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ALGEBRAIC FIBER SPACES AND CURVATURE OF HIGHER DIRECT IMAGES

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Abstract Let $p : X \to Y$ be an algebraic fiber space, and let L be a line bundle on X. In this article, we obtain a curvature formula for the higher direct images of $\Omega^i_{X/Y} \otimes L$ restricted to a suitable Zariski open subset of X. Our results are particularly meaningful if L is semi-negatively curved on X and strictly negative or trivial on smooth fibers of p. Several applications are obtained, including a new proof of a result by Viehweg–Zuo in the context of a canonically polarized family of maximal variation and its version for Calabi–Yau families. The main feature of our approach is that the general curvature formulas we obtain allow us to bypass the use of ramified covers – and the complications that are induced by them.

Keywords: higher direct imags of relative canonical bundle; Hodge theory; posititivity; curvture; singular metrics

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Contents

1. Introduction

Let $p : X \to Y$ be a surjective holomorphic map of relative dimension n between two non-singular, compact Kähler manifolds. We denote by $\Delta \subset Y$ the union of codimension one components of the set of singular values of p, and we will use the same notation for the associated divisor. The reduced divisor corresponding to the support of the inverse $p^{-1}(\Delta)$ will be denoted by W. In order to formulate our results, we will assume from the very beginning that Δ and W have simple normal crossings.

We denote by $\Omega^1_X\langle W\rangle$ and $\Omega^1_Y\langle\Delta\rangle$ the logarithmic cotangent bundle associated to (X, W) and (Y, Δ), respectively (cf. [11] and the references therein). The pullback of forms with log poles along Δ gives an injective map of sheaves

$$p^\star\Omega^1_Y\langle\Delta\rangle \to \Omega^1_X\langle W\rangle. \tag{1.1}$$

Let $\Omega^1_{X/Y}\langle W\rangle$ be the cokernel of map (1.1).
We consider a Hermitian line bundle (L, h_L) on X. The metric h_L is assumed to be smooth when restricted to the generic fibers of p. The curvature of (L, h_L) will be denoted by Θ_L.

In this article, our primary goal is to explore the differential geometric properties of the higher direct images

$$\mathcal{R}^i p_\star\left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right), \tag{1.2}$$

where $n \geqslant i \geqslant 1$ together with some applications. We will proceed in two steps.

(A) *Restriction to the set of regular values* First, we will analyze the properties of (1.2) when restricted to the complement of a closed analytic subset of Y, which we now describe. By a fundamental result of Grauert [2], sheaves (1.2) are coherent; thus they are locally free on a Zariski open subset U of Y.

The set of singular values of the map p will be denoted by $\Sigma \subset Y$. Then the complement $\Sigma \setminus \Delta$ is a union of algebraic subsets of Y of codimension at least two. For each $i =$

$1, \dots, n$, we consider the function

$$t \to h^i(X_t, \Omega^{n-i}_{X_t} \otimes L_t) \tag{1.3}$$

defined on $Y_0 := Y \setminus \Sigma$. These functions are upper semi-continuous; hence they are constant on a Zariski open subset of Y_0 (in the analytic topology), which will be denoted by $\mathcal{B}$.

The set $p^{-1}(\mathcal{B}) \subset X$ will be denoted by $\mathcal{X}$, and we will use the same symbol $p : \mathcal{X} \to \mathcal{B}$ for the restriction to $\mathcal{X}$ of our initial map. It is a proper holomorphic submersion, and the restriction of (1.2) to $\mathcal{B}$ is equal to

$$\mathcal{R}^i p_\star\big(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\big)|_{\mathcal{B}} = \mathcal{R}^i p_\star\big(\Omega^{n-i}_{\mathcal{X}/\mathcal{B}} \otimes L\big). \tag{1.4}$$

(1.4) thus defines a vector bundle, say $\mathcal{H}^{i,n-i}$, over $\mathcal{B}$. Since we have now restricted to the set where (1.3) is constant, its fiber at t is isomorphic to the Dolbeault cohomology space $H^i(X_t, \Omega^{n-i}_{X_t} \otimes L_t)$ (see [2, Theorem 4.12]). In § 3, we will give another description of this vector bundle, which is more suitable for our computations.

Let $[u] \in \Gamma(V, \mathcal{H}^{i,n-i})$ be a local holomorphic section of (1.4), where $V \subset \mathcal{B}$ is a coordinate open subset and u is a $\bar\partial$-closed $(0,i)$-form with values in the bundle $\Omega^{n-i}_{\mathcal{X}/\mathcal{B}} \otimes L|_{p^{-1}(V)}$. The restriction u_t of u to X_t is a $\bar\partial$-closed L_t-valued $(i, n-i)$-form on X_t. We denote by $[u_t]$ its $\bar\partial$-cohomology class. Let ω be a smooth $(1,1)$-form on $\mathcal{X}$, whose restriction to the fibers of p is positive definite. Then one may use $\omega|_{X_t}$ and h_L to define a Hermitian norm on $\mathcal{H}^{i,n-i}$. More precisely, the norm of a class $[u_t]$ will be defined as the L^2-norm of its unique $\bar\partial$-harmonic representative, say $u_{t,h}$, i.e.,

$$\|[u]\|^2_t := \int_{\mathcal{X}_t} |u_{t,h}|^2 e^{-\varphi} \omega^n/n!. \tag{1.5}$$

By a fundamental result of Kodaira–Spencer, the variation of $u_{t,h}$ with respect to t is smooth.

Since the bundle $\Omega^{n-i}_{\mathcal{X}/B} \otimes L$ is a quotient of $\Omega^{n-i}_{\mathcal{X}} \otimes L$, we can construct (cf. § 2) a smooth $(0,i)$-form $\widetilde{u}$ with values in $\Omega^{n-i}_{\mathcal{X}} \otimes L|_{p^{-1}(V)}$, which is $\bar\partial$-closed on the fibers of p and whose restriction to each X_t belongs to the cohomology class $[u_t]$. In what follows, $\widetilde{u}$ will be called a *representative* of $[u]$.

Summing up, $\mathcal{H}^{i,n-i}$ is a holomorphic Hermitian vector bundle. The sign of the corresponding curvature form is of fundamental importance in applications. Certainly, the curvature tensor is an intrinsic object, completely determined by the complex and the Hermitian structure of the bundle. Nevertheless, it can be expressed in several ways, depending on the choice of the representatives $\widetilde{u}$ mentioned above. So an important question would be to choose 'the best' representative for the holomorphic sections of $\mathcal{H}^{i,n-i}$.

We first assume that the curvature form Θ_L of (L, h_L) is semi-negative on X and strictly negative on the fibers of $p : \mathcal{X} \to \mathcal{B}$. In this case, we can take $\omega := -\Theta_L$. Another consequence of this assumption is the existence of a *vertical representative* corresponding to any section u (cf. § 5). The vertical representatives are used in the proof of our first main result, namely Theorem 6.1. We show that the curvature of $\mathcal{H}^{i,n-i}$ evaluated in a direction u can be written as the difference of two semi-positive forms on the base $\mathcal{B}$. Roughly speaking, the positive contribution in the expression of the curvature is given by the contraction (or cup product) with the Kodaira–Spencer class.

Rather than stating Theorem 6.1 here, we mention one of its consequences, which will be important in applications. Let $\mathcal{K}^i$ be the kernel of the map

$$\mathcal{R}^i p_\star\big(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\big) \to \mathcal{R}^{i+1} p_\star\big(\Omega^{n-i-1}_{X/Y}\langle W\rangle \otimes L\big) \otimes \Omega^1_Y\langle\Delta\rangle \tag{1.6}$$

defined by contraction with the Kodaira–Spencer class. By shrinking $\mathcal{B}$, we can assume that the restriction $\mathcal{K}^i|_{\mathcal{B}}$ is a sub-bundle of $\mathcal{R}^i p_\star\big(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\big)|_{\mathcal{B}}$. We have the following statement.

Theorem 1.1. *We assume that $\Theta_L \leqslant 0$ on X and that $\Theta_L|_{\mathcal{X}_t} < 0$ for any $t \in \mathcal{B}$. Then the bundle $\mathcal{K}^i|_{\mathcal{B}}$ endowed with the metric induced from (1.5) is semi-negative in the sense of Griffiths, provided that we choose $\omega|_{\mathcal{X}_t} = -\Theta_L|_{\mathcal{X}_t}$.*

We list next a few other results related to Theorem 6.1. For example, assume that the canonical bundle $K_{\mathcal{X}_t}$ of $\mathcal{X}_t$ is ample (i.e., the family p is canonically polarized). Then, by a crucial theorem of Aubin and Yau [1, 47], we can construct a metric $h_t = e^{-\psi_t}$ on $K_{\mathcal{X}_t}$ such that $\omega_t = \sqrt{-1}\partial\bar\partial\psi_t$ is Kähler–Einstein. By a result of Schumacher [32], the metric on the relative canonical bundle $K_{\mathcal{X}/\mathcal{B}}$ induced by the family $\big(e^{-\psi_t}\big)_{t\in\mathcal{B}}$ is semi-positively curved. We then consider the induced metric on $L := -K_{\mathcal{X}/\mathcal{B}}$. Our formula in this case coincides with earlier results of Siu [34], Schumacher [32] and To–Yeung [37], and following the arguments in [32, 37] (with some variations), we show that this leads to the construction of a metric of strictly negative sectional curvature on the base manifold Y.

During the (long) preparation of this paper, the article [26] by Naumann was posted on arXiv. In this work, Theorem 6.1 is also established. Naumann shows that the expression of the curvature of $\mathcal{H}^{i,n-i}$ can be derived by a method similar to Schumacher's [32]. Our approach here is very different and perhaps lighter technically. It can be seen as a generalization of the computations in [3, 4] for the case of $\mathcal{H}^{n,0}$ (or better say, $\mathcal{H}^{0,n}$).

In § 7, we adapt the previous arguments to the case when the curvature of the bundle L is not strictly negative but instead identically zero on the fibers and semi-negative on the total space. We then establish a variant of Theorem 6.1 in this setting, Theorem 7.5. This formula generalizes the classical formula of Griffiths for the case when L is trivial. As it turns out, the only difference between our formula and that of Griffiths is an additional term coming from the curvature of L on the total space. In previous work, Nannicini [25] has generalized Siu's computations from [34] for $\mathcal{H}^{n-1,1}$ under related conditions when L is the relative canonical bundle. Assuming also that the fibers have a trivial canonical bundle, Wang [44] has simplified Nannicini's argument by using Griffiths's theorem instead of Siu's method. Here we follow the same argument as Wang, replacing Griffiths's theorem by Theorem 7.5 to be able to treat also the case when the canonical bundles of the fibers are only assumed to be flat. We also mention that the general case, when the curvature of L is semi-negative on fibers, is treated in a somewhat different way by the third author in [45].

A projective family $p : X \to Y$ is called *Calabi–Yau* if $c_1(X_y) = 0$, which is equivalent to the fact that some multiple of the canonical bundle of the generic fibers is trivial. In this case, $L = -K_{\mathcal{X}/\mathcal{B}}$ is fiberwise flat, and we show that L has a metric that is moreover semi-negative on the total space. We say that p has *maximal variation* if the Kodaira–Spencer map is injective on $Y \setminus \Sigma$.

The notion of *Kobayashi hyperbolicity* describes in a very accurate manner whether a complex manifold contains 'large' disks. In this setting, we have the following statement, obtained as a corollary of our curvature formulas combined with a few ideas from [32, 37].

Theorem 1.2. *Let $p : X \to Y$ be a canonically polarized or Calabi–Yau family. We assume that p has maximal variation. Then there is no non-constant holomorphic curve $f : \mathbb{C} \to Y \setminus \Sigma$.*

Our result 1.2 is a direct consequence of a more general statement we establish in § 8.3. We note that Theorem 1.2 (and a few other results we prove in this article) has also been announced by Yeung and To in their joint project [43].

(B) *Extension across the singularities.* The kernel

$$\mathcal{K}^i \subset \mathcal{R}^i p_\star\big(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\big) \tag{1.7}$$

of map (1.6) is a coherent sheaf of $\mathcal{O}_Y$-modules whose sections are called *quasi-horizontal* in [14, p. 26]. Unlike in the case $L = \mathcal{O}_X$, it is not clear whether $\mathcal{K}^i$ is torsion-free or not (cf. [49]). Fortunately, this is not a problem for us. We define

$$\mathcal{K}^i_f := \mathcal{K}^i / T(\mathcal{K}^i), \tag{1.8}$$

where $T(\mathcal{K}^i) \subset \mathcal{K}^i$ is the torsion subsheaf. It turns out that the metric induced on $\mathcal{K}^i_f|_{\mathcal{B}}$ is also semi-negatively curved in the sense of Griffiths (the quotient map $\mathcal{K}^i \to \mathcal{K}^i_f$ is an isometry).

Our goal is to show that the metric defined on the bundles $\mathcal{K}^i_f|_{\mathcal{B}}$ extends as a semi-negatively curved singular Hermitian metric (in the sense of [5, 28]) on the torsion-free sheaf $\mathcal{K}^i_f$, provided that the metric h_L satisfies one of the two conditions below.

($\mathcal{H}_1$) We have $\Theta_L \leqslant 0$ on X, and moreover $\Theta_L|_{X_y} = 0$ for each y in the complement of some Zariski closed set.

($\mathcal{H}_2$) We have $\Theta_L \leqslant 0$ on X, and moreover there exists a Kähler metric ω_Y on Y such that we have $\Theta_L \wedge p^\star\omega_Y^m \leqslant -\varepsilon_0\omega \wedge p^\star\omega_Y^m$ on X (where m is the dimension of the base).

Thus the first condition requires L to be semi-negative and trivial on fibers, whereas in ($\mathcal{H}_2$), we assume that L is uniformly strictly negative on fibers in the sense that we have

$$\Theta_{h_L}(L)|_{X_y} \leqslant -\varepsilon_0\omega|_{X_y} \tag{1.9}$$

for any regular value y of the map p.

In this context, we have the following result.

Theorem 1.3. *Let $p : X \to Y$ be an algebraic fiber space, and let (L, h_L) be a Hermitian line bundle that satisfies hypothesis $(\mathcal{H}_i)$ above. We assume that the restriction of h_L to the generic fiber of p is non-singular. Then, we have the following:*

(i) *For each $i \geqslant 1$, the sheaf $\mathcal{K}_f^i$ admits a semi-negatively curved singular Hermitian metric.*

(ii) *We assume that the curvature form of L is smaller than $-\varepsilon_0 p^\star \omega_Y$ on the p-inverse image of some open subset $\Omega \subset Y$ of Y. Then the metric on $\mathcal{K}_f^i$ is strongly negatively curved on Ω (and semi-negatively curved outside a codimension greater than two subset of Y).*

As a consequence of point (i), we infer, e.g., that the dual sheaf $\mathcal{K}_f^{i\star}$ is weakly semi-positive in the sense of Viehweg; cf. [7, 40, 49]. We note that similar statements appear in various contexts in algebraic geometry; cf. [7, 13, 16, 18, 20, 22, 23, 29, 30, 41, 42, 49] among many others. The fundamental results of Cattani *et al.* [9] are an indispensable tool in the arguments of most of the articles quoted above. Unfortunately, the analogue of the period mapping in our context is not defined (because of the twisting with the bundle L); so we do not have these techniques at our disposal.

Part of the motivation for the analysis of the curvature properties of the kernels $\mathcal{K}^i$ is the existence of a non-trivial map

$$\mathcal{K}_f^{i\star} \to \mathrm{Sym}^i\, \Omega_Y^1\langle\Delta\rangle$$

for some i such that $\mathcal{K}_f^{i\star} \neq 0$ provided that $L^\star := \det \Omega_{X/Y}^1\langle W\rangle$ and p is either Calabi–Yau or canonically polarized with maximal variation. This is a well-known and very important fact. We refer to [29, 42] and the references therein for related results.

The point is that here we construct the so-called *Viehweg–Zuo sheaf* $\mathcal{K}^{i\star}$ in a very direct and explicit manner without the use of ramified covers as in the original approach. Indeed, in the case of a canonically polarized or Calabi–Yau family p with maximal variation, we show in § 9 that the bundle $\det \Omega_{X/Y}^1\langle W\rangle$ has a property that is very similar to the hypothesis $(\mathcal{H}_i)$ above. Hence Theorem 1.3 applies, and we obtain a new proof of the existence of the Viehweg–Zuo sheaf; cf. [42]. Our arguments work as well in the case of Calabi–Yau families, where we obtain a similar result. We remark that the proof is much simpler than in the canonically polarized case.

On the other hand, in article [29], Popa–Schnell obtain a vast generalization of the result by Viehweg–Zuo [42] in the case of a family p whose generic fiber is of general type. To this end, they use deep results from the theory of Hodge modules. It would be very interesting to see if our explicit considerations here could provide an alternative argument.

This paper is organized as follows.

2. Preliminaries

As in the introduction, we let $p : \mathcal{X} \to \mathcal{B}$ be a smooth proper fibration of relative dimension n over an m-dimensional base $\mathcal{B}$. We let $L \to \mathcal{X}$ be a holomorphic line bundle, equipped with a smooth metric ϕ. Mostly, we will assume that ϕ has semi-negative curvature and that $\Omega : -i\partial\bar\partial\phi$ is strictly positive on each fiber X_t. Sometimes we let ω_t denote the restriction of Ω to the fiber X_t; sometimes we just write ω.

We consider maps $t \to u_t$ defined for t in $\mathcal{B}$, where u_t is an n form on X_t for each t. We call such a map a *section* of the bundle of n-forms on the fiber. We will need to define what it means for such a map to be smooth.

If $t = (t_1, \dots, t_m)$ are local coordinates, we can, via the map p, consider t_j to be functions on $\mathcal{X}$, and similarly we have the forms dt_j, $d\bar t_j$, $dt = dt_1 \wedge \cdots dt_m$ and $d\bar t$ on $\mathcal{X}$. That p is a smooth fibration means that its differential is surjective everywhere. Therefore, locally near a point x in $\mathcal{X}$, we can complete the t_j's with functions $z_1, \dots, z_n$ to get local coordinates on $\mathcal{X}$. Then, for t fixed, z_j are local coordinates on X_t, and we can write

$$u_t = \sum_{|I|=p,|J|=q} c_{I,J}(t,z)\, dz_I \wedge d\bar z_J.$$

We say that u_t is smooth if the coefficients $c_{I,J}$ are smooth. Since this amounts to saying that the form on $\mathcal{X}$

$$u_t \wedge dt \wedge d\bar t$$

is smooth, it does not depend on the choice of coordinates.

Lemma 2.1. *A section u_t is smooth if and only if there is a smooth n-form, $\tilde u$ on $\mathcal{X}$, such that the restriction of $\tilde u$ to each X_t equals u_t.*

Proof. Locally, this is clear from the discussion above: we may take $\tilde u = \sum c_{I,J}\, dz_I \wedge d\bar z_J$. The global case follows via a partition of unity. □

These definitions extend to forms with values in L. We will call a form $\tilde u$ as in the lemma a *representative* of the section u_t.

Lemma 2.2. *Two forms, $\tilde u$ and $\tilde u'$, on $\mathcal{X}$ have the same restriction to the fiber X_t, i.e., represent the same section u_t, if and only if*

$$(\tilde u - \tilde u') \wedge dt \wedge d\bar t = 0,$$

on X_t, where $dt = dt_1 \wedge \cdots dt_m$ for some choice of local coordinates on the base. This holds if and only if

$$\tilde u' = \tilde u + \sum_1^m a_j \wedge dt_j + \sum_1^m b_j \wedge d\bar t_j$$

for some $(n-1)$-forms a_j and b_j on $\mathcal{X}$.

Proof. Both statements follow by choosing local coordinates (t, z) as above. □

Notice that the forms a_j and b_j are not uniquely determined, but their restriction to the fibers are uniquely determined. This follows since if

$$\sum a_j \wedge dt_j + \sum b_j \wedge d\bar t_j = 0,$$

e.g., wedging with $\widehat{dt_j} \wedge d\bar t$, we get

$$a_j \wedge dt \wedge d\bar t = 0,$$

which means that a_j vanishes on fibers by the lemma.

We will mainly be interested in the case when u_t are $\bar\partial$-closed of a fixed bidegree (p,q) and in their cohomology classes $[u_t]$ in $H^{p,q}(L_t)$. By Hodge theory, each such class has a unique harmonic representative for the $\bar\partial$-Laplacian $\Box''$ defined by the fiber metric ϕ (restricted to X_t) and the Kähler form ω_t. Since $\Box''=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$ and X_t is compact, a form u is harmonic if and only if $\bar\partial u=0$ and $\bar\partial^* u=0$. In the next lemma, we give an alternative characterization of harmonic forms that will be useful in our computations.

Lemma 2.3. *Let X be an n-dimensional compact complex manifold and $L\to X$ a holomorphic line bundle with a negatively curved smooth metric ϕ, i.e., $\omega:=-i\partial\bar\partial\phi>0$ on X. Let u be a (p,q)-form on X with values in L, where $p+q=n$. Then u is harmonic for the Kähler metric ω on X and the metric ϕ on L if and only if $\bar\partial u=0$ and $\partial^\phi u=0$, where $\partial^\phi=e^\phi\partial e^{-\phi}$ is the $(1,0)$-part of the connection on L induced by the metric ϕ. Moreover, a harmonic (p,q)-form with $p+q=n$ is primitive, i.e., satisfies $\omega\wedge u=0$.*

Proof. Let $\Box'=\partial^\phi(\partial^\phi)^*+(\partial^\phi)^*\partial^\phi$ be the $(1,0)$-Laplacian. By the Kodaira–Nakano formula,

$$\Box''=\Box'+[i\partial\bar\partial\phi,\Lambda_\omega].$$

Since $i\partial\bar\partial\phi=-\omega$ and the commutator $[\omega,\Lambda_\omega]=0$ on n-forms, we see that $\Box'=\Box''$ on n-forms. Hence, $\partial^\phi u=0$ if u is harmonic, which proves one direction in the lemma. We also have, when $\bar\partial u=\partial^\phi u=0$, that $\bar\partial\partial^\phi u+\partial^\phi\bar\partial u=0$, which gives that $\omega\wedge u=0$; so u is primitive.

Then $*u=cu$, where $c\in\mathbb{C}$ such that $|c|=1$; so when $\partial^\phi u=0$, $\bar\partial^* u=-*\partial^\phi * u=0$. Hence $\Box'' u=0$ if $\bar\partial u=0$ and $\partial^\phi u=0$, which proves the converse direction of the lemma. □

A similar result of course holds (with the same proof) if L is positive and we use $\omega=i\partial\bar\partial\phi$ as our Kähler metric.

At this point, we also insert an estimate that will be useful later when we simplify the curvature formula. It generalizes a formula in [4] (see § 4 in that paper).

Proposition 2.4. *With notation and assumptions as in Lemma 2.3, let g be an L-valued form of bidegree (p,q) and f an L-valued form of bidegree $(p+1,q-1)$, where $p+q=n$. Assume*

$$\partial^\phi g=\bar\partial f$$

and that $f\perp Range(\bar\partial)$ and $g\perp Ker(\partial^\phi)$. Then

$$\|f\|^2-\|g\|^2=\langle(\Box''+1)^{-1}f,f\rangle.$$

Moreover, $g\in R(\Box''-1)$ and

$$\|f\|^2-\|g\|^2=\langle(\Box''-1)^{-1}g,g\rangle.$$

Proof. Since g is orthogonal to the space of harmonic forms, g lies in the domain of $(\Box'')^{-1}$. Since the spectrum of $\Box''$ is discrete, g also does lie in the domain of $(\Box''-\lambda)^{-1}$ for λ sufficiently small and positive. We have

$$\begin{aligned}\lambda\langle(\Box''-\lambda)^{-1}g,g\rangle+\|g\|^2&=\langle\Box''(\Box''-\lambda)^{-1}g,g\rangle\\&=\langle\Box'(\Box'-\lambda)^{-1}g,g\rangle=\langle(\Box'-\lambda)^{-1}\partial^\phi g,\partial^\phi g\rangle=\langle(\Box'-\lambda)^{-1}\bar\partial f,\bar\partial f\rangle.\end{aligned}$$

Here we have used in the first equality on the second line that $\Box' = \partial^\phi(\partial^\phi)^* + (\partial^\phi)^*\partial^\phi$ and g is orthogonal to $R(\partial^\phi)$. Since $\Box' = \Box'' + 1$ on forms of degree $(n+1)$, this equals

$$\langle \bar\partial^*\bar\partial(\Box'' + 1 - \lambda)^{-1} f, f\rangle = \langle \Box''(\Box'' + 1 - \lambda)^{-1} f, f\rangle,$$

where the last equality follows since f is orthogonal to $R(\bar\partial)$. Clearly, this expression stays bounded as λ increases to 1; so the expansion of g in eigenforms of $\Box''$ cannot have any terms corresponding to eigenvalues less than or equal to 1. Hence g lies in the domain of $(\Box'' - \lambda)^{-1}$ for all $0 \leqslant \lambda \leqslant 1$ and

$$\lambda\langle(\Box'' - \lambda)^{-1} g, g\rangle + \|g\|^2 = \langle \Box''(\Box'' + 1 - \lambda)^{-1} f, f\rangle.$$

Putting $\lambda = 1$, we get

$$\|g\|^2 + \langle(\Box'' - 1)^{-1} g, g\rangle = \|f\|^2,$$

and putting $\lambda = 0$, we get

$$\|g\|^2 = \langle \Box''(\Box'' + 1)^{-1} f, f\rangle = \|f\|^2 - \langle(\Box'' + 1)^{-1} f, f\rangle. \qquad \Box$$

Note that in the case of positive curvature, when $i\partial\bar\partial\phi = \omega$, we get with the same proof that

$$\|g\|^2 - \|f\|^2 = \langle(\Box'' + 1)^{-1} g, g\rangle,$$

when $\bar\partial f = \partial^\phi g$, g is orthogonal to the range of ∂^ϕ and f is orthogonal to the kernel of $\bar\partial$.

For future use in the case of Calabi–Yau families, we also record the following counterpart of the previous proposition, which is proved in a similar (but simpler) way.

Proposition 2.5. *Let (X, ω) be a compact Kähler manifold and $(L, e^{-\phi})$ a Hermitian holomorphic line bundle over X with $i\partial\bar\partial\phi = 0$. Let g be an L-valued form of bidegree (p, q) and f be L-valued of bidegree $(p+1, q-1)$, where $p+q = n$. Assume $f \perp Range(\bar\partial)$ and $g \perp Ker(\partial^\phi)$ and that*

$$\partial^\phi g = \bar\partial f.$$

Then,

$$\|f\|^2 = \|g\|^2.$$

The next lemma gives in particular a simple way to compute norms of harmonic forms.

Lemma 2.6. *Let $c_n = (i)^{n^2}$. If u is a primitive (p, q)-form, with $p+q = n$ on an n-dimensional Kähler manifold (X, ω),*

$$(-1)^q c_n u \wedge \bar u = |u|^2_\omega \omega^n/n!.$$

The proof can be found in [15].

This means that the L^2-norms of primitive forms u_t over a fiber X_t can be expressed in terms of the fiberwise integrals of $u_t \wedge \bar u_t$. When we differentiate such integrals with respect to t, it is convenient to express them as pushforwards.

Lemma 2.7. *If u_t is a smooth section of primitive (p,q)-forms on X_t, where $p+q=n$, we have*

$$\int_{X_t} |u_t|^2_{\omega_t} \omega_t^n/n!\, e^{-\phi} = (-1)^q c_n p_*(\tilde{u} \wedge \bar{\tilde{u}} e^{-\phi})$$

if $\tilde{u}$ is any representative of u_t.

We will also need the next lemma (follows from the fundamental theorem of Kodaira–Spencer in [19]).

Lemma 2.8. *Let $\mathcal{X} \to \mathcal{B}$ be a smooth proper fibration, and $L \to \mathcal{X}$ a holomorphic line bundle. Assume that the dimension of $H^{p,q}(L_t)$ is a constant. Let $t \to u_t$ be a smooth section of (p,q)-forms, $\bar\partial$-closed on X_t, and let $u_{t,h}$ be the harmonic representatives of the classes $[u_t]$ in $H^{p,q}(L_t)$. Then $t \to u_{t,h}$ is smooth.*

In the computations below, we will frequently use the *interior multiplication* of a form with a vector (field), defined as the adjoint of exterior multiplication under the natural duality between forms and the exterior algebra of tangent vectors. If V is a vector and θ is a form, it is denoted as $V\rfloor\theta$. Interior multiplication is an antiderivation, so $V\rfloor(a\wedge b) = (V\rfloor a)\wedge b + (-1)^{deg(a)} a \wedge (V\rfloor b)$. More generally, we can take the interior product of a wedge product of vectors with a form, and it holds that

$$(V_k \wedge \cdots V_1)\rfloor\theta = V_1\rfloor \cdots (V_k\rfloor\theta).$$

3. The bundles $\mathcal{H}^{n-q,q}$

We will now consider the vector bundles $\mathcal{H}^{n-q,q}$ over the base $\mathcal{B}$ whose fibers are

$$\mathcal{H}^{n-q,q}_t := H^{n-q,q}(L_t).$$

A smooth section of this bundle can be written as $t \to [u_t]$ such that $t \to u_{t,h}$ (see Lemma 2.8) is a smooth section of the bundle of $(n-q,q)$-forms on X_t as in Lemma 2.1 (when we write $[u_t]$, it always means that u_t is $\bar\partial$-closed on X_t). We will first give a way to express the $(0,1)$-part of the connection on $\mathcal{H}^{n-q,q}$.

For this, we let $\tilde{u}$ be any representative of the section $[u_t]$, i.e., an $(n-q,q)$-form on $\mathcal{X}$ whose restriction to each X_t is cohomologous to u_t. Since $\bar\partial\tilde{u} = 0$ on fibers, Lemma 2.2 shows that

$$\bar\partial\tilde{u} = \sum_1^m dt_j \wedge \eta_j + \sum_1^m d\bar{t}_j \wedge \nu_j$$

for some forms η_j and ν_j. Taking $\bar\partial$ of this equation and wedging with $\widehat{d\bar{t}_j} \wedge dt$, we see that $\bar\partial\nu_j = 0$ on fibers. We can then define

$$D''[u_t] = \sum [\nu_j|_{X_t}] \otimes d\bar{t}_j. \tag{3.1}$$

For this to make sense, we have to check that it does not depend on the various choices made. First, if $\tilde{u}'$ is smooth and restricts to each X_t to a form u_t' cohomologous to u_t, we can write $u_t' = u_t + \bar\partial v_t$, where v_t can be taken to depend smoothly on t. Then, if $\tilde{v}$ is a representative of v_t, then $\tilde{u}' - \bar\partial\tilde{v}$ is a representative of u_t. We therefore only have to check that the definition does not depend on the choice of representatives in the sense of Lemma 2.1. But if we change a representative $\tilde{u}$ by adding $\sum a_j \wedge dt_j + \sum b_j \wedge d\bar{t}_j$, ν_j changes only by the $\bar\partial$-exact term $\bar\partial b_j$; so the cohomology class $[\nu_j|_{X_t}]$ does not change. Thus, the definition of $D''[u_t]$ does not depend on the choice of representative of the section u_t nor on the choice of representative in the cohomology class.

Proposition 3.1. *The operator D'' defines an integrable complex structure on $\mathcal{H}^{n-q,q}$.*

Proof. Let $[u_t]$ be a smooth section and $\tilde{u}$ a representative of $[u_t]$. Then

$$D''[u_t] = \sum [\nu_j] \otimes d\bar{t}_j,$$

where

$$\bar\partial \tilde{u} = \sum dt_j \wedge \eta_j + \sum d\bar{t}_j \wedge \nu_j.$$

Since $\bar\partial \nu_j = 0$ on fibers, we get

$$\bar\partial \nu_j = \sum dt_k \wedge \eta_{jk} + \sum d\bar{t}_k \wedge \nu_{jk},$$

so

$$D''[\nu_j] = \sum [\nu_{jk}] \otimes d\bar{t}_k.$$

Using that $\bar\partial^2 \tilde{u} = 0$, we get that $\nu_{jk} = \nu_{kj}$ on fibers, which gives $(D'')^2[u_t] = 0$. □

We now return to the setting in the introduction and let $p : X \to Y$ be a surjective holomorphic map. We take $\mathcal{B}$ to be the Zariski open subset of Y, defined in the introduction (immediately after (1.3)). Then p is a proper submersion over $\mathcal{B}$, and the qth direct image

$$\mathcal{R}^q p_\star(\Omega^{n-q}_{X/Y} \otimes L)|_{\mathcal{B}} \tag{3.2}$$

is the sheaf of sections of a vector bundle with fibers $H^q(X_t, \Omega^{n-q}_{X_t} \otimes L_t)$. These spaces are naturally identified with the spaces $H^{n-q,q}(X_t, L_t)$, the fibers of the bundle $\mathcal{H}^{n-q,q}$. We will next show that the holomorphic structure introduced above on $\mathcal{H}^{n-q,q}$ coincides with the usual complex structure of the higher direct image.

By definition, the *holomorphic* sections of the vector bundle (3.2) on a coordinate open subset $V \subset \mathcal{B}$ are elements of

$$H^q\left(p^{-1}(V), \Omega^{n-q}_{\mathcal{X}/\mathcal{B}} \otimes L|_{p^{-1}(V)}\right). \tag{3.3}$$

Let (u_j) be a set of $\bar\partial$-closed $(0,q)$–forms with values in $\Omega^{n-q}_{\mathcal{X}/\mathcal{B}} \otimes L|_{p^{-1}(V)}$ such that their corresponding Dolbeault cohomology classes $[u_j]$ give a local frame for (3.2) over V. Then for each $t \in V$, the restriction $(u_j|_{X_t})$ induces a basis for $H^q(X_t, \Omega^{n-q}_{X_t} \otimes L)$. In particular, we infer that the space of smooth sections of (3.2) coincides with the space of smooth sections of $\mathcal{H}^{n-q,q}$ (as defined at the beginning of the current section) since we can express $[u_t]$ as a linear combination of $\left[u_j|_{X_t}\right]$ for $t \in V$.

Lemma 3.2. *Under the above identifications, the holomorphic structure on the bundle $\mathcal{H}^{n-q,q}$ coincides with the holomorphic structure on the qth direct image bundle.*

Proof. Let $[u]$ be one of the elements in the frame $([u_j])$. By restriction to fibers, it defines a smooth section $[u_t]$ of $\mathcal{H}^{q,n-q}$. We first show that

$$D''([u_t]) = 0. \tag{3.4}$$

This can be seen as follows. By a partition of unity, we can construct a smooth $(n-q,q)$-form $\widetilde{u}$ with values in L whose image by the quotient map

$$\Omega^{n-q}_{\mathcal{X}} \otimes L|_{p^{-1}(V)} \to \Omega^{n-q}_{\mathcal{X}/\mathcal{B}} \otimes L|_{p^{-1}(V)} \tag{3.5}$$

is equal to u. This implies that $\widetilde{u}$ is a representative of $[u]$, and moreover we have

$$\widetilde{u} \wedge dt = u \wedge dt. \tag{3.6}$$

Note that in (3.6), the expression $u \wedge dt$ is a well-defined $(n-q+m,q)$-form with values in L. Now we apply $\bar\partial$ to (3.6) and get

$$\bar\partial \widetilde{u} \wedge dt = 0 \tag{3.7}$$

since $\bar\partial u \wedge dt$ equals zero. Thus the ν_j defined by the representative $\tilde{u}$ are even zero, so $D''[u_t] = 0$. Hence any holomorphic section of the qth direct image bundle is holomorphic as a section of $\mathcal{H}^{n-q,q}$.

Let now $[u_t]$ be a smooth section of $\mathcal{H}^{n-q,q}$ such that (3.4) holds at each point of V. We use the holomorphic frame $[u_j]$ of $\mathcal{R}^q p_\star(\Omega^{n-q}_{X/Y} \otimes L)|_{\mathcal{B}}$. We have already shown that the $[(u_j)_t]$ satisfy $D''[(u_j)_t] = 0$, so they form a holomorphic frame for the bundle $\mathcal{H}^{n-q,q}$ as well. Then

$$[u_t] = \sum_j f_j[(u_j)_t] \tag{3.8}$$

for some smooth functions f_j defined on V. We have

$$0 = D''([u_t]) = \sum_j [(u_j)_t] \otimes \bar\partial f_j, \tag{3.9}$$

so f_j are holomorphic. Then clearly $[u_t]$ is holomorphic as a section of $\mathcal{R}^q p_\star(\Omega^{n-q}_{X/Y} \otimes L)|_{\mathcal{B}}$ as well. □

We next equip $\mathcal{H}^{p,q}$ with the Hermitian metric

$$\|[u_t]\|^2_t = \int_{X_t} |u_{t,h}|^2_\omega e^{-\phi} \omega^n/n!$$

(where we have written ω instead of ω_t), where $u_{t,h}$ is the harmonic representative of $[u_t]$. By Lemmas 2.3 and 2.6, this equals

$$(-1)^q c_n \int_{X_t} u_{t,h} \wedge \bar{u}_{t,h} e^{-\phi},$$

which by Lemma 2.7 also can be written as

$$\|[u_t]\|^2_t = (-1)^q c_n p_*(\tilde{u} \wedge \bar{\tilde{u}} e^{-\phi}) \tag{3.10}$$

for any choice of representative of the smooth section $u_{t,h}$.

Let now $\tilde{u}$ be any representative of a smooth section u_t such that each u_t is harmonic on X_t. By Lemma 2.3, this means that $\bar\partial u_t = 0$ on X_t and $\partial^\phi u_t = 0$ on X_t. We then have that

$$\partial^\phi \tilde{u} = \sum dt_j \wedge \mu_j + \sum d\bar{t}_j \wedge \xi_j,$$

by Lemma 2.2, since the left-hand side vanishes on fibers. We thus have two sets of equations for $\tilde{u}$,

$$\bar\partial \tilde{u} = \sum_1^m dt_j \wedge \eta_j + \sum_1^m d\bar{t}_j \wedge \nu_j \tag{3.11}$$

and

$$\partial^\phi \tilde{u} = \sum dt_j \wedge \mu_j + \sum d\bar{t}_j \wedge \xi_j. \tag{3.12}$$

Taking $\bar\partial$ of the first equation and wedging with appropriate forms on the base, we see that η_j and ν_j are $\bar\partial$-closed on fibers, and taking ∂^ϕ of the second equation, we see that μ_j and ξ_j are ∂^ϕ-closed. We stress that the forms η_j etc., and even their restrictions to fibers, depend on the choice of representative $\tilde{u}$, but the cohomology classes they define do not.

We now polarize (3.10) and use it to compute

$$\bar\partial \langle u_t, v_t\rangle_t = (-1)^q c_n \left(\sum p_*(d\bar{t}_j \wedge \nu_j \wedge \bar{\tilde{v}} e^{-\phi}) + \sum (-1)^n p_*(\tilde{u} \wedge \overline{dt_j \wedge \mu_j} e^{-\phi}) \right)$$

(the η_j's and the ξ_j's give no contribution for bidegree reasons).

From this, we see that $\sum [\mu_j] \otimes dt_j = D'[u_t]$; so we now also have an expression for the $(1,0)$-part of the connection on $\mathcal{H}^{p,q}$ in terms of representatives.

To compute the curvature of the connection $D = D' + D''$, we will compute $i\partial\bar\partial \|u_t\|_t^2$ and use the classical formula

$$-\langle \Theta^E_{W,\bar{W}} u_t, u_t\rangle_t + \|D'_W u_t\|_t^2 = \bar{W} \rfloor (W \rfloor \partial\bar\partial \|u_t\|_t^2),$$

which is valid for any holomorphic section of a Hermitian vector bundle E and any $(1,0)$ vector in the base, where

$$\Theta^E_{W,\bar{W}} := \bar{W} \rfloor (W \rfloor \Theta^E), \quad D'_W := W \rfloor D', \tag{3.13}$$

and the Chern curvature Θ^E denotes the square of the Chern connection of the Hermitian vector bundle E (in our case, $E = \mathcal{H}^{p,q}$). Here it will be convenient to choose our representatives in a special way; so we first discuss this issue in the following sections.

4. The horizontal lift

Recall that we assume that $\Omega = -i\partial\bar\partial\phi$ is strictly positive on fibers. In this situation, we can, following Schumacher [32], define the *horizontal lift* of a vector field on the base. This represents a variation of the notion of *harmonic lift*, introduced by Siu in [34].

Proposition 4.1. *Let W be a $(1,0)$ vector field on the base B. If $\Omega > 0$ on fibers, there is a unique vector field $V = V_W$ on $\mathcal{X}$ such that*

$$dp(V) = W \tag{4.1}$$

and

$$\Omega(V,\bar{U})=0 \tag{4.2}$$

for any field U on $\mathcal{X}$ that is vertical, i.e., satisfies $dp(U)=0$. V is called the horizontal lift of W.

Proof. We begin with uniqueness. Assume V and V' both satisfy (4.1) and (4.2). Then $V-V'$ is vertical, so

$$\Omega(V-V',\overline{V-V'})=\Omega(V,\overline{V-V'})-\Omega(V',\overline{V-V'})=0.$$

Since we have assumed that Ω is strictly positive on fibers, $V-V'=0$.

For the existence, we note that the proposition is a purely pointwise, linear algebra statement. Given an arbitrary lift V' at a point x in $\mathcal{X}$, a general lift can be written as $V=V'+U$, where U is vertical. We therefore have n equations for n unknowns; so uniqueness implies existence. □

Expressed differently, condition (4.2) means that

$$V\rfloor\Omega(\bar{U})=0$$

for all vertical U; so

$$V_W\rfloor\Omega=\sum b_j d\bar{t}_j =: b \tag{4.3}$$

for some functions b_j if t_j are local coordinates on the base. Clearly, b depends linearly on W; so if $W=\sum W_j\partial/\partial t_j$,

$$V_W\rfloor\Omega=i\sum c_{j,k}(\Omega)W_j\, d\bar{t}_k=V_W\rfloor C(\Omega),$$

where

$$C(\Omega)=i\sum c_{j,k}(\Omega)\, dt_j\wedge d\bar{t}_k.$$

(Here we have also used that $V_W\rfloor dt_j=W_j$.) We shall see below that $C(\Omega)$ is globally well defined. We say that a $(1,0)$ vector (field) on $\mathcal{X}$ is horizontal if it is the horizontal lift of some vector (field) on the base. The dimension of the space of horizontal vectors at a point is the dimension of the base, and the intersection between the horizontal vectors and the vertical vectors is just the zero vector. Hence any vector at a point in $\mathcal{X}$ has a unique decomposition as a sum of a vertical and a horizontal vector. Similarly, a $(0,1)$ vector (field) V is horizontal, respectively, vertical if $\bar{V}$ is horizontal or vertical. We next introduce the corresponding notions for forms.

Definition 1. A form θ is *horizontal* if $U\rfloor\theta=0$ for any vertical U, and θ is *vertical* if $V\rfloor\theta=0$ for any horizontal vector (field) V.

To make this slightly more concrete, let V_j be the horizontal lift of $\partial/\partial t_j$, where t_j are local coordinates on the base for $j=1,\dots,m$. Let W_k for $k=1,\dots,n$ be a system of vertical $(1,0)$-fields such that V_j and W_k together form a basis for the space of $(1,0)$ vectors near a given point x in $\mathcal{X}$. Then we may find local coordinates (s,z) near x such that $W_k=\partial/\partial z_k$ for $k=1,\dots,n$ and $V_j=\partial/\partial s_j$ for $j=1,\dots,m$ at x. (Then, still at x,

$ds_j(V_k) = \delta_{jk}$ and $ds_j(W_k) = 0$, from which it follows that $ds_j = p^*(dt_j)(= dt_j)$.) Then $V_j \rfloor dz_k = 0$; so all dz_k are vertical forms at x, and similarly $W_k \rfloor ds_j = 0$, and so $dt_j = ds_j$ are horizontal (which is easy to see directly). Therefore the horizontal forms are of the type

$$\sum a_{J,K} dt_J \wedge d\bar{t}_K,$$

and vertical forms are of the type

$$\sum b_{J,K} dz_J \wedge d\bar{z}_K.$$

In this terminology, $C(\Omega)$ is horizontal, and the relation $V_W \rfloor \Omega = V_W \rfloor C(\Omega)$ for all horizontal W implies that $\Omega = C(\Omega) + \Omega_v$, where Ω_v is vertical. This implies that $C(\Omega)$ does not depend on the choice of local coordinates. Indeed, if $C'(\Omega)$ is defined using different local coordinates, $C(\Omega) - C'(\Omega)$ is both horizontal and vertical and hence zero.

Next, let $V_1, V_2, \ldots$. be the horizontal lifts of $\partial/\partial t_1, \partial/\partial t_2, \ldots$, where t_j are local coordinates on the base. Let $\mathcal{V} = V_m \wedge \cdots V_1$ and put for an arbitrary form θ on $\mathcal{X}$,

$$P_v(\theta) = (\bar{\mathcal{V}} \wedge \mathcal{V}) \rfloor (dt \wedge d\bar{t} \wedge \theta). \tag{4.4}$$

Clearly, $P_v(\theta)$ is always vertical and does not depend on the choice of local coordinates. Moreover, if θ is already vertical, then $P_v(\theta) = \theta$; so P_v is a projection from the space of all forms on $\mathcal{X}$ to the space of vertical forms.

Proposition 4.2. *If θ is an arbitrary form on $\mathcal{X}$, $P_v(\theta) - \theta$ vanishes on fibers. If a vertical form vanishes on fibers, it vanishes identically.*

Proof. The first statement means that

$$P_v(\theta) \wedge dt \wedge d\bar{t} = \theta \wedge dt \wedge d\bar{t}.$$

This follows from the definition of P_v if we use that contraction is an antiderivation and $(\bar{\mathcal{V}} \wedge \mathcal{V}) \rfloor (dt \wedge d\bar{t}) = 1$.

For the second statement, we use that if θ is vertical, $P_v(\theta) = \theta$. On the other hand, $P_v(\theta) = 0$ if θ vanishes on fibers since $dt \wedge d\bar{t} \wedge \theta = 0$ then. □

Finally, we note that since $C(\Omega)^{m+1} = 0$ and $\Omega_v^{n+1} = 0$,

$$\Omega^{n+m}/(n+m)! = C(\Omega)^m/m! \wedge \Omega_v^n/n! = C(\Omega)^m/m! \wedge \Omega^n/n!.$$

Thus, if $m = 1$ and we write $C(\Omega) = c(\Omega) i\, dt \wedge d\bar{t}$, we have the following proposition.

Proposition 4.3. *If $m = 1$,*

$$c(\Omega) = \frac{\Omega^{n+1}/(n+1)!}{\Omega^n/n! \wedge i\, dt \wedge d\bar{t}}.$$

This is the well-known geodesic curvature of the curve of metrics ϕ_t in Mabuchi space (if ϕ_t depends only on the real part of t, and by extension in general).

5. Vertical representatives

Let $[u]$ be a section of the bundle of smooth n-forms on the fibers. Recall that this means that for each t in the base, u_t is an n-form on the fiber X_t, and there is a smooth n-form $\tilde{u}$ on the total space $\mathcal{X}$ (a representative of u_t), which restricts to u_t on each X_t.

Proposition 5.1. *Any smooth section u_t has a unique vertical representative.*

Proof. Let $\tilde{u}$ be an arbitrary representative and let $\hat{u} := P_v(\tilde{u})$. Then Proposition 4.2 implies that $\hat{u}$ is also a representative of u_t (since $\hat{u} - \tilde{u}$ vanishes on fibers), and $\hat{u}$ is by definition vertical. Uniqueness also follows from Proposition 4.2. □

In the next proposition, we shall consider sections that are primitive on fibers, i.e., are such that on each X_t, $u_t \wedge \Omega = 0$. In terms of representatives, this means that

$$\Omega \wedge \tilde{u} \wedge dt \wedge d\bar{t} = 0$$

if $dt = dt_1 \wedge \cdots dt_m$ for a system of local coordinates on the base.

Proposition 5.2. *If $\hat{u}$ is a vertical representative of a section that is primitive on fibers, then*

$$\Omega \wedge \hat{u} = C(\Omega) \wedge \hat{u}.$$

Proof. The form

$$\theta := (\Omega - C(\Omega)) \wedge \hat{u}$$

is a product of vertical forms and hence vertical. Since $C(\Omega)$ vanishes on fibers and $\hat{u}$ is primitive on fibers, θ vanishes on fibers. Hence $\theta = 0$ by Proposition 4.2. □

Note that this means in particular that $\Omega \wedge \hat{u} \wedge dt = \Omega \wedge \hat{u} \wedge d\bar{t} = 0$ since $C(\Omega) \wedge dt = C(\Omega) \wedge d\bar{t} = 0$.

Proposition 5.3. *Let $\hat{u}$ be a vertical representative of a section u_t that satisfies the equation*

$$(\bar{\partial} + \partial^{\phi}) u_t = 0$$

on each fiber X_t. Let $\mathcal{D} := (\bar{\partial} + \partial^{\phi})$ on the total space $\mathcal{X}$. Then

$$\mathcal{D}\hat{u} = \sum a_j \wedge dt_j + \sum b_j \wedge d\bar{t}_j,$$

where a_j and b_j are primitive and satisfy $(\bar{\partial} + \partial^{\phi}) a_j = (\bar{\partial} + \partial^{\phi}) b_j = 0$ on fibers.

Proof. Since $(\bar{\partial} + \partial^{\phi})\hat{u}$ vanishes on fibers,

$$\mathcal{D}\hat{u} = \sum a_j \wedge dt_j + \sum b_j \wedge d\bar{t}_j$$

for some n-forms a_j and b_j. Since $\mathcal{D}^2 = \Omega\wedge$, we get

$$\Omega \wedge \hat{u} = \sum \mathcal{D}a_j \wedge dt_j + \sum \mathcal{D}b_j \wedge d\bar{t}_j.$$

Wedging with $\widehat{dt_j} \wedge d\bar{t}$, we find that the restriction of $\mathcal{D}a_j$ to fibers vanishes since $\hat{u} \wedge \Omega \wedge d\bar{t} = 0$, and similarly, we find that $\mathcal{D}b_j$ vanishes on fibers. Hence $\mathcal{D}^2 a_j = \mathcal{D}^2 b_j$ also vanish when restricted to fibers, which gives that a_j and b_j are primitive. □

Note the particular case when u_t has pure bidegree (p, q). Then the assumption means that $\bar\partial\hat u = \partial^\phi \hat u = 0$ on fibers, i.e., that u_t is harmonic on each fiber. Then the first conclusion is that

$$\bar\partial\hat u = \sum_1^m dt_j \wedge \eta_j + \sum_1^m d\bar t_j \wedge \nu_j \tag{5.1}$$

and

$$\partial^\phi \hat u = \sum dt_j \wedge \mu_j + \sum d\bar t_j \wedge \xi_j, \tag{5.2}$$

where all forms η_j, ν_j, μ_j and ξ_j are primitive. This follows from Proposition 5.3 since e.g., $(-1)^n a_j = \eta_j + \mu_j$ and η_j and μ_j have different bidegrees so that each of them must be primitive.

We also get

$$\partial^\phi \eta_j = -\bar\partial \mu_j \tag{5.3}$$

and

$$\partial^\phi \nu_j = -\bar\partial \xi_j \tag{5.4}$$

from $(\bar\partial + \partial^\phi)a_j = 0$ and $(\bar\partial + \partial^\phi)b_j = 0$.

5.1. Vertical representatives and the Kodaira–Spencer tensor

We first recall the definition of the Kodaira–Spencer class. Let W be a holomorphic $(1, 0)$-vector field on $\mathcal{B}$ and let V be a smooth lift of W to $\mathcal{X}$ (so that $dp(V) = W$). In general, unless the fibration is locally trivial, we cannot find a holomorphic lift, and the Kodaira–Spencer class is an obstruction to this. First, we define

$$\kappa_V = \bar\partial V$$

if V is any smooth lift of W. Then κ_V is a $(0, 1)$-form on $\mathcal{X}$ with coefficients in $T^{(1,0)}(\mathcal{X})$. Since $dp(V) = W$ is holomorphic, it actually takes values in the sub-bundle, $F = T(\mathcal{X}/\mathcal{B})$, of $T^{(1,0)}$ of vertical vectors, i.e., vectors V' such that $dp(V') = 0$. Any other lift, say V', of W can be written as $V + V''$, where V'' is vertical and

$$\kappa_{V'} = \kappa_V + \bar\partial V''.$$

Therefore, the cohomology class of κ_V in $H^{(0,1)}(\mathcal{X}, F)$ is well defined. We will call it $[\kappa_W]$. Similarly, we let $[\kappa^t_W]$ be the cohomology class of κ_V restricted to X_t.

In the sequel, we will assume that the lift of W is taken as the horizontal lift. If $W = \partial/\partial t_j$ for a given system of coordinates on the base, we will sometimes write V_j for the horizontal lift of $\partial/\partial t_j$ and just κ_j for κ_{V_j}. This depends of course on Ω; so it would be more proper to write $\kappa_j(\Omega)$, but we will use the lighter notation instead.

Locally, using local coordinates $x = (t, z)$ such that $p(x) = t$, κ_V can be written as

$$\kappa_V = \sum_1^n Z_j \otimes d\bar z_j + \sum_1^m T_k \otimes d\bar t_k,$$

where Z_j and T_k are vector fields tangential to fibers. Its restriction to a fiber X_t is

$$\kappa^t_V = \sum_1^n Z_j \otimes d\bar z_j,$$

where we interpret z_j as local coordinates on X_t. If θ is a form on $\mathcal{X}$, κ_V operates on θ by letting the vector part of κ_V operate by contraction, followed by taking the wedge product with the form part. The result is called $\kappa_V \cup \theta$, which we sometimes write as $\kappa_V.\theta$. In the same way, κ_V^t operates on forms on a fiber X_t. Note that since the vector parts of the Kodaira–Spencer forms are vertical, the cup product commutes with restriction to fibers: $(\kappa_V \cup \theta)|_{X_t} = \kappa_V^t \cup (\theta|_{X_t})$.

Proposition 5.4. *Let u_t be a smooth section of (p,q)-forms such that u_t is harmonic on each fiber X_t. Let η_j and ξ_j be defined as in (5.1) and (5.2), where $\tilde{u} = \hat{u}$ is the vertical representative. Then, on each fiber*

$$\eta_j = \kappa_j^t \cup u_t$$

and

$$\xi_j = \overline{\kappa_j^t} \cup u_t.$$

Proof. In the proofs here, we will use the easily verified formulas $\bar\partial(V \rfloor \theta) = (\bar\partial V).\theta - V \rfloor (\bar\partial \theta)$ and $\partial^\phi(\bar V \rfloor \theta) = (\partial \bar V).\theta - \bar V \rfloor (\partial^\phi \theta)$ for any form θ if V is of type $(1,0)$. (Note however that there are no similar formulas when V is of type $(0,1)$.)

We have $V_j \rfloor \hat{u} = 0$; so taking $\bar\partial$, we get

$$(\bar\partial V_j).\hat{u} = V_j \rfloor \bar\partial \hat{u} = \eta_j + \cdots ,$$

where the dots indicate forms that contain dt_k or $d\bar t_j$ that therefore vanish when restricted to fibers. Restricting to fibers, we then get (in view of the remarks above) that

$$\kappa_j^t \cup u_t = \eta_j.$$

For the second statement, we use that $\bar V_j \rfloor \hat{u} = 0$ and take ∂^ϕ. Then

$$(\partial \bar V_j) \rfloor \hat{u} = \bar V_j \rfloor \partial^\phi \hat{u} = \xi_j + \cdots ,$$

which gives that $\xi_j = \overline{\kappa_j^t} \cup \hat{u}$. □

The final result in this section is a reflection of the familiar fact that the Kodaira–Spencer class defines a holomorphic section of the bundle with fibers $H^{0,1}(T^{1,0}(X_t))$.

Proposition 5.5. *Let $[u_t]$ be a holomorphic section of the bundle $\mathcal{H}^{n-q,q}$. Let W be a holomorphic vector field on B. Then*

$$[\kappa_W^t \cup u_t]$$

is a holomorphic section of $\mathcal{H}^{n-q-1,q+1}$.

Proof. It is enough to prove this when $W = \partial/\partial t_j$ so that $\kappa_j^t \cup u_t = \eta_j$ by Proposition 5.4. That u_t is holomorphic means that we can find some representative $\tilde{u}$ such that

$$\bar\partial \tilde{u} = \sum dt_j \wedge \eta_j + \sum d\bar t_j \wedge \nu_j,$$

and each ν_j is zero on fibers, i.e., we can write

$$\nu_j = \sum dt_k \wedge a_j^k + \sum d\bar{t}_k \wedge b_j^k.$$

Taking $\bar\partial$ of $\bar\partial\tilde{u}$, we get that

$$0 = \sum dt_j \wedge \bar\partial\eta_j + \sum d\bar{t}_j \wedge \bar\partial\nu_j.$$

Thus each $\bar\partial\eta_j$ is zero on fibers, and we can write

$$\bar\partial\eta_j = \sum dt_k \wedge \eta_j^k + \sum d\bar{t}_k \wedge \nu_j^k.$$

Thus we have

$$\begin{aligned} 0 = &\sum dt_j \wedge dt_k \wedge \eta_j^k + \sum dt_j \wedge d\bar{t}_k \wedge \nu_j^k \\ &+ \sum dt_k \wedge d\bar{t}_j \wedge \bar\partial a_j^k + \sum d\bar{t}_k \wedge d\bar{t}_j \wedge \bar\partial b_j^k, \end{aligned}$$

which implies that $\nu_j^k = -\bar\partial a_k^j$ on fibers. Thus each ν_j^k are $\bar\partial$-exact on fibers and thus η_j defines a holomorphic section of $\mathcal{H}^{n-q-1,q+1}$. □

As an example of this, we consider the case when $L = -K_{\mathcal{X}/B}$. Then $\mathcal{H}^{n,0}$ has a canonical trivializing section, u_t^0, defined as follows. On any complex manifold X, the bundle $K_X - K_X$ is of course trivial and has a canonical trivializing section defined locally as $s = dz \otimes (dz)^{-1}$. Applying this to each fiber X_t, we get a trivializing section u_t^0 of the line bundle $\mathcal{H}^{n,0}$. Wedging with the corresponding trivializing section of the base, which is locally $dt \otimes (dt)^{-1}$, we get an $(n+m,0)$-form U^0 on $\mathcal{X}$ with values in $K_{\mathcal{X}/\mathcal{B}}^{-1} \otimes K_{\mathcal{B}}^{-1} = K_{\mathcal{X}}^{-1}$ (by the adjunction isomorphism). The form U^0 is easily seen to be the canonical trivializing section of $K_{\mathcal{X}} - K_{\mathcal{X}}$; in particular, it is holomorphic. This means that $\bar\partial u^0 \wedge dt = 0$, so $\bar\partial u^0 = \sum dt_j \wedge \eta_j$. Hence u^0 is a holomorphic section of $\mathcal{H}^{n,0}$.

Applying the proposition iteratively, we get holomorphic sections of $\mathcal{H}^{n-q,q}$ as

$$\kappa_{i_1}^t \cup \cdots \kappa_{i_q}^t \cup u_t^0$$

for any multi-index $I = (i_1, \ldots, i_q)$.

6. The curvature formulas

We now have all the ingredients to begin computing the curvature of $\mathcal{H}^{n-q,q}$ with the L^2-metric. Let u_t be a smooth section of this bundle, which now means that $\bar\partial u_t = 0$ on X_t for each t and $\partial^\phi u_t = 0$ on X_t for each t. We take $\hat{u}$ to be a vertical representative of the section u_t so that

$$\bar\partial\hat{u} = \sum dt_j \wedge \eta_j + \sum d\bar{t}_j \wedge \nu_j \tag{6.1}$$

and

$$\partial^\phi\hat{u} = \sum dt_j \wedge \mu_j + \sum d\bar{t}_j \wedge \xi_j. \tag{6.2}$$

Then ν_j and μ_j have bidegree (p,q), η_j have bidegree $(p-1,q+1)$ and ξ_j have bidegree $(p+1,q-1)$ (forms of negative degree should be interpreted as zero). We decompose each of these forms on each fiber into one harmonic part and one part that is orthogonal to harmonic forms so that e.g., $(\eta_j)_h$ is the harmonic part of η_j and $(\mu_j)_\perp$ is the part of μ_j that is orthogonal to harmonic forms.

Our main curvature formula is as follows.

Theorem 6.1. *Let $L \to \mathcal{X}$ be a line bundle with metric $e^{-\phi}$, where $i\partial\bar\partial\phi < 0$ on fibers, and define the Hermitian structure on $\mathcal{H}^{n-q,q}$ using the Kähler forms $\omega_t = -i\partial\bar\partial\phi|_{X_t}$ on fibers. Then*

$$\begin{aligned}\langle\Theta_{\partial/\partial t_j,\partial/\partial \bar t_k} u_t, u_t\rangle_t \\ = -\langle(\Box''+1)^{-1}(\mu_j)_\perp, (\mu_k)_\perp\rangle_t - \langle(\Box''+1)^{-1}(\xi_k), (\xi_j)\rangle_t + \langle(\eta_j)_h, (\eta_k)_h\rangle_t \\ - c_n(-1)^q \int_{X_t} c_{j,k}(\Omega) u_t \wedge \bar u_t e^{-\phi},\end{aligned}$$

where μ, ξ, η are defined in formulas (5.1) and (5.2), using a vertical representative of u_t.

Recall that by Proposition 5.4, $\eta_j = \kappa_j \cup u_t$, and that $(\eta_j)_h$ is the harmonic representative of η_j. Similarly, $\xi_j = \overline{\kappa_j} \cup u_t$. As for the term containing μ_j, we can use the second part of Proposition 2.4 and write

$$\langle(\Box''+1)^{-1}(\mu_j)_\perp, (\mu_k)_\perp\rangle = \langle(\Box''-1)^{-1}(\eta_j)_\perp, (\eta_k)_\perp\rangle.$$

In this way, we see that the whole formula for the curvature can be expressed in terms of u_t and κ.

Also, we have

$$\langle(\Box''-1)^{-1}(\eta_j)_\perp, (\eta_k)_\perp\rangle - \langle(\eta_j)_h, (\eta_k)_h\rangle_t = \langle(\Box''-1)^{-1}(\eta_j), (\eta_k)\rangle,$$

so the formula we obtain here is similar to the one in [26].

Note that all the terms in the curvature formula are negative except the one coming from the harmonic part of η. This 'bad term' is clearly zero when $(p, q) = (0, n)$, for bidegree reasons. It also vanishes if the harmonic part of κ vanishes so that the fibration is isotrivial. More generally, if $W = \sum a_j \partial/\partial t_j$ is a vector at a point t in the base, and the cohomology class defined by $[\kappa_W] \cup u_t = 0$, then $\sum a_j (\eta_j)_h = 0$. Therefore, we get that

$$\langle\Theta_{W,\bar W} u_t, u_t\rangle_t \leqslant 0.$$

In particular, if $\kappa_j \cup u_t = 0$ in cohomology for all j, then the full curvature operator acting on u_t is negative. This will play an important role in § 9.

In the proof of Theorem 6.1, we may assume that the base dimension m is equal to 1. Indeed, this means that the formula for the curvature form implicit in Theorem 6.1 holds when restricted to any disk, and then it must hold on all of the base. We start from the classical fact that if u_t is a holomorphic section of a Hermitian holomorphic vector bundle E, then

$$\frac{\partial^2}{\partial t \partial \bar t}\|u_t\|_t^2 = -\langle\Theta^E_{\partial/\partial t,\partial/\partial \bar t} u_t, u_t\rangle_t + \|D'_{\partial/\partial t} u_t\|_t^2. \tag{6.3}$$

Thus we need to compute $i\partial\bar\partial\|u_t\|_t^2$ when u_t is a holomorphic section and u_t is harmonic on fibers. We choose some representative $\tilde u$. By Lemma 2.7, we have

$$\|u_t\|_t^2 = (-1)^q c_n p_*(\tilde u \wedge \bar{\tilde u} e^{-\phi}).$$

Since the pushforward commutes with $\bar\partial$, we get

$$\bar\partial p_*(\tilde u \wedge \overline{\tilde u} e^{-\phi}) = p_*(\bar\partial \tilde u \wedge \overline{\tilde u} e^{-\phi}) + (-1)^n p_*(\tilde u \wedge \overline{\partial^\phi \tilde u} e^{-\phi}).$$

We claim that the first term vanishes. For this, we use that $\bar\partial \tilde u = dt \wedge \eta + d\bar t \wedge \nu$. The term containing η gives no contribution for bidegree reasons. For the second term, we use that ν is $\bar\partial$-exact on fibers, which gives that the fiber integral

$$\int_{X_t} \tilde u \wedge \bar\nu e^{-\phi} = 0$$

since $\tilde u = u_t$ on X_t is harmonic on X_t. Thus

$$\bar\partial p_*(\tilde u \wedge \overline{\tilde u} e^{-\phi}) = (-1)^n p_*(\tilde u \wedge \overline{\partial^\phi \tilde u} e^{-\phi}).$$

Taking ∂, we find

$$\partial\bar\partial p_*(\tilde u \wedge \overline{\tilde u} e^{-\phi}) = (-1)^n p_*(\partial^\phi \tilde u \wedge \overline{\partial^\phi \tilde u} e^{-\phi}) + p_*(\tilde u \wedge \overline{\bar\partial \partial^\phi \tilde u} e^{-\phi}) =: A + B.$$

For the first term, we recall that

$$\partial^\phi \tilde u = dt \wedge \mu + d\bar t \wedge \xi.$$

Clearly, the mixed terms containing $\mu \wedge dt \wedge \bar\xi \wedge dt$ vanish, so

$$A = [p_*(\mu \wedge \bar\mu e^{-\phi}) - p_*(\xi \wedge \bar\xi e^{-\phi})]\, dt \wedge d\bar t. \tag{6.4}$$

For the B-term, we use $\bar\partial\partial^\phi = -\partial^\phi\bar\partial + \partial\bar\partial\phi$. Hence

$$B = -p_*(\tilde u \wedge \overline{\partial^\phi \bar\partial \tilde u} e^{-\phi}) - p_*(\partial\bar\partial\phi \wedge \tilde u \wedge \overline{\tilde u} e^{-\phi}).$$

As we have seen,

$$p_*(\tilde u \wedge \overline{\bar\partial \tilde u} e^{-\phi}) = 0.$$

Taking $\bar\partial$ of this, we get

$$p_*(\bar\partial \tilde u \wedge \overline{\bar\partial \tilde u} e^{-\phi}) = (-1)^{n+1} p_*(\tilde u \wedge \overline{\partial^\phi \bar\partial \tilde u} e^{-\phi}),$$

so

$$B = -p_*(\partial\bar\partial\phi \wedge \tilde u \wedge \overline{\tilde u} e^{-\phi}) + (-1)^n p_*(\bar\partial \tilde u \wedge \overline{\bar\partial \tilde u} e^{-\phi}).$$

Now we use that

$$\bar\partial \tilde u = dt \wedge \eta + d\bar t \wedge \nu.$$

As before, the mixed terms give no contribution, so

$$(-1)^n p_*(\bar\partial \tilde u \wedge \overline{\bar\partial \tilde u} e^{-\phi}) = p_*(\eta \wedge \bar\eta e^{-\phi}) - p_*(\nu \wedge \bar\nu e^{-\phi}).$$

Putting all this together (and using that $\Omega = -i\partial\bar\partial\phi$), we finally get

$$\begin{aligned} &i\partial\bar\partial \|u_t\|_t^2 \\ &\quad = (-1)^q c_n [p_*(\mu \wedge \bar\mu e^{-\phi}) - p_*(\xi \wedge \bar\xi e^{-\phi}) + p_*(\eta \wedge \bar\eta e^{-\phi}) - p_*(\nu \wedge \bar\nu e^{-\phi})] i\, dt \wedge d\bar t \\ &\qquad + (-1)^q c_n p_*(\Omega \wedge \tilde u \wedge \overline{\tilde u} e^{-\phi}). \end{aligned}$$

Up to this point, the formula holds for any choice of representative. Now we take $\tilde{u} = \hat{u}$ to be the vertical representative. Then, by Proposition 5.3 and the statements following it, all forms η, μ, ξ and ν are primitive; so by Lemma 2.6, the pushforward terms can be expressed as norms. Since μ and ν are of bidegree (p, q) whereas ξ is $(p+1, q-1)$ and η is $(p-1, q+1)$, we get

$$i\partial\bar{\partial}\|u_t\|_t^2 = \left(\|\mu\|_t^2 + \|\xi\|_t^2 - \|\nu\|_t^2 - \|\eta\|_t^2 + c_n(-1)^q \int_{X_t} c(\Omega)u_t \wedge \bar{u}_t e^{-\phi}\right) i\, dt \wedge d\bar{t}, \quad (6.5)$$

where we have also used Proposition 5.2 in the last term.

Here we decompose $\mu = \mu_h + \mu_\perp$, where μ_h is harmonic and $\mu_\perp$ is orthogonal to harmonic forms. Then $\mu_h = D'_{\partial/\partial t} u_t$; so comparing with the general curvature formula (6.3), we get

$$\langle \Theta_{\partial/\partial t, \partial/\partial \bar{t}}\, u_t, u_t\rangle_t = -\|\mu_\perp\|^2 - \|\xi\|_t^2 + \|\nu\|_t^2 + \|\eta\|_t^2 - c_n(-1)^q \int_{X_t} c(\Omega)u_t \wedge \bar{u}_t e^{-\phi}. \quad (6.6)$$

This formula contains two positive contributions to the curvature, one coming from the norm of ν and the other coming from the norm of η. We shall now see that the first of these can be eliminated and the second can be improved by replacing η by its harmonic part.

For this, we decompose the forms η, ξ and ν, as we did with μ, into one harmonic part and one part orthogonal to harmonic forms. Since ν is exact, $\nu = \nu_\perp$. We now use formulas (5.3) and (5.4), which clearly also hold for the non-harmonic parts of η, ξ, μ and ν. By Proposition 2.4, we can now rewrite formula (6.6).

First, we note that since $\bar{\partial}\eta_\perp = 0$ and $\bar{\partial}\nu_\perp = 0$, they are not only orthogonal to harmonic forms but also to all of the kernels of ∂^ϕ. In the same way, $\mu_\perp$ and $\xi_\perp$ are orthogonal to the range of $\bar{\partial}$. Therefore, we can apply Proposition 2.4 with $g = \eta_\perp$ and $f = -\mu_\perp$, and to $g = \nu_\perp$ and $f = -\xi_\perp$. Finally, we use that

$$\langle (\Box'' + 1)^{-1}\xi_\perp, \xi_\perp\rangle_t + \|\xi_h\|^2 = \langle (\Box'' + 1)^{-1}\xi, \xi\rangle_t.$$

Inserting this in formula (6.6), we obtain Theorem 6.1.

We finally record the counterpart of Theorem 6.1 for positively curved bundles. This can be proved in the same way as Theorem 6.1 (using the analogue of Proposition 2.4 for positively curved metrics as stated in § 2) or by Serre duality, using Theorem 6.1.

Theorem 6.2. *Let $L \to \mathcal{X}$ be a line bundle with metric $e^{-\phi}$, where $i\partial\bar{\partial}\phi > 0$ on fibers, and define the Hermitian structure on $\mathcal{H}^{n-q,q}$ using the Kähler forms $\omega_t = i\partial\bar{\partial}\phi|_{X_t}$ on fibers. Then*

$$\begin{aligned}
&\langle \Theta_{\partial/\partial t_j, \partial/\partial \bar{t}_k}\, u_t, u_t\rangle_t \\
&\quad = \langle (\Box'' + 1)^{-1}\eta_j, \eta_k\rangle_t + \langle (\Box'' + 1)^{-1}\nu_k, \nu_j\rangle_t \\
&\qquad - \langle (\xi_k)_h, (\xi_j)_h\rangle_t + c_n(-1)^q \int_{X_t} c_{j,k}(\Omega)u_t \wedge \bar{u}_t e^{-\phi},
\end{aligned}$$

where μ, ξ, η are defined in formulas (5.1) and (5.2), using a vertical representative of u_t.

This formula generalizes [4, Theorem 1.2], which deals with the case $p = n, q = 0$. Then $\xi = 0$ (since it is of bidegree $(p+1, q-1)$) and $\nu = 0$ since ν is of bidegree $(n, 0)$ and therefore must vanish identically if u_t is a holomorphic section. This corresponds to the simplest case of Theorem 6.1 when $(p, q) = (0, n)$. Then the only contribution with 'bad sign' in that theorem, coming from η_h, must vanish for bidegree reasons.

7. Fiberwise flat metrics

In this section, we will consider the case when the bundle L has a metric $e^{-\phi}$ with $i\partial\bar\partial\phi \leqslant 0$ on $\mathcal{X}$ and $i\partial\bar\partial\phi = 0$ on each fiber X_t. We then have to make an assumption given an auxiliary Kähler form Ω on $\mathcal{X}$. (When we assumed $i\partial\bar\partial\phi < 0$ on fibers, we could take $\Omega = -i\partial\bar\partial\phi$ or $\Omega = -i\partial\bar\partial\phi + p^*(\beta)$, where β is a local Kähler form on the base.) We denote by ω_t, or just ω, the restriction of Ω to X_t.

The main difference as compared to the previous case is that the cohomology is no longer necessarily primitive (with respect to ω). Let us first pause to discuss the notion of primitive cohomology classes in $H^{p,q}(X, L)$, where (X, ω) is a compact Kähler manifold and $(L, e^{-\phi})$ is a Hermitian holomorphic line bundle. If $p + q = n - k$, we say that a class $[u]$ in $H^{p,q}(X, L)$ is primitive if the class $[\omega^{k+1} \wedge u]$ vanishes. This means that $\omega^{k+1} \wedge u = \bar\partial v$ for some L-valued form v of degree $n + k + 1$. By the pointwise Lefschetz theorem, $v = \omega^{k+1} \wedge v'$, and it follows that

$$\omega^{k+1} \wedge (u - \bar\partial v') = 0.$$

Hence we could equivalently have taken as our definition that the class $[u]$ has a representative that is pointwise primitive. Thus, there is a natural notion of primitive classes also for cohomology with values in a line bundle. However, in general, the Lefschetz decomposition on forms does not induce a Lefschetz decomposition on cohomology.

We now assume that $\partial\bar\partial\phi = 0$ on X. Then it follows from the Kodaira–Nakano formula that $\Box'_\phi = \Box''_\phi =: \Box$. Moreover, the commutator $[\Box, \omega\wedge]$ vanishes. Indeed, this is well known when L is trivial and $\phi = 0$. Since the statement is local, it also holds when $\partial\bar\partial\phi = 0$ since we can always find a local trivialization with $\phi = 0$ then. The first formula implies that a harmonic form u satisfies $\partial^\phi u = 0$, and the second formula shows that if a class $[u]$ is primitive, then its harmonic representative is (pointwise) primitive. Indeed,

$$\omega^{k+1} \wedge u_h = \bar\partial v$$

implies $\omega^{k+1} \wedge u_h = 0$ since $\omega^{k+1} \wedge u_h$ is harmonic. For later use, we also point out that if a cohomology class $[u]$ is primitive for the Kähler form ω, it is also primitive for any other Kähler form $\omega' \in [\omega]$. Indeed, $\omega' = \omega + \bar\partial v$. Hence

$$(\omega')^{k+1} \wedge u = \omega^{k+1} \wedge u + \bar\partial(v' \wedge u),$$

and so $(\omega')^{k+1} \wedge u$ is $\bar\partial$-exact if $\omega^{k+1} \wedge u$ is $\bar\partial$-exact.

We then have a Lefschetz decomposition of cohomology classes

$$H^{p,q} = P^{p,q} + \omega \wedge P^{p-1,q-1} + \omega^2 \wedge P^{p-2,q-2} + \cdots,$$

where $P^{l,m}$ denotes the space of primitive classes from the corresponding decomposition of harmonic forms. This implies

$$H^{p,q} = P^{p,q} + \omega \wedge H^{p-1,q-1}$$

and in particular $h^{p,q} = p^{p,q} + h^{p-1,q-1}$ for the dimensions of the corresponding spaces.

We now apply this fiberwise to our fiber space $p : \mathcal{X} \to \mathcal{B}$, where $h_t^{p,q}$ is constant over $\mathcal{B}$ and $p+q=n$. Then $h_t^{p-1,q-1}$ is upper semi-continuous and since $P^{p,q}$ is the kernel of a smoothly varying homomorphism, its dimension is also upper semi-continuous. Therefore, since their sum is constant, both $p_t^{p,q}$ and $h_t^{p-1,q-1}$ are constant on $\mathcal{B}$. This implies in particular the following proposition.

Proposition 7.1. *$\mathcal{P}^{p,q}$ is a smooth sub-bundle of $\mathcal{H}^{p,q}$.*

We also have the next proposition.

Proposition 7.2. *$\mathcal{P}^{p,q}$ is a complex sub-bundle of $\mathcal{H}^{p,q}$.*

Proof. For this, it is enough to show that if u_t is in $\Gamma(\mathcal{P}^{p,q})$, then $D'' u_t \in \Gamma^{0,1}(\mathcal{P}^{p,q})$. Let $\hat{u}$ be the vertical representative of u_t with respect to the Kähler form Ω, as defined in § 5. Recall that

$$D'' u_t = \sum [\nu_j]\, d\bar{t}_j,$$

where

$$\bar{\partial}\hat{u} = \sum \eta_j \wedge dt_j + \sum \nu_j \wedge d\bar{t}_j.$$

We have

$$\nu_j \wedge dt \wedge d\bar{t} \wedge \Omega = \pm \bar{\partial}\hat{u} \wedge \widehat{d\bar{t}_j} \wedge dt \wedge \Omega = 0$$

since $\Omega \wedge dt \wedge \hat{u} = 0$. Hence ν_j is a primitive representative of $[\nu_j]$. □

It follows in the same way that η_j are primitive. As before, we can now define a Hermitian metric on our bundle $\mathcal{P}^{p,q}$ by

$$\|[u]\|_t^2 = (-1)^q c_n \int_{X_t} u_h \wedge \bar{u}_h e^{-\phi_t},$$

i.e., the norm of a class is the norm of its harmonic representative, which is given by the above integral since the harmonic representative is primitive.

As we have seen, if $[u_t]$ is a section of our bundle, then $\partial^\phi (u_t)_h = 0$ on fibers. Hence, if $\hat{u}$ is the vertical representative of $(u_t)_h$,

$$\partial^\phi \hat{u} = \sum \mu_j \wedge dt_j + \sum \xi_j \wedge d\bar{t}_j$$

as before, and we see that μ_j and ξ_j are primitive on fibers in the same way that we proved that $\nu_j|_{X_t}$ is primitive.

Lemma 7.3. *$\partial\bar{\partial}\phi = p^*(C(\phi))$, where $C(\phi)$ is a $(1,1)$-form on the base.*

Proof. Choose local coordinates (t, z) on $\mathcal{X}$ such that $p(t,z) = t$. Since $\partial\bar{\partial}\phi$ vanishes on fibers, we have

$$\partial\bar{\partial}\phi = \partial\bar{\partial}_{t,\bar{t}}\phi + \partial\bar{\partial}_{t,\bar{z}}\phi + \partial\bar{\partial}_{z,\bar{t}}\phi.$$

Since $i\partial\bar{\partial}\phi \leqslant 0$, it follows from Cauchy's inequality that the mixed terms vanish; so

$$\partial\bar{\partial}\phi = \sum \phi_{jk}(t,z) dt_j \wedge d\bar{t}_k.$$

Finally, the condition that $\partial\bar\partial\phi$ is d-closed gives that the coefficients are independent of z, which proves the lemma. □

Proposition 7.4. *On each fiber,*

$$\bar\partial\mu_j + \partial^\phi\eta_j = 0$$

and

$$\bar\partial\xi_j + \partial^\phi\nu_j = 0.$$

Proof. We have

$$(\bar\partial\mu_j + \partial^\phi\eta_j)\wedge dt\wedge d\bar t = (\bar\partial\partial^\phi + \partial^\phi\bar\partial)\hat u\wedge\widehat{dt_j}\wedge d\bar t = \partial\bar\partial\phi\wedge\hat u\wedge\widehat{dt_j}\wedge d\bar t = 0$$

by the previous lemma. □

For a moment, we now assume that the base is one-dimensional and obtain exactly as in § 6 that formula (6.5) still holds,

$$\langle\Theta u, u\rangle = (-\|\mu\|^2 - \|\xi\|^2 + \|\nu\|^2 + \|\eta\|^2)\, dt\wedge d\bar t - c_n(-1)^q p_*(C(\phi)\wedge\hat u\wedge\bar{\hat u}e^{-\phi}).$$

We then decompose the forms μ, ξ, η and ν into a harmonic part and another part that is orthogonal to harmonic forms. The harmonic part of ν is zero since the section is holomorphic, and the harmonic part of μ is also zero if we assume that $D'u_t = 0$ at the given point. We then apply Proposition 2.5 and conclude that $\|\mu_\perp\| = \|\eta_\perp\|$ and $\|\xi_\perp\| = \|\nu_\perp\|$. Hence, the curvature formula becomes

$$\langle\Theta u, u\rangle = -\|\xi_h\|^2 + \|\eta_h\|^2 - c_n(-1)^q p_*(C(\phi)\wedge\hat u\wedge\bar{\hat u}e^{-\phi}).$$

Since this holds for the restriction of Θ to any line in the base, we finally get the curvature formula.

Theorem 7.5. *Let $L\to\mathcal X$ be a line bundle with Hermitian metric $e^{-\phi}$, where $i\partial\bar\partial\phi\leqslant 0$ and $i\partial\bar\partial\phi = 0$ on fibers. Then the curvature of the L^2 metric on $\mathcal P^{p,q}$, $p+q=n$, is*

$$\langle\Theta u, u\rangle_t = \sum\langle(\eta_j)_h, (\eta_k)_h\rangle_t\, dt_j\wedge d\bar t_k - \sum\langle(\xi_j)_h, (\xi_k)_h\rangle_t\, dt_j\wedge d\bar t_k + C(\phi)\|u\|_t^2.$$

We could now have continued and applied similar arguments to $\mathcal H^{p-i,q-i}$. Summing up the results, we get that the same curvature formula holds for the entire bundle $\mathcal H^{p,q}$. Note that when L is trivial and $\phi = 0$, this is a classical formula of Griffiths [16]. Here, however we will be content with the sub-bundle $P^{p,q}$ since that is enough for our applications.

8. Estimates of the holomorphic sectional curvature and hyperbolicity

Let $p : X\to Y$ be a surjective holomorphic map between two compact Kähler manifolds. As in the introduction part, we denote by $Y_0 := Y\setminus\Sigma$ the set of regular values of p so that if $X_0 := p^{-1}(Y_0)$, the restriction $p : X_0\to Y_0$ becomes a proper submersion. Our goal in this section is to analyze the properties of the base Y_0 induced by the variation of the complex structure of the fibers X_y of p together with the semi-positivity properties of the canonical bundle of the said fibers. Under a certain hypothesis, we will construct a Finsler metric on the subset $\mathcal{B} \subset Y_0$ (cf. § 1) whose holomorphic sectional curvature is bounded from above by a negative constant. To this end, we follow the same line of arguments as in the work by To–Yeung [37] and Schumacher [32]. We repeat them here (with some modifications) to see how they adapt in our more general setting. The main conclusion is that when the base is one-dimensional, we get a metric on the base with curvature bounded from above by a negative constant, provided that our metrics satisfy an integral bound that is automatic in the Kähler–Einstein case. We also take the opportunity to include the case of families of Calabi–Yau manifolds. This was also recently treated by To–Yeung [38]. For simplicity of formulation, we will assume that the base is one-dimensional.

8.1. The canonically polarized case

We start with the case of a family of canonically polarized manifolds; $p : \mathcal{X} \to \mathcal{B}$, and we assume here that $\mathcal{B}$ is of dimension 1. We let $L = -K_{\mathcal{X}/\mathcal{B}}$ and let $h = e^{-\phi}$ be a smooth metric on L with $\Omega = -i\partial\bar{\partial}\phi$ strictly positive on each fiber and semi-positive on the total space. By a result of Schumacher, this holds if ϕ is a (normalized) potential of the Kähler–Einstein metric on each fiber, but for the moment, we make no such assumption. Let t be a local coordinate on the base $\mathcal{B}$, V be the horizontal lift of $\partial/\partial t$, and κ be the section of the bundle with fibers $Z^{0,1}(X_t, T^{1,0}(X_t))$, defined by $\kappa_t = \bar{\partial}_{X_t} V$. As in the last paragraph of § 5, we let u^0 be the canonical trivializing section of $\mathcal{H}^{n,0}$ (the bundle with fibers $H^{n,0}(X_t, -K_{X_t}) \sim \mathbb{C}$), and then define u^q inductively by $u^q = (\kappa \cup u^{q-1})_h$. As we have seen in § 5, u^q is a holomorphic section of $\mathcal{H}^{n-q,q}$.

It follows from Theorem 6.1 that

$$\langle i\Theta u^q, u^q\rangle \leqslant (-\|\xi_h\|^2 + \|\eta_h\|^2)(i\, dt \wedge d\bar{t}) \tag{8.1}$$

since $\langle (1+\Box)^{-1}\xi, \xi\rangle \geqslant \|\xi_h\|^2$ and $c(\Omega) \geqslant 0$ since we have assumed that $\Omega \geqslant 0$. Recall that $\eta = \kappa \cup u^q$, $\xi = \bar{\kappa} \cup u^q$, and the subscript h means that we have taken the harmonic part. We start with the following important observation by To–Yeung, [37].

Proposition 8.1.

$$\|\xi_h\|^2 \geqslant \frac{\|u^q\|^4}{\|u^{q-1}\|^2}$$

if $u^{q-1} \neq 0$.

Proof. We have

$$\langle \xi_h, u^{q-1}\rangle = \langle \bar{\kappa} \cup u^q, u^{q-1}\rangle = \|u^q\|^2.$$

Hence Cauchy's inequality gives

$$\|u^q\|^2 \leqslant \|\xi_h\|\|u^{q-1}\|,$$

which yields the claim. □

Next we introduce the notation

$$\phi_q = \log \|u^q\|^2.$$

Then

$$i\partial\bar\partial\phi_q \geqslant -\langle i\Theta u^q, u^q\rangle/\|u^q\|^2.$$

Therefore, formula (8.1) together with the proposition gives that

$$i\partial\bar\partial\phi_q \geqslant (e^{\phi_q-\phi_{q-1}} - e^{\phi_{q+1}-\phi_q})(i\,dt\wedge d\bar t) \tag{8.2}$$

since $\|\eta_h\|^2 = e^{\phi_{q+1}}$.

A few comments are in order. Note that ϕ_q is only locally defined since it depends on the local coordinate t in the base. Changing local coordinates, we see that e^{ϕ_q} transforms as a metric on the qth power of the tangent bundle of $\mathcal{B}$, and $e^{\phi_q-\phi_{q-1}}$ defines a metric on the tangent bundle. In particular, e^{ϕ_1} is the generalized Weil–Petersson metric on $\mathcal{B}$, and it is the genuine Weil–Petersson metric when the metric ϕ on $K_{\mathcal{X}/\mathcal{B}}$ is the (normalized) Kähler–Einstein potential on each fiber. Note also that u^q may be identically 0 on all fibers. If this happens for some q, we let m be the maximal q such that u^q is not identically 0. Since u^m is a holomorphic section of a vector bundle, it can then only vanish on an analytic set. Hence e^{ϕ_m} defines a singular metric on the mth power of the tangent bundle of $\mathcal{B}$. We will assume that the family is effectively parametrized, which means that m is at least equal to 1.

Multiplying (8.2) by q and summing from 1 to m, we get

$$i\partial\bar\partial\sum_1^m q\phi_q \geqslant \sum_1^m (e^{\phi_q-\phi_{q-1}})(i\,dt\wedge d\bar t).$$

If $a_q > 0$ and $\sum a_q = 1$, we get by the convexity of the exponential function that the right-hand side here is greater than

$$\sum_1^m a_q e^{\phi_q-\phi_{q-1}} \geqslant e^{\sum a_q(\phi_q-\phi_{q-1})}.$$

Now take $a_q = c(m + (m-1) + \cdots q)$, with c chosen so that $\sum_1^m a_q = 1$. Then $a_{q-1} - a_q = c(q-1)$; so since $\sum_1^m a_q(\phi_q - \phi_{q-1}) = a_m\phi_m + (a_{m-1} - a_m)\phi_{m-1} + \cdots - a_1\phi_0$, we get $\sum a_q(\phi_q - \phi_{q-1}) = c\sum q\phi_q - a_1\phi_0$. Hence

$$i\partial\bar\partial\sum_1^n cq\phi_q \geqslant ce^{c\sum q\phi_q}e^{-a_1\phi_0}(i\,dt\wedge d\bar t).$$

Moreover, since $e^{\phi_q-\phi_{q-1}}$ is a metric on the tangent bundle of $\mathcal{B}$,

$$e^{\sum a_q(\phi_q-\phi_{q-1})} = e^{c\sum q\phi_q}e^{-a_1\phi_0}$$

is also a metric on the tangent bundle of $\mathcal{B}$, and so is $e^{c\sum q\phi_q} =: e^{\Phi}$, since ϕ_0 is a function. In conclusion, there is a metric with fundamental form $e^{\Phi} i\,dt\wedge d\bar t$ on $\mathcal{B}$, which satisfies

$$i\partial\bar\partial\Phi \geqslant ce^{\Phi}e^{-a_1\phi_0} i\,dt\wedge d\bar t,$$

and thus has curvature bounded from above by

$$-ce^{-a_1\phi_0},$$

and so by a fixed negative constant if

$$\phi_0 = \log c_n \int_{X_t} u^0 \wedge \bar{u}^0 e^{-\phi}$$

is bounded from above. The last requirement is automatic if ϕ is a normalized Kähler–Einstein potential, which means that

$$u^0 \wedge \bar{u}^0 e^{-\phi} = (i\partial\bar{\partial}\phi)^n / n!$$

on each fiber.

8.2. The Calabi–Yau case

Let $\Omega > 0$ be an arbitrary Kähler form on the total space X of the fibration p. As before, t denotes a local coordinate on the base, and we let V be the horizontal lift of $\partial/\partial t$ with respect to Ω. The Kodaira–Spencer representative κ and the holomorphic sections u^q of $\mathcal{H}^{n-q,q}$ are defined as in the canonically polarized case.

We will use the fact that the fibers X_t are Calabi–Yau as follows.

Proposition 8.2. *There exists a unique metric $e^{-\phi_{X/Y}}$ on the relative canonical bundle $K_{X/Y}$ such that $\partial\bar{\partial}\phi_t := \partial\bar{\partial}\phi_{X/Y}|_{X_t} = 0$ and*

$$c_n \int_{X_t} u^0 \wedge \bar{u}^0 e^{\phi_t} = 1 \tag{8.3}$$

for each $t \in Y_0$. This metric satisfies $i\partial\bar{\partial}\phi_{X/Y} \geqslant 0$.

Proof. We first remark that by [39], for any fixed fiber X_{t_0}, there is a positive integer m such that $mK_{X_{t_0}}$ is trivial. This implies that $h^0(mK_{X_t})$ is then equal to 1 for every t. Indeed, in the complement of a closed analytic subset of the base Y, this is true by general semi-continuity arguments; cf. [2]. At special points τ, we have $h^0(mK_{X_\tau}) \geqslant 1$. But, since $c_1(X_\tau) = 0$, a holomorphic section can never vanish. Therefore mK_{X_τ} is trivial; so $h^0(mK_{X_\tau}) = 1$ at τ as well.

In conclusion, there exists a positive integer m such that $h^0(mK_{X_t}) = 1$ for all $t \in Y_0$. Moreover, the group $H^0(X_t, mK_{X_t})$ is generated by a nowhere-vanishing section, which extends locally near t.

We show next that the metric $e^{-\phi_{X/Y}}$ with the properties stated in 8.2 is simply the m-Bergman metric; cf. [5].

Let $t \in Y_0$ be a regular value of the map p, and let $x \in X_t$ be a point of the fiber X_t. If $(z_1, \dots, z_n)$ is a coordinate system on X_t centered at x, then we obtain a coordinate system $(z_1, \dots, z_n, t)$ of the total space X_0 at x by adding t. This induces in particular a trivialization of the relative canonical bundle with respect to which the expression of the m-Bergman metric becomes

$$e^{\phi_{X/Y}(z,t)} = \frac{|f(z,t)|^{2/m}}{\int_{X_t} |u|^{2/m}}, \tag{8.4}$$

where the notations are as follows: u is any non-zero section of mK_{X_t} and $|u|^{2/m}$ is the corresponding volume element on X_t. As we have seen, u admits an extension on the p-inverse image of a small open set centered at t. Locally near x, we write $u = f\frac{(dz\wedge dt)^{\otimes m}}{(dt)^{\otimes m}}$ for some holomorphic function f. This identifies with $f\,(dz)^{\otimes m}$; so we see that we have

$$c_n u^0 \wedge \bar{u}^0 e^{\phi_t(z)} = \frac{|f(z,t)|^{2/m}}{\int_{X_t}|u|^{2/m}} d\lambda(z) \tag{8.5}$$

from which the normalization condition (8.3) follows.

It follows from [5] that $\phi_{X/Y}$ is plurisubharmonic (this will also be a consequence of the remark below), and by (8.4), it is moreover smooth since f is non-vanishing. It also follows from (8.4) that $i\partial\bar{\partial}\phi_{X/Y}|_{X_t} = 0$. Uniqueness follows since any other metric that is flat on fibers must have the form $\phi = \phi_{X/Y} + p^*(\chi)$, where χ is a function on the base. Condition (8.3) then implies that $\chi = 0$; so Proposition 8.2 is proved. □

Remark. Yoshikawa [48] has noted that the curvature form $i\partial\bar{\partial}\phi_{X/Y}$ is actually the pullback of the Kähler form ω_{WP} of the Weil–Petersson metric on the base. Apart from being interesting in itself, this gives another proof of the semi-positivity. Below we indicate how Yoshikawa's result can be obtained from our Theorem 7.5 (which we had not observed before learning about Yoshikawa's work).

The Weil–Petersson metric for a Calabi–Yau family $p : \mathcal{X} \to Y$ is defined as follows (cf. [36, 44] for the case when the fibers have a trivial canonical bundle).

Let as before Ω be a Kähler form on the total space $\mathcal{X}$. Normalizing, we may assume that

$$\int_{X_y} \Omega^n/n! = 1$$

for all fibers. Fix a point y in the base. By Yau's theorem [47], there is a unique Kähler form $\omega_y \in [\Omega|_{X_y}]$ such that $\omega_y^n/n!$ is the unique Ricci-flat volume element on X_y. If $[\kappa] \in H^{0,1}(X_y, T^{1,0}(X_y))$, we define (slightly abusively)

$$\|[\kappa]\|^2_{y,WP} := \int_{X_y} |\kappa_h|^2_{\omega_y}\omega_y^n/n!,$$

where κ_h is the harmonic representative of $[\kappa]$ for the metric ω_y. (More correctly, if $\kappa = \bar{\partial}V$, where V is a lift of a field ∂_t on the base, this is the Weil–Petersson norm of ∂_t.) A priori, this definition depends on the choice of Ω, but we shall see that it does not.

First, we claim that the Weil–Petersson norm as we have defined it coincides with the norm that we have used above, i.e.,

$$\|[\kappa]\|^2_{y,WP} = -c_n \int_{X_y} (\kappa\cup u^0)_h \wedge \overline{(\kappa\cup u^0)_h} e^{\phi_{X/Y}}. \tag{8.6}$$

To see this, we note that

$$\omega_y^n/n! = c_n u^0 \wedge \overline{u^0} e^{\phi_{X/Y}} \tag{8.7}$$

since both sides are Ricci-flat volume elements of total volume 1. Consider the map

$$\tau : \mathcal{E}^{0,1}(X_y, T^{1,0}(X_y)) \to \mathcal{E}^{n-1,1}(X_y, -K_{X_y}),$$

defined by $\tau(\kappa) = \kappa \cup u^0$. This map sends $\bar\partial$-closed forms to $\bar\partial$-closed forms and $\bar\partial$-exact forms to $\bar\partial$-exact forms; so it induces a map $\tilde\tau$ on the corresponding cohomology groups. One also verifies that it is an isometry in the sense that

$$\int_{X_y} |\kappa|^2_{\omega_y} \omega_y^n/n! = \int_{X_y} (\kappa \cup u^0) \wedge * \overline{(\kappa \cup u^0)} e^{\phi_{X/Y}}.$$

(This uses (8.7).) Hence, τ sends harmonic forms to harmonic forms. By our discussion of primitivity at the beginning of § 7, $(\kappa \cup u^0)_h$ is primitive; so (8.6) follows from the previous formula.

We now apply Theorem 7.5 with $(p, q) = (n, 0)$. The bundle $\mathcal{P}^{n,0} = \mathcal{H}^{n,0}$ is trivial even metrically (by (8.3), the holomorphic section u^0 has norm constant equal to 1); so its curvature is 0. Moreover, the term involving ξ vanishes for bidegree reasons. It follows that if ∂_t is a tangent vector in the base, V is a lift of ∂_t and $\kappa = \bar\partial V$, then

$$\|[\kappa]\|^2_{y,WP} = C(\phi_{X/Y})(\partial_t, \bar\partial_t).$$

Therefore, $C(\phi_{X/Y})$ is the Kähler form of the Weil–Petersson metric, ω_{WP}, i.e.,

$$i\partial\bar\partial\phi_{X/Y} = p^*(\omega_{WP}). \tag{8.8}$$

In particular, the Weil–Petersson metric does not depend on the choice of Ω. □

The rest of the argument now goes as in the canonically polarized case. We put

$$\phi_q = \log \|u^q\|^2.$$

Then, $\phi_0 = 0$ by construction, and for $q \geqslant 1$, we get from Theorem 7.5 that

$$i\partial\bar\partial\phi_q \geqslant (e^{\phi_q - \phi_{q-1}} - e^{\phi_{q+1} - \phi_q})(i\, dt \wedge d\bar t).$$

Defining $\Phi = c \sum_1^n q\phi_q$ again, we find that

$$i\partial\bar\partial\Phi \geqslant ce^{\Phi} i\, dt \wedge d\bar t$$

and that e^Φ defines a metric on the tangent bundle of Y with curvature bounded from above by a strictly negative constant.

Remark. We refer the reader to the preprint [46] by the third author, where it is showed that the curvature formula in the fiberwise flat case can be used in a different way to produce a Hermitian metric on Y of strictly negative *bisectional* curvature. □

8.3. Hyperbolicity

A direct consequence of the results in §§ 8.1 and 8.2, respectively, is the following.

Theorem 8.3. *We assume that the hypotheses of Theorem 1.2 are satisfied. Then there exists a subset $\mathcal{B} \subset Y_0$ such that we have the following:*

(1) *The complement $Y_0 \setminus \mathcal{B}$ is a closed analytic set, say S.*
(2) *There exists a Finsler metric on $\mathcal{B}$ locally bounded from above at each point of S and whose holomorphic sectional curvature is bounded from above by $-C$.*

Let us first show how to use the above theorem to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is based on ideas in [33, Proposition 12]. Let $f:\mathbb{C}\to Y\setminus\Sigma$ be an entire curve in $Y\setminus\Sigma := Y_0$. By the above theorem, there exists a smooth strictly negatively curved Finsler metric on a Zariski open subset $\mathcal{B} := Y_0\setminus S$, which extends to a singular metric on Y_0.

First, let us assume that f does not lie in S; thus the image of f contains at least one point in $\mathcal{B}$. Then two cases can arise: f is a constant or there exists $t\in\mathbb{C}$ such that $f(t)\in\mathcal{B}$ and $df(t)\neq 0$. Let us prove that the second case can never arise. Consider the pullback along f of our Finsler metric on Y_0. The second case would give a singular metric, say $e^{\phi(t)}idt\wedge d\bar{t}$, on $\mathbb{C}$ such that ϕ is smooth on an open set in $\mathbb{C}$ and

$$\phi_{t\bar{t}} \geqslant ce^{\phi(t)}$$

in the sense of distribution on $\mathbb{C}$, where c is a positive constant. But the Ahlfors–Schwarz lemma (see [12, p. 17]) implies that

$$e^{\phi(t)} \leqslant \frac{2}{c}\frac{R^{-2}}{(1-|t|^2/R^2)^2} \to 0, \quad \text{as } R\to\infty.$$

Thus $\phi\equiv-\infty$, and we get a contradiction (since ϕ should be locally bounded from above).

If the entire curve f is contained in S, we argue as follows. We can assume that S is the Zariski closure of the image of f (as if this is not the case, then we simply replace S with the said Zariski closure). If S is non-singular, then we are done by the argument in the first part of this proof. If not, let

$$\pi:\widehat{S}\to S$$

be a desingularization of S, and let $\widehat{p}:\widehat{X}\to\widehat{S}$ be the family obtained by base change and desingularization of the total space.

The properties that $\widehat{p}$ inherits from the initial map p are as follows.

(a) *The generic fiber of $\widehat{p}$ is isomorphic to the generic fiber of p.* This is indeed clear since S is not contained in the singular locus of p.

(b) *The map $\widehat{p}$ has maximal variation.* Again, this is immediate because the desingularization map π is generically isomorphic.

(c) *The entire curve f lifts to $\widehat{p}$.* This is the case since S is the Zariski closure of the image of f.

Points (a)–(c) together with the arguments already invoked in the first part of our proof allow us to conclude. □

Remark. If M is a complex manifold, the *Kobayashi–Royden infinitesimal metric* is a Finsler metric on T_M defined as

$$k(x,v) := \inf\{\lambda>0 : \exists\,\gamma:\mathbb{D}\to M, \gamma(0)=x, \gamma'(0)=1/\lambda v\}. \tag{8.9}$$

We take $M := Y_0$, where we recall that $Y_0 := Y\setminus\Sigma$ was the set of regular values of the map p. Then the above proof implies that

$$\frac{k(x,v)}{|v|} \geqslant C_0 > 0, \tag{8.10}$$

where the norm of the vector v in (8.10) is measured with respect to our Finsler metric on a dense subset $\mathcal{B}$ of Y_0. By a result of Royden (cf. [31]), the function k is upper semi-continuous on T_{Y_0}. Thus (8.10) also holds true on Y_0.

Proof of Theorem 8.3. We define the set $\mathcal{B} \subset Y_0$ as in the introduction such that the restriction of $\mathcal{H}^{n-i,i}|_{\mathcal{B}}$ is a vector bundle, whose fiber at t is $H^{n-i,i}(X_t, L|_{X_t})$. The Finsler metric defined in 8.1 and 8.2 has the required curvature property. The only thing to be checked is that the functions ϕ_q are locally bounded from above near each point of $Y_0 \setminus \mathcal{B}$. In fact, each function ϕ_q is the norm of the qth contraction of the tautological section u_0 with $\bar{\partial} V$, where V is the horizontal lift of $\frac{\partial}{\partial t}$. The norm of the harmonic representative of a cohomology class is smaller than the norm of any other representative, and since our fibration p is smooth on Y_0, the boundedness statement in (2) follows. □

9. Extension of the metric

In this section, the setup is as follows. We are given a surjective map $p : X \to Y$ between two smooth projective manifolds such that the fibers of p are connected (this last condition is not really necessary...). This will be referred to as *algebraic fiber space*. Such a map p will not be a submersion in general, so we cannot use directly the results obtained in the previous sections. In order to formulate our next results, we will recall next the notion of *logarithmic tangent bundle* (cf. [11]), which offers the right context to deal with the (eventual) singularities of p.

9.1. Notations, conventions and statements

Let $(W_\alpha)_{\alpha \in J}$ be a finite set of non-singular hypersurfaces of X, which have transverse intersections. The logarithmic tangent bundle $T_X\langle W\rangle$ is the vector bundle whose local frame in a coordinate set U is given by

$$z_1 \frac{\partial}{\partial z_1}, \dots, z_k \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n}, \tag{9.1}$$

where $z_1, \dots, z_n$ are coordinates defined on U such that $W_j \cap U = (z_j = 0)$ for $j = 1, \dots, k$. Hence we assume implicitly that only k among the hypersurfaces W_α intersects the coordinate set U. We remark that the logarithmic tangent bundle is a subsheaf of T_X; its dual is the logarithmic cotangent bundle $\Omega_X\langle W\rangle$.

Throughout the current section, we will observe the following conventions.

• We denote by $\Sigma \subset Y$ the set of singular values of p. Let $\Delta \subset \Sigma$ be the codimension one subset of Σ. We assume that the components of Δ are smooth and have transverse intersections. Note that the closure of the difference $\Sigma \setminus \Delta$ is a set of codimension at least two.

• Let $B \subset Y \setminus \Sigma$ be the Zariski open subset of Y for which the dimension of the relevant cohomology groups (1.3) is constant.

• We assume that there exists a Zariski open subset $\mathcal{Y}_0 \subset Y$ whose complement has codimension greater than or equal to two such that if we denote $\mathcal{X}_0 := p^{-1}(\mathcal{Y}_0)$, then the following holds.

For any x_0 and y_0 in $\mathcal{X}_0$ and $\mathcal{Y}_0$, respectively, such that $p(x_0) = y_0$, we have local coordinates $(z_1, \ldots, z_{n+m})$ centered at x_0 and $(t_1, \ldots, t_m)$ centered at y_0 with respect to which the map p is given by

$$(z_1, \ldots, z_{m+n}) \to \left(z_{n+1}, \ldots, z_{n+m-1}, z_{n+m}^{b_{n+m}} \prod_{j=1}^{q} z_j^{b_j} \right), \tag{9.2}$$

where $b_l \geqslant 1$ above are strict positive integers and $q \leqslant n$. Moreover, locally near y_0, the set Δ is given by the equation $t_m = 0$. We remark that this assumption induces no loss of generality. Indeed, if the p-inverse image of the regular set of Δ has simple normal crossing support, then (9.2) is automatic. In general, we can simply use a log-resolution of the divisor $p^{-1}(\Delta)$ and reduce ourselves to the previous case. We refer to [21] for all that one needs to know about such resolutions.

Remark 1. Given an algebraic fiber space $f : \widetilde{X} \to Y$, part of the properties above can be achieved modulo the modification of the total space $\pi : X \to \widetilde{X}$ along the inverse image of the discriminant of f. Also, if the discriminant Δ is not a simple normal crossing divisor, we consider a log-resolution of (Y, Δ) together with the corresponding fibered product. Hence up to such transformations, we can assume without loss of generality that the bullets above hold true for the maps p we are considering. □

We will denote by $T_Y\langle\Delta\rangle$ the logarithmic tangent bundle of (Y, Δ) described locally as in (9.1); let $\Omega_Y\langle\Delta\rangle$ be its dual.

We have the following easy and well-known statement, which is a consequence of the bullets above.

Lemma 9.1. *We have a natural morphism of vector bundles*

$$p^\star\left(\Omega_Y\langle\Delta\rangle\right) \to \Omega_X\langle W\rangle, \tag{9.3}$$

which is defined and injective on $\mathcal{X}_0$.

Proof. The verification is immediate: in the complement of the divisor Δ, things are clear. Let x_0 and y_0 in X and Y, respectively, be two points as in the third bullet above. We have the coordinates (z_i) and (t_j) with respect to which the map p can be written as in (9.2). Then we have

$$p^\star(dt_j) = dz_{n+j} \tag{9.4}$$

for $j = 1, \ldots, m-1$ and

$$p^\star\left(\frac{dt_m}{t_m}\right) = b_{n+m}\frac{dz_{n+m}}{z_{n+m}} + \sum_{i=1}^{q} b_i \frac{dz_i}{z_i}. \tag{9.5}$$

Thus the lemma is proved, given that the right-hand side of (9.5) is a local section of $\Omega_X\langle W\rangle$. □

We note that map (9.3) is an injection of sheaves on X, but in order to obtain an injection of vector bundles, in general, we have to restrict to $\mathcal{X}_0$ (as one sees by considering the blow-up of a point in $\mathbb{C}^2$).

Let $\Omega_{X/Y}\langle W\rangle$ be the cokernel of (9.3) so that we have the exact sequence

$$0 \to p^\star(\Omega_Y\langle\Delta\rangle) \to \Omega_X\langle W\rangle \to \Omega_{X/Y}\langle W\rangle \to 0 \tag{9.6}$$

on the open set $\mathcal{X}_0 \subset X$.

For further use, we will give next the expression of a local frame of the bundle $\Omega_{X/Y}\langle W\rangle$ with respect to the coordinates z and t considered above. Let $U \subset \mathcal{X}_0$ be the open set on which the functions z_j are defined. Then the local frame of $\Omega_X\langle W\rangle|_U$ is given by

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_q}{z_q}, dz_{q+1}, \ldots, dz_{n+m-1}, \frac{dz_{n+m}}{z_{n+m}}. \tag{9.7}$$

Thus, the local frame of $\Omega_{X/Y}\langle W\rangle|_U$ is given by the symbols

$$\frac{dz_1}{z_1}, \ldots, \frac{dz_q}{z_q}, dz_{q+1}, \ldots, dz_n, \frac{dz_{n+m}}{z_{n+m}} \tag{9.8}$$

modulo the relation

$$b_{n+m}\frac{dz_{n+m}}{z_{n+m}} + \sum_{j=1}^{q} b_j \frac{dz_j}{z_j} = 0. \tag{9.9}$$

The edge morphism corresponding to the direct image of the dual of (9.3) gives the analogue of the Kodaira–Spencer map in a logarithmic setting, as follows:

$$ks : T_Y\langle\Delta\rangle \to \mathcal{R}^1 p_\star T_{X/Y}\langle W\rangle. \tag{9.10}$$

It turns out that we have

$$T_{X/Y}\langle W\rangle \simeq \Omega^{n-1}_{X/Y}\langle W\rangle \otimes K^{-1}_{X/Y} \otimes \mathcal{O}(p^\star(\Delta) - W) \tag{9.11}$$

on $\mathcal{X}_0$. Hence we can rewrite (9.10) as follows:

$$ks : T_Y\langle\Delta\rangle \to \mathcal{R}^1 p_\star\left(\Omega^{n-1}_{X/Y}\langle W\rangle \otimes L\right) \tag{9.12}$$

provided that the twisting bundle equals

$$L := K^{-1}_{X/Y} \otimes \mathcal{O}\left(p^\star(\Delta) - W\right). \tag{9.13}$$

Hence, this fits perfectly into the framework developed in the previous sections of this article.

In general, given any bundle L and an index $i \geqslant 0$, we have a map

$$\tau^i : \mathcal{R}^i p_\star\left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right) \to \mathcal{R}^{i+1} p_\star\left(\Omega^{n-i-1}_{X/Y}\langle W\rangle \otimes L\right) \otimes \Omega_Y\langle\Delta\rangle \tag{9.14}$$

obtained by contraction with (9.10). A local section of $\mathcal{R}^i p_\star\left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right)$ is represented by a $(0, i)$-form with values in the $(n-i)$ exterior power of the relative logarithmic cotangent bundle twisted with L. We couple this form with the $(0, 1)$-form with values in the logarithmic tangent bundle of X corresponding to a local section of $T_Y\langle\Delta\rangle$ via (9.10). The result is a $(0, i+1)$-form with values in the $(n-i-1)$ exterior power of the logarithmic cotangent bundle twisted with L. This is map (9.14); as we see, it is only defined in the complement of a set of codimension at least two in Y.

Definition 2. We denote by $\mathcal{K}^i$ the kernel of (9.14). It is a subsheaf of $\mathcal{R}^i p_\star \left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right)$. Also, let $\mathcal{K}^i_f$ be the quotient of $\mathcal{K}^i$ by its torsion subsheaf.

We can assume that the restriction $\mathcal{K}^i|_B$ is a sub-bundle of $\mathcal{R}^i p_\star \left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right)|_B$ by shrinking the set B.

Remark 2. In view of our main formula (cf. Theorem 6.1), it is clear that the curvature of $\mathcal{K}^i|_B$ has better chances to be semi-negative than the full bundle $\mathcal{R}^i p_\star \left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right)$. Indeed, if the section $[u]$ in 6.1 is a local holomorphic section of $\mathcal{K}^i|_B$, then the fiberwise projection of the η_j's on the space of harmonic forms is identically zero. □

The sheaves $\mathcal{K}^i$ have been extensively studied in algebraic geometry; cf. [7, 49] and the references therein. We will be concerned here with their differential geometric properties. To this end, we formulate the requirements below concerning the Hermitian bundle (L, h_L).

$(\mathcal{H}_1)$ We have $i\Theta_{h_L}(L) \leqslant 0$ on X, and moreover $i\Theta_{h_L}(L)|_{X_y} = 0$ for each y in the complement of some Zariski closed set.

$(\mathcal{H}_2)$ We have $i\Theta_{h_L}(L) \leqslant 0$ on X, and moreover there exists a Kähler metric ω_Y on Y such that we have $i\Theta_{h_L}(L) \wedge p^\star\omega_Y^m \leqslant -\varepsilon_0\omega \wedge p^\star\omega_Y^m$ on X.

Thus the first condition requires L to be semi-negative and trivial on fibers, whereas in $(\mathcal{H}_2)$, we assume that L is uniformly strictly negative on fibers in the sense that we have

$$i\Theta_{h_L}(L)|_{X_y} \leqslant -\varepsilon_0\omega|_{X_y} \tag{9.15}$$

for any regular value y of the map p.

We recall that we denote by $\mathcal{K}^i_f$ the quotient of $\mathcal{K}^i$ by its torsion subsheaf (cf. Definition 2). In this context, we have the following result.

Theorem 9.2. *Let $p : X \to Y$ be an algebraic fiber space, and let (L, h_L) be a Hermitian line bundle that satisfies hypothesis $(\mathcal{H}_i)$ above. We assume that the restriction of h_L to the generic fiber of p is non-singular. Then we have the following:*

(a) *For each $i \geqslant 1$, the bundle $\mathcal{K}^i_f$ admits a semi-negatively curved singular Hermitian metric. Moreover, this metric is smooth on a Zariski open subset of Y.*

(b) *We assume that the curvature form of L is smaller than $-\varepsilon_0 p^\star\omega_Y$ on the p-inverse image of some open subset $\Omega \subset Y$ of Y. Then the metric on $\mathcal{K}^i_f$ is strongly negatively curved on Ω (and semi-negatively curved in the complement of a codimension greater than two subset of Y).*

The method of the proof of Theorem 9.2 also gives the following statement, which is potentially important in the analysis of families of holomorphic disks tangent to the pair (Y, Δ). Prior to stating this result, we define the 'iterated Kodaira–Spencer map' (cf. [32])

$$ks^{(i)} : \mathrm{Sym}^i\, T_Y\langle\Delta\rangle \otimes \mathcal{R}^0 p_\star \left(\Omega^n_{X/Y}\langle W\rangle \otimes L\right) \to \mathcal{R}^i p_\star \left(\Omega^{n-i}_{X/Y}\langle W\rangle \otimes L\right) \tag{9.16}$$

by contracting successively the sections of $\Omega^{\bullet}_{X/Y}\langle W\rangle \otimes L$ with the $T_{X/Y}\langle W\rangle$-valued $(0,1)$-forms given by ks in (9.10) (we remark that the contraction operations are commutative, and this is the reason why map (9.16) is defined on Sym^i rather than $\otimes^i$ of the log tangent bundle of the base).

In Proposition 9.3, we denote by s_Δ the section whose vanishing equals the support of Δ. Also, we denote by $\|ks^{(i)}\|$ the operator norm of map (9.16).

Proposition 9.3. *We assume that the singularities of the map p are contained in the snc divisor Δ. Let $\omega_{\mathcal{P},X}$ and $\omega_{\mathcal{P},Y}$ be two Kähler metrics with Poincaré singularities on (X,W) and (Y,Δ), respectively. We assume that the metric h_L of L is smooth when restricted to the p-inverse image of a Zariski open subset of Y and that its weights are bounded from below. Then for each $i \geqslant 1$, we have*

$$\|ks^{(i)}\|_y^{2/i} \leqslant C \log^N \frac{1}{|s_\Delta|_y^2}, \tag{9.17}$$

where norm (9.17) is induced by the metrics $\omega_{\mathcal{P},X}$ and $\omega_{\mathcal{P},Y}$. The constants C, N depend on everything but the point $y \in Y$.

Remark 3. The proof of Proposition 9.3 is very similar to the arguments we will give for Theorem 9.2, and we have decided that we can afford to skip it. We will however highlight the main points after completing the arguments for 9.2. □

Assume that the bundle (L, h_L) verifies hypothesis $(\mathcal{H}_i)$ and that moreover we have

$$H^0\left(X, \Omega^n_{X/Y}\langle W\rangle \otimes L\right) \neq 0. \tag{9.18}$$

If σ is a holomorphic section of $\Omega^n_{X/Y}\langle W\rangle \otimes L$, then we have a holomorphic map

$$ks^{(1)}_\sigma : \mathcal{O}_Y \to \mathcal{R}^1 p_\star \left(\Omega^{n-1}_{X/Y}\langle W\rangle \otimes L\right) \otimes \Omega_Y\langle\Delta\rangle \tag{9.19}$$

induced by the map $ks^{(1)}$; cf. (9.16).

We formulate yet another hypothesis.

$(\mathcal{H}_3)$ There exists a holomorphic section σ of $\Omega^n_{X/Y}\langle W\rangle \otimes L$ such that the map $ks^{(1)}_\sigma$ is not identically zero when restricted to a non-empty open subset of $\mathcal{B}$.

The following result is a corollary of Theorem 9.2.

Theorem 9.4. *Let $p : X \to Y$ be an algebraic fiber space, and let (L, h_L) be a Hermitian line bundle that satisfies hypothesis $(\mathcal{H}_j)$ for $j = 1, 2$. Moreover, we assume that $(\mathcal{H}_3)$ is equally satisfied. Then there exist $s \leqslant n = \dim(X_y)$ and a non-trivial map*

$$\mathcal{K}_f^{s\star} \to \mathrm{Sym}^s\,\Omega_Y\langle\Delta\rangle. \tag{9.20}$$

In addition, if the curvature of L is smaller than $-\varepsilon_0 p^\star\omega_Y$ on the pre-image of some non-empty open subset V of Y, then there exists an ample line bundle A on Y such that the bundle

$$\otimes^M \Omega_Y\langle\Delta\rangle \otimes A^{-1}$$

has a (non-identically zero) global section for some $M \gg 0$.

Our next results will show that in some interesting geometric circumstances, a version of hypothesis $(\mathcal{H}_2)$ is verified.

If the canonical bundle of the generic fiber of $p : X \to Y$ is ample, then we say that the family defined by p is *canonically polarized*. If the Chern class of the canonical bundle of the generic fiber of p equals zero, then we say that p defines a *Calabi–Yau* family. Next, a canonically polarized or Calabi–Yau family p has *maximal variation* if the Kodaira–Spencer map (9.10) is injective when restricted to a non-empty open subset of Y_0.

Then we have the following result, which provides a (geometric) sufficient condition under which the anticanonical bundle of p has strong curvature properties.

Theorem 9.5. *Let $p : X \to Y$ be a family of canonically polarized manifolds of maximal variation, and let ω be a reference metric on X. Then there exist an effective divisor Ξ in X and a metric $h_{X/Y}$ on the twisted relative canonical bundle $K_{X/Y} + \Xi$ such that the following hold true:*

- (i) *The curvature corresponding to $h_{X/Y}$ is greater than $\varepsilon_0\omega$ for some $\varepsilon_0 > 0$, and the restriction $h_{X/Y}|_{X_y}$ is non-singular for all y in the complement of a Zariski closed subset of Y.*
- (ii) *The codimension of the direct image $p(\Xi)$ is greater than two.*

Remark 4. Given that K_{X_y} is ample, a natural choice for the construction of a positively curved metric on $K_{X/Y}$ would be the fiberwise KE metric (cf. § 8) – this is well defined in the complement of the singular locus of p, and it extends across the singularities (cf. [27]). However, it is not clear to us whether it is possible to obtain a *useful* lower bound of the eigenvalues of the KE metric with respect to a fixed Kähler metric on X as we approach the singular locus of p. This information is critical in the process of extending the metric, as we will see below. □

In order to derive an interesting consequence of Theorem 9.5, we recall the following result for which we refer to [8, Theorem 2.3].

Theorem 9.6 [8]. *We assume that the hypotheses of Theorem 9.5 are satisfied. Then there exists a metric $h^{(1)}_{X/Y}$ on the relative canonical bundle $K_{X/Y}$ with the following properties.*

- (i) *The restriction of $h^{(1)}_{X/Y}$ to the fiber $X_y := p^{-1}(y)$ is smooth. It is induced by the sections of mK_{X_y} for some $m \gg 0$, for all y in the complement of a proper algebraic subset $Z \subset Y$.*
- (ii) *The corresponding curvature current Θ is positive definite on each compact subset of $p^{-1}(Y \setminus Z)$, and moreover we have*

$$\Theta \geqslant \sum (t^j - 1)[W_j] \tag{9.21}$$

in the sense of currents on X.

In statement (9.6), we denote by t^j the multiplicity of the hypersurface W_j in the inverse image $p^{-1}(\Delta)$. By combining Theorems 9.5 and 9.6, we obtain the following result.

Corollary 9.7. *Let $\eta > 0$ be a positive real number. We assume that the hypotheses of Theorem 9.5 are satisfied, and we define the following family of metrics:*

$$\psi_{X/Y}^{(\eta)} := (1-\eta)\varphi_{X/Y} + \eta\varphi_{X/Y}^{(1)} - \sum(t^j - 1)\log|f_j|^2 \tag{9.22}$$

on the bundle $K_{X/Y} - \sum(t^j - 1)W_j$, where $\varphi_{X/Y}$ and $\varphi_{X/Y}^{(1)}$ are the weights given by Theorems 9.5 and 9.6, respectively, and f_j is a local equation of the hypersurface W_j. The resulting metrics $h^{(\eta)}$ have the following properties.

(a) *The curvature current Θ_η verifies*

$$\Theta_\eta \geqslant (1-\eta)\varepsilon_0\omega - (1-\eta)\sum(t^j-1)[W_j] - (1-\eta)[\Xi].$$

(b) *Each of the metrics $h^{(\eta)}$ is smooth when restricted to the generic fiber of p.*

(c) *For each compact subset $K \subset X \setminus p^{-1}(\Delta)$, we have $\Theta_\eta|_K \geqslant C_K\omega|_K$, where $C_K > 0$ is independent of η.*

In Corollary 9.7, we denote by $\varphi_{X/Y}$ the weight of the metric induced on $K_{X/Y}$ by the metric we construct in Theorem 9.5. The effect is the presence of the divisor $(1-\eta)[\Xi]$ in (a) with a negative sign.

Thus, in the setup of Theorem 9.5, point (a) of Corollary 9.7 shows that a *version* of hypothesis $(\mathcal{H}_2)$ is satisfied. Indeed, we consider the bundle $L := -K_{X/Y} + \sum(t^j-1)W_j$ endowed with the metric induced by $h^{(1)}$. This metric does not necessarily verify the requirements in $(\mathcal{H}_2)$. Nevertheless, it is the limit of $h^{(\eta)}$ as $\eta \to 1$, and the restriction to fibers of Θ_η is uniformly positive; cf. (a). We will show at the end of this section that this suffices to infer that Theorem 9.2 holds true.

The following important result due to Viehweg–Zuo (cf. [42]) can now be seen as a direct consequence of the results 9.2, 9.4 and 9.5.

Theorem 9.8 [42]. *Let $p : X \to Y$ be a family of canonically polarized manifolds of maximal variation. Then there exists a positive integer $q \leqslant \dim(Y)$ such that the bundle $\mathrm{Sym}^q\,\Omega_Y\langle\Delta\rangle$ contains a non-trivial big coherent subsheaf.*

We refer to *loc. cit.* for the notion of 'big subsheaf' $\mathcal{F}$: it implies that for any ample line bundle H, the bi-dual of $\mathrm{Sym}^m\,\mathcal{F} \otimes H^{-1}$ is generated by global sections on some open subset $U \subset X$ for all $m \gg 0$.

As a by-product of our methods, it turns out that a completely similar result holds in the context of Calabi–Yau families.

Theorem 9.9. *Let $p : X \to Y$ be a Calabi–Yau family. Assume that p has maximal variation. Then there exists a positive integer $q \leqslant \dim(Y)$ such that the bundle $\mathrm{Sym}^q\,\Omega_Y\langle\Delta\rangle$ contains a non-trivial big coherent subsheaf.*

Remark 5. As was pointed out to us by the anonymous referee, Theorem 9.9 becomes completely wrong in the absence of the Kähler hypothesis for the total space X. A counterexample is provided by the twistor spaces: the map p is smooth, its fibers are hyperkähler manifolds and the family has maximal variation but the base Y is the projective line. □

In the following subsections, we will establish the results stated above.

9.2. Proof of Theorem 9.2

Our first task will be to construct a metric on the quotient

$$\mathcal{K}^i_f := \mathcal{K}^i / T(\mathcal{K}^i) \tag{9.23}$$

of the kernel by its torsion subsheaf. This will be naturally induced by the metric on the direct image thanks to the following simple observation.

Lemma 9.10. *The support of the torsion subsheaf $T(\mathcal{K}^i)$ is contained in $Y \setminus B$.*

This is clear since $\mathcal{K}^i|_B$ is a vector bundle.

Next, the sheaf $\mathcal{K}^i_f$ is coherent and torsion-free. Therefore, it is locally free on an open subset $\mathcal{Y}_0$ whose codimension in Y is at least two. We define next a metric on the restriction $\mathcal{K}^i_f|_B$ as follows.

Let s_1, s_2 be local sections of $\mathcal{K}^i_f$ defined on an open set V such that the sequence

$$0 \to \Gamma\left(V, T(\mathcal{K}^i)\right) \to \Gamma\left(V, \mathcal{K}^i\right) \to \Gamma\left(V, \mathcal{K}^i_f\right) \to 0 \tag{9.24}$$

is exact. We consider two sections $u_1, u_2 \in \Gamma\left(V, \mathcal{K}^i\right)$ projecting into s_1 and s_2, respectively, via (9.24). Then for each $t \in B \cap V$, we define

$$\langle s_1, s_2 \rangle_t := \langle u_1, u_2 \rangle_t. \tag{9.25}$$

We note that this does not depend on the u_i's by Lemma 9.10, and therefore we have a well-defined Hermitian structure on $\mathcal{K}^i_f|_B$.

Next, we will use the Leray isomorphism in order to construct a representative of a local section $[u]$ of $\mathcal{K}^i$. Since this is slightly different from the convention adopted in the previous sections, we give a few precisions in what follows. Assume that $[u]$ is defined on a small enough open subset $V \subset Y$ containing a smooth point of the divisor Δ. In particular, $[u]$ corresponds to an element of the cohomology group

$$H^i\left(p^{-1}(V), \Omega^{n-i}_{X/Y}\langle W \rangle \otimes L|_{p^{-1}(V)}\right). \tag{9.26}$$

Let $\mathcal{U} = (U_\alpha)$ be a finite cover of X such that the intersections $U_I := U_{\alpha_0} \cap \cdots \cap U_{\alpha_s}$ are contractible for all multi-indexes $I = (\alpha_0 < \cdots < \alpha_s)$ having $s+1$ components and all s.

The section $[u]$ corresponds to a collection of holomorphic sections (u_I) of the bundle

$$\Omega^{n-i}_{X/Y}\langle W \rangle \otimes L|_{U_I}$$

whose Cech coboundary is equal to zero (i.e., it is a cocycle). Here the multi-index I has $i+1$ components. We can assume that the map

$$\Omega_X^{n-i}\langle W\rangle \otimes L|_{U_I} \to \Omega_{X/Y}^{n-i}\langle W\rangle \otimes L|_{U_I} \tag{9.27}$$

is surjective at the level of sections. Let $\widetilde{u}_I$ be a lifting of the section u_I via map (9.27). Each $\widetilde{u}_I$ is an L-valued $n-i$-form with log poles on U_I. We note that in general, the collection $(\widetilde{u}_I)$ is not necessarily a cocycle. However, this is the case for the restriction $(\widetilde{u}_I|_{U_I\cap X_t})$ for each $t\in B$, and this identifies with the class $[u_t]$.

We apply the Leray isomorphism procedure to $(\widetilde{u}_I)$; this gives a $(0,i)$-form with values in $\Omega_X^{n-i}\langle W\rangle \otimes L|_{p^{-1}(V)}$, which we denote by $\widetilde{u}$. The exact formula is as follows:

$$\widetilde{u} = \sum_I \rho_{\alpha_s}\widetilde{u}_{\alpha_0\cdots\alpha_s}\bar\partial\rho_{\alpha_0}\wedge\cdots\wedge\bar\partial\rho_{\alpha_{s-1}}, \tag{9.28}$$

where (ρ_α) is a partition of unit subordinate to the Leray cover $\mathcal{U}$. In the usual formula, we do not have the factor ρ_{α_s} in (9.28) since the forms defined on the open sets U_{α_s} coincide on intersections. As already mentioned, here $(\widetilde{u}_I)$ is not necessarily a cocycle, but this is the case for the projection (u_I). Since $\sum\rho_j=1$, the image of $\widetilde{u}$ on the space of $(0,i)$-forms with values in $\Omega_{X/Y}^{n-i}\langle W\rangle\otimes L|_{p^{-1}(V)}$ is $\bar\partial$-closed, and it represents the class $[u]$.

Convention. For the rest of this section, we will call $\widetilde{u}$ a representative of $[u]$. Also, the representative of a section u of $\mathcal{K}_f^i$ will be by definition the representative of a section of $\mathcal{K}^i$ projecting into u.

Let $\widetilde{u}$ be a fixed representative of a local section of the bundle $\mathcal{K}_f^i$, which is defined near a smooth point $y_0\in\Delta$. Thus $\widetilde{u}$ is a $(0,i)$-form with values in the bundle $\Omega_X^{n-i}\langle W\rangle\otimes L|_{p^{-1}(V)}$ such that the following properties are satisfied for any $t\in B\cap V$.

- The restriction of the form $\widetilde{u}$ to the fibers X_t is $\bar\partial$-closed.
- The cup product of $\widetilde{u}$ with any element in the image of ks is cohomologically zero when restricted to X_t.

We first assume that the bundle (L,h_L) verifies hypothesis $(\mathcal{H}_2)$. Let $\theta := -\Theta_{h_L}(L)$ be the corresponding curvature form. Then we have

$$\sqrt{-1}c_{jk}(\theta)dt_j\wedge d\bar t_k \geqslant 0, \tag{9.29}$$

and given that $\widetilde{u}$ is a section of the kernel, the fiberwise projection of the (troublesome) forms η_j on the space of harmonic forms is equal to zero (as above, we assume that $t\in B$). Thus, for any local holomorphic section u of $\mathcal{K}_f^i$ defined on an open set V centered at the point y_0, the function

$$t\to \log\|[u]\|_t^2 \tag{9.30}$$

is psh on $V\setminus\Delta$ as a consequence of Theorem 6.1 (the bracket notation in (9.30) has the same meaning as in § 2). This is not strictly correct because Theorem 6.1 can only be applied by restriction to $\mathcal{B}$. However, as we see from the arguments to follow, extending function (9.30) across Δ is the most subtle part of the proof.

We will show next that we have

$$\sup_{t\in V\setminus\Delta} \log \|[u]\|_t^2 < \infty, \tag{9.31}$$

and then Theorem 9.2 follows by elementary properties of psh functions.

In order to establish (9.31), by the definition of the norm of a section of $\mathcal{H}^{p,q}$, we have the inequality

$$\|[u]\|_t^2 \leqslant \int_{X_t} \left|\widetilde{u}_{|X_t}\right|^2_{\theta,h_L} \theta^n \tag{9.32}$$

for each $t \in V \setminus \Delta$. Hence it would be enough to bound the right-hand-side term of (9.32). Unfortunately, we cannot do this directly (as the following computations will show, the technical reason is that we do not have an upper bound for the eigenvalues of θ at our disposal), and we will proceed as follows.

By hypothesis $(\mathcal{H}_2)$, the form θ is semi-positive on X and greater than $\varepsilon_0\omega$ on the fibers of p. For each j, let s_j be the global section whose set of zeros is the hypersurface W_j (recall that the W_j are the support of the inverse image $p^{-1}(\Delta)$). They correspond locally on the open set U (cf. (9.2)) to the coordinates $z_1, \dots, z_q, z_{m+n}$. We have the decomposition $U := \bigcup U_j$, where $U_j \subset U$ is defined as follows:

$$U_j := \left\{z \in U : |z_j| \leqslant |z_i|, \text{ for all } i = 1, \dots, q \text{ and } i = n+m\right\}. \tag{9.33}$$

Here j belongs to the set $1, \dots, q$ together with $j = n+m$. The motivation for introducing this stratification will become completely clear in (9.46).

If $j = 1, \dots, q$, then on the open set $U_j \cap X_t$ of the fiber X_t, we take the local coordinates $z_1, \dots, z_{j-1}, z_{n+m}, z_{j+1}, \dots, z_n$. On the set $X_t \cap U_{n+m}$, the coordinates are $z_1, \dots, z_n$.

There exists a constant $C > 0$ such that for each $\varepsilon > 0$, on $X_t \cap U_{n+m}$ we have

$$\theta - \varepsilon \sum_j dd^c \log\log \frac{1}{|s_j|^2} \geqslant C\sqrt{-1}\left(\sum_{j=1}^n dz_j \wedge d\overline{z}_j + \varepsilon \sum_{j=1}^q \frac{dz_j \wedge d\overline{z}_j}{|z_j|^2 \log^2 |z_j|^2}\right). \tag{9.34}$$

Inequality (9.34) is a direct consequence of the fact that θ is greater than $\varepsilon_0\omega$ on the fibers of p by hypothesis.

In order to apply Theorem 6.1, we perturb the metric of L as

$$h_{L,\varepsilon} := \left(\prod_j \log^{\varepsilon} \frac{1}{|s_j|^2}\right)^{-1} e^{C\sqrt{\varepsilon}|t|^2} h_L, \tag{9.35}$$

and let θ_ε be the opposite of the resulting curvature form (whose restriction to fibers is none other than the left-hand side of (9.34)). The constant $C \gg 0$ in (9.35) is chosen so that θ_ε still verifies $(\mathcal{H}_2)$ for each positive $\varepsilon > 0$. This is quickly seen as follows. We have

$$-dd^c \log\log \frac{1}{|s_j|^2} = \frac{\sqrt{-1}}{\pi} \frac{\langle D's_j, D's_j\rangle}{|s_j|^2 \log^2 |s_j|^2} + \frac{1}{\log |s_j|^2} \Theta(W_j), \tag{9.36}$$

so we remark that the negativity induced by the first factor of (9.35) in the expression of θ_ε is of order $\sum_j \frac{\varepsilon}{\log 1/|s_j|^2}\omega$. This is the reason why we introduce the weight $C\sqrt{\varepsilon}|t|^2$ in (9.35): given the strict positivity of θ on the fibers of p, the curvature of $h_{L,\varepsilon}$ will be semi-negative on X.
In conclusion, we can change the metric of L in order to have Poincaré singularities on the divisor $W = \sum W_j$.
We clearly have

$$\lim_{\varepsilon\to 0} \|[u]\|^2_{\varepsilon,t} = \|[u]\|^2_t \tag{9.37}$$

for every $t \in V \setminus \Delta$. In (9.37), we denote by $\|\cdot\|^2_{\varepsilon,t}$ the norm induced by the perturbed metric of L. Equality (9.37) is a consequence of the usual elliptic theory (cf. [19]) applied for each t.

We formulate the following claim, where we recall that $\widetilde{u}$ is a representative of the section u.

Claim. For each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that we have

$$\sup_{t\in V\setminus\Delta} |t_m|^{2\varepsilon} \int_{X_t} \left|\widetilde{u}_{|X_t}\right|^2_{\theta_\varepsilon, h_{L,\varepsilon}} \theta_\varepsilon^n \leqslant C_\varepsilon < \infty. \tag{9.38}$$

If we are able to show that (9.38) holds true, then we are done (despite the fact that the constant C_ε above may not be bounded as $\varepsilon \to 0$). Indeed, this would imply that the log of the expression

$$t \to |t_m|^{2\varepsilon}\|[u]\|^2_{\varepsilon,t} \tag{9.39}$$

defines a psh function on V for each $\varepsilon > 0$. We write the corresponding mean inequality

$$\log\left(|\tau|^{2\varepsilon}\|[u]\|^2_{\varepsilon,(t',\tau)}\right) \leqslant \varepsilon C + \int_0^{2\pi} \log\|[u]\|^2_{\varepsilon,(t',\tau_\theta)}\frac{d\theta}{2\pi}, \tag{9.40}$$

where we use the notation $\tau_\theta := \tau + re^{\sqrt{i}\theta}$ under the integral sign in (9.40), and $0 < r \ll 1$.
Now, the concavity of the log function implies

$$\int_0^{2\pi} \log\|[u]\|^2_{\varepsilon,(t',\tau_\theta)}\frac{d\theta}{2\pi} \leqslant \log\left(\int_0^{2\pi} \|[u]\|^2_{\varepsilon,(t',\tau_\theta)}\frac{d\theta}{2\pi}\right), \tag{9.41}$$

and we obviously have

$$\|[u]\|^2_{\varepsilon,(t',\tau_\theta)} \leqslant \int_{X_t} |\widetilde{u}_{|X_t}|^2_{\theta_\varepsilon, h_{L,\varepsilon}}\theta_\varepsilon^n, \tag{9.42}$$

where t in (9.42) is equal to (t', τ_θ). Inequality (9.42) is true because the norm on the left-hand side is given by the *harmonic* representative and this is clearly smaller than the norm of the representative $\widetilde{u}$.
On the other hand, if $|t_m| = \delta_0 > 0$, then we have

$$\sup_{|t_m|=\delta_0,\ t\in V} \int_{X_t} |\widetilde{u}_{|X_t}|^2_{\theta_\varepsilon, h_{L,\varepsilon}}\theta_\varepsilon^n \leqslant C_0 \tag{9.43}$$

uniformly with respect to ε, since we are 'away' from the singularities of the map p. The mean inequality (9.40) combined with (9.37) and (9.43) gives the uniform boundedness of the initial norm, which is what we wanted.

In conclusion, the argument is that we will be using the non-effective estimate (9.38) in order to infer that function (9.39) is log-psh. Then the uniform boundedness of our initial norm follows by the mean inequality, together with the convergence as $\varepsilon \to 0$ property (9.37).

Proof (of the Claim). To establish the claim, we will use the local frame (9.8) in order to carry out a few local computations. Without loss of generality, we can assume that z belongs to the coordinate set $U_{n+m} \subset U$. For each $t \in V \setminus \Delta$, the local coordinates on $X_t \cap U_{n+m}$ will be $z_1, \dots, z_n$.

Our representative $\widetilde{u}$ can be expressed as follows:

$$\widetilde{u} = \sum_{I,J} \tau_{I\overline{J}}(z) \frac{dz_\alpha}{z_\alpha} \wedge dz_\beta \wedge d\overline{z}_J \otimes e_L, \tag{9.44}$$

where α, β are multi-indexes whose union equals I such that we have $\alpha \subset \{1, \dots, q, m+n\}$ and $\beta \subset \{q+1, \dots, m+n-1\}$. The set J is contained in $\{1, \dots, m+n\}$, and it has length i. The coefficients $\tau_{I\overline{J}}$ are differentiable functions, in particular uniformly bounded.

The restriction of $\widetilde{u}$ to $X_t \cap U_{n+m}$ reads as follows:

$$\widetilde{u}|_{X_t} = \sum_{I,J} \xi_{I\overline{J}}(z,t) \frac{dz_\alpha}{z_\alpha} \wedge dz_\beta \wedge d\overline{z}_J \otimes e_L, \tag{9.45}$$

where now α, β in (9.45) are multi-indexes whose union is equal to I. Moreover, $\alpha \subset \{1, \dots, q\}$ and $\beta \subset \{q+1, \dots, n\}$. The lengths of I and J are $n-i$ and i, respectively.

The important remark is that the absolute value of the coefficients $\xi_{I\overline{J}}(\cdot, t)$ of restriction (9.45) is bounded uniformly with respect to $t \in V \setminus \Delta$. This is quickly seen thanks to relation (9.9), which expresses the logarithmic form $\frac{dz_{n+m}}{z_{n+m}}$ as a linear combination of $\frac{dz_j}{z_j}$, and the fact that the quotients $|z_{n+m}|/|z_j|$ are bounded from above on U_{n+m}. In particular, we have

$$d\overline{z}_{n+m} = \sum_{j=1}^{q} \mu_j d\overline{z}_j, \tag{9.46}$$

where the μ_j are uniformly bounded.

In general, given two Kähler metrics g_1 and g_2 such that $g_1 \geqslant g_2$, then for any form γ of type $(n-i, i)$, we have

$$|\gamma|_{g_1} \leqslant |\gamma|_{g_2}, \tag{9.47}$$

and in our setting, this implies the inequality

$$\left|\widetilde{u}_{|X_t}\right|^2_{\theta_\varepsilon, h_{L,\varepsilon}} \leqslant C \frac{e^{-\varphi_{L,\varepsilon}}}{\varepsilon^2} \sum_{I,J} |\xi_{I\overline{J}}|^2 \prod_{j \in \alpha} \log^2 \left|z_j\right|^2 \tag{9.48}$$

at each point of the intersection $U_{n+m} \cap X_t$; cf. (9.34). This explains the reason why we need to work with a metric having Poincaré singularities.

Next we remark that the local weights of the metric h_L are bounded from below, given the curvature hypothesis; this is also true for the perturbed metric $h_{L,\varepsilon}$ (and actually the lower bound is independent of ε).

On the other hand, for each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that we have

$$|t_m|^{2\varepsilon} \prod_{j\in\alpha} \log^2 |z_j|^2 \leqslant C_\varepsilon \tag{9.49}$$

for any $z \in X_t \cap U$; cf. (9.2). We thus obtain

$$|t_m|^{2\varepsilon} \left|\widetilde{u}_{|X_t}\right|^2_{\theta_\varepsilon, h_{L,\varepsilon}} \leqslant C_\varepsilon(\xi), \tag{9.50}$$

where $C_\varepsilon(\xi)$ is uniform with respect to the point $t \in V \setminus \Delta$ as a consequence of (9.48) and (9.49).

All in all, we infer the existence of a constant $C_\varepsilon(\xi)$ such that we have

$$|t_m|^{2\varepsilon} \int_{X_t} \left|\widetilde{u}_{|X_t}\right|^2_{\theta_\varepsilon, h_{L,\varepsilon}} \theta_\varepsilon^n \leqslant C_\varepsilon(\xi) \tag{9.51}$$

because the volume of each fiber $(X_t, \theta_\varepsilon)$ is bounded from above by a positive constant independent of t (and ε). Therefore, the claim is proved and so is Theorem 9.2 if the line bundle (L, h_L) verifies hypothesis $\mathcal{H}_2$.

If the curvature of L is trivial when restricted to fibers of p, then we can use an arbitrary Kähler metric on X to define the norm on $\mathcal{K}^i$. Given this, the proof is completely identical, and we will not provide further details.

As we have remarked right after (9.30), the set $\mathcal{B}$ may be strictly smaller than $V \setminus \Delta$. However, the complement is contained in the smooth loci of the map p, and the boundedness of function (9.30) follows from basic Hodge theory. □

Remark 6. One can also prove the claim by a slightly different argument, which avoids the use of restriction (9.45). The observation is that we have

$$\left|\widetilde{u}_{|X_t}\right|^2_{\theta_\varepsilon, h_{L,\varepsilon}} \leqslant |\widetilde{u}|^2_{\theta_\varepsilon, h_{L,\varepsilon}} \tag{9.52}$$

at each point of the fiber X_t. This follows by a simple linear algebra calculation. The right-hand side of (9.52) can be bounded from above by using the fact that θ_ε is a metric with Poincaré singularities when restricted to U. In inequality (9.43) however, we have to work with the left-hand-side term of (9.52) directly, mainly because we have to obtain an upper bound that is independent of ε, and the metric h_L is only assumed to be strictly positively curved in the fibers' directions. □

Remark 7. The proof of Theorem 9.2 is much easier than the one establishing the *positivity* of the direct image sheaf

$$\mathcal{F} := p_\star(K_{X/Y} + L)$$

if (L, h_L) is semi-positively curved. The reason is that in order to show that the natural metric of $\mathcal{F}$ extends across the eventual singularities of the map p, one has to show that the function

$$t \to \|[u_t]\|_t^2$$

is bounded *from below* by a strictly positive constant as soon as we have $[u_t]_{t_0} \notin m_0 \mathcal{F}_{t_0}$, where $t_0 \in V \cap \Delta$ and $m_0 \subset \mathcal{O}_{Y,t_0}$ is the maximal ideal associated to this point.
On the other hand, in Theorem 9.2, we assume that L is endowed with a metric whose curvature satisfies very strong uniformity requirements (with respect to the fibers of p). As we will see in Theorem 9.5, it is not always very easy to construct such metrics, e.g., if $L = -K_{X/Y}$. □

As already mentioned, the proof of Proposition 9.3 is practically contained in the arguments invoked for Theorem 9.2. The only additional tool would be the following statement, generalizing [11].

Lemma 9.11. *Let $\Omega \subset Y$ be a small coordinate set centered at the point y_0. For each $i = 1, \dots, m$, there exists a vector field v_i defined on $p^{-1}(\Omega)$ with the following properties.*

(i) *We have $dp(v_i) = \frac{\partial}{\partial t_i}$ for $i = 1, \dots, m-1$ and $dp(v_m) = t_m \frac{\partial}{\partial t_m}$ on Ω.*

(ii) *On the open set $U \subset X$ as above, we have*

$$v_i = \frac{\partial}{\partial z_{n+i}} + \sum_{j=1}^{q} \psi_i^j \left(b_{m+n} z_j \frac{\partial}{\partial z_j} - b_j z_{m+n} \frac{\partial}{\partial z_{m+n}} \right) + \sum_{l=q+1}^{n} \psi_i^l \frac{\partial}{\partial z_l} \tag{9.53}$$

for $i = 1, \dots, m-1$ as well as

$$v_m = \frac{1}{b_{m+n}} z_{m+n} \frac{\partial}{\partial z_{n+m}} + \sum_{j=1}^{q} \psi_m^j \left(b_{m+n} z_j \frac{\partial}{\partial z_j} - b_j z_{m+n} \frac{\partial}{\partial z_{m+n}} \right) + \sum_{l=q+1}^{n} \psi_m^l \frac{\partial}{\partial z_l}, \tag{9.54}$$

where the functions ψ_j^i are smooth on U.

Proof. The proof is straightforward: locally the lifting in (i) and (ii) clearly exists, given the structure of the map p. We glue them by a partition of unit. As the logarithmic tangent bundle is invariant with respect to a change of coordinates, the conclusion follows. □

We remark that in order to construct the vector fields as in Lemma 9.11, it is enough to assume the following:

(s_1) The singular values of the map p are contained in an snc divisor Δ.

(s_2) The inverse image $p^{-1}(\Delta)$ is a divisor W with snc support.

9.3. Proof of Theorem 9.4

We follow the method of Viehweg–Zuo; cf. [42].
In order to simplify the writing, we introduce the notation

$$E^s_{X/Y} := \mathcal{R}^s p_\star \left(\Omega^{n-s}_{X/Y} \langle W \rangle \otimes L \right) \tag{9.55}$$

for $s \geqslant 0$.

Then for each $s \geqslant 0$, we have the holomorphic map

$$\tau^s : E^s_{X/Y} \to E^{s+1}_{X/Y} \otimes \Omega^1_Y\langle\Delta\rangle \tag{9.56}$$

as recalled at the beginning of the current section. By hypothesis, there exists a section σ of the bundle

$$K_{X/Y} + W - p^\star(\Delta) + L \tag{9.57}$$

such that the map

$$ks^{(1)}_\sigma : \mathcal{O}_Y \to E^1_{X/Y} \otimes \Omega^1_Y\langle\Delta\rangle \tag{9.58}$$

given by the cup product of σ with the Kodaira–Spencer class is non-identically zero on a non-empty open subset of Y.

Then the proof of Theorem 9.4 is obtained as follows. Let

$$ks^{(j)}_\sigma : \mathcal{O}_Y \to E^j_{X/Y} \otimes \mathrm{Sym}^j\, \Omega^1_Y\langle\Delta\rangle \tag{9.59}$$

be the map deduced from the iterated Kodaira–Spencer map (9.16).

We denote by Π_j the projection

$$E^j_{X/Y} \otimes \mathrm{Sym}^j\, \Omega^1_Y\langle\Delta\rangle \to \left(E^j_{X/Y}/\mathcal{K}^j\right) \otimes \mathrm{Sym}^j\, \Omega^1_Y\langle\Delta\rangle, \tag{9.60}$$

and let i be the smallest integer such that $\Pi_i \circ ks^{(i)}_\sigma$ is identically zero. This means that the image of $ks^{(i)}_\sigma$ is contained in

$$\mathcal{K}^i \otimes \mathrm{Sym}^i\, \Omega^1_Y\langle\Delta\rangle \tag{9.61}$$

and that $ks^{(i)}_\sigma$ is not identically zero. Indeed, if $ks^{(i)}_\sigma$ would be zero, then the image of $ks^{(i-1)}_\sigma$ is contained in $\mathcal{K}^{i-1} \otimes \mathrm{Sym}^{i-1}\, \Omega^1_Y\langle\Delta\rangle$. This contradicts our choice of i.

Note that we have $i \geqslant 1$ thanks to assumption $(\mathcal{H})_3$. We obtain a non-identically zero section of bundle (9.61). This in turn defines a non-trivial map

$$\mathcal{K}^{i\star}_f \to \mathrm{Sym}^i\, \Omega^1_Y\langle\Delta\rangle \tag{9.62}$$

since the dual of $\mathcal{K}^i$ is the same as the dual of $\mathcal{K}^i_f$. We remark that this already shows that the bundle $\mathrm{Sym}^i\, \Omega^1_Y\langle\Delta\rangle$ has a semi-positively curved subsheaf (i.e., the image of map (9.62)).

If the curvature of (L, h_L) satisfies hypothesis $(\mathcal{H}_2)$ and if moreover there exists an open subset $\Omega \subset Y$ together with a positive $\varepsilon_0 > 0$ such that

$$\Theta_{h_L}(L) \leqslant -\varepsilon_0 p^\star(\omega_Y)$$

on $p^{-1}(\Omega)$, then the curvature properties of the kernels $\mathcal{K}^i s$ are considerably better:

- The sheaf $\mathcal{K}^i$ admits a singular semi-negatively curved Hermitian metric, which is smooth on a Zariski open subset of Y.
- For any local holomorphic section $[u_t]$ of the bundle $\mathcal{K}^i|_\Omega$, we have

$$dd^c \log \|[u]\|^2 \geqslant \varepsilon_0 \sum_{j=1}^m \sqrt{-1}dt_j \wedge d\bar{t}_j. \tag{9.63}$$

In this context, we recall the following result, which concludes the proof of the last part of Theorem 9.4.

Let Y be a smooth projective variety, and let $\mathcal{F}$ be a torsion-free coherent sheaf on Y. We consider $\mathbb{P}(\mathcal{F}) := \mathcal{P}roj(\bigoplus_{m\geqslant 0} S^m(\mathcal{F}))$ the scheme over Y associated to $\mathcal{F}$, together with the projection $\pi : \mathbb{P}(\mathcal{F}) \to Y$. We denote by $\mathcal{O}_{\mathcal{F}}(1)$ the tautological line bundle on $\mathbb{P}(\mathcal{F})$. Let $Y_1 \subset Y$ be a Zariski open subset on which $\mathcal{F}$ is locally free (in particular, $\mathbb{P}(\mathcal{F})$ is smooth over Y_1) and $\operatorname{codim}_Y(Y \setminus Y_1) \geqslant 2$.

The following statement is established in [28].

Theorem 9.12 [28]. *We suppose that $\mathcal{O}_{\mathcal{F}}(1)|_{\pi^{-1}(Y_1)}$ admits a singular Hermitian metric g with semi-positive curvature and that there exists a point $y \in Y_1$ such that $\mathcal{I}(g^k|_{\mathbb{P}(\mathcal{F}_y)}) = \mathcal{O}_{\mathbb{P}(\mathcal{F}_y)}$ for any $k > 0$, where $\mathbb{P}(\mathcal{F}_y) = \pi^{-1}(y)$. Assume moreover that there exists an open neighborhood Ω of y such that $\Theta_g(\mathcal{O}(1)) - \pi^\star\omega_Y \geqslant 0$ on $\pi^{-1}(\Omega)$. Then $\mathcal{F}$ is big.*

This result together with the properties of the kernels $\mathcal{K}_f^j$ summarized in the bullets above shows that each of $\mathcal{K}_f^{j\star}$ is big; in particular, this applies to $\mathcal{K}_f^{i\star}$. Let A be an ample line bundle. We obtain a section of $\operatorname{Sym}^M \mathcal{K}_f^{i\star} \otimes A^{-1}$ for $M \gg 0$, which induces a section of $\otimes^{iM}\Omega_Y^1\langle\Delta\rangle \otimes A^{-1}$ thanks to (9.62). □

Remark 8. Hypothesis $(\mathcal{H}_i)$ (for $i = 1, 2$) together with the existence of a section σ of $K_{X/Y} + W - p^\star(\Delta) + L$ has a few consequences on the map p itself, which we will now discuss. We write

$$K_{X/Y} = L_1 - L, \tag{9.64}$$

where $L_1 := \big(K_{X/Y} + W - p^\star(\Delta) + L\big) + p^\star(\Delta) - W$. In particular, L_1 is an effective line bundle. As for $-L$, it is semi-positive and strictly positive/trivial on the fibers of p. Therefore, the relative canonical bundle $K_{X/Y}$ is relatively big in the first case and pseudo-effective in the second.

If moreover, the curvature of L verifies the requirements in the second part of Theorem 9.4, then for any m large enough, the bundle $\det p_\star(mK_{X/Y})$ is big. From this perspective, Theorem 9.4 is similar to (but slightly weaker than) the main statement of Popa–Schnell in [29]. □

9.4. Metrics on the relative canonical bundle

In this section, we will establish the following result; afterward, we will show that it implies Theorem 9.5.

Theorem 9.13. *Let $p : X \to Y$ be an algebraic fiber space, and let ω be a fixed reference metric on X. We assume that the following properties are satisfied.*

1. *The canonical bundle of the generic fiber of p is ample.*
2. *There exists a positive $m \gg 0$ such that the line bundle*

$$\det p_\star(mK_{X/Y})$$

is big.

Then there exist an effective, p-contractible divisor Ξ on X and a singular metric $h_{X/Y} = e^{-\varphi_{X/Y}}$ on $K_{X/Y} + \Xi$ such that:

(i) *There exists $\varepsilon_0 > 0$ for which we have $dd^c\varphi_{X/Y} \geqslant \varepsilon_0\omega$ in the sense of currents on X, i.e., the curvature of $(K_{X/Y} + \Xi, h_{X/Y})$ is strongly positive.*

(ii) *The restriction $h_{X/Y}|_{X_y}$ is non-singular for each $y \in B$, where B is a Zariski open subset of Y (which can be described in a precise manner).*

Before presenting the arguments of the proof, we give a few comments about the preceding statement.

Remark 9. If instead of hypothesis (1) above we assume that K_{X_y} is big, then the metric $e^{-\varphi_{X/Y}}$ can be constructed so that it has the same singularities as the *metric with minimal singularities* on this bundle. It would be really interesting to develop the techniques in sections (1)–(6) under the assumption that the bundle L is endowed with a sequence of smooth metrics h_ε such that the curvature θ_ε admits a lower bound

$$\theta_\varepsilon \geqslant (\varepsilon_0 - \lambda_\varepsilon)\omega,$$

where $\varepsilon_0 > 0$ and λ_ε is a family of positive functions uniformly bounded from above and converging to zero almost everywhere. □

Remark 10. If the Kodaira dimension of the fibers of p is not maximal, we can derive a similar result – of course, conclusion (i) has to be modified accordingly. However, in the absence of hypothesis (2), it is not clear what we can expect. □

Proof. The arguments that follow rely on two results. We recall next the first one; cf. [8].

Theorem 9.14 [8]. *We assume that K_X is $\mathbb{Q}$-effective when restricted to the generic fiber of p. For any positive integer m, there exist a real number $\eta_0 > 0$ and a divisor Ξ on X such that the codimension of $p(\Xi)$ is at least two and such that the difference*

$$K_{X/Y} + \Xi - \eta_0 p^\star\left(\det\left(p_\star(mK_{X/Y})\right)\right) \tag{9.65}$$

is pseudo-effective.

Actually, what we will use is not this result in itself but its proof – this will be needed in order to establish (ii) above. Hence the plan for the remaining part of this subsection is to review the main parts of the proof as in [8] (see also the references therein) and to extract the statement we want – this concerns the singularities of the current in (9.65).

• If the map p is a smooth submersion, then the arguments in [8] are borrowed from Viehweg's weak semi-stability theorem proof; cf. [40]. The first observation is that given any vector bundle E of rank r, we have a canonical injection

$$\det(E) \to \otimes^r E, \tag{9.66}$$

and hence a nowhere-vanishing section of the bundle

$$\otimes^r E \otimes (\det(E))^\star. \tag{9.67}$$

We apply this to the direct image

$$E := p_\star\left(m_0 K_{X/Y}\right), \tag{9.68}$$

which is indeed a vector bundle by the invariance of plurigenera [35]. Here m is a positive integer, large enough so that the multiple $m_0 K_{X_t}$ is very ample, and r is the rank of the direct image (9.68).

In this case, the bundle $\otimes^r E$ can be interpreted as a direct image of the relative pluricanonical bundle of the map

$$p^{(r)} : X^{(r)} \to Y, \tag{9.69}$$

where $X^{(r)} := X \times_Y \cdots \times_Y X$ is the rth fibered product corresponding to the map $p : X \to Y$. Note that $X^{(r)} \subset \times^r X$ is a non-singular submanifold of the r-fold product of X with itself (thanks to the assumption that p is a submersion).

The important observation is that we have the formulas

$$p_\star^{(r)}\left(m_0 K_{X^{(r)}/Y}\right) = \otimes^r p_\star\left(m_0 K_{X/Y}\right) \tag{9.70}$$

as well as

$$K_{X^{(r)}/Y} = \prod \pi_i^\star(K_{X/Y}), \tag{9.71}$$

where $\pi_i : X^{(r)} \to X$ is induced by the projection on the ith factor.

By relations (9.67) and (9.70), we have a section, say σ, of the bundle

$$m_0 K_{X^{(r)}/Y} - p^{(r)\star}\left(\det p_\star\left(m_0 K_{X/Y}\right)\right). \tag{9.72}$$

Formula (9.71) shows that the restriction of the bundle $K_{X^{(r)}/Y}$ to the diagonal $X \subset X^{(r)}$ is precisely $rK_{X/Y}$, so all in all, the restriction of the bundle in (9.72) to X coincides with (9.65). However, we cannot use directly the section σ in order to conclude since by construction, we have $\sigma|_X \equiv 0$. Indeed this can be seen as follows. In order to simplify the writing, let us assume that the rank of E in (9.68) is two. If E is any rank two vector bundle, the section of (9.67) can be expressed as

$$(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (e_1 \wedge e_2)^{-1}, \tag{9.73}$$

where e_1, e_2 is any basis of E. Now if we take E as in (9.68), the section $\sigma_{x,y}$ will be a multiple of $f(x)g(y)(e_x \otimes e_y - e_y \otimes e_x) \otimes (s_1 \wedge s_2)^{-1}_{(x,y)}$, and this vanishes identically on the diagonal $x = y$. We denote here by e the local frame for $m_0 K_{X/Y}$ and s_1, s_2 a basis of the space of sections of this bundle over the fiber containing x (and y).

To bypass this difficulty, we will use the following excellent trick invented by Viehweg; cf. [40]. Let $\varepsilon_0 > 0$ be a small enough positive rational number such that the pair

$$\left(X^{(r)}, B\right) \tag{9.74}$$

is klt, where the boundary $B := \varepsilon_0 Z_\sigma$ is the ε_0 multiple of the divisor $Z_\sigma := (\sigma = 0)$.

The next step is to apply the results in [6] and deduce that there exists a fixed ample line bundle A_Y on Y such that for any $k \geqslant 1$ divisible enough, the restriction map

$$H^0\left(X^{(r)}, k\left(K_{X^{(r)}/Y} + B\right) + p^{(r)\star} A_Y\right) \to H^0(X_y^{(r)}, k(K_{X_y^{(r)}} + B|_{X_y^{(r)}})) \tag{9.75}$$

is surjective for any $y \in Y \setminus \Delta$. Now the bundle on the right-hand side of (9.75) is equal to

$$k(1+\varepsilon_0 m_0)\prod_{i=1}^{r}\pi_i^\star\left(K_{X_y}\right), \tag{9.76}$$

so it has many sections, given that K_{X_y} is ample. Moreover, all the sections in question are automatically $L^{2/k}$ integrable, given that $(X^{(r)}, B)$ and its restriction to the generic fiber are klt. For the complete argument concerning the surjectivity of map (9.75), we refer to [8] (and the references therein).

In particular, for each k divisible enough, the bundle

$$k\left(K_{X^{(r)}/Y}+B\right)+p^{(r)\star}A_Y \tag{9.77}$$

admits a holomorphic section whose restriction to the diagonal $X \subset X^{(r)}$ is not identically zero. Indeed, we have (many) sections of bundle (9.75) on fiber $X_y \times \cdots \times X_y$ whose restriction to the diagonal $X_y \subset X_y \times \cdots \times X_y$ is not identically zero.

In conclusion, this shows the existence of a positively curved metric on the bundle

$$r(1+\varepsilon_0 m_0)K_{X/Y}-\varepsilon_0 p^\star \det\left(p_\star(m_0 K_{X/Y})\right)+\frac{1}{k}p^\star A_Y, \tag{9.78}$$

whose restriction to the generic fiber X_y of p is induced by the sections of multiples of K_{X_y}.

• The general case where p is simply a surjective map is much more involved; the complications arise from the fact that the fibered product $X^{(r)}$ is no longer smooth. However, as shown in [8], one obtains a similar result, modulo adding the divisor Ξ, which projects in codimension greater than two. We will not reproduce here the proof but just mention that modulo the singularities of $X^{(r)}$, the structure of the argument is identical to the case detailed in the preceding bullet – in particular, we control the singularities of the restriction of the metric on (9.78) to the generic fiber X_y.

Remark 11. It would be really interesting to have a more direct proof of Theorem 9.14, i.e., without using Viehweg's trick. □

Recall that we have an integer m_0 above such that the following conditions are satisfied.

(i) The bundle $m_0 K_{X_y}$ is very ample.

(ii) The bundle $\det\left(p_\star(m_0 K_{X/Y})\right)$ is big.

By point (ii) above, there exists a positive integer $m_1 \gg 0$ such that

$$m_1 \det\left(p_\star(m_0 K_{X/Y})\right) \simeq A_Y + E_Y, \tag{9.79}$$

where E_Y is an effective divisor on Y.

The properties of bundle (9.78) combined with our previous considerations show the existence of a metric $e^{-\psi_{X/Y}}$ on the bundle $K_{X/Y}+\Xi$ with the following properties.

(a) The metric $e^{-\psi_{X/Y}}$ is semi-positively curved, and it has algebraic singularities (meaning that its local weights are of the form log of a sum of squares of holomorphic functions, modulo a smooth function). Moreover, the restriction $e^{-\psi_{X/Y}}|_{X_y}$ is smooth for any y belonging to a Zariski open subset of Y.

(b) The form $dd^c\psi_{X/Y}$ is smooth and definite positive on the p-inverse image of a (non-empty) Zariski open subset.

We see that the main difference between properties (a) and (b) of the metric $e^{-\psi_{X/Y}}$ and the conclusion of Theorem 9.13 is strict positivity, i.e., the existence of ε_0 as in (i). In order to finish the proof, we will use a result due to Nakamaye (cf. [24]) concerning the augmented base loci of nef and big line bundles.

As a preparation for this, we blow up the ideal corresponding to the singularities of $e^{-\psi_{X/Y}}$; let $\pi : \widetilde{X} \to X$ be the associated map. We write

$$\pi^\star\left(K_{X/Y} + \Xi\right) \simeq \mathcal{L}_1 + \mathcal{L}_2 \tag{9.80}$$

such that $\mathcal{L}_j$ above are $\mathbb{Q}$-line bundles induced by the decomposition of the inverse image of the curvature of $e^{-\psi_{X/Y}}$ into smooth and singular parts denoted by θ_1 and θ_2, respectively. By properties (a) and (b) above, θ_1 is smooth and definite positive in the complement of an algebraic set that projects into a strict subset of Y. As for θ_2, it is simply the current of integration on an algebraic subset of $\widetilde{X}$ that equally projects properly into Y.

In particular, the bundle $\mathcal{L}_1$ is nef and big (given that it admits a metric whose curvature is θ_1). We denote by $\mathrm{B}_+(\mathcal{L}_1)$ the stable base locus of the $\mathbb{Q}$-bundle $\mathcal{L}_1 - A$, where A is any small enough ample on $\widetilde{X}$. We recall the following result (cf. [24]).

Theorem 9.15 [24]. *The algebraic set* $\mathrm{B}_+(\mathcal{L}_1)$ *is the union of* $W \subset \widetilde{X}$ *such that* $\int_W \theta_1^d = 0$, *where* d *is the dimension of* W.

Given the positivity properties of the form θ_1, we infer that the set $p \circ \pi\left(\mathrm{B}_+(\mathcal{L}_1)\right)$ is strictly contained in Y. This then implies that we can endow $\mathcal{L}_1$ with a metric whose curvature can be written as the sum of a Kähler metric (given by the ample A above) and a closed positive current whose singularities are $p \circ \pi$-vertical. By combining it with the metric on $\mathcal{L}_2$, we get a metric with the same properties on the inverse image $\pi^\star(K_{X/Y} + \Xi)$.

The metric $h_{X/Y}$ is obtained by pushforward, and Theorem 9.13 is proved. □

9.5. Proof of Theorem 9.5

The hypothesis of 9.5 together with the results in [17, 20, 29] that for each $m \gg 0$ the bundle

$$\det\left(p_\star(mK_{X/Y})\right) \tag{9.81}$$

is big. This can equally be obtained along the line of arguments in this paper as follows. It is shown in [4] that the curvature form of the vector bundle $p_\star(mK_{X/Y})$ has no zero eigenvalue – as if not, the 'maximal variation' hypothesis would be contradicted. Thus the curvature form of its determinant is semi-positive and strictly positive at some point. The fact that it is big follows, e.g., by holomorphic Morse inequalities. We now apply Theorem 9.13, and Theorem 9.5 is proved.

9.6. Proof of Theorem 9.8

This is almost a linear combination of the results obtained in the previous sections: we define L as

$$L := -K_{X/Y} + \sum (t^i - 1) W_i \tag{9.82}$$

so that bundle (9.57) is trivial (in particular, it has a section).

We intend to use Theorem 9.4 in order to conclude, but we do not seem to prove that bundle (9.82) admits a metric satisfying hypothesis $(\mathcal{H}_2)$. However, in Corollary 9.7, we construct a family of metrics whose curvature properties represent an approximation of this hypothesis. We will show next that we can adapt the proof of Theorem 9.4 and obtain the same conclusion by using Corollary 9.7 instead of $(\mathcal{H}_2)$.

We will denote by h_L the metric on (9.82) induced by the metric $h^{(1)}_{X/Y}$. Note that the curvature current corresponding to this metric is semi-positive on X and strictly positive on generic fibers of p by Theorem 9.6.

Let $[u_t]$ be a local holomorphic section of the bundle $\mathcal{K}^i$, defined on the open coordinate subset $V \subset Y$ centered at a generic point of Δ. We have to show that the quantity

$$\sup_{t \in V \setminus \Delta} \log \|[u_t]\|^2_t < \infty \tag{9.83}$$

is finite. To this end, we follow the same path as in the proof of 9.2; so we will only highlight the main differences next.

By Corollary 9.7, we have

$$h_L|_{X_t} = \lim_{\eta \to 0} h^{(\eta)}_{X/Y}|_{X_t} \tag{9.84}$$

for each $t \in V \setminus \Delta$. For each $0 < \varepsilon \ll \eta$, we define the perturbation $h_{L,(\varepsilon,\eta)}$ of the metric $h^{(\eta)}_{X/Y}$ precisely as in (9.35) (meaning that h_L in (9.35) is replaced with $h^{(\eta)}_{X/Y}$ in the actual context). Then we have

$$\sup_{t \in V \setminus \Delta} |t_m|^{C(1-\eta)} \|[u_t]\|^2_{t,(\varepsilon,\eta)} \leqslant C(\varepsilon, u) < \infty, \tag{9.85}$$

where C in (9.85) is a fixed constant, which is large enough compared with the multiplicities b^i in the expression of the divisor $p^{-1}(\Delta)$. Inequality (9.85) is established by the same procedure as the proof of the Claim – the only slight difference here is the presence of a factor of order $\mathcal{O}(1-\eta)$ with the wrong sign in the metric $h^{(\eta)\star}_{X/Y}$, accounting for the first negative term in the curvature estimate (a) in Corollary 9.7. This term is tamed by the factor $|t_m|^{C(1-\eta)}$. Inequality (9.85) together with the mean inequality shows that (9.83) holds true.

Hence, under the hypothesis of Theorem 9.8, the conclusions of Theorems 9.2 and 9.4 hold true provided that the bundle (L, h_L) is chosen as in (9.82).

9.7. Proof of Theorem 9.9

Let $p : X \to Y$ be a Calabi–Yau family, which has maximal variation. By the results in [17, 20, 29], we infer that for all $m \gg 0$ divisible enough, the bundle

$$\mathcal{L} := \big(p_\star(mK_{X/Y})\big)^{\star\star} \tag{9.86}$$

is big. Actually, the metric version of this statement is true, as we will see after recalling a few facts.

Let $y \in Y$ be a regular value of the map p. Since $c_1(X_y) = 0$, there exists a positive integer m such that the bundle mK_{X_y} admits a nowhere-vanishing section, say, s_y. The invariance of plurigenera [35] shows that for any coordinate open subset V containing y, there exists a section s of the bundle $mK_{X/Y}|_{p^{-1}(V)}$ whose restriction to the fiber X_y equals s_y.

By shrinking V, we can assume that s is nowhere-vanishing, and therefore it gives a trivialization of the bundle $\mathcal{L}|_V$. With respect to this trivialization, the local weight of the metric on $\mathcal{L}$ in [5, 28] is given by the function

$$\varphi_{\mathcal{L}}(w) = \log \int_{X_w} |s|^{2/m}, \tag{9.87}$$

where the expression under the integral in (9.87) is the volume element on X_w induced by the restriction of s.

It is proved in [28] that metric (9.87) is semi-positively curved. Under the hypothesis of 9.9, the following stronger result holds true.

Lemma 9.16. *We assume that the Calabi–Yau family p has maximal variation. Then the curvature of $(\mathcal{L}, e^{-\varphi_{\mathcal{L}}})$ is positive definite on each compact subset of a non-empty Zariski open subset of Y. In particular, $\mathcal{L}$ is big.*

Proof. We observe that the metric $e^{\varphi_{\mathcal{L}}}$ is smooth on a Zariski open subset $Y_0 \subset Y$. Let $\Theta_{\mathcal{L}}$ be the corresponding curvature form. We claim that $\Theta_{\mathcal{L}}^{\dim(Y)} > 0$ on $Y_0 \cap U$, where $U \subset Y$ is such that the restriction of the Kodaira–Spencer map $ks|_U$ to U is injective. The argument goes as follows: if we have $\Theta_{\mathcal{L}}^{\dim(Y)} = 0$, then the kernel of $\Theta_{\mathcal{L}}$ would define a foliation that is not necessarily holomorphic but whose leaves are holomorphic. The fact that the leaves are holomorphic is basically a consequence of the fact that $\Theta_{\mathcal{L}}$ is of (1,1)–type. We refer to [20] and the references therein for a complete explanation.

We consider a holomorphic disk $\mathcal{D}$ contained in a generic leaf of this (local) foliation. Then the family $f : X_{\mathcal{D}} \to \mathcal{D}$ induced by p is a submersion, where $X_{\mathcal{D}} := p^{-1}(\mathcal{D})$, and moreover the curvature of the direct image $f_\star(mK_{X_{\mathcal{D}}/\mathcal{D}})$ is equal to zero. If $m = 1$, we invoke the results in [10] to infer that this forces the vanishing of the Kodaira–Spencer class of f. If $m \geqslant 2$, we argue as follows. By the definition of the metric $\varphi_{X/Y}$ in § 8.2 together with the fact that the curvature of $f_\star(mK_{X_{\mathcal{D}}/\mathcal{D}})$ is zero, we infer that

$$i\partial\bar{\partial}\varphi_{X/Y} = 0.$$

This implies the vanishing of ω_{WP}; cf. (8.8). We therefore obtain a contradiction; so Lemma 9.16 is proved. □

As a consequence, we obtain the following statement.

Corollary 9.17. *Let $p : X \to Y$ be a Calabi–Yau family that has maximal variation. Then the relative canonical bundle $K_{X/Y}$ has the following property. There exists a positive integer m such that the curvature current Θ of the m-Bergman metric $e^{-\varphi_{X/Y}}$*

is semi-positive on X and greater than the inverse image of a metric on the pre-image of a non-empty open subset of Y. Moreover, we have

$$\Theta \geqslant \sum_i (t^i - 1)[W_i] \tag{9.88}$$

on X.

Proof. Indeed, in our setup, the m–Bergman metric has the expression

$$e^{\varphi_{X/Y}(x)} = \frac{|s|_x^{2/m}}{\int_{X_y} |s|^{2/m}}, \tag{9.89}$$

where $y = p(x)$. Therefore, we have $\Theta = p^\star(\Theta_{\mathcal{L}})$, and our statement follows from Lemma 9.16. □

Therefore, the conclusion is that Theorem 9.9 is a direct consequence of Theorem 9.2 together with Theorem 9.4.

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