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Approximating the Transition Probability Function Corresponding to the Solution of Stochastic Optimal Velocity Dynamical Model

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***Abstract*— In this paper a stochastic optimal velocity dynamical model in considered. The probabilistic behavior of the solution of such dynamics can be explained by its transition density function. We investigate an explicit approximation of this transition function through an iterative method.**

I. INTRODUCTION AND RELATED WORKS

Optimal Velocity (OV) dynamics proposed by Bando et al. [1] is used to explain the drivers' reaction in the sense of acceleration or deceleration, in response to the behavior of the other vehicles. More accurately,

$$\ddot{X}^{(n)}(t) = \alpha \left\{ V\left(\frac{X^{(n-1)}(t) - X^{(n)}(t)}{d}\right) - \dot{X}^{(n)}(t) \right\}, \qquad (1)$$

where $X^{(n)}(t)$ is the location of the n-th vehicle, $\alpha > 0$ is a constant associated with the drivers' sensitivity to any change, d is a scaling parameter and the real valued function V is a monotonically increasing and bounded function which will be explicitly defined in the next section. In this model, the optimal velocity is calculated for each vehicle and the comparison of the optimal velocity with current speed decides the acceleration or deceleration of the vehicle under consideration.

Optimal Velocity model is extended in different directions. [2], [3] consider the delay in reaction of the drivers with respect to sudden changes. Simple stochastic versions of this model have also been studied in the literature. Drivers' uncertain behavior is discussed in [4, section 12] and references therein. Stochasticity which causes the traffic breakdown is investigated in [5] and more recently [6], [7] discuss the stochastic stability of OV models. The asymptotic behavior of the deterministically and stochastically perturbed OV models is studied in [8]. The rate of convergence to the limiting approximations for these perturbed models is investigated in [9].

For N vehicles moving in one line, we can apply the following change of variables in model (1)

$$\begin{aligned} z^{(2n-1)}(t) &\overset{\text{def}}{=} X^{(1)}(t) - X^{(n)}(t) = v_0 t - X^{(n)}(t) \\ z^{(2n)}(t) &\overset{\text{def}}{=} \dot{X}^{(1)}(t) - \dot{X}^{(n)}(t) = v_0 - \dot{X}^{(n)}(t), \end{aligned} \qquad (2)$$

for $n \in \{1, \cdots, N\}$, where v_0 is the constant velocity of the first car. In fact, in this model the difference of the locations and the velocities with the first vehicle is considered (see [8] for more details). Such presentation will be very helpful in interpreting the solution of the new dynamics as will be explained later in this section. Applying the modification (2), equation (1) can be written in the form of

$$\begin{aligned} &\dot{z}^{(2n-1)}(t) = z^{(2n)}(t) \\ &\dot{z}^{(2n)}(t) = -\alpha \left\{ V\left(\frac{z^{(2n-1)}(t) - z^{(2n-3)}(t)}{d}\right) - v_0 + z^{(2n)}(t) \right\}. \end{aligned} \qquad (3)$$

In this research we are interested in the stochastic version of the OV model (3). We refer the interested readers to [7], [6] as well as [8] for importance and the applications of these models. We consider (controllable) noises with constant intensity in this work. However, the results of the analysis can be carried over to more general noises (non-constant diffusion terms) in a straightforward manner[1].

In particular, let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a probability space with complete filtration. For simplicity of notations and arguments, the result will be established for the first two vehicles and it would become clear that the extension to any number of vehicles will be immediate. The stochastic OV model of interest is in the form of

$$\begin{aligned} dX_t^\varepsilon &= Y_t^\varepsilon dt \\ dY_t^\varepsilon &= \beta(X_t^\varepsilon, Y_t^\varepsilon)dt + \varepsilon dW_t, \\ (X_0^\varepsilon, Y_0^\varepsilon) &= (x, y), \end{aligned} \qquad (4)$$

on $[0,T]$ for any $T > 0$. The drift term in this model is defind as

$$\beta(u,v) \overset{\text{def}}{=} -\alpha \left\{ V\left(\frac{u}{d}\right) + v - v_0 \right\}, \quad (u,v) \in \mathbb{R}^2$$

where the optimal velocity function is considered as a hyperbolic tangent function

$$V(u) \overset{\text{def}}{=} \tanh(u+2) - \tanh(2).$$

Constant $\alpha > 0$ is introduced in equation (1), $(W_t)_{t\geq 0}$ denotes a standard $\mathscr{F}_t$- Brownian Motion and $\varepsilon > 0$ is constant intensity of the Brownian motion.

The solutions of stochastic differential equation (SDE) (4) is a random process $(X_t^\varepsilon, Y_t^\varepsilon)$ in $\mathbb{R}^2$ that is interpreted as the distance and the difference of velocities between the two consecutive vehicles respectively. Therefore, the aim

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[1]It should be noted that, non-constant diffusion terms are required to satisfy some regularities, e.g. smooth and boundedness.

is to understand the probabilistic behavior of this process which represents the solution of the stochastic OV model. In particular, it is known that such a solution is indeed an $\mathscr{F}_t$-adapted Markov process and hence the corresponding transition probability (density) function explains its desired probabilistic behavior.

Our Contributions. In this paper, we study the solution of the stochastic dynamical model (4) by approximating its associated transition probability function in an explicit form. This transition density function explains what behavior to expect from the solutions in probabilistic sense. The construction of this transition function is by iteratively evolving a simple Gaussian density and its convolution with the solution of a Volterra equation.

Moreover, we show that the transition density function can be approximated in a bounded domain rather than $\mathbb{R}^2$ due to the structure of our problem. Such observation is crucial in practice since calculating the integrals in unbounded domains is impractical.

Finally, by rigorous study of probabilistic properties of the solution, we establish a powerful tool for further analysis of stochastic OV models.

Why OV model. The dynamical model (1) is a highly descriptive model in terms of the interaction between the leading and the following vehicles. Many analytical results applicable to this model can be directly extended to other optimal velocity dynamics. In addition, modifications of this basic model can suitably explain many recent technological advances such as adaptive cruise control and the dynamics of autonomous vehicles.

Organization of this paper is as follows. In section II we consider two special cases of the drift terms and we calculate their transition density functions. These two cases will motivate the general nonlinear drift term. In section III we consider the general case of the drift term and the transition density function for this case will be constructed iteratively. In section **??** we simulate some of the iterations of the constructed method. Finally, we discuss the results and possible future directions.

II. Motivation and Special Cases

In this section we study two special cases of the drift term $\beta(x,y)$. Discussing the transition density function in these cases will be insightful in constructing the density function of the general drift term in the next section.

A. Transition probability Function for Time-Dependent Drift Term

Let us suppose that the drift term β in (4) depends only on the time variable. In other words, the drift term can be approximated as a function of the form $\beta(t)$.

Such consideration is only from the theoretical point of view and we will explain how studying this case can be helpful in understanding the transition probability with nonlinear drift term (see remark 2.2).

In this case the solution of stochastic differential equation (4) can be presented by

$$\begin{pmatrix} X_t^\varepsilon \\ Y_t^\varepsilon \end{pmatrix} = \begin{pmatrix} x+yt+\int_0^t\int_0^s \beta(r)drds+\varepsilon\int_0^t W_s ds \\ y+\int_0^t \beta(s)ds+\varepsilon W_t \end{pmatrix} \tag{5}$$

on time interval $[0,T]$. Therefore, the solution of the system is a Gaussian process with mean vector

$$\mu_t \stackrel{\text{def}}{=} \begin{pmatrix} \mathbb{E}X_t^\varepsilon \\ \mathbb{E}Y_t^\varepsilon \end{pmatrix} = \begin{pmatrix} x+yt+\int_0^t\int_0^s \beta(r)drds \\ y+\int_0^t \beta(s)ds \end{pmatrix}.$$

To calculate the covariance matrix of the solution we need the following direct calculations

$$\mathbb{E}(W_t W_s) = s\wedge t,$$

$$\begin{aligned}\mathbb{E}\left\{\left(\int_0^t W_s ds\right)^2\right\} &= \mathbb{E}\left\{\int_0^t W_s ds \int_0^t W_r dr\right\} \\ &= \int_0^t\int_0^t \mathbb{E}\{W_s W_r\}dsdr = \frac{1}{3}t^3,\end{aligned}$$

$$\mathbb{E}\left\{W_t\int_0^t W_s ds\right\} = \int_0^t \mathbb{E}\{W_t W_s\}ds = \int_0^t sds = \frac{1}{2}t^2,$$

where $a\wedge b \stackrel{\text{def}}{=} \min\{a,b\}$. Therefore, the covariance matrix denoted by $A=(a_{i,j})$ is:

$$A(t) = \varepsilon^2\begin{pmatrix} \frac{1}{3}t^3 & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & t \end{pmatrix}, \quad \det A(t) = \frac{\varepsilon^4 t^4}{12}, \tag{6}$$

and so the inverse matrix will be:

$$\hat{A}(t) = (\hat{a}_{ij}) \stackrel{\text{def}}{=} A^{-1}(t) = \varepsilon^{-2}\begin{pmatrix} \frac{12}{t^3} & -\frac{6}{t^2} \\ -\frac{6}{t^2} & \frac{4}{t} \end{pmatrix}, \quad \det\hat{A}(t) = \frac{12}{\varepsilon^4 t^4}. \tag{7}$$

Let

$$z=(x,y)\in\mathbb{R}^2, \quad \xi=(\xi_1,\xi_2)\in\mathbb{R}^2,$$

then the transition probability function of the process in this case is a standard Gaussian density of the form

$$p(t,z,\xi) \stackrel{\text{def}}{=} \frac{\sqrt{\det\hat{A}(t)}}{2\pi}\exp\left\{-\frac{1}{2}\left(\xi-\mu_t,\hat{A}(t)(\xi-\mu_t)\right)_{\mathbb{R}^2}\right\} \tag{8}$$

where, $(a,b)_{\mathbb{R}^2}$ denotes the inner product in $\mathbb{R}^2$. Therefore, we can calculate the desired statistics of the trajectory $(X_t^\varepsilon, Y_t^\varepsilon)$ by using $p(t,z,\xi)$.

Remark 2.1: Equation (5) reveals an important interaction between the solutions X_t^ε and Y_t^ε. In fact, the perturbation in the dynamics of Y_t^ε is as the result of Brownian motion, while the perturbation in the dynamics of X_t^ε is generated by perturbation in Y_t^ε. This results in the variance of order t in y direction and order of t^3 in x direction. As we shall see later, this causes some degeneracy in the system.

B. Transition probability Function for Linear Drift Term

As the second case we consider the linearization of the drift term β in (4) along the trajectory (X_t, Y_t). That is,

$$\beta(X_t^\varepsilon, Y_t^\varepsilon) \approx \beta(X_t, Y_t) + \partial_x \beta(X_t, Y_t)(X_t^\varepsilon - X_t) + \partial_y \beta(Y_t^\varepsilon - Y_t). \tag{9}$$

It is not difficult to rewrite the dynamical system (4) as a linear SDE of the from

$$dZ_t^\varepsilon = \left(f(t) + F(t) Z_t^\varepsilon\right) dt + \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} dW_t, \tag{10}$$

where, $Z_t^\varepsilon = \begin{pmatrix} X_t^\varepsilon \\ Y_t^\varepsilon \end{pmatrix}$ and W_t is a standard 1-dim $\mathscr{F}_t$ Brownian motion and for some vector-valued function $f(t)$ and a matrix-valued function $F(t)$. Equation (10) is a linear SDE with initial value of $Z_0^\varepsilon = z = (x, y)$. Let us assume that $\Phi(t)$, the fundamental matrix solution of homogeneous form of (10), exists. For instance, if we consider

$$(X_t, Y_t)^\top = (X_\infty, 0)^\top \stackrel{\text{def}}{=} (d \cdot V^{-1}(v_0), 0)^\top,$$

which is the equilibrium solution of the OV model, then $\Phi(t) = e^{Ft}$ and can be explicitly calculated. We denote by

$$\hat{\Phi}(t) = \left(\hat{\phi}_{ij}(t)\right) \stackrel{\text{def}}{=} \Phi^{-1}(t).$$

Then, the solution of linear SDE (10) will be

$$Z_t^\varepsilon = \Phi(t) \left(z_0 + \int_0^t \hat{\Phi}(s) f(s) ds + \int_0^t \hat{\Phi}(s) B_\varepsilon dW_s \right),$$

which implies that solution $(Z_t^\varepsilon)_{t \geq 0}$ is a Gaussian process and hence it can be fully characterized by the mean vector

$$\mathbb{E} Z_t^\varepsilon = \Phi(t) \left(z_0 + \int_0^t \hat{\Phi}(s) f(s) ds \right). \tag{11}$$

and the covariance matrix

$$\begin{aligned} A(t) = \Phi(t) \mathbb{E} \Bigg\{ & \left(\int_0^t \hat{\Phi}(s) B_\varepsilon dW_s \right) \\ & \times \left(\int_0^t \hat{\Phi}(s) B_\varepsilon dW_s \right)^\top \Bigg\} \Phi^\top(t) \\ = \Phi(t) \mathbb{E} & \{H(t)\} \Phi^\top(t), \end{aligned} \tag{12}$$

where, $C^\top$ denotes the transpose of a matrix C. Elements of matrix $H = (H_{ij})$ can be calculated by

$$\begin{aligned} \mathbb{E} H_{ij}(t) = \mathbb{E} H_{ji}(t) &= \mathbb{E} \left\{ \left(\int_0^t \hat{\phi}_{ij}(s) dW_s \right) \left(\int_0^t \hat{\phi}_{ji}(s) dW_s \right) \right\}, \\ &= \int_0^t \hat{\phi}_{ij}(s) \hat{\phi}_{ji}(s) ds, \quad i, j = 1, 2, \end{aligned} \tag{13}$$

where all terms are directly calculated from the isometry of the Gaussian white noise.

Remark 2.2: The study of both previous cases shows that by focusing on the dominant interactions between X_t^ε and Y_t^ε rather than the nonlinear drift term, the transition probability function of the process is a Gaussian density. This implies that even in the presence of the nonlinear drift term we might be able to find the transition probability function by *evolving* some *Gaussian density* associated to our problem. In the next section we show that this is in fact the case. ◇

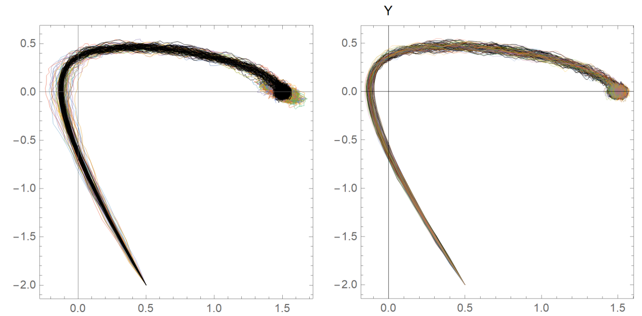

Fig. 1: Left figure: Trajectories of degenerate system (Dark trajectories) as well as the trajectories of approximate system (colored trajectories) for $\sigma = 0.05$.
Right figure: The same set of trajectories for $\sigma = 0.01$. For smaller noises the trajectories almost coincide.
Other common parameters are $\alpha = 2$, $v_0 = 0.5$, $d = 1$, $(X_0, Y_0) = (0.5, -1)$, $\varepsilon = 0.05$, and $T = 30$; Fifty trajectories are generated for each dynamical model.

III. TRANSITION PROBABILITY FUNCTION IN THE PRESENCE OF NONLINEAR DRIFT TERM

In this section we find an explicit approximation for transition probability function associated with the Markov process that represents the solution of (4). As mentioned in remark 2.1, this stochastic dynamical model is degenerate in the sense that it does not include any perturbation in the first equation. Such degeneracy creates some analytical complexities in bounding the growth rate of some terms. While it is possible to analyze the degenerate problem directly, in this paper to avoid analytical complexities we consider the following non-degenerate (elliptic) approximate dynamical system

$$\begin{aligned} dX_t &= Y_t dt + \sigma dU_t \\ dY_t &= \beta(X_t, Y_t) dt + \varepsilon dW_t, \end{aligned} \tag{14}$$

for $t \in [0, T]$ and where U_t and W_t are independent standard Wiener processes. The parameters σ and ε are positive intensity constants and nonlinear drift $\beta(x, y)$ is introduced before. We consider the initial condition to be $(X_0, Y_0) = (x, y)$.

It can be shown rigorously (see [8, Theorem 3] for a similar argument) that for sufficiently small intensity σ, the dynamical model (14) is a good approximation of dynamical model (4). More accurately, if $Z_t^{\sigma,\varepsilon} = (X_t^{\sigma,\varepsilon}, Y_t^{\sigma,\varepsilon})^\top$ denotes the solution of (14) and $Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon)^\top$ the solution of (4) on time interval $[0, T]$, then we have

$$\mathbb{P} \left\{ \sup_{t \in [0,T]} \left| Z_t^{\sigma,\varepsilon} - Z_t^\varepsilon \right| > \delta \right\} \leq \delta^{-2} \sigma^2 u(T), \tag{15}$$

for some bounded real valued function u on $[0, T]$. This means that for sufficiently small value of σ, for the main part $Z_t^{\varepsilon,\delta}$ will remain in δ-neighborhood of Z_t^ε on $[0, T]$, and the probability of residing outside this neighborhood is very small. Figure 1 illustrates the comparison between the trajectories of degenerate and non-degenerate systems and shows that for sufficiently small σ-noise the trajectories of the two models coincide with high probability (the right figure).

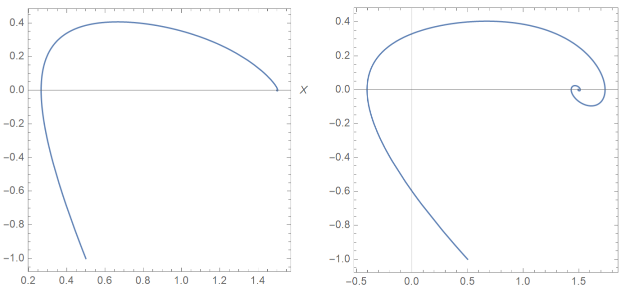

Fig. 2: Trajectory of Deterministic OV Model with $v_0 = 0.5$, $d = 1$, $(X_0, Y_0) = (0.5, -1)$ and $\alpha = 2$ and 0.5 respectively; $T = 50$.

Therefore, in this paper we consider the non-degenerate (elliptic) approximate model and will discuss its transition probability function in what follows.

A. *Presentation of Fundamental solution*

It is known that the transition probability function of a Markov process satisfies Kolmogorov equations in the form of an initial value partial differential equations (PDE) (c.f. (16)). In this section, we will derive an explicit form of such transition density functions by discussing the solution of initial value Kolmogorov equations.

Before we proceed to the main results, we need to recall some properties of the OV dynamics that helps in defining a proper PDE approximation model. In particular, for the deterministic model

$$\dot{X}_t = Y_t$$
$$\dot{Y}_t = \beta(X_t, Y_t),$$

we can study the energy level of the system by introducing the Hamiltonian

$$H(x,y) \stackrel{\text{def}}{=} \frac{1}{2}y^2 + W(x),$$

with

$$W(x) \stackrel{\text{def}}{=} \alpha \int_0^x V(x'/d)dx' - \alpha v_0 x,$$

which implies that

$$H(X_t, Y_t) - H(X_0, Y_0) \leq -\alpha \int_0^t Y_s^2 ds.$$

The negative value on the right hand side implies that the energy level of the system *dissipates* with time. Taking such behavior into consideration, the solution of the system is proven to be bounded in deterministic case ([8, Theorems 1]). Figure 2 illustrates the trajectory of the deterministic OV model for two different values of α while other parameters are fixed. Although the trajectory can be affected by parameters, the solution will remain bounded. Therefore, [8, Theorem 3] and (15) suggest that with high probability the solution of stochastic OV model remains in a bounded region for sufficiently small perturbations on time interval $[0, T]$. Figure 3 provides visualization of the boundedness of the stochastic trajectories for two sets of parameters (ε, σ). Moreover, Figure 4 shows the 95% confidence interval for generated trajectories which illustrates the neighborhood in which the stochastic trajectories reside in for sufficiently small noises.

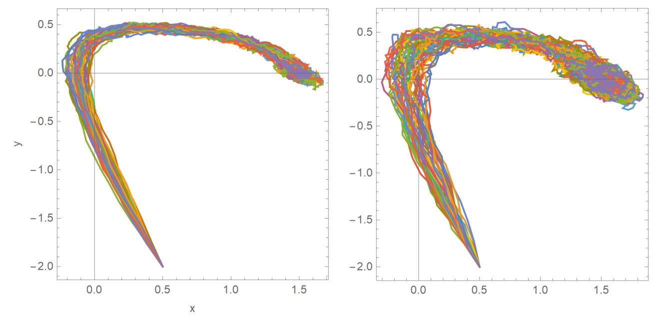

Fig. 3: Brownian perturbation of the OV model. Left Figure: $(\varepsilon, \sigma) = (0.05, 0.05)$. Right Figure: $(\varepsilon, \sigma) = (0.1, 0.1)$; $T = 50$ and fifty trajectories have been generated. Other parameters are fixed and the same as the deterministic case.

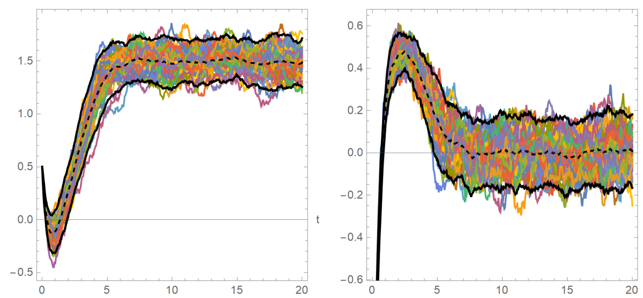

Fig. 4: Brownian perturbation of the OV model with 95% Confidence Interval for fifty generated trajectories. The Dashed line is the mean trajectory of the generated data. $(\varepsilon, \sigma) = (0.1, 0.1)$.

Putting all together, it is reasonable to approximate the solution of (14) in a sufficiently large bounded domain rather than the entire $\mathbb{R}^2$. Such consideration will remove the complexity of dealing with unbounded coefficients and integral domains from both theoretical and practical point of view and provides a good approximation of the solution. We define set D to be an open and bounded domain which is sufficiently large, and hence the transition probability function of the solution (14) (which is known to be a Markov process) can be approximated by the solution of the following backward Kolmogorov equations:

$$\begin{aligned} &\partial_t u(t,z) = Lu(t,z), \quad z = (x,y) \in D, \quad t \in (0,T] \\ &\lim_{t \searrow 0} u(t,z) = \delta_\xi(z), \end{aligned} \tag{16}$$

for any $\xi = (\xi_1, \xi_2) \in D$ and where δ_ξ is the Dirac distribution concentrated at ξ. The differential operator L (the generator of the Markov process) is defined as

$$Lu \stackrel{\text{def}}{=} \frac{1}{2}\sigma^2 \partial_{xx} u + \frac{1}{2}\varepsilon^2 \partial_{yy} u + y\partial_x u + \beta(x,y)\partial_y u,$$

which is defined based on coefficients of stochastic dynamical model (14). We refer interested readers to any standard textbook in stochastic differential equations like [10] for more detail on the infinitesimal operator of the Markov processes associated with the solution of the stochastic dynamical model of interest.

Therefore, our goal is to discuss an explicit solution of (16). We define the *parabolic operator* in the form

$$\mathscr{L} \stackrel{\text{def}}{=} L - \partial_t.$$

Definition 3.1: The solution of equations (16) is called fundamental solution of operator $\mathscr{L}$ or fundamental solution of $\mathscr{L}u = 0$.

Remark 3.1: Motivated by previous special drift cases, the idea is to start with a Gaussian density and (iteratively) evolve this kernel to solve (16). Such iterative scheme traces back to Levy and several papers discuss this method under different regularity assumptions, see for example [11] and references therein. In this paper, we customize this method and the proofs to serve the particular structure of our problem. $\diamond$

Let us consider only part of the differential operator L and define:

$$L_0 u \stackrel{\text{def}}{=} \frac{1}{2}\sigma^2 \partial_{xx} u + \frac{1}{2}\varepsilon^2 \partial_{yy} u,$$

and the corresponding parabolic operator $\mathscr{L}_0 = L_0 - \partial_t$. The fundamental solution of $\mathscr{L}_0 u = 0$ is then a Gaussian density with mean vector and covariance matrix

$$\mu = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = t \begin{pmatrix} \sigma^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$$

respectively, as we desired. We denote such Gaussian kernel by:

$$p^G(t,z,\xi) = \frac{(\sigma\varepsilon)^{-1}}{2\pi t} \exp\left\{ -\frac{\mathscr{K}(z,\xi)}{2t} \right\} \tag{17}$$

where,

$$\mathscr{K}(z,\xi) \stackrel{\text{def}}{=} \sigma^{-2}(\xi_1 - x)^2 + \varepsilon^{-2}(\xi_2 - y)^2,$$

and for any $\xi \in D$. In the next section, we evolve this kernel to get the (fundamental) solution of operator $\mathscr{L}$, i.e. transition probability function of the solution of (14).

B. Construction of Fundamental Solution

Before proceeding to the detailed proof, let us briefly mention the main results. In this section, we show that the fundamental solution of $\mathscr{L}u = 0$ (i.e. the transition density function of the solution of stochastic OV model) can be presented explicitly by:

$$\begin{aligned} p(t,z,\xi) &= p^G(t,z,\xi) + \int_0^t \int_D p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds \\ &\stackrel{\text{def}}{=} p^G(t,z,\xi) + H(t,z,\xi). \end{aligned} \tag{18}$$

Function $\Phi(t,z,\xi)$ can be calculated from the iterative method of the form

$$\Phi(t,z,\xi) = \sum_{r=1}^{\infty} \phi_r(t,z,\xi) \tag{19}$$

$$\phi_1(t,z,\xi) \stackrel{\text{def}}{=} \mathscr{L}p^G(t,z,\xi),$$

$$\phi_{r+1}(t,z,\xi) \stackrel{\text{def}}{=} \int_0^t \int_D \mathscr{L}p^G(t-s,z,\zeta)\phi_r(s,\zeta,\xi)d\zeta ds$$

and

$$\begin{aligned} \mathscr{L}p^G(t,z,\xi) &= Lp^G(t,z,\xi) - \partial_t p^G(t,z,\xi) \\ &= y\partial_x p^G(t,z,\xi) + \beta(z)\partial_y p^G(t,z,\xi). \end{aligned} \tag{20}$$

To show these results in detail, suppose we apply operator $\mathscr{L}$ on equation (18). Despite the singularity at $t = 0, z = \xi$, it can be shown that (see Appendix A for the rigorous proof)

$$\begin{aligned} \mathscr{L}H(t,z,\xi) = \int_0^t \int_D \mathscr{L}p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds \\ - \Phi(t,z,\xi). \end{aligned} \tag{21}$$

Kernel $p(t,z,\xi)$ should solve (16) as a solution and so $\mathscr{L}p = 0$ (this will be proven in theorem 3.3). Therefore, function Φ should satisfy a Volterra quation of the form

$$\begin{aligned} \Phi(t,z,\xi) &= \mathscr{L}p^G(t,z,\xi) \\ &\quad + \int_0^t \int_D \mathscr{L}p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds. \end{aligned} \tag{22}$$

where, $\mathscr{L}p^G(t,z,\xi)$ is calculated in (20). Volterra equation (22) suggests that there exists a function Φ that can be calculated recursively. The next theorem shows that this is in fact the case.

Theorem 3.2: For any $\xi \in D$, function $\Phi(t,z,\xi)$ in (19) solves the Volterra equation (22).

Proof: In what follows $C > 0$ is considered to be a generic constant. First, we show that summation (19) converges. For any $t \in (0,T]$, $z = (x,y) \in D$ and $\xi \in D$, equation (17) implies that

$$\begin{aligned} \left|\partial_x p^G(t,z,\xi)\right| &\le Ct^{-2+\frac{1}{2}} \left(\frac{(\xi_1 - x)^2}{t} \right)^{1/2} \exp\left\{ -\frac{\mathscr{K}(z,\xi)}{2t} \right\} \\ &\le Ct^{-1/2} p^G(t,\hat{z},\hat{\xi}), \end{aligned} \tag{23}$$

where, $\hat{z} = \lambda_\circ^{1/2} z$ and $\hat{\xi} = \lambda_\circ^{1/2}\xi$, for any $\lambda_\circ \in (0,1)$. The same bound can be established for $\partial_y p^G(t,z,\xi)$. Therefore,

$$|\phi_1(t,z,\xi)| = |\mathscr{L}p^G(t,z,\xi)| \le Ct^{-1/2} p^G(t,\hat{z},\hat{\xi}). \tag{24}$$

Now, in the second iteration by considering the Chapman-Kolmogorov equation we have

$$\begin{aligned} |\phi_2(t,z,\xi)| &\le C \int_0^t (t-s)^{-1/2} s^{-1/2} \int_D p^G(t-s,\hat{z},\hat{\zeta}) \\ &\qquad\qquad \times p^G(s,\hat{\zeta},\hat{\xi}) d\hat{\zeta} ds \\ &\le Cp^G(t,\hat{z},\hat{\xi}) \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)}. \end{aligned}$$

Here, Γ is the Gamma function. By induction we can show that for any $t \in (0,T]$ and $z, \xi \in D$ we have

$$|\phi_n(t,z,\xi)| \le C^n t^{\frac{n}{2}-1} p^G(t,\hat{z},\hat{\xi}) \frac{1}{\Gamma(n/2)}.$$

This implies that the series (19) is absolutely convergent. Moreover, on the interval $[\tau_\circ, \tau_1] \subset (0,T]$ the series is uniformly convergent.

The bound on each term of the series suggests that

$$|\Phi(t,z,\xi)| \le |\mathscr{L}p^G(t,z,\xi)| \le Ct^{-1/2} p^G(t,\hat{z},\hat{\xi}). \tag{25}$$

Replacing function $\Phi(t,z,\xi)$ in the integral term of (22) shows that

$$\sum_{r=1}^{\infty}\int_0^t\int_D \mathscr{L}p^G(t-s,z,\zeta)\phi_r(s,\zeta,\xi)d\zeta ds = \Phi(t,z,\xi) - \mathscr{L}p^G(t,z,\xi),$$

which proves the claimed result. ∎

The next step is to show that $p(t,x,\xi)$ is the desired transition probability function.

Theorem 3.3: Transition function $p(t,z,\xi)$ defined in (18) is a fundamental solution of $\mathscr{L}u=0$.

Proof: First we show that $\mathscr{L}p=0$ for $t>0, z,\xi\in D$. This can be seen directly by applying operator $\mathscr{L}$ on (18) and considering definition of function Φ in theorem 3.2. It remains to show that

$$\lim_{t\to 0} p(t,z,\xi) = \delta_\xi(z),$$

as a distribution. In other words, we show that

$$\lim_{t\to 0}\int_D p(t,z,\xi)\varphi(z)dz = \varphi(\xi), \quad \forall \varphi\in C_c^\infty(D).$$

From equation (18) we have that

$$p(t,z,\xi) = p^G(t,z,\xi) + H(t,z,\xi).$$

For the first term, we know that p^G is fundamental solution of $\mathscr{L}_0$ and hence

$$\lim_{t\to 0} p^G(t,z,\xi) = \delta_\xi(z).$$

It remains to show that

$$\mathscr{I} \stackrel{\text{def}}{=} \lim_{t\to 0}\int_D H(t,z,\xi)\varphi(z)dz = 0, \quad \forall\varphi\in C_c^\infty(D).$$

From (25) and definition of function $H(t,z,\xi)$ in (17) we have that

$$\begin{aligned}\mathscr{I} &\le C\lim_{t\to 0}\int_D\left(\int_{s=0}^t s^{-1/2}\int_D p^G(t-s,\hat z,\hat\zeta)p^G(s,\hat\zeta,\hat\xi)d\zeta ds\right) \\ &\qquad \times\varphi(z)dz \\ &\le C\lim_{t\to 0}\int_D t^{1/2}p(t,\hat z,\hat\xi)dz \le C\lim_{t\to 0} t^{1/2} = 0.\end{aligned}$$

∎

IV. CONCLUSION AND FUTURE WORKS

In this paper we discussed and iterative method which can be used in constructing the transition probability function for the solution of the stochastic OV model. Transition density functions contain all the statistical properties of the process and hence they explain the behavior of the solution.

The OV model, considered in this paper, in its current form is highly dependant on the parameters of the model which makes it unstable in some situations. As our future work we are considering a modified version of this model in which such dependence has been addressed by incorporating some terms to the dynamical system. We shall extend our probabilistic analysis to this new model to study the behavior of its solution.

APPENDIX

A. Differentiation of the Convolution Term

For $z=(x,y)\in D$, $\xi\in D$ and $t\in(0,T]$, we defined

$$H(t,z,\xi) = \int_0^t\int_D p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds,$$

and function $\Phi(t,z,\xi)$ to be determined. The goal is to show that

$$\mathscr{L}H(t,z,\xi) = \int_0^t\int_D \mathscr{L}p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds - \Phi(t,z,\xi)$$

This is equivalent to showing that the following theorem holds true.

Theorem 1.1: Suppose function $\Phi(t,z,\xi)$ is differentiable with respect to $(t,z)\in(0,T)\times D$, and the function and its first order derivatives are continuous on $[\tau_\circ,\tau_1]\times\bar D$ for any fixed $[\tau_\circ,\tau_1]\subset(0,T]$. Moreover, we suppose that $\Phi\in L^1((0,T]\times D)$ in the sense of Lebesgue integration. Then, we have

1) $\partial_x H(t,z,\xi)$ exists, continuous and

$$\partial_x H(t,z,\xi) = \int_0^t\int_D \partial_x p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds$$

The same result holds true for derivative with respect to y,

2) $\partial_{yy}H(t,z,\xi)$ exists, is continuous and

$$\partial_{xx}H(t,z,\xi) = \int_0^t\int_D \partial_{xx}p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds.$$

The same result holds true for the second partial derivative with respect to y.

3) $\partial_t H(t,z,\xi)$ exists, is continuous and

$$\partial_t H(t,z,\xi) = \int_0^t\int_D \partial_t p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds - \Phi(t,z,\xi).$$

Proof: Let $t\in(0,T]$ and $z,\xi\in D$. The first statement is obvious by (23) and (25).

For the second statement we calculate

$$\begin{aligned}|\partial_{xx}p^G(t,z,\xi)| &\le t^{-2}\exp\left\{-\frac{\mathscr{K}(z,\xi)}{2t}\right\} \\ &\le Ct^{-1}p^G(t,\hat z,\hat\xi),\end{aligned} \tag{26}$$

where, $\hat z = \lambda_\circ^{1/2}z$ and similarly for $\hat\xi$, for any $\lambda_\circ\in(0,1)$. This implies that due to singularity $t=0, z=\xi$ the bound on the second order derivatives of $p^G(t,z,\xi)$ is not sufficient for proving the integrability and hence we need more delicate discussions to show that the second statement of the theorem is valid. To do so, we split the integral's domain in the form of

$$\begin{aligned}F(t,z,\xi) &\stackrel{\text{def}}{=} \left(\int_0^{t/2}\int_D + \int_{t/2}^t\int_D\right)\partial_x p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds \\ &= \mathscr{I}(t,z,\xi) + \mathscr{J}(t,z,\xi).\end{aligned}$$

In fact, on each of these integrals either $p^G(t-s,z,\zeta)$ or the kernel $\Phi(s,\zeta,\xi)$ posses some smoothness which helps us proving the result.

For the first integral, $p^G(t-s,z,\zeta)$ is smooth with no singular point and hence differentiation can be applied under the integral sign. For the second integral we need to define some notations first. Let

$$\mathscr{J}^{(x)}(t,z,\xi) \stackrel{\text{def}}{=} \int_{t/2}^{t}\int_{D} \partial_{xx} p^G(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds,$$

Next, we define

$$\mathscr{J}_\delta(t,z,\xi) \stackrel{\text{def}}{=} \int_{t/2}^{t}\int_{D} \partial_{x} p^G_\delta(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds,$$

where,

$$p^G_\delta(t,z,\xi) \stackrel{\text{def}}{=} p^G(t,z,\xi)\eta\left(\frac{|\hat{M}(\zeta-z)|}{\delta}\right),$$

and

$$\hat{M} \stackrel{\text{def}}{=} \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}.$$

Smooth (cut-off) function η is considered such that

$$\eta(r) \stackrel{\text{def}}{=} \begin{cases} 0 & , r<1 \\ 1 & , r>2, \end{cases}$$

and $\eta'(r) \in (0,2)$.

Therefore, $\mathscr{J}_\delta$ is smooth and clearly

$$\partial_x \mathscr{J}_\delta(t,z,\xi) = \int_{t/2}^{t}\int_{D} \partial_{xx} p^G_\delta(t-s,z,\zeta)\Phi(s,\zeta,\xi)d\zeta ds.$$

The aim is to show that

$$\mathscr{J}_\delta \to \mathscr{J}, \quad \text{as} \quad \delta \to 0, \tag{27}$$

uniformly with respect to z on compact subsets of D, and

$$\partial_x \mathscr{J}_\delta \to \mathscr{J}^{(x)} \quad \text{as} \quad \delta \to 0. \tag{28}$$

uniformly with respect to z on compact subsets of D. Upon proving, these results imply that $\mathscr{J}(t,z,\xi)$ is continuously differentiable with respect to z and the second statement of the theorem holds true.

To show these results, it is more insightful to consider the following bounds

$$|\partial_x p^G(t,z,\xi)| \le \frac{C}{t^\gamma}\frac{1}{|\xi-z|^{3-2\gamma}}, \quad \gamma \in (1/2,1) \tag{29}$$

$$|\partial_{xx} p^G(t,z,\xi)| \le \frac{C}{t^\gamma}\frac{1}{|\xi-z|^{4-2\gamma}}, \tag{30}$$

which can be directly calculated from the definition of $p^G(t,z,\xi)$ and its derivatives by some algebraic manipulation.

To show the uniform convergence in (27) considering the assumed properties of kernel Φ, change of variable $\kappa = \hat{M}(\zeta - z)$, and definition of function η we can write

$$|\mathscr{J}_\delta(t,z,\xi) - \mathscr{J}(t,z,\xi)| \le C \sup_{\substack{\tau\in[t/2,T] \\ \nu\in\bar{D}}} |\Phi(\tau,\nu,\xi)| \times \int_{t/2}^{t}(t-s)^{-1}ds\int_{|\kappa|<2\delta}\left|\partial_\kappa\left\{\left(\eta\left(\frac{|\kappa|}{\delta}\right)-1\right) \times \exp\left\{-\frac{|\kappa|^2}{2(t-s)}\right\}\right\}\right|d\kappa.$$

Then by considering bound (29) and using polar coordinate integration, we have

$$|\mathscr{J}_\delta(t,z,\xi) - \mathscr{J}(t,z,\xi)| \le C \sup_{\substack{\tau\in[t/2,T] \\ \nu\in\bar{D}}} |\Phi(\tau,\nu,\xi)|C(\delta),$$

where, $C(\delta) \to 0$ as $\delta \to 0$.

To follow the same argument for proving the uniform convergence in (28), we need to use smoothness of kernel Φ. In fact, we write

$$\begin{aligned}\mathscr{J}^{(x)}(t,z,\xi) = \int_{t/2}^{t}\int_{D} \partial_{xx} p^G(t-s,z,\zeta)(\Phi(s,\zeta,\xi) \\ -\Phi(s,z,\xi))d\zeta ds \\ + \int_{t/2}^{t} \Phi(s,\zeta,\xi)\int_{D} \partial_{xx} p^G(t-s,z,\xi)d\zeta ds.\end{aligned}$$

Intuitively, in the first integral by using the fact that

$$|\Phi(s,\zeta,\xi) - \Phi(s,z,\xi)| \le K|\zeta - z|,$$

we can improve the bound (30) as

$$|\partial_{xx} p^G(t-s,z,\zeta)||\Phi(s,\zeta,\xi) - \Phi(s,z,\xi)| \le \frac{C}{(t-s)^\gamma|\zeta-z|^{2-2\gamma}}$$

The second integral will vanish in $\partial_x\mathscr{J} - \mathscr{J}_{(x)}$ by applying divergence theorem. Employing the change of variable $\kappa = \hat{M}(\zeta - z)$, improved bound (30) and using polar coordinate integration the result follows.

The third statement of the theorem can be proven in a similar way. ■

Remark 1.2: Using the definition (19) and breaking the integral limits as in previous theorem, we can show that the assumptions on $\Phi(t,z,\xi)$ in the statement of the theorem are in fact valid.

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