UCC-HIM with the network influence matrix D and local transition matrix A, each element of the reduced-order Markov chain R is calculated as

$$R_{p,q} = P(s^r[k+1] = q|s^r[k] = p) = C_q \prod_{m=1}^{M} \left(X_{pq}\mathbf{e}_m\right), \tag{14}$$

where $1 \le p, q \le r$, C_q is the qth entry of length r column vector C, X_{pq} is a length M row vector, and $\mathbf{e}_m$ is a length M column vector of zeros, except a single '1' at its mth entry.

$$C_q = \frac{N!}{\prod_{m=1}^{M}\left(S_q^R[k+1]\mathbf{e}_m\right)!}, \tag{15}$$

$$X_{pq} = \left(\frac{1}{N}S_p^R[k]A\right)^{\circ S_q^R[k+1]}. \tag{16}$$

Proof: To obtain the conditional probability of $s^r[k+1] = q$ given $s^r[k] = p$, we start with the conditional probability of individual sites' next statuses given $s^r[k] = p$. Let $P_{i|p}[k+1]$ denote the conditional PMF of site i's next status given $s^r[k] = p$ and $S_p^R[k]\mathbf{e}_m$ indicates the mth entry of $S_p^R[k]$. According to the influence model's 4-step update rule in Section II-A, at time k, site i chooses a site in status m as its determining site with probability $\frac{1}{N}$, where $m \in \{1, \ldots, M\}$. Then, the probability of its next status in n based on the status of the determining site can be calculated as $\frac{1}{N}a_{m,n}$. With $S_p^R[k]\mathbf{e}_m$ sites in status m and $m \in \{1, \ldots, M\}$, the probability of site i's next status in n, $P(s_i[k+1] = n|s^r[k] = p)$, can be calculated as $\sum_{m=1}^{M} \frac{S_p^R[k]\mathbf{e}_m}{N}a_{m,n}$. Hence we have

$$P_{i|p}[k+1] = \begin{bmatrix} P(s_i[k+1] = 1|s^r[k] = p) \\ P(s_i[k+1] = 2|s^r[k] = p) \\ \vdots \\ P(s_i[k+1] = M|s^r[k] = p) \end{bmatrix}' = \begin{bmatrix} \sum_{m=1}^{M} \frac{S_p^R[k]\mathbf{e}_m}{N}a_{m,1} \\ \sum_{m=1}^{M} \frac{S_p^R[k]\mathbf{e}_m}{N}a_{m,2} \\ \vdots \\ \sum_{m=1}^{M} \frac{S_p^R[k]\mathbf{e}_m}{N}a_{m,M} \end{bmatrix}' = \frac{1}{N}S_p^R[k]A. \tag{17}$$

Because of the symmetric network topology of the UCC-HIM, i.e., all the elements in D are identical, each site i shares the same conditional PMF of their next status given $s^r[k] = p$, which is $P_{i|p}[k+1]$.

Because the statuses of all sites evolve independently, $R_{p,q}$ can be obtained as follows. First, choose $S_q^R[k+1]\mathbf{e}_1$ sites from N sites and assign them local status 1 with probability $P(s_i[k+1] = 1|s^r[k] = p)^{(S_q^R[k+1]\mathbf{e}_1)}$. Next, choose $S_q^R[k+1]\mathbf{e}_2$ sites from $N - S_q^R[k+1]\mathbf{e}_1$ sites and assign them local status 2 with probability $P(s_i[k+1] = 2|s^r[k] = p)^{(S_q^R[k+1]\mathbf{e}_2)}$. This process continues, and eventually, choose $S_q^R[k+1]\mathbf{e}_M$ sites from the rest sites and assign them local status M with probability $P(s_i[k+1] = M|s^r[k] = p)^{(S_q^R[k+1]\mathbf{e}_M)}$. Hence we have

$$\begin{aligned} R_{p,q} &= P(s^r[k+1] = q|s^r[k] = p) \\ &= \binom{N}{S_q^R[k+1]\mathbf{e}_1}(P_{i|p}[k+1]\mathbf{e}_1)^{\left(S_q^R[k+1]\mathbf{e}_1\right)} \\ &\quad \binom{N - S_q^R[k+1]\mathbf{e}_1}{S_q^R[k+1]\mathbf{e}_2}(P_{i|p}[k+1]\mathbf{e}_2)^{\left(S_q^R[k+1]\mathbf{e}_2\right)} \cdots \\ &\quad \binom{N - \sum_{l=1}^{M-1}\left(S_q^R[k+1]\mathbf{e}_l\right)}{S_q^R[k+1]\mathbf{e}_M}(P_{i|p}[k+1]\mathbf{e}_M)^{\left(S_q^R[k+1]\mathbf{e}_M\right)} \\ &= \prod_{m=1}^{M}\binom{N - \sum_{l=1}^{m-1}\left(S_q^R[k+1]\mathbf{e}_l\right)}{S_q^R[k+1]\mathbf{e}_m}(P_{i|p}[k+1]\mathbf{e}_m)^{\left(S_q^R[k+1]\mathbf{e}_m\right)} \\ &= \frac{N!}{\prod_{m=1}^{M}(S_q^R[k+1]\mathbf{e}_m)!}\prod_{m=1}^{M}\left(\left(\frac{1}{N}S_p^R[k]A\right)^{\circ S_q^R[k+1]}\right)\mathbf{e}_m \\ &= C_q\prod_{m=1}^{M}(X_{pq}\mathbf{e}_m). \end{aligned} \tag{18}$$

The penultimate equality is established based on the fact that $\sum_{l=1}^{M} S_q^R[k+1]\mathbf{e}_l = N$ and $(N-N)! = 0! = 1$. ■

C. The Mapping Relationship Between R and G

In this section, we show the mapping relationship between R and its corresponding master Markov chain G. We start with constructing the state-transfer vector T which captures the relationship between the states of R and G, and then show that R and G have a one-to-one mapping relationship.

Lemma 3: The M^N states in the master Markov chain G and the r states in the reduced-order Markov chain R have a many-to-one mapping, captured by the state-transfer vector $T \in \mathbb{R}^{M^N \times 1}$ whose row index is s^g while each entry indicates the corresponding s^r. T can be obtained by calculating the scalar index of each row in B_u using (13) where

$$B_u = B(\mathbf{1}_N \otimes I_M). \tag{19}$$

$\mathbf{1}_N$ is an all-one column vector of length N, and I_M is an M-dimensional identity matrix.

$$B = [B_1, \ldots, B_i, \ldots, B_N], \tag{20}$$

$$B_i = \mathbf{1}_{M^{i-1}} \otimes I_M \otimes \mathbf{1}_{M^{N-i}}. \tag{21}$$

Proof: Each row of the event matrix B indicates one of the M^N states of the master Markov chain G. B_u is constructed as in (19) to store the number of sites at each local status for all the rows in B, i.e., each row of B_u indicates one of the r states $S^R[k]$ of the reduced-order Markov chain R with repetition. Then the entries of T, $s^r[k]$, can be obtained using (13) given each row of B_u, $S^R[k]$. ■

Note that the number of repetitions of $S^R[k]$ in B_u is captured by the length r column vector C in Theorem 2, named as the state-count vector herein.

For example, in the $N = 3$ and $M = 2$ network shown in Figure 1,

$$B_u = \begin{bmatrix} 3 & 2 & 2 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \end{bmatrix}',$$

$$C = \begin{bmatrix} 1 & 3 & 3 & 1 \end{bmatrix}',$$

and

$$T = \begin{bmatrix} 1 & 2 & 2 & 3 & 2 & 3 & 3 & 4 \end{bmatrix}'.$$

$s^g[k] = 2$, 3, and 5 all correspond to $s^r[k] = 2$.

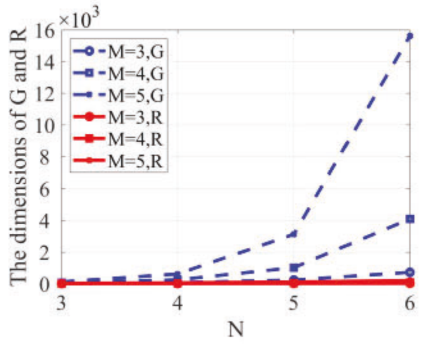

Fig. 2. The dimensions of G and R with different N and M.

Theorem 3: For a UCC-HIM with the network influence matrix D and local transition matrix A, G and R have a one-to-one mapping as:

$$R = WGV, \tag{22}$$

$$G = VRU, \tag{23}$$

where $V \in \mathbb{R}^{M^N \times r}$ is constructed based on the state-transfer vector T as

$$V_{k,l} = \begin{cases} 1 & if\ l = T_k \\ 0 & otherwise \end{cases} \quad 1 \leq k \leq M^N, 1 \leq l \leq r. \tag{24}$$

$W \in \mathbb{R}^{r \times M^N}$ is obtained by transposing V and then leaving only the left first '1' in each row and setting the rest '1's to '0'. $U \in \mathbb{R}^{r \times M^N}$ is constructed based on V and the state-count vector C by $U_{l,k} = \frac{V_{k,l}}{C_l}$.

Proof: We show the proof by explaining the construction of the auxiliary matrices V, W and U.

Because all elements of the network influence matrix D are identical, we find that in the state-transfer vector T, the indices of the identical elements indicate the column indexes and row indexes with identical state transition probabilities in G. To obtain R based on G, V is constructed to add up G's columns according to T. W is constructed to delete the repeated rows of G according to T. Equation (22) is thus proven.

Reversely, the state transition probability in G can also be uniquely determined by R because of the symmetric network structure. To obtain G from R, V is constructed to duplicate R's rows according to T, and U is constructed to expand the information in R's columns according to T and the state-count vector C. To reconstruct the columns of G, we only need to divide the columns in R with corresponding states by their numbers of repetitions in G which are recorded in C. Therefore, we have (23). ■

According to Theorem 3, R is a reduced-order representation of G for the UCC-HIM, obtained by discarding and merging redundant information in G. Because G is uniquely determined from the observation sequences, it also implies that R is unique.

The dimension of the reduced Markov chain R is $r = \frac{(M+N-1)!}{(M-1)!N!}$, which is far less than the dimension of master Markov chain G, M^N. Figure 2 shows a comparison between the dimensions of the master Markov chain G and the reduced-order Markov chain R. With the increase of network size N and the number of statuses M, the dimension of G increases dramatically, while the dimension growth of R is very slow, indicating the significant dimension and computational cost reduction using R.

D. The Estimation Algorithm for A Based on R

In this subsection, we develop an efficient estimation algorithm for A from observation sequences of the UCC-HIM.

First, we show that the reduced-order Markov chain R can be uniquely constructed from observation sequences by counting the corresponding state transition frequencies based on the law of large numbers [19]. Because G can be uniquely determined from the observation sequences, and G and R have a one-to-one mapping, R can also be uniquely determined. This result is summarized in the following lemma.

Lemma 4: Given the observation sequences of a UCC-HIM, O, with $L, K \to \infty$, the reduced-order Markov chain R can be uniquely constructed.

The next theorem shows the estimation for A based on the reduced-order Markov chain R.

Theorem 4: Given the reduced-order Markov chain R of the UCC-HIM, the elements of A can be uniquely determined as

$$a_{m,n} = \sqrt[N]{R_{r+1-\sum_{k=0}^{M-m}\binom{N+k-1}{k},\, r+1-\sum_{k=0}^{M-n}\binom{N+k-1}{k}}}. \tag{25}$$

Proof: According to (13), $r + 1 - \sum_{k=0}^{M-m}\binom{N+k-1}{k}$ denotes the state of the reduced-order Markov chain R where all sites are in local status m, i.e., $S^R[k] = [0 \ \cdots \ N \ \cdots \ 0]$, with all positions filled with '0's except an 'N' at the mth position. According to (14), (16) and (15), we have

$$\begin{aligned} &R_{r+1-\sum_{k=0}^{M-m}\binom{N+k-1}{k},\, r+1-\sum_{k=0}^{M-n}\binom{N+k-1}{k}} \\ &= 1\ a_{m,1}^0 \cdots a_{m,n}^N \cdots a_{m,M}^0 = a_{m,n}^N. \end{aligned} \tag{26}$$

Therefore, (25) is derived. ■

Note that the estimation of A from R has the same accuracy as the estimation from G according to (11) and (25).

IV. SIMULATION STUDIES

To demonstrate the results developed in Section III, two simulation studies are conducted. Example 1 verifies the practicability and efficiency of the estimation algorithm. Example 2 is a real-world application that models the decision making process in a social network using the UCC-HIM.

A. Example 1: Estimation of A in the UCC-HIM

To verify efficiency of the proposed estimation algorithm, we compare the performance of our algorithm with the estimation algorithm using the master Markov chain approach. We consider a UCC-HIM with 5 sites and 3 statuses for each sites. The network influence matrix D has element $d_{i,j} = \frac{1}{5}\ \forall i, j \in \{1, \ldots, 5\}$. The local transition matrix A is given as

$$A = \begin{bmatrix} 0.36 & 0.32 & 0.32 \\ 0.33 & 0.40 & 0.27 \\ 0.35 & 0.25 & 0.40 \end{bmatrix}.$$

An observation sequence of length 400, 000 is generated. Then G and R are computed by finding the state transition frequencies respectively. According to Theorem 4, the estimated local transition matrix $\hat{A}$ through R is

$$\hat{A} = \begin{bmatrix} 0.3347 & 0.3395 & 0.3258 \\ 0.3083 & 0.4106 & 0.2812 \\ 0.3132 & 0.2538 & 0.4330 \end{bmatrix},$$

which is identical to the estimation through G. The mean squared errors (MSE) for $\hat{A}$ is 4.6860×10^{-4}, showing the accuracy of the estimation. The execution time of our reduced-order algorithm based on R is about half of that based on G, indicating the improved efficiency.

B. Example 2: Support or Oppose?

In social network models, agents interact with each other and update their opinions based on rules that capture the influences from other agents [20]. When no individual takes the role of opinion leaders and all individuals exert the same influences to the whole team, the opinion propagation can be captured as the UCC-HIM.

In a leaderless social network, discussion and voting are two common steps in a decision-making process. In this example, a group of 5 people meet to discuss whether to act on a problem. There are two statuses for each person, support or oppose. Their opinions may change during the discussion due to the influences they receive from others. The mutual influences in the group are identical. The ones who oppose have a greater impact than those who support. After thorough discussions, people vote for the final decision. We can model this process using a UCC-HIM. For example, $d_{i,j} = \frac{1}{5}\ \forall i, j \in \{1, \ldots, 5\}$ and $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}$ due to the tendency to oppose.

According to Theorem 2, the reduced-order Markov chain R has 6 states and can be computed as

$$R = \begin{bmatrix} 0.1681 & 0.3602 & 0.3087 & 0.1323 & 0.0283 & 0.0024 \\ 0.0778 & 0.2592 & 0.3456 & 0.2304 & 0.0768 & 0.0102 \\ 0.0313 & 0.1562 & 0.3125 & 0.3125 & 0.1562 & 0.0313 \\ 0.0102 & 0.0768 & 0.2304 & 0.3456 & 0.2592 & 0.0078 \\ 0.0024 & 0.0283 & 0.1323 & 0.3087 & 0.3602 & 0.1681 \\ 0.0003 & 0.0064 & 0.0512 & 0.2048 & 0.4096 & 0.3277 \end{bmatrix}.$$

The state s^r ranges from 1 to 6, with $s^r - 1$ denoting the number of people who oppose. Using the reduced Markov chain R, we can effectively predict the final voting result. The steady-state distribution of R is determined by the left eigenvector of R associated with the eigenvalue '1', which is [0.0217 0.0978 0.2162 0.2976 0.2565 0.1102]. Hence, the probability of 3 people opposing is the largest (0.2976). The probability that more than 2 people oppose is 0.6643, indicating that more than half of the people are more likely to oppose this action eventually. According to the majority rule, the final decision is more probable to be 'oppose'. If using the master Markov chain to obtain the steady-state results, the eigen-analysis of a 32×32 Markov chain is required, which incurs more computation.

V. CONCLUSION

In this letter, we study the reduced-order estimation of IM. For UCC-HIM, a canonical class of IM, we prove that it is identifiable. Then we construct a reduced-order Markov chain R to facilitate the estimation study. The dimension of R is far less than the master Markov chain G. We find the one-to-one mapping between R and G. By using R, an efficient parameter estimation algorithm for A is developed. Compared with the master Markov chain approach, the same accuracy is achieved but with significant reduction of computational load. Simulation studies verify efficiency of our proposed parameter estimation algorithm and demonstrate its practical value in real applications. In the future, we will explore reduced-computation solutions for more general IMs.

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