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THE $\theta = \infty$ CONJECTURE IMPLIES THE RIEMANN HYPOTHESIS

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Abstract. We show that the $\theta = \infty$ conjecture implies the Riemann hypothesis.

§1. *Introduction.* Since the work of Levinson [4], it has been known that one can obtain lower bounds for the proportion of zeros of the Riemann zeta-function on the critical line by computing upper bounds for the mollified second moment

$$I_N(T_1, T_2) := \int_{T_1}^{T_2} |M_N(\tfrac{1}{2} + \mathrm{i}t)|^2 |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^2 \, dt, \tag{1.1}$$

where $M_N(s)$ is a mollifier roughly of the form

$$M_N(s) := \sum_{n \leqslant N} \frac{\mu(n)}{n^s} \bigg(1 - \frac{\log n}{\log N}\bigg),$$

with $N \geqslant 2$ an integer. Levinson [4] computed the asymptotic formula

$$\lim_{T \to \infty} \frac{I_{T^\theta}(0, T)}{T} = 1 + \frac{1}{\theta} \tag{1.2}$$

for $0 < \theta < 1/2$, and used this result to deduce that $\kappa > 1/3$, where

$$\kappa := \frac{\#\{\rho \mid \zeta(\rho) = 0,\ 0 < \Im\rho < T,\ \Re\rho = \frac{1}{2}\}}{\#\{\rho \mid \zeta(\rho) = 0,\ 0 < \Im\rho < T\}}$$

is the proportion of the non-trivial zeros of $\zeta(s)$ that lie on the critical line. Conrey [1] later proved that (1.2) (with a slightly different mollifier) remains valid for $\theta < 4/7$, and thereby deduced that $\kappa > 2/5$.

Initially, it was believed (see [2]) that (1.2) does not hold when $\theta > 1$. However, Farmer [2] produced a heuristic argument suggesting that it holds for every $\theta > 0$, and called this the "$\theta = \infty$ conjecture". Moreover, he proved that this conjecture implies that $\kappa = 1$, in other words, that 100% of the non-trivial zeros of $\zeta(s)$ lie on the critical line. He also argued that a slightly stronger form of the conjecture implies Montgomery's pair correlation conjecture. More recently, Radziwiłł [6] showed that, as $\theta \to \infty$, $M_{T^\theta}(t)$ is essentially the best possible mollifier of length T^θ for $\zeta(s)$. In particular, his work implies that Levinson's

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method can give $\kappa = 1$ only if it is used with mollifiers of length T^θ, where θ is arbitrarily large.

The purpose of this note is to show that the $\theta = \infty$ conjecture actually implies the Riemann hypothesis. Indeed, we show that even an upper bound of the form $I_N(0, T) \ll T^{1+\varepsilon}$, for some $\theta > 1$ and all N in the range $2 \leqslant N \leqslant T^\theta$, implies a zero-free region for the zeta-function of the form $\Re s > 1 - \delta$, for some $\delta > 0$ depending on θ: in other words, a quasi-Riemann hypothesis.

THEOREM 1. *Let $\theta > 0$ and assume that, for every $\varepsilon > 0$, $I_N(0, T) \ll_\varepsilon T^{1+\varepsilon}$ for N in the range $2 \leqslant N \leqslant T^\theta$. Then $\zeta(s)$ has no zero in the half-plane $\Re s > 1/2 + 1/2\theta$. In particular, if $I_N(0, T) \ll_\varepsilon T^{1+\varepsilon}$ for $2 \leqslant N \leqslant T^\theta$ with θ arbitrarily large, then the Riemann hypothesis is true.*

In a number of recent works on mean values of L-functions in the t-aspect, the integral is taken over $[T, 2T]$ rather than over $[0, T]$. Thus, it is natural to ask whether one can obtain a version of Theorem 1 for the interval $[T, 2T]$. Usually, there is no difficulty in passing from one interval to the other. In our case, however, the problem for $[T, 2T]$ is more subtle because one needs an Ω-result for $M_N(t)$ that is uniform in t. Using ideas from [**5**] and [**3**], we prove the following.

THEOREM 2. *Let $\theta > 0$ and assume that, for every $\varepsilon > 0$, $I_N(T, 2T) \ll_\varepsilon T^{1+\varepsilon}$ for N in the range $2 \leqslant N \leqslant T^\theta$. Then $\zeta(s)$ has no zero in the half-plane $\Re s > 1/2 + 2/\theta$. In particular, if $I_N(T, 2T) \ll_\varepsilon T^{1+\varepsilon}$ for $2 \leqslant N \leqslant T^\theta$ with θ arbitrarily large, then the Riemann hypothesis is true.*

Notice that Theorem 2 only implies a quasi-Riemann hypothesis when $\theta > 4$, so, in this respect, it is weaker than Theorem 1. However, Theorem 2, whose proof is more difficult than that of Theorem 1, is, in a certain sense, best possible. If, for example, one assumes that $\zeta(s)$ has a unique simple zero $\rho_0 = \beta_0 + i\gamma_0$ such that $\gamma_0 > 0$ and $\beta_0 > 1/2$, one can show that

$$I_N(T, 2T) = c_1 \frac{N^{2\beta_0 - 1}}{T^3} \frac{\log T}{\log^2 N}\left(1 + \Re\left(N^{2i\gamma_0}\frac{|\zeta'(\rho_0)|^2}{\zeta'(\rho_0)^2}\right) + o(1)\right) + O\left(T^{1+\varepsilon} + \frac{N^{\beta_0 - 1/2+\varepsilon}}{T}\right)$$

for some constant $c_1 > 0$ as $T \to \infty$, and this is consistent with the assumption that $I_{T^\theta}(T, 2T) \ll T^{1+\varepsilon}$ if $\theta < 4$. For the sake of comparison, we note that, with the same zero configuration,

$$I_N(0, T) = \frac{N^{2\beta_0 - 1}}{\log^2 N}(C(N) + o(1)) + O(T^{1+\varepsilon} + N^{\beta_0 - 1/2+\varepsilon} T^\varepsilon)$$

for some positive function $C(N)$ bounded away from zero, so that $I_{T^\theta}(0, T) \ll T^{1+\varepsilon}$ implies that $\beta_0 \leqslant 1/2 + 1/2\theta$, which is consistent with Theorem 1.

§2. *Proof of the theorems.* We will prove Theorems 1 and 2 at the same time. It should be pointed out, however, that an easier argument would suffice for the former.

We begin by extending our earlier definition of $M_N(s)$ slightly by writing

$$M_x(s)\log x = \sum_{n\leqslant x} \frac{\mu(n)}{n^s}\log(x/n) \tag{2.1}$$

for $x > 0$ (with $M_1(s) := 0$). Notice that the right-hand side is zero when $0 < x \leqslant 1$ and that this also allows us to extend the definition of $I_N(T_1, T_2)$ in (1.1) to $I_x(T_1, T_2)$. Now, for $t \in \mathbb{R}$,

$$M_x(\tfrac{1}{2} + \mathrm{i}t)\log x = \frac{1}{2\pi\mathrm{i}}\int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} \frac{x^z}{\zeta(\frac{1}{2}+\mathrm{i}t+z)}\frac{dz}{z^2}.$$

Thus, by Mellin inversion, we see that

$$H_t(w) := \int_1^\infty M_x(\tfrac{1}{2}+\mathrm{i}t)(\log x)x^{-w}\,dx = \frac{1}{(w-1)^2\zeta(w-\frac{1}{2}+\mathrm{i}t)}$$

for $\Re w > 3/2$. Next, assuming that $\rho_0 = \beta_0 + \mathrm{i}\gamma_0$ is a fixed zero of $\zeta(w)$ with $\beta_0 \geqslant 1/2$, we define

$$G_t(w) := \frac{(w-1)^2(w-\frac{3}{2}+\mathrm{i}t)\zeta(w-\frac{1}{2}+\mathrm{i}t)}{(w+1)^2(w-\frac{1}{2}+\mathrm{i}t-\rho_0)(w+\mathrm{i}t+1)^4}.$$

In the half-plane $\Re w \geqslant 0$, $G_t(w)$ is holomorphic and satisfies $G_t(w) \ll (1 + |w+\mathrm{i}t|)^{-5/2}$. Thus, setting

$$g_t(u) = \frac{1}{2\pi\mathrm{i}}\int_{3-\mathrm{i}\infty}^{3+\mathrm{i}\infty} G_t(w)u^{-w}\,dw$$

for $u > 0$,

$$g_t(u) = \begin{cases} 0 & \text{if } u > 1, \\ O(1) & \text{if } 0 \leqslant u \leqslant 1, \end{cases} \tag{2.2}$$

as can be seen by moving the line of integration to $\Re w = +\infty$ when $u > 1$, and to $\Re w = 0$ when $0 \leqslant u \leqslant 1$.

Now consider the integral

$$\begin{aligned} J_t(x) &:= \frac{1}{2\pi\mathrm{i}}\int_{3-\mathrm{i}\infty}^{3+\mathrm{i}\infty} G_t(w)H_t(w)x^w\,dw \\ &= \frac{1}{2\pi\mathrm{i}}\int_{3-\mathrm{i}\infty}^{3+\mathrm{i}\infty} \frac{(w-\frac{3}{2}+\mathrm{i}t)x^w}{(w+1)^2(w-\frac{1}{2}+\mathrm{i}t-\rho_0)(w+\mathrm{i}t+1)^4}\,dw, \end{aligned} \tag{2.3}$$

where, from this point on, we assume that $x \geqslant 2$. On the one hand, by the convolution formula for products of Mellin transforms, and since $M_y(1/2 + it)\log y = 0$ when $0 < y \leqslant 1$,

$$J_t(x) = \int_1^\infty M_y(\tfrac{1}{2} + \mathrm{i}t)(\log y) g_t(y/x)\, dy.$$

Thus, by (2.2),

$$J_t(x) \ll \int_1^x |M_y(\tfrac{1}{2} + \mathrm{i}t)| \log y\, dy \tag{2.4}$$

for $x \geqslant 2$. On the other hand, moving the line of integration in (2.3) to $\Re w = 0$, we see that

$$J_t(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} G_t(w) H_t(w) x^w\, dw + \frac{x^{\rho_0+1/2-\mathrm{i}t}(\rho_0 - 1)}{(\frac{3}{2} + \rho_0 - \mathrm{i}t)^2(\rho_0 + \frac{3}{2})^4}. \tag{2.5}$$

The integral on the right is $O(1)$ since $H_t(w)G_t(w) \ll (1+|w|)^{-2}$ for $\Re w = 0$. Thus, from (2.4) and (2.5), we deduce that

$$\frac{x^{\beta_0+1/2}}{(1+|t|)^2} + 1 \ll \int_1^x |M_y(\tfrac{1}{2} + \mathrm{i}t)| \log y\, dy.$$

It follows from the Cauchy–Schwarz inequality that

$$\frac{x^{2\beta_0}}{(1+|t|)^4} + \frac{1}{x} \ll \int_1^x |M_y(\tfrac{1}{2} + \mathrm{i}t)|^2 \log^2 y\, dy$$

for $x \geqslant 2$. Multiplying both sides by $|\zeta(1/2 + \mathrm{i}t)|^2$ and integrating with respect to t over the interval $[T_1, T_2]$, where $0 \leqslant T_1 \leqslant T_2/2$, we obtain

$$\begin{aligned}
&\int_{T_1}^{T_2} |\zeta(\tfrac{1}{2} + \mathrm{i}t)|^2 \left(\frac{x^{2\beta_0}}{(1+t)^4} + \frac{1}{x}\right) dt \\
&\quad \ll \int_1^x \log^2 y \int_{T_1}^{T_2} |M_y(\tfrac{1}{2} + \mathrm{i}t)\zeta(\tfrac{1}{2} + \mathrm{i}t)|^2\, dt\, dy \\
&\quad \leqslant \log^2 x \int_1^x I_y(T_1, T_2)\, dy.
\end{aligned}$$

Now $\int_{T_1}^{T_2} |\zeta(1/2 + \mathrm{i}t)|^2\, dt \gg T_2 \log(T_2 + 2)$ for $0 \leqslant T_1 \leqslant T_2/2$, so

$$\frac{x^{2\beta_0}\log(T_1 + 2)}{|1 + T_1|^3} + \frac{T_2 \log(T_2 + 2)}{x} \ll \log^2 x \int_1^x I_y(T_1, T_2)\, dy.$$

Thus, if $I_N(0, T) \ll_\varepsilon T^{1+\varepsilon}$ holds for $2 \leqslant N \leqslant T^\theta$ and for every $\varepsilon > 0$, then, taking $T_1 = 0$, $T_2 = T$ and $x = T^\theta$, we obtain

$$T^{2\beta_0\theta} \ll_\varepsilon T^{1+\varepsilon+\theta}.$$

Letting $T \to \infty$ and letting $\varepsilon > 0$ be sufficiently small, we obtain $\beta_0 \leqslant 1/2 + 1/2\theta$, as claimed in Theorem 1. Theorem 2 follows in the same way on taking $T_1 = T$ and $T_2 = 2T$.

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