Generalized Gröbner Bases and New Properties of Multivariate Difference Dimension Polynomials

Alexander Levin
levin@cua.edu
The Catholic University of America
Washington, D. C. 20064, USA

ABSTRACT

We present a method of Gröbner bases with respect to several term orderings and use it to obtain new results on multivariate dimension polynomials of inversive difference modules. Then we use the difference structure of the module of Kähler differentials associated with a finitely generated inversive difference field extension of a given difference transcendence degree to describe the form of a multivariate difference dimension polynomial of the extension.

CCS CONCEPTS

• **Computing methodologies → Symbolic and algebraic manipulation**.

KEYWORDS

Inversive difference field; Inversive difference module; Module of Kähler differentials; Gröbner basis

ACM Reference Format:
Alexander Levin. 2021. Generalized Gröbner Bases and New Properties of Multivariate Difference Dimension Polynomials. In *Proceedings of the 2021 International Symposium on Symbolic and Algebraic Computation (ISSAC'21), July 18–23, 2021, Virtual Event, Russian Federation.* ACM, New York, NY, USA, 8 pages. https://doi.org/10.1145/3452143.3465544

1 INTRODUCTION

The role of difference dimension polynomials in difference algebra is similar to the role of Hilbert polynomials in commutative algebra and algebraic geometry, as well as to the role of Kolchin differential dimension polynomials in differential algebra. In particular, as it is shown in [7] (see also [9, Chapter 7]), the univariate difference dimension polynomial of a system of algebraic difference equations expresses the A. Einstein's strength of the system, that is, the difference counterpart of the concept of strength of a system of partial differential equations introduced in [1]. This fact determines the importance of the study of difference dimension polynomials and methods of their computation for the qualitative theory of difference equations. Furthermore, dimension polynomials of finitely generated difference and inversive difference field extensions carry certain invariants, i. e., numbers that are independent of a system of difference generators and therefore characterize the corresponding difference algebraic structure. Also, properties of difference dimension polynomials associated with prime reflexive difference ideals provide a powerful tool in the dimension theory of difference rings, see [10], [4, Chapter 7] and [9, Sections 3.6 and 4.6]. In 2007 the author proved the existence theorems and found invariants of multivariate difference and difference-differential polynomials of difference and difference-differential modules and field extensions associated with given partitions of basic sets of operators (translations and derivations), see [6] and [7]. Based on these results, one can associate with a system of algebraic difference (or difference-differential) equations a family of multivariate dimensional polynomials that carry essentially more characteristics of the system than their univariate counterpart. Similar results for inversive difference modules and field extensions were obtained in [8] and [9, Sections 3.5 and 4.2]. While the theorems on multivariate difference and difference-differential dimension polynomials were obtained via the developed technique of Gröbner bases with respect to several term orderings in free modules over rings of difference and difference-differential operators, the corresponding results in the inversive difference case were proved by the method of generalized characteristic sets. Even though this approach allows one to prove theorems on multivariate dimension polynomials, it does not give an algorithm for their computation. In this connection, one should mention a work [12] where the authors introduced the concept of a relative Gröbner bases in a free difference-differential module with respect to two term orderings and gave a new proof of the theorem on a difference-differential dimension polynomial of a finitely generated difference-differential module.

In this paper we present a method of Gröbner bases with respect to several term orderings in a free inversive difference module over an inversive difference field that allows one to obtain an algorithm for computing multivariate dimension polynomials of inversive difference modules and inversive difference field extensions. (In particular, we extend the algorithmic technique of [12] to the case of several term orderings associated with partitions of the sets of translations.) We also present new results on finitely generated inversive difference modules with certain multivariate dimension polynomials and prove (using properties of modules of Kähler differentials) the existence of a special difference transcendence basis of a finitely generated inversive difference field extension that provides a nice representation of the multivariate dimension polynomial of the extension.

2 PRELIMINARIES

Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the sets of all non-negative integers, integers, and rational numbers, respectively.

ISSAC'21, July 18–23, 2021, Virtual Event, Russian Federation

ACM ISBN 978-1-4503-8382-0/21/07…$15.00
https://doi.org/10.1145/3452143.3465544

$\mathbb{Q}[t_1,\dots,t_p]$ will denote the ring of polynomials in p variables $t_1,\dots,t_p$ over $\mathbb{Q}$. If $B = A_1\times\cdots\times A_p$ is a Cartesian product of ordered sets with orders $<_1,\cdots <_p$, respectively, then by the product order on B we mean a partial order $<_P$ such that $(a_1,\dots,a_p) <_P (a'_1,\dots,a'_p)$ if and only if $a_i <_i a'_i$ for $i=1,\dots,p$. This notation will be used, in particular, in the sets $\mathbb{N}^p$ and $\mathbb{Z}^p$.

By a *difference ring* we mean a commutative ring R together with a finite set $\sigma=\{\alpha_1,\dots,\alpha_m\}$ of mutually commuting injective endomorphisms of R. The set σ is called a *basic set* of R and its elements are called *translations*. We also say that R is a σ-ring. If all translations of R are automorphisms, we set $\sigma^* = \{\alpha_1,\dots,\alpha_m,\alpha_1^{-1},\dots,\alpha_m^{-1}\}$ and say that R is an *inversive difference ring* or a σ^*-ring. If a difference (respectively, inversive difference) ring R is a field, it is called a *difference* (or σ-) field (respectively, an *inversive difference* (or σ^*-) field). In what follows all σ^*-rings and σ^*-fields will be considered with the above set of m translations and Γ will denote the free commutative group of all power products $\gamma=\alpha_1^{k_1}\dots\alpha_m^{k_m}$ where $k_i\in\mathbb{Z}$ $(1\le i\le m)$. All fields are assumed to have characteristic zero.

If K is an inversive difference (σ^*-) field and its subfield K_0 is closed with respect to every $\alpha\in\sigma^*$, then K_0 is said to be an inversive difference (or σ^*-) subfield of K or that K is a σ^*-overfield of K_0 and we have a σ^*-field extension K/K_0. In this situation, if $S\subseteq K$, then the smallest σ^*-subfield of K containing K_0 and S will be denoted by $K_0\langle S\rangle$. If $K=K_0\langle S\rangle$, then the set S is called the set of σ^*-generators of K over K_0 or the set of σ^*-generators of the extension K/K_0. As a field, $K_0\langle S\rangle = K_0(\gamma a\,|\,\gamma\in\Gamma,\ a\in S)$. If the set S is finite, we say that K/K_0 is a finitely generated inversive difference (or σ^*-) field extension.

Let K be an inversive difference field with a basic set of automorphisms $\sigma=\{\alpha_1,\dots,\alpha_m\}$. Suppose that the set σ is represented as the union of p disjoint subsets ($p\ge 1$):

$$\sigma=\sigma_1\cup\cdots\cup\sigma_p \tag{1}$$

$$\text{where }\ \sigma_1=\{\alpha_1,\dots,\alpha_{m_1}\},\ \sigma_2=\{\alpha_{m_1+1},\dots,\alpha_{m_1+m_2}\},\ \dots,$$
$$\sigma_p=\{\sigma_{m_1+\cdots+m_{p-1}+1},\dots,\alpha_m\}\quad (m_1+\cdots+m_p=m).$$

If $\gamma=\alpha_1^{k_1}\dots\alpha_m^{k_m}\in\Gamma$ $(k_j\in\mathbb{Z})$ then the order of γ with respect to a set σ_i $(1\le i\le p)$, denoted by $\mathrm{ord}_i\,\gamma$, is defined as $\sum_{\nu=m_1+\cdots+m_{i-1}+1}^{m_1+\cdots+m_i}|k_\nu|$. (If $i=1$, the last sum is replaced by $\sum_{\nu=1}^{m_1}|k_\nu|$.) The number $\mathrm{ord}\,\gamma=\sum_{j=1}^{m}|k_j|$ is called the order of γ. Also, for any $r_1,\dots,r_p\in\mathbb{N}$, we set

$$\Gamma(r_1,\dots,r_p)=\{\gamma\in\Gamma\mid \mathrm{ord}_i\,\gamma\le r_i\ \ (i=1,\dots,p)\}.$$

Let R be an inversive difference (σ^*-) ring. An expression of the form $\sum_{\gamma\in\Gamma}a_\gamma\gamma$, where $a_\gamma\in R$ for any $\gamma\in\Gamma$ and only finitely many elements a_γ are different from 0, is called an *inversive difference* (or σ^*-) *operator* over R. Two σ^*-operators $\sum_{\gamma\in\Gamma}a_\gamma\gamma$ and $\sum_{\gamma\in\Gamma}b_\gamma\gamma$ are considered to be equal if and only if $a_\gamma=b_\gamma$ for any $\gamma\in\Gamma$. The set of all σ^*-operators over R can be equipped with a ring structure if one uses the natural structure of a left R-module on this set, and defines the multiplication by setting $\gamma a=\gamma(a)\gamma$ for any $a\in R$, $\gamma\in\Gamma$ and extending the operation by distributivity. The resulting ring is called *the ring of inversive difference (or σ^*-) operators over R* and is denoted by $\mathcal{E}$.

A left $\mathcal{E}$-module is called an *inversive difference R-module* (or a *σ^*-R-module*). In other words, an R-module M is a σ^*-R-module if elements of σ^* act on M as mutually commuting endomorphisms of the additive group of M such that $\alpha(ax)=\alpha(a)\alpha(x)$ and $\alpha(\alpha^{-1}x)=x$ for any $a\in R$, $\alpha\in\sigma^*$.

In what follows K will denote a σ^*-field of characteristic zero, $\mathcal{E}$ will denote the ring of σ^*-operators over K, and we will use the term "σ^*-K-module" for a left $\mathcal{E}$-module M. If M is a finitely generated $\mathcal{E}$-module, then the maximal number of elements of M that are linearly independent over $\mathcal{E}$ is called the σ^*-dimension of M; it is denoted by $\sigma^*\text{-}\dim_K M$. We also assume that partition (1) of σ is fixed.

We will consider p orders $<_1,\dots,<_p$ on Γ (the free commutative group generated by σ) defined as follows: $\gamma=\alpha_1^{k_1}\dots\alpha_m^{k_m} <_i \gamma'=\alpha_1^{l_1}\dots\alpha_m^{l_m}$ if and only if $(\mathrm{ord}_i\,\gamma,\mathrm{ord}_1\,\gamma,\dots,\mathrm{ord}_{i-1}\,\gamma,\mathrm{ord}_{i+1}\,\gamma,\dots,\mathrm{ord}_p\,\gamma,k_{m_1+\cdots+m_{i-1}+1},\dots,k_{m_1+\cdots+m_i},k_1,\dots,k_{m_1+\cdots+m_{i-1}},$ $k_{m_1+\cdots+m_i+1},\dots,k_m)$ is less than the corresponding vector for γ' with respect to the lexicographic order on $\mathbb{Z}^{m+p}$. Clearly, Γ is well-ordered with respect to each of the orders $<_1,\dots,<_p$.

For any $r_1,\dots,r_p\in\mathbb{N}$, the vector K-subspace of $\mathcal{E}$ generated by $\Gamma(r_1,\dots,r_p)$ will be denoted by $\mathcal{E}_{r_1,\dots,r_p}$.
Setting $\mathcal{E}_{r_1,\dots,r_p}=\{0\}$ for any $(r_1,\dots,r_p)\in\mathbb{Z}^p\setminus\mathbb{N}^p$, we get a family $\{\mathcal{E}_{r_1,\dots,r_p}|(r_1,\dots,r_p)\in\mathbb{Z}^p\}$ of vector K-subspaces of $\mathcal{E}$ which is called the *standard p-dimensional filtration* of $\mathcal{E}$. Clearly, $\mathcal{E}_{r_1,\dots,r_p}\subseteq\mathcal{E}_{s_1,\dots,s_p}$ if $(r_1,\dots,r_p)\le_P(s_1,\dots,s_p)$ ($\le_P$ denotes the product order on $\mathbb{Z}^p$) and if $(i_1,\dots,i_p)$,
$(j_1,\dots,j_p)\in\mathbb{N}^p$, then $\mathcal{E}_{i_1,\dots,i_p}\mathcal{E}_{j_1,\dots,j_p}=\mathcal{E}_{i_1+j_1,\dots,i_p+j_p}$.

Definition 2.1. *If M is a σ^*-K-module, then a family $\{M_{r_1,\dots,r_p}|(r_1,\dots,r_p)\in\mathbb{Z}^p\}$ is said to be a p-dimensional filtration of M if the following four conditions hold.*

(i) *$M_{r_1,\dots,r_p}\subseteq M_{s_1,\dots,s_p}$ for any p-tuples $(r_1,\dots,r_p)$, $(s_1,\dots,s_p)\in\mathbb{Z}^p$ such that $(r_1,\dots,r_p)\le_P(s_1,\dots,s_p)$.*

(ii) $\displaystyle\bigcup_{(r_1,\dots,r_p)\in\mathbb{Z}^p}M_{r_1,\dots,r_p}=M.$

(iii) *There exists a p-tuple $(r_1^{(0)},\dots,r_p^{(0)})\in\mathbb{Z}^p$ such that $M_{r_1,\dots,r_p}=0$ if $r_i<r_i^{(0)}$ for at least one index i $(1\le i\le p)$.*

(iv) *$\mathcal{E}_{r_1,\dots,r_p}M_{s_1,\dots,s_p}\subseteq M_{r_1+s_1,\dots,r_p+s_p}$ for any p-tuples $(r_1,\dots,r_p),(s_1,\dots,s_p)\in\mathbb{Z}^p$.*

If every vector K-space $M_{r_1,\dots,r_p}$ is finite-dimensional and there exists an element $(h_1,\dots,h_p)\in\mathbb{Z}^p$ such that $\mathcal{E}_{r_1,\dots,r_p}M_{h_1,\dots,h_p}=M_{r_1+h_1,\dots,r_p+h_p}$ for any $(r_1,\dots,r_p)\in\mathbb{N}^p$, the p-dimensional filtration is called excellent.

Clearly, if $z_1,\dots,z_k$ is a finite system of generators of a vector σ^*-K-space M, then $\{\sum_{i=1}^{k}\mathcal{E}_{r_1,\dots,r_p}z_i|(r_1,\dots,r_p)\in\mathbb{Z}^p\}$ is an excellent p-dimensional filtration of M.

NUMERICAL POLYNOMIALS OF SUBSETS OF $\mathbb{Z}^m$

Definition 2.2. *A polynomial $f(t_1,\dots,t_p)\in\mathbb{Q}[t_1,\dots,t_p]$ ($p\ge 1$) is called* numerical *if $f(r_1,\dots,r_p)\in\mathbb{Z}$ for all sufficiently large $(r_1,\dots,r_p)\in\mathbb{Z}^p$ (i. e., there exist $(s_1,\dots,s_p)\in\mathbb{Z}^p$ such that the inclusion $f(r_1,\dots,r_p)\in\mathbb{Z}$ holds for all $(r_1,\dots,r_p)\in\mathbb{Z}^p$ such that $(s_1,\dots,s_p)\le_P(r_1,\dots,r_p)$).*

Clearly, a polynomial with integer coefficients is numerical. As an example of a numerical polynomial in p variables with non-integer coefficients ($p \geq 1$) one can consider $\prod_{i=1}^{p} \binom{t_i}{m_i}$ where $m_1, \dots, m_p \in \mathbb{N}$. (As usual, $\binom{t}{k}$ ($k \geq 1$) denotes the polynomial $\frac{t(t-1)\dots(t-k+1)}{k!}$, $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if $k < 0$.)

The following theorem proved in [4, Corollary 2.1.5] gives the "canonical" representation of a numerical polynomial.

Theorem 2.3. *Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables $t_1, \dots, t_p$, and let $\deg_{t_i} f = m_i$ ($m_1, \dots, m_p \in \mathbb{N}$). Then $f(t_1, \dots, t_p)$ can be represented in the form*

$$f(t_1, \dots t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \quad (2)$$

with uniquely defined integer coefficients $a_{i_1 \dots i_p}$.

In what follows (until the end of the section), we deal with subsets of $\mathbb{Z}^m$ ($m \geq 1$) and a fixed partition of the set $\mathbb{N}_m = \{1, \dots, m\}$ into p disjoint subsets ($p \geq 1$):

$$\mathbb{N}_m = N_1 \cup \dots \cup N_p \quad (3)$$

where $N_1 = \{1, \dots, m_1\}, \dots, N_p = \{m_1 + \dots + m_{p-1} + 1, \dots, m\}$ ($m_1 + \dots + m_p = m$).

If $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$, we denote the numbers $\sum_{i=1}^{m_1} |a_i|$, $\sum_{i=m_1+1}^{m_1+m_2} |a_i|, \dots, \sum_{i=m_1+\dots+m_{p-1}+1}^{m} |a_i|$ by $\text{ord}_1\, a, \dots, \text{ord}_p\, a$, respectively. Furthermore, $\mathbb{Z}^m$ will be considered as the union

$$\mathbb{Z}^m = \bigcup_{1 \leq j \leq 2^m} \mathbb{Z}_j^{(m)} \quad (4)$$

where $\mathbb{Z}_1^{(m)}, \dots, \mathbb{Z}_{2^m}^{(m)}$ are all different Cartesian products of m sets each of which is either $\mathbb{N}$ or $\mathbb{Z}_- = \{a \in \mathbb{Z} \,|\, a \leq 0\}$. We assume that $\mathbb{Z}_1^{(m)} = \mathbb{N}^m$ and call $\mathbb{Z}_j^{(m)}$ the *jth orthant* of $\mathbb{Z}^m$, $1 \leq j \leq 2^m$. (Clearly, $(0, \dots, 0)$ belongs to all orthants.)

The set $\mathbb{Z}^m$ will be considered as a partially ordered set with the order $\trianglelefteq$ such that $(e_1, \dots, e_m) \trianglelefteq (e'_1, \dots, e'_m)$ if and only if $(e_1, \dots, e_m)$ and $(e'_1, \dots, e'_m)$ belong to the same orthant $\mathbb{Z}_k^{(m)}$ and the m-tuple $(|e_1|, \dots, |e_m|)$ is less than $(|e'_1|, \dots, |e'_m|)$ with respect to the product order on $\mathbb{N}^m$.

In what follows, for any set $A \subseteq \mathbb{Z}^m$, W_A will denote the set of all elements of $\mathbb{Z}^m$ that do not exceed any element of A with respect to the order $\trianglelefteq$. (Thus, $w \in W_A$ if and only if there is no $a \in A$ such that $a \trianglelefteq w$.) Also, for any $r_1, \dots, r_p \in \mathbb{N}$, $A(r_1, \dots, r_p)$ will denote the set $\{x = (x_1, \dots, x_m) \in A \,|\, \text{ord}_i\, x \leq r_i \ (i = 1, \dots, p)\}$.

The above notation can be naturally applied to subsets of $\mathbb{N}^m$ (treated as subsets of $\mathbb{Z}^m$). If $E \subseteq \mathbb{N}^m$ and $s_1, \dots, s_p \in \mathbb{N}$, then $E(s_1, \dots, s_p)$ will denote the set of all m-tuples $e = (e_1, \dots, e_m) \in E$ such that $\text{ord}_i\, e \leq s_i$ for $i = 1, \dots, p$. Furthermore, V_E will denote the set of all m-tuples $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ that are not greater than or equal to any m-tuple from E with respect to the product order on $\mathbb{N}^m$. (Clearly, an element $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ belongs to V_E if and only if for any element $(e_1, \dots, e_m) \in E$, there exists $i \in \mathbb{N}, 1 \leq i \leq m$, such that $e_i > v_i$.)

The following two theorems proved in [4, Chapter 2] generalize the well-known Kolchin's result on the numerical polynomials associated with subsets of $\mathbb{N}^m$ (see [3, Chapter 0, Lemma 16]) and give explicit formulas for multivariate numerical polynomials associated with finite subsets of $\mathbb{N}^m$.

Theorem 2.4. *Let $E \subseteq \mathbb{N}^m$ where $m = m_1 + \dots + m_p$ for some non-negative integers $m_1, \dots, m_p$ ($p \geq 1$). Then there exists a numerical polynomial $\omega_E(t_1, \dots, t_p)$ such that*

(i) *$\omega_E(r_1, \dots, r_p) = \text{Card}\, V_E(r_1, \dots, r_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$.* (As usual, $\text{Card}\, M$ denotes the number of elements of a finite set M.)

(ii) *The total degree $\deg\, \omega_E$ of the polynomial ω_E does not exceed m and $\deg_{t_i} \omega_E \leq m_i$ for all $i = 1, \dots, p$.*

(iii) *$\deg\, \omega_E = m$ if and only if $E = \emptyset$. Then $\omega_E(t_1, \dots, t_p) = \prod_{i=1}^{p} \binom{t_i + m_i}{m_i}$.*

Definition 2.5. *The polynomial $\omega_E(t_1, \dots, t_p)$ is called the dimension polynomial of the set $E \subseteq \mathbb{N}^m$ associated with the partition $(m_1, \dots, m_p)$ of m.*

Theorem 2.6. *Let $E = \{e_1, \dots, e_q\}$ ($q \geq 1$) be a finite subset of $\mathbb{N}^m$ and let a partition* (3) *of the set $\mathbb{N}_m$ into p disjoint subsets $N_1, \dots, N_p$ be fixed. Let $e_i = (e_{i1}, \dots, e_{im})$ $(1 \leq i \leq q)$ and for any $l \in \mathbb{N}$, $0 \leq l \leq q$, let $\Theta(l, q)$ denote the set of all l-element subsets of the set $\mathbb{N}_q = \{1, \dots, q\}$. Let $\bar{e}_{\emptyset j} = 0$ and for any $\theta \in \Theta(l, q)$, $\theta \neq \emptyset$, let $\bar{e}_{\theta j} = \max\{e_{ij} \,|\, i \in \theta\}$, $1 \leq j \leq m$. (That is, if $\theta = \{i_1, \dots, i_l\}$, then $\bar{e}_{\theta j}$ denotes the greatest jth coordinate of the elements $e_{i_1}, \dots, e_{i_l}$.) Furthermore, let $b_{\theta k} = \sum_{h \in N_k} \bar{e}_{\theta h}$ ($k = 1, \dots, p$). Then*

$$\omega_E(t_1, \dots, t_p) = \sum_{l=0}^{q} (-1)^l \sum_{\theta \in \Theta(l,q)} \prod_{j=1}^{p} \binom{t_j + m_j - b_{\theta j}}{m_j} \quad (5)$$

Remark. It is clear that if $E \subseteq \mathbb{N}^m$ and E^* is the set of all minimal elements of the set E with respect to the product order on $\mathbb{N}^m$, then the set E^* is finite and $\omega_E(t_1, \dots, t_p) = \omega_{E^*}(t_1, \dots, t_p)$. Thus, Theorem 2.6 gives an algorithm that allows one to find a numerical polynomial associated with any subset of $\mathbb{N}^m$ (and with a given partition of the set $\{1, \dots, m\}$): one should first find the set of all minimal points of the subset and then apply Theorem 2.6.

The following result can be obtained precisely in the same way as Theorem 3.4 of [5].

Theorem 2.7. *Let $A \subseteq \mathbb{Z}^m$ and let partition* (3) *of the set $\mathbb{N}_m$ be fixed. Then there exists a numerical polynomial $\phi_A(t_1, \dots, t_p)$ in p variables such that*

(i) *$\phi_A(r_1, \dots, r_p) = \text{Card}\, W_A(r_1, \dots, r_p)$ for all sufficiently large p-tuples $(r_1, \dots, r_p) \in \mathbb{N}^p$.*

(ii) *$\deg \phi_A \leq m$ and $\deg_{t_i} \phi_A \leq m_i$ ($1 \leq i \leq p$). Also, if $\phi_A(t_1, \dots, t_p)$ is written in the form* (2), *then $2^m | a_{m_1 \dots m_p}$.*

(iii) *If $A = \emptyset$, then*

$$\phi_A(t_1, \dots, t_p) = \prod_{j=1}^{p} \left[\sum_{i=0}^{m_j} (-1)^{m_j - i} 2^i \binom{m_j}{i} \binom{t_j + i}{i} \right]. \quad (6)$$

Theorem 2.8. *With the notation of Theorem 2.7, take the mapping* $\rho : \mathbb{Z}^m \longrightarrow \mathbb{N}^{2m}$ *defined by*

$$\rho((e_1, \dots, e_m) = (\max\{e_1, 0\}, \dots, \max\{e_m, 0\}, \max\{-e_1, 0\},$$
$$\dots, \max\{-e_m, 0\}).$$

Let $B = \rho(A) \bigcup \{\bar{e}_1, \dots, \bar{e}_m\}$ *where* $\bar{e}_i$ $(1 \le i \le m)$ *is a 2m-tuple in* $\mathbb{N}^{2m}$ *whose ith and* $(m+i)$*th coordinates are equal to 1 and all other coordinates are equal to 0. Then*

$$\phi_A(t_1, \dots, t_p) = \omega_B(t_1, \dots, t_p)$$

where $\omega_B(t_1, \dots, t_p)$ *is the dimension polynomial of the set B (see* Definition 2.5*) associated with the partition* $\mathbb{N}_{2m} = N'_1 \cup \dots \cup N'_p$ *where* $N'_i = N_i \cup \{j+m \mid j \in N_i\}$, $1 \le i \le p$ *(*N_i *is the ith component of the partition* (3)*).*

The polynomial $\phi_A(t_1, \dots, t_p)$ is called the *dimension polynomial* of the set $A \subseteq \mathbb{Z}^m$ associated with partition (3).

Note that Theorem 2.8, together with Theorem 2.6, give an algorithm for computing multivariate dimension polynomials of subsets of $\mathbb{Z}^m$ associated with partitions of $\mathbb{N}_m$.

3 GENERALIZED GRÖBNER BASES IN INVERSIVE DIFFERENCE MODULES

Let K be an inversive difference (σ^*-) field, $\sigma = \{\alpha_1, \dots, \alpha_m\}$, Γ the free commutative group generated by σ, and $\mathcal{E}$ the ring of σ^*-operators over K. Furthermore, we assume that a partition (1) of the set σ is fixed.

In what follows, a free $\mathcal{E}$-module is also called a *free* σ^**-K-module*. If such a module F has a finite family $\{f_1, \dots, f_n\}$ of free generators, it is called a finitely generated free σ^*-K-module. In this case the elements of the form γf_ν $(\gamma \in \Gamma, 1 \le \nu \le n)$ are called *terms* while the elements of the group Γ are called *monomials*. The set of all terms is denoted by Γf; it is easy to see that this set generates F as a vector space over the field K. By the order of a term $u = \gamma f_\nu$ with respect to σ_i (it is denoted by $\mathrm{ord}_i\, u$, $1 \le i \le p$) we mean the order of the monomial γ with respect to σ_i.

We shall consider p orderings of the set Γf that correspond to the orderings of the group Γ introduced above. These orderings are denoted by the same symbols $<_1, \dots, <_p$ and defined as follows: if $\gamma f_\mu, \gamma' f_\nu \in \Gamma f$, then $\gamma f_\mu <_i \gamma' f_\nu$ if and only if $\gamma <_i \gamma'$ in Γ or $\gamma = \gamma'$ and $\mu < \nu$. As before, we consider the representation (4) of the set $\mathbb{Z}^m$; it implies that the group Γ and the set of terms Γf can be represented as the unions

$$\Gamma = \bigcup_{j=1}^{2^m} \Gamma_j \quad \text{and} \quad \Gamma f = \bigcup_{j=1}^{2^m} \Gamma_j f$$

where $\Gamma_j = \{\alpha_1^{k_1} \dots \alpha_m^{k_m} | (k_1, \dots, k_m) \in \mathbb{Z}_j^{(m)}\}$ and $\Gamma_j f = \{\gamma f_i \mid \gamma \in \Gamma_j,\ 1 \le i \le m\}$.

Two elements $\gamma, \gamma' \in \Gamma$ are said to be *similar* if they belong to the same set Γ_j $(1 \le j \le 2^m)$. In this case we write $\gamma \backsim \gamma'$ or $\gamma \backsim_j \gamma'$. Note that $\backsim$ is not a transitive relation on Γ.

Let F be a finitely generated free σ^*-K-module and let $f_1, \dots, f_n$ be linearly independent generators of F over the ring of σ^*-operators $\mathcal{E}$. An element $\gamma \in \Gamma$ and a term $\gamma' f_i \in \Gamma f$ $(\gamma' \in \Gamma, 1 \le i \le n)$ are called similar if $\gamma \backsim_j \gamma'$ for some j $(1 \le j \le 2^m)$. It is written as $\gamma \backsim \gamma' f_i$ or $\gamma \backsim_j \gamma' f_i$. Furthermore, we say that two terms $\gamma f_i, \gamma' f_k \in \Gamma f$ $(1 \le i, k \le n)$ are similar and write $\gamma f_i \backsim \gamma' f_k$ or $\gamma f_i \backsim_j \gamma' f_k$, if $\gamma \backsim_j \gamma'$ for some $j = 1, \dots, 2^m$. It is easy to see that if $\gamma \backsim u$ for some term $u \in \Gamma f$ (or $\gamma \backsim \gamma'$ for some $\gamma' \in \Gamma$), then $\mathrm{ord}_\nu(\gamma u) = \mathrm{ord}_\nu \gamma + \mathrm{ord}_\nu u$ (respectively, $\mathrm{ord}_\nu(\gamma\gamma' u) = \mathrm{ord}_\nu \gamma + \mathrm{ord}_\nu \gamma'$) for $\nu = 1, \dots, p$.

Definition 3.1. *Let* $\gamma_1, \gamma_2 \in \Gamma$. *We say that* γ_1 *is a transform of* γ_2 *and write* $\gamma_2 \mid \gamma_1$, *if* $\gamma_1 \backsim_j \gamma_2$ $(1 \le j \le 2^m)$ *and there exists* $\gamma \in \Gamma_j$ *such that* $\gamma_1 = \gamma\gamma_2$. (In this case we write $\gamma = \dfrac{\gamma_1}{\gamma_2}$.) *Furthermore, we say that a term* $u = \gamma_1 f_i$ *is a transform of a term* $v = \gamma_2 f_k$ *and write* $v \mid u$, *if* $i = k$ *and* γ_1 *is a transform of* γ_2. (If $u = \gamma v$, we write $\gamma = \dfrac{u}{v}$.)

Since the set Γf is a basis of F over K, any nonzero element $h \in F$ has a unique representation in the form

$$h = a_1\gamma_1 f_{i_1} + \dots + a_l \gamma f_{i_l} \tag{7}$$

where $\gamma_\nu \in \Gamma$, $a_\nu \in K$, $a_\nu \ne 0$ $(1 \le \nu \le l)$, $1 \le i_1, \dots, i_l \le n$, and $\gamma_\nu f_{i_\nu} \ne \gamma_\mu f_{i_\mu}$ whenever $\nu \ne \mu$ $(1 \le \nu, \mu \le l)$.

Definition 3.2. *Let* $h \in F$ *be written in the form* (7) *and* $k \in \{1, \dots, p\}$. *Then the greatest with respect to* $<_k$ *term* $\gamma_\nu f_{i_\nu}$ $(1 \le \nu \le l)$ *is called the k-leader of h. It is denoted by* $u_h^{(k)}$. *The coefficient of* $u_h^{(k)}$ *in* (7) *is called the k-leading coefficient of h and denoted by* $\mathrm{lc}_k(h)$.

Remark. Let an element $h \in F$ be written in the form (7). Then for every $j \in \{1, \dots, 2^m\}$, there is a unique term v_j in h ($v_j = \gamma_\nu f_{i_\nu}$ for some ν, $1 \le \nu \le l$) such that $u_{\gamma h}^{(1)} = \gamma v_j$ for every $\gamma \in \Gamma_j$. Indeed, suppose that there are two terms, v_j and w_j in h such that $\gamma_1 v_j = u_{\gamma_1 h}^{(1)}$ and $\gamma_2 w_j = u_{\gamma_2 h}^{(1)}$ for some elements $\gamma_1, \gamma_2 \in \Gamma_j$. Then $\gamma_2\gamma_1 v_j$ is the 1-leader of the element $\gamma_2\gamma_1 h$ and $\gamma_1\gamma_2 w_j$ is also the 1-leader of this element. It follows that $\gamma_2\gamma_1 v_j = \gamma_1\gamma_2 w_j$ whence $v_j = w_j$. The term v_j with the above property is denoted by $\mathrm{lt}_j(h)$.

Definition 3.3. *Let* $f, g \in F$ *and let* $k, i_1, \dots, i_l$ *be distinct elements in the set* $\{1, \dots, p\}$. *Then the element f is said to be* $(<_k, <_{i_1}, \dots <_{i_l})$*-reduced with respect to g if f does not contain any transform* $\gamma u_g^{(k)}$ *such that* $\mathrm{ord}_{i_\nu} \gamma + \mathrm{ord}_{i_\nu} u_g^{(i_\nu)} \le \mathrm{ord}_{i_\nu} u_f^{(i_\nu)}$ $(\nu = 1, \dots, l)$.

An element $f \in F$ *is said to be* $(<_k, <_{i_1}, \dots <_{i_l})$*-reduced with respect to a set* $G \subseteq F$, *if f is* $(<_k, <_{i_1}, \dots <_{i_l})$*-reduced with respect to every element of G.*

With the above notation, let us consider $p-1$ new symbols $z_1, \dots, z_{p-1}$ and the free commutative semigroup Λ of all power products $\lambda = \gamma z_1^{l_1} \dots z_{p-1}^{l_{p-1}}$ with $\gamma \in \Gamma; l_1, \dots, l_{p-1} \in \mathbb{N}$. Let $\Lambda f = \{\lambda f_j \mid \lambda \in \Lambda, 1 \le j \le n\} = \Lambda \times \{f_1, \dots, f_n\}$. Furthermore, for any element $f \in F$, let $d_i(f) = ord_i u_f^{(i)} - ord_i u_f^{(1)}$ $(2 \le i \le p)$ and let $\rho : F \to \Lambda f$ be defined by $\rho(f) = z_1^{d_2(f)} \dots z_{p-1}^{d_{p(f)}} u_f^{(1)}$

Definition 3.4. *Let N be a* $\mathcal{E}$*-submodule of F. A finite set* $G = \{g_1, \dots, g_t\} \subseteq N$ *will be called a Gröbner basis of N with respect to the orders* $<_1, \dots, <_p$ *if for any* $f \in N$, *there exists* $g_i \in G$ *such that* $\rho(g_i) \mid \rho(f)$ *in* Λf. *(It means that* $u_{g_i}^{(1)} \mid u_f^{(1)}$ *and* $d_j(g_i) \le d_j(f)$ *for* $j = 2, \dots, p$.*)*

Remark. The above condition $\rho(g_i) \mid \rho(f)$ means that f is not $(<_1, \dots, <_p)$-reduced with respect to g_i, since the equality $d_j(g_i) \le$

$d_j(f)$ for $j = 2, \dots, p$ means that $u_f^{(1)} = \gamma u_{g_i}^{(1)}$ ($\gamma \backsim u_{g_i}^{(1)}$) and $\mathrm{ord}_j \gamma + \mathrm{ord}_j u_{g_i}^{(j)} \leq \mathrm{ord}_j u_f^{(j)}$. The expression of this property with the use of the divisibility of elements of Λf is convenient because it also shows that the existence of the Gröbner basis in the sense of Definition 3.4 immediately follows from the Dickson's lemma.

For any $f, g, h \in F$, with $g \neq 0$, we say that the element f $(<_k, <_{i_1}, \dots, <_{i_l})$-reduces to h modulo g in one step and write $f \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g} h$ if and only if $u_g^{(k)} | w$ for some term w in f with a coefficient a, $w = \gamma u_g^{(k)}$ $(\gamma \in \Gamma)$, $h = f - a(\gamma(lc_k(g)))^{-1}\gamma g$ and $\mathrm{ord}_{i_\nu} \gamma + \mathrm{ord}_{i_\nu} u_g^{(i_\nu)} \leq \mathrm{ord}_{i_\nu} u_f^{(i_\nu)}$ $(1 \leq \nu \leq l)$. If $f, h \in F$ and $G \subseteq F$, then we say that the element f $(<_k, <_{i_1}, \dots, <_{i_l})$-reduces to h modulo G and write $f \xrightarrow[<_k; <_{i_1}, \dots, <_{i_l}]{G} h$ if and only if there exist two sequences $g^{(1)}, g^{(2)}, \dots, g^{(q)} \in G$ and $h_1, \dots, h_{q-1} \in F$ such that

$$f \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(1)}} h_1 \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(2)}} \dots \xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(q-1)}} h_{q-1}$$
$$\xrightarrow[<_k, <_{i_1}, \dots, <_{i_l}]{g^{(q)}} h.$$

Let $G = \{g_1, \dots, g_r\}$ be a finite set of elements in the free σ^*-K-module F. Then the following algorithm (whose termination is obvious) shows that for every $f \in F$ there exist elements $g \in F$ and $Q_1, \dots, Q_r \in \mathcal{E}$ such that $f - g = \sum_{i=1}^{r} Q_i g_i$ and g is $(<_1, \dots, <_p)$-reduced with respect to G.

Algorithm 3.5. *$(f, r, g_1, \dots, g_r;\ g;\ Q_1, \dots, Q_r)$*

Input: $f \in E$, a positive integer r, $G = \{g_1, \dots, g_r\} \subseteq E$ where $g_i \neq 0$ for $i = 1, \dots, r$

Output: An element $g \in F$ and elements $Q_1, \dots, Q_r \in \mathcal{E}$ such that $g = f - (Q_1 g_1 + \dots + Q_r g_r)$ and g is $(<_1, \dots, <_p)$-reduced with respect to G

Begin

$Q_1 := 0, \dots, Q_r := 0, g := f$

While there exist i, $1 \leq i \leq r$, and a term w, that appears in g with a nonzero coefficient $c(w)$, such that $u_{g_i}^{(1)} | w$ and $\mathrm{ord}_j(\frac{w}{u_{g_i}^{(1)}} u_{g_i}^{(j)}) \leq \mathrm{ord}_j u_g^{(j)}$ *for* $j = 2, \dots, p$ **do**

z:= the greatest (with respect to $<_1$*)* of the terms w that satisfies the above conditions.

k:= the smallest number i for which $u_{g_i}^{(1)}$ is the greatest (with respect to $<_1$*)* 1-leader of an element $g_i \in G$ such that $u_{g_i}^{(1)} | z$ and $ord_j(\frac{z}{u_{g_i}^{(1)}} u_{g_i}) \leq ord_j u_g^{(j)}$ *for* $j = 2, \dots, p$.

$$Q_k := Q_k + c(z) \left(\frac{z}{u_{g_k}^{(1)}} (lc_1(g_k)) \right)^{-1} \frac{z}{u_{g_k}^{(1)}} g_k$$

$$g := g - c(z) \left(\frac{z}{u_{g_k}^{(1)}} (lc_1(g_k)) \right)^{-1} \frac{z}{u_{g_k}^{(1)}} g_k$$

End

We define the least common multiple of two terms $u = \gamma u_1 f_i = \alpha_1^{k_1} \dots \alpha_m^{k_m} f_i$ and $v = \gamma' f_j = \alpha_1^{l_1} \dots \alpha_m^{l_m}$ in Γf, as follows: $\mathrm{lcm}(u, v) = 0$ if either $i \neq j$ or $i = j$ and the m-tuples $(k_1, \dots, k_m)$ and $(l_1, \dots, l_m)$ do not belong to the same orthant $Z_j^{(m)}$ of $\mathbb{Z}^m$. If $i = j$ and the m-tuples of exponents of u and v belong to the same orthant, then $\mathrm{lcm}(u, v) = \alpha_1^{\epsilon_1 \max\{|k_1|, |l_1|\}} \dots \alpha_m^{\epsilon_m \max\{|k_m|, |l_m|\}} f_i$ where $(\epsilon_1, \dots, \epsilon_m)$ is an m-tuple with entries 1 and -1 that belongs to the same orthant $\mathbb{Z}_j^{(m)}$.

Now, for any nonzero elements $f, g \in F$, we can define the *rth S-polynomial of f and g* $(1 \leq r \leq p)$ as the element $S_r(f, g) =$

$$\left(\frac{\mathrm{lcm}(u_f^{(r)}, u_g^{(r)})}{u_f^{(r)}} (\mathrm{lc}_r(f)) \right)^{-1} \frac{\mathrm{lcm}(u_f^{(r)}, u_g^{(r)})}{u_f^{(r)}} f$$
$$- \left(\frac{\mathrm{lcm}(u_f^{(r)}, u_g^{(r)})}{u_g^{(r)}} (\mathrm{lc}_r(g)) \right)^{-1} \frac{\mathrm{lcm}(u_f^{(r)}, u_g^{(r)})}{u_g^{(r)}} g.$$

The following two statements can be obtained by mimicking the proof of the corresponding statements in [7, Section 4] (Proposition 4.8 and Theorem 4.11).

Proposition 3.6. *Let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis of an $\mathcal{E}$-submodule N of F with respect to the orders $<_1, \dots, <_p$. Then*

(i) *$f \in N$ if and only if $f \xrightarrow[<_1, <_2, \cdots <_p]{G} 0$.*

(ii) *If $f \in N$ and f is $(<_1, <_2, \cdots <_p)$-reduced with respect to G, then $f = 0$.*

Proposition 3.7. *Let $G = \{g_1, \dots, g_t\}$ be a Gröbner basis of an $\mathcal{E}$-submodule N of F with respect to each of the following sequences of orders: $<_p$; $<_{p-1}, <_p$; $\dots$; $<_{r+1}, \dots, <_p$ $(1 \leq r \leq p-1)$. Furthermore, suppose that*

$S_r(g_i, g_j) \xrightarrow[<_r, <_{r+1}, \cdots <_p]{G} 0$ for any $g_i, g_j \in G$.

Then G is a Gröbner basis of N with respect to the sequence of orders $<_r, <_{r+1}, \dots, <_p$.

Clearly, the last proposition, together with Algorithm 3.5, gives an algorithm for the computation of Gröbner bases of a σ^*-K-submodule of a finitely generated free σ^*-K-module. This algorithm (together with Algorithm 3.5) is currently being implemented in MAPLE and PYTHON.

4 MULTIVARIATE DIMENSION POLYNOMIALS IN THE INVERSIVE DIFFERENCE CASE

Using the above results about Gröbner bases in finitely generated free σ^*-K- (that is, $\mathcal{E}$-) modules (we keep the above notation and conventions), one can obtain the following theorem on a multivariate dimension polynomial of a σ^*-K-submodule that was proved in [9, Section 3.5] with the use of the characteristic set technique. The use of Gröbner bases has an obvious advantage because we have an algorithm for their computation, so the theorem gives an algorithm for computing multivariate dimension polynomials.

Theorem 4.1. *Let M be a finitely generated $\mathcal{E}$-module with a system of generators $\{h_1, \dots, h_n\}$, F a free $\mathcal{E}$-module with a basis $f_1, \dots, f_n$, and $\pi : F \longrightarrow M$ the natural $\mathcal{E}$-epimorphism of F onto M $(\pi(f_i) = h_i$ for $i = 1, \dots, n)$. Let $N = Ker\, \pi$ and let $G = \{g_1, \dots, g_d\}$*

be a Gröbner basis of N with respect to $(<_1, \dots, <_p)$. Furthermore, for any $r_1, \dots, r_p \in \mathbb{N}$, let $M_{r_1\dots r_p} = \sum_{i=1}^{p} \mathcal{E}_{r_1\dots r_p} f_i$, and let

$V_{r_1\dots r_p} = \{u \in \Gamma f \mid \mathrm{ord}_i\, u \le r_i$ for $i = 1, \dots, p$, and u is not a transform of any $u^{(1)}_{g_i}$ $(1 \le i \le d)$ $\}$,

$W_{r_1\dots r_p} = \{u \in \Gamma f \setminus V_{r_1 r_p} \mid \mathrm{ord}_i\, u \le r_i$ for $i = 1, \dots, p$ and whenever $u = \gamma u^{(1)}_g$ is a transform of some $u^{(1)}_g$ $(g \in G, \gamma \in \Gamma)$, there exists i, $2 \le i \le p$, such that $\mathrm{ord}_i\, \gamma + ord_i u^{(i)}_g > r_i\}$,

and $U_{r_1\dots r_p} = V_{r_1\dots r_p} \bigcup W_{r_1\dots r_p}$. Then

(i) *For any $(r_1, \dots, r_p) \in \mathbf{N}^p$, the set $\pi(U_{r_1\dots r_p})$ is a basis of the vector K-space $M_{r_1\dots r_p}$.*

(ii) *There exist numerical polynomials $\psi_M(t_1, \dots, t_p)$ and $\chi_M(t_1, \dots, t_p)$ such that $\psi_M(r_1, \dots, r_p) = \mathrm{Card}\, V_{r_1\dots r_p}$ and $\chi_M(r_1, \dots, r_p) = \mathrm{Card}\, W_{r_1\dots r_p}$ for all sufficiently large $(r_1 \dots r_p) \in \mathbb{N}^p$, so that the polynomial $\Phi_M(t_1, \dots, t_p) = \psi_M(t_1, \dots, t_p) + \chi_M(t_1, \dots, t_p)$ has the property that $\Phi_M(r_1, \dots, r_p) = \dim_K M_{r_1\dots r_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$.*

(iii) *$\deg_{t_i} \Phi_M \le m_i$ $(1 \le i \le p)$, so that $\deg \Phi_M \le m$ and the polynomial $\Phi_M(t_1, \dots, t_p)$ can be represented as*

$$\Phi_M(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1\dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \quad (8)$$

where $a_{i_1\dots i_p} \in \mathbb{Z}$ for all $i_1, \dots, i_p$, and $a_{m_1\dots m_p} = d2^m$ where $d = \sigma^*$-$\dim_K M$.

(iv) *Let $E_j = \{(k_1, \dots, k_m) \in \mathbb{Z}^m \mid \alpha_1^{k_1} \dots \alpha_m^{k_m} f_j$ is the 1-leader of some element of $G\}$ $(1 \le j \le n)$. Then $\psi_M = \sum_{j=1}^{m} \phi_{E_j}$ where ϕ_{E_j} is the dimension polynomial of E_j (see Theorem 2.7). Clearly, the coefficient of $\binom{t_1+m_1}{m_1} \dots \binom{t_p+m_p}{m_p}$ in the canonical representation of ψ_M (formula (2)) is equal to the number of empty sets among $E_1, \dots, E_n$.*

(v) *$\chi_M(t_1, \dots, t_p)$ is an alternating sum of polynomials of the form*

$$\chi_{j;k_1,\dots,k_q} = \binom{t_1 + m_1 - b_{1j}}{m_1} \dots \binom{t_{k_1-1} + m_{k_1-1} - b_{k_1-1,j}}{m_{k_1-1}}$$
$$\left[\binom{t_{k_1} + m_{k_1} - a_{k_1,j}}{m_{k_1}} - \binom{t_{k_1} + m_{k_1} - b_{k_1,j}}{m_{k_1}}\right] \cdot$$
$$\binom{t_{k_1+1} + m_{k_1+1} - b_{k_1+1,j}}{m_{k_1+1}} \dots \binom{t_{k_q-1} + m_{k_q-1} - b_{k_q-1,j}}{m_{k_q-1}} \cdot$$
$$\left[\binom{t_{k_q} + m_{k_q} - a_{k_q,j}}{m_{k_q}} - \binom{t_{k_q} + m_{k_q} - b_{k_q,j}}{m_{k_q}}\right] \dots$$
$$\binom{t_p + m_p - b_{pj}}{m_p}, \quad \text{so that } \deg \chi_M < m.$$

Definition 4.2. *The polynomial $\Phi_M(t_1, \dots, t_p)$ is called a σ^*-dimension polynomial of the σ^*-K-module M associated with the given excellent p-dimensional filtration $\{M_{r_1\dots r_p} \mid (r_1, \dots, r_p) \in \mathbb{N}^p\}$ of this module.*

Remark. The last theorem combines several results from [9, Section 3.5] (as we have mentioned, one can use Gröbner bases with respect to p orderings instead of generalized characteristic sets used in [9, Section 3.5]). The only exception is part (iv) that follows from the description of the sets $V_{r_1\dots r_p}$ and Theorem 2.7. (Therefore, Proposition 3.7, together with Theorems 2.8 and 2.6, give a method of computation of multivariate σ^*-dimension polynomials.) Also, as it is shown in [9, Theorem 3.5.38], $\Phi_M(t_1, \dots, t_p)$ carries several invariants of the σ^*-K-module M, i. e., integers that are independent of the set of finite generators of M over $\mathcal{E}$ (in what follows we will use only one of such invariants, $d = \sigma^*$-$\dim_K M$).

Example. Let K be an inversive (σ^*-) field with $\sigma = \{\alpha_1, \alpha_2\}$, $\mathcal{E}$ the ring of σ^*-operators over K, and M a σ^*-K-module with one generator y and one defining relation $\omega y = 0$ where $\omega = c_1(\alpha_1 + \alpha_1^{-1}) + c_2(\alpha_2 + \alpha_2^{-1}) \in \mathcal{E}$ (c_1 and c_2 are constants in K, that is, $\alpha_i(c_j) = c_j$, $1 \le i, j \le 2$). Thus, M can be treated as a factor module of a free $\mathcal{E}$-module F with free generator f by its $\mathcal{E}$-submodule $N = \mathcal{E}\omega f$. Then $\{g_1 = \omega f, g_2 = \alpha_1^{-1} g_1\}$ is an $(<_1, <_2)$-Gröbner basis of N. (In this case $u^{(1)}_{g_1} = \alpha_1 f$ and $u^{(1)}_{g_2} = \alpha_1^{-2} f$; the fact that for every $h \in N$, there is $i \in \{1, 2\}$ such that $\rho(g_i) \mid \rho(h)$ (see Definition 3.4) follows from [4, Corollary 6.5.4].) With the notation of Theorems 4.1 and 2.7, for any $r_1, r_2 \in \mathbb{N}$, $V_{r_1 r_2} = W_{\{(1,0).(-2,0)\}}$ and $W_{r_1 r_2} = \{(i, j) \in \mathbb{Z}^2 \mid 1 \le |i| \le r_1, j \in \{r_2, -r_2\}\}$. Using Theorems 2.8 and 2.6 we obtain that $\mathrm{Card}\, V_{r_1 r_2} = 4r_2$ and $\mathrm{Card}\, W_{r_1 r_2} = 4r_1$, so the σ^*-dimension polynomial of M associated with the generator y is $\Phi(t_1, t_2) = 4t_1 + 4t_2$.

The next theorem gives a property of a multivariate filtration of a finitely generated σ^*-K-module whose σ^*-dimension polynomial has the simplest possible form.

Theorem 4.3. *Let K be an inversive difference field with a basic set of automorphisms $\sigma = \{\alpha_1, \dots, \alpha_m\}$ and let the partition (1) of σ be fixed. Let M be a finitely generated σ^*-K-module with σ^*-generators $x_1, \dots, x_n$ and let $\{M_{r_1,\dots,r_p} \mid (r_1, \dots, r_p) \in \mathbb{N}^p\}$ be the standard filtration of M associated with this system, that is, $M_{r_1,\dots,r_p} = \sum_{i=1}^{n} \mathcal{E}_{r_1,\dots,r_p} x_i$ for every $(r_1, \dots, r_p) \in \mathbb{N}^p$. Let $\Phi_M(t_1, \dots, t_p)$ be the σ^*-dimension polynomial associated with this filtration be of the form*

$$\Phi_M(t_1, \dots, t_p) = d \prod_{j=1}^{p} \left[\sum_{i=0}^{m_j} (-1)^{m_j - i} 2^i \binom{m_j}{i} \binom{t_j + i}{i}\right]$$

where $d = \sigma^*$-$\dim_K M$. Without loss of generality we can assume that $x_1, \dots, x_d$ are σ^*-linearly independent over K, that is, linearly independent over $\mathcal{E}$. Let $N = \sum_{i=1}^{d} \mathcal{E} x_i$ and for any $(r_1, \dots, r_p) \in \mathbb{N}^p$, let $N_{r_1,\dots,r_p} = \sum_{i=1}^{d} \mathcal{E}_{r_1,\dots,r_p} x_i$. Then $M_{r_1,\dots,r_p} = N_{r_1,\dots,r_p}$ for all $(r_1, \dots, r_p) \in \mathbb{N}^p$.

Proof. Clearly, the statement is true for $m = 0$, so we can assume that $m > 0$ and $p \ge 1$. By part (iv) of Theorem 4.1 and part (iii) of Theorem 2.7 (see formula (6)),

$$\dim N_{r_1,\dots,r_p} = d \prod_{j=1}^{p} \left[\sum_{i=0}^{m_j} (-1)^{m_j - i} 2^i \binom{m_j}{i} \binom{t_j + i}{i}\right] =$$

$\dim M_{r_1,\dots,r_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$.

Let $r^{(0)} = (r_1^{(0}, \dots, r_p^{(0)}) \in \mathbb{N}^p$ be a minimal with respect to the product order $\leq_P$ on $\mathbb{N}^p$ element such that the last equality holds for all p-tuples $r \in \mathbb{N}^p$ with $r^{(0)} \leq_P r$ (we use the fact that every subset of $\mathbb{N}^p$ contains finitely many elements minimal with respect to the product order). We should prove that $r^{(0)} = (0, \dots, 0)$.

Assume for contradiction that this is not true. Without loss of generality we can assume that $r_p^{(0)} > 0$. Then the last equality does not hold for the p-tuple $s^{(0)} = (r_1^{(0}, \dots, r_p^{(0)} - 1)$, so there exists $\gamma_0 x_i \in M_{s^{(0)}} \subseteq M_{r^{(0)}} = N_{r^{(0)}}$ such that $\gamma_0 x_i \notin N_{s^{(0)}}$. Since $\gamma_0 x_i \in N_{r^{(0)}} \setminus N_{s^{(0)}}$, we can write

$$\gamma_0 x_i = \sum_{j=1}^{d} \sum_{k=1}^{e_j} a_{jk} \gamma_{jk} x_j, \tag{9}$$

where $0 \neq a_{jk} \in K$, $\gamma_{jk} \in \Gamma(r^{(0)})$ $(1 \leq j \leq d,\ 1 \leq k \leq e_j$. Also, the last sum contains a term $a_{j'k'}\gamma_{j'k'}x_{j'}$ such that $a_{j'k'} \neq 0$ and $\gamma_{j'k'} \in \Gamma(r^{(0)}) \setminus \Gamma(s^{(0)})$.

Let Σ' be the sum of all terms $a_{jk}\gamma_{jk}x_j$ in (9) such that $\mathrm{ord}_p\, \gamma_{jk} = r_p^{(0)}$ let Σ'' be the sum of the remaining terms. Let $a_{j'k'}\gamma_{j'k'}x_{j'}$ be a term in Σ' such that if we write it as $\gamma_{j'k'} = \gamma_{j'k'}^{(1)} \dots \gamma_{j'k'}^{(p)}$, where $\gamma_{j'k'}^{(i)}$ $(1 \leq i \leq p)$ is a power product of elements σ_i and $\gamma_{j'k'}^{(p)} = \alpha_{m_1+\dots+m_{p-1}+1}^{q_{m_1+\dots+m_{p-1}+1}} \dots \alpha_m^{q_m}$, then the m_p-tuple $(q_{m_1+\dots+m_{p-1}+1}, \dots, q_m)$ is the largest possible one with respect to the following order $\preccurlyeq$ on $\mathbb{Z}^{m_p}$: $(i_1, \dots, i_{m_p}) \preccurlyeq (j_1, \dots, j_{m_p})$ if and only if the $2m_p$-tuple $(|i_1|, \dots, |i_{m_p}|, i_1, \dots, i_{m_p})$ is less than the $2m_p$-tuple $(|j_1|, \dots, |j_{m_p}|, j_1, \dots, j_{m_p})$ with respect to the lexicographic order on $\mathbb{Z}^{2m_p}$. Since $\mathrm{ord}_p\, \gamma_{j'k'} > 0$, we can choose the smallest index $\nu \geq m_1 + \dots + m_{p-1} + 1$ such that $q_\nu \neq 0$. Let α denote α_ν if $q_\nu > 0$ and α_ν^{-1} if $q_\nu < 0$. Then

$$\alpha\gamma_0 x_i = \sum_{j=1}^{d} \sum_{k=1}^{e_j} \alpha(a_{jk}) \gamma_{jk} x_j, \tag{10}$$

$\alpha\gamma_0 x_i \in M_{r^{(0)}} = N_{r^{(0)}}$ and $\mathrm{ord}_p(\alpha\gamma_{j'k'}) = r_p^{(0)} + 1$. Since $\alpha\gamma_0 x_i \in N_{r^{(0)}}$, $\alpha\gamma_0 x_i$ can be written as a linear combination of elements of the set $\{\gamma x_j \mid \gamma \in \Gamma(r^{(0)}), 1 \leq j \leq d\}$ with coefficients in K. If we write the equality of such a linear combination and expression (10), then the term $\alpha(a_{j'k'})\gamma_{j'k'}x_{j'}$ cannot be cancelled. This contradicts the σ^*-linear independence of elements $x_1, \dots, x_d$ over K. Thus, $r^{(0)} = (0, \dots, 0)$ and the theorem is proved. □

The following result gives an intersection property of a finitely generated σ^*-K-module.

Theorem 4.4. *Let K be an inversive difference (σ^*-) field with a basic set $\sigma = \{\alpha_1, \dots, \alpha_m\}$ and M a finitely generated σ^*-K-module, with σ^*-generators $x_1, \dots, x_n$ (that is, generators of M as a module over the ring $\mathcal{E}$ of σ^*-operators over K). Let $d = \sigma^*\text{-}\dim_K M$. Then there exist $i_1, \dots, i_d \in \{1, \dots, n\}$ with $i_1 < \dots < i_d$ such that $x_{i_1}, \dots, x_{i_d}$ are σ^*-linearly independent over K and if N is the $\mathcal{E}$-module generated by $x_{i_1}, \dots, x_{i_d}$, $M_{r_1,\dots,r_p} = \sum_{k=1}^{n} \mathcal{E}_{r_1,\dots,r_p} x_k$, and $N_{r_1,\dots,r_p} = \sum_{j=1}^{d} \mathcal{E}_{r_1,\dots,r_p} x_{i_j}$ $(r_1, \dots, r_p \in \mathbb{N})$, then*

$$N_{r_1,\dots,r_p} = M_{r_1,\dots,r_p} \bigcap N \quad \text{for all} \quad r_1, \dots, r_p \in \mathbb{N}.$$

Proof. Let F be the free $\mathcal{E}$-module of rank n with free generators $f_1, \dots, f_n$. Let $\phi : F \to M$ be the homomorphism of $\mathcal{E}$-modules such that $\phi(f_i) = x_i$ for $i = 1, \dots, n$. Let $W = \mathrm{Ker}\, \phi$ and let G be a Gröbner basis of W with respect to $(<_1, \dots, <_p)$. For every $j = 1, \dots, n$, let

$E_j = \{e = (e_1, \dots, e_m) \in \mathbb{Z}^m \mid \delta_1^{e_1} \dots \delta_n^{e_n}$ is a 1-leader of some element of $G\}$.

Then there exists a polynomial $\Phi(t_1, \dots, t_p) \in \mathbb{Q}[t_1, \dots, t_p]$ such that $\Phi(r_1, \dots, r_p) = \dim_K M_{r_1,\dots,r_p}$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$ and this polynomial can be written as

$$\Phi(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1,\dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p}$$

where $a_{i_1,\dots i_p} \in \mathbb{Z}$ for all p-tuples $(i_1, \dots, i_p) \in \mathbb{N}^p$ with $(i_1, \dots, i_p)$ $\leq_P$ $(m_1, \dots, m_p)$, and $2^m | a_{m_1,\dots m_p}$. Furthermore,

$\Phi(t_1, \dots, t_p) = \sum_{j=1}^{n} \phi_{E_j}(t_1, \dots, t_p) +$ a polynomial of degree less than m,

where $\phi_{E_j}(t_1, \dots, t_p)$ is the dimension polynomial of the set E_j associated with the partition of the set $\mathbb{N}_m = \{1, \dots, m\}$ that corresponds to the given partition of σ.

Since $\deg \phi_{E_j} = m$ if and only if $E_j = \emptyset$ and then $\phi_{E_j}(t_1, \dots, t_p) = \prod_{j=1}^{p} \left[\sum_{i=0}^{m_j} (-1)^{m_j - i} 2^i \binom{m_j}{i} \binom{t_j + i}{i} \right]$, there are exactly d indices j for which $E_j = \emptyset$. Without loss of generality we can assume that these indices are $1, \dots, d$. It means that the leaders of elements of G contain only generators f_k of F with indices $k \in \{d + 1, \dots, n\}$.

Suppose that there exists $r = (r_1, \dots, r_p) \in \mathbb{N}^p$ such that $N_{r_1,\dots,r_p} \not\supseteq M_{r_1,\dots,r_p} \bigcap N$. Then there exists an element $A \in \left(M_{r_1,\dots,r_p} \bigcap N\right) \setminus N_{r_1,\dots,r_p}$ of the lowest (with respect to $(<_1, \dots, <_p)$) possible rank among all such elements. Then $A = \sum_{j=1}^{d} D_j x_j = \sum_{i=1}^{n} D_i' x_i$ where $D_j \in \mathcal{E}$ $(1 \leq j \leq d)$ and $D_i' \in \mathcal{E}_{r_1,\dots,r_p}$ $(1 \leq i \leq n)$. It follows that if we set $P = \sum_{j=1}^{d} D_j f_j - \sum_{i=1}^{n} D_i' f_i$, then $P \in W$.

Let $q = (q_1, \dots, q_p)$ be the smallest with respect to the product order element of $\mathbb{N}^p$ such that $r \leq_P q$, $D_j, D_i' \in \mathcal{E}_{q_1,\dots,q_p}$ $(1 \leq j \leq d$, $1 \leq i \leq n)$ and let q be the smallest such p-tuple with respect to the lexicographic order on $\mathbb{N}^p$.

Then $(r_1, \dots, r_p) <_P (q_1, \dots, q_p)$ (if $r = q$, then $A \in N_{r_1,\dots,r_p}$). Let $v = \theta_1 f_k$ be the 1-leader of P. Then $\mathrm{ord}_i\, \theta_1 = q_i$ and $1 \leq k \leq d$. Indeed, if v appears in one of D_i', then $(\mathrm{ord}_1\, \theta_1, \dots, \mathrm{ord}_p\, \theta_1) \leq_P (r_1, \dots, r_p) <_P (q_1, \dots, q_p)$ while some D_j $(1 \leq j \leq d)$ contains a term $\theta' f_\nu$ with $(\mathrm{ord}_1\, \theta', \dots, \mathrm{ord}_p\, \theta') = (q_1, \dots, q_p)$.

Let P' be the result of the $(<_1, \dots, <_p)$-reduction of P with respect to G, so P' is reduced with respect to G. Since the reduction uses transforms of leaders of elements of G and these leaders contain only f_i with $d + 1 \leq i \leq n$, P' must contain v. Therefore, $P' \neq 0$. On the other hand, $P' \in W$ and P' is $(<_1, \dots, <_p)$-reduced with respect to G. This contradiction (see Proposition 3.6) shows that $N_{r_1,\dots,r_p} = M_{r_1,\dots,r_p} \bigcap N$ for all $r_1,\dots,r_p \in \mathbb{N}$. □

The last theorem allows one to obtain an interesting result about σ^*-dimension polynomials of σ^*-field extensions (see Theorem 4.5 below) that is a difference version of [2, Theorem 3.1]. Recall (see [9, Theorem 4.2.17]) that if $L = K\langle\eta_1,\dots,\eta_n\rangle$ is a σ^*-field extension ($\operatorname{Char} K = 0$) with a finite set of generators $\eta = \{\eta_1,\dots,\eta_n\}$, then (given that the partition (1) of σ is fixed) there exists a polynomial in p variables $\Psi_{\eta|K} \in \mathbb{Q}[t_1,\dots,t_p]$ such that $\Psi_{\eta|K}(r_1,\dots,r_p) = \text{tr.}\deg_K K(\{\gamma\eta_i \mid \tau \in \Gamma(r_1,\dots,r_p),\ 1 \le j \le n\})$ for all sufficiently large $(r_1,\dots,r_p) \in \mathbb{N}^p$. This polynomial, which is called the σ^*-dimension polynomial of the extension L/K associated with the set of σ^*-generators η, carries a number of invariants of the extension. In particular, if it is written in the canonical form (2), then $d = a_{m_1\dots m_p}/2^m$ is the σ^*-transcendence degree of L/K (denoted by $\sigma^*\text{-tr.}\deg_K L$), that is, the maximal number of elements $x_1,\dots,x_k$ of L such that the set $\{\gamma(x_i) \mid \gamma \in \Gamma, 1 \le i \le k\}$ is algebraically independent over K. (In this case we say that $x_1,\dots,x_k$ are σ^*-algebraically independent over K. Any maximal set of such elements has d elements; it is called a σ^*-transcendence basis of L/K. As in the classical field theory, every set of σ^*-generators of L/K contains a σ^*-transcendence basis of this extension.) Properties of σ^*-dimension polynomials of σ^*-field extensions and their applications to the analysis of systems of algebraic difference equations can be found in [9, Chapters 4 and 7]. In this connection, in the future research we plan to develop and implement an algorithm for computing multivariate σ^*-dimension polynomials of systems of algebraic difference equations and apply the obtained results to the computation of a difference analog of the Einstein's strength of concrete such systems.

Theorem 4.5. *Let $L = K\langle\eta_1,\dots,\eta_n\rangle$ be a σ^*-field extension generated by a finite set $\eta = \{\eta_1,\dots,\eta_n\}$ and let $d = \sigma^*\text{-tr.}\deg_K L$. Then the set η contains a σ^*-transcendence basis $B = \{\eta_{i_1},\dots\eta_{i_d}\}$ of L over K $(1 \le i_1 < \dots < i_d \le n)$ such that if η' denotes the set $\eta \setminus B$, then*

$$\Psi_{\eta'|K\langle B\rangle}(t_1,\dots,t_p) = \Psi_{\eta|K}(t_1,\dots,t_p) - d\prod_{j=1}^{p}\left[\sum_{i=0}^{m_j}(-1)^{m_j-i}2^i\binom{m_j}{i}\binom{t_j+i}{i}\right]. \tag{11}$$

Moreover, $\text{tr.}\deg_K K(\{\gamma\eta_i \mid \tau \in \Gamma(r_1,\dots,r_p),\ 1 \le j \le n\})$

$$= d\prod_{j=1}^{p}\left[\sum_{i=0}^{m_j}(-1)^{m_j-i}2^i\binom{m_j}{i}\binom{r_j+i}{i}\right] +$$

$\text{tr.}\deg_{K\langle B\rangle} K\langle B\rangle(\{\gamma\eta_i \mid \tau \in \Gamma(r_1,\dots,r_p),\ 1 \le j \le n\})$ *for all* $(r_1,\dots,r_p) \in \mathbb{N}^p$.

Proof. Clearly we can assume $m = \operatorname{Card}\sigma > 0$ (for $m = 0$ the statement is obvious). Let $\Omega_{L|K}$ denote the module of Kähler differentials associated with the extension L/K. Then $\Omega_{L|K}$ can be treated as a σ^*-L-module where the action of the elements of σ^* is defined in such a way that $\alpha(d\zeta) = d\alpha(\zeta)$ for any $\zeta \in L$, $\alpha \in \sigma^*$ (see [9, Lemma 4.2.8]). Furthermore, by [9, Theorem 4.2.9], we have $\sigma^*\text{-}\dim_L \Omega_{L|K} = \sigma^*\text{-tr.}\deg_K L$.

Let $M = \Omega_{L|K}$ and for any $(r_1,\dots,r_p) \in \mathbb{N}^p$, let $M_{r_1,\dots,r_p}$ denote the vector L-space generated by all elements $d\zeta$ where $\zeta \in K(\bigcup_{i=1}^{n}\Gamma(r_1,\dots,r_p)\eta_i)$. It is easy to check that $\{M_{r_1\dots r_p} \mid (r_1,\dots,r_p) \in \mathbb{Z}^p\}$ ($M_{r_1,\dots,r_p} = 0$ if $(r_1,\dots,r_p) \in \mathbb{Z}^p \setminus \mathbb{N}^p$) is an excellent p-dimensional filtration of the σ^*-L-module M. Also, $\dim_K M_{r_1,\dots,r_p} = \text{tr.}\deg_K K(\{\gamma\eta_i \mid \tau \in \Gamma(r_1,\dots,r_p),\ 1 \le j \le n\})$ for all $(r_1,\dots,r_p) \in \mathbb{N}^p$ (see[11, Proposition 23.17]).
By Theorem 4.4, there exists a set $B' = \{d\eta_{i_1},\dots d\eta_{i_d}\} \subseteq M$ $(1 \le i_1 < \dots < i_d \le n)$ such that the elements of B' are linearly independent over $\mathcal{E}$ and if $N = \sum_{\nu=1}^{d}\mathcal{E}d\eta_{i_\nu}$ and $N_{r_1,\dots,r_p} = \sum_{\nu=1}^{d}\mathcal{E}_{r_1,\dots,r_p}d\eta_{i_\nu}$, then $N_{r_1,\dots,r_p} = M_{r_1,\dots,r_p} \bigcap N$ for all $(r_1,\dots,r_p) \in \mathbb{N}^p$. Therefore, the elements of the set $B = \{\eta_{i_1},\dots\eta_{i_d}\}$ are σ^*-algebraically independent over K and $\sigma^*\text{-tr.}\deg_{K\langle B\rangle} L = 0$. Now we can repeat the arguments of the proof of [2, Theorem 3.2] to show that if we set $L_{r_1,\dots,r_p} = K(\{\gamma\eta_i \mid \tau \in \Gamma(r_1,\dots,r_p),\ 1 \le j \le n\})$, $F = K\langle B\rangle$, and $F_{r_1,\dots,r_p} = K(\{\gamma\eta_{i_\nu} \mid \tau \in \Gamma(r_1,\dots,r_p),\ 1 \le \nu \le d\})$, then for any $(r_1,\dots,r_p) \in \mathbb{N}^p$ there is an exact sequence of σ^*-L-modules

$0 \to L \bigotimes_{F_{r_1,\dots,r_p}} \Omega_{F_{r_1,\dots,r_p}|K} \to L \bigotimes_{L_{r_1,\dots,r_p}} \Omega_{L_{r_1,\dots,r_p}|K}$

$\to L \bigotimes_{L_{r_1,\dots,r_p}} \Omega_{L_{r_1,\dots,r_p}|F_{r_1,\dots,r_p}} \to 0,$

$\dim_L L \bigotimes_{L_{r_1,\dots,r_p}} \Omega_{L_{r_1,\dots,r_p}|K} = \text{tr.}\deg_K L_{r_1,\dots,r_p},$

$\dim_L L \bigotimes_{F_{r_1,\dots,r_p}} \Omega_{F_{r_1,\dots,r_p}|K} =$

$d\prod_{j=1}^{p}\left[\sum_{i=0}^{n_j}(-1)^{n_j-i}2^i\binom{n_j}{i}\binom{t_j+i}{i}\right]$, and

$\dim_L L \bigotimes_{L_{r_1,\dots,r_p}} \Omega_{L_{r_1,\dots,r_p}|F_{r_1,\dots,r_p}} =$

$\text{tr.}\deg_F F(\{\gamma\eta_i | \gamma \in \Gamma(r_1,\dots,r_p), 1 \le j \le n\}).$

Since $\dim_L$ is an additive function for exact sequences, we obtain the statement of our theorem. □

5 ACKNOWLEDGES

This research was supported by the NSF grant CCF−1714425.

REFERENCES

[1] A. Einstein. *The Meaning of Relativity. Appendix II (Generalization of gravitation theory)*, 153–165. Princeton University Press, Princeton, NJ, 1953.
[2] J. Johnson and W. Sit. On the differential transcendence polynomials of finitely generated differential field extensions. *Amer. J. Math.*, 101 (1979), 1249–1263.
[3] E. R. Kolchin. *Differential Algebra and Algebraic Groups.* Academic Press, 1973.
[4] M. V. Kondrateva, A. B. Levin, A. V. Mikhalev, and E. V. Pankratev. *Differential and Difference Dimension Polynomials.* Kluwer Acad. Publ., 1999.
[5] A. B. Levin. Reduced Groebner bases, free difference-differential modules and difference-differential dimension polynomials. *J. Symb. Comput.*, 30 (2000), 357–382.
[6] A. B. Levin. Gröbner bases with respect to several orderings and multivariable dimension polynomials. *J. Symb. Comput.*, 42 (2007), no. 5, 561–578.
[7] A. B. Levin Computation of the Strength of Systems of Difference Equations via Generalized Gröbner Bases. *Grobner Bases in Symbolic Analysis*, Walter de Gruyter, 2007, 43–73.
[8] A. B. Levin. Multivariate dimension polynomials of inversive difference field extensions *Algebraic and algorithmic aspects of differential and integral operators. Lecture Notes in Comput. Sci.*, 8372 (2014), 146–163.
[9] A. B. Levin *Difference Algebra.* Springer, New York, 2008.
[10] A. B. Levin and A. V. Mikhalev. Type and Dimension of Finitely Generated G-algebras. *Contemporary Mathematics*, 184 (1995), 275–280.
[11] P. Morandi. *Field and Galois Theory.* Springer-Verlag, New York, 1996.
[12] M. Zhou and F. Winkler Computing difference-differential dimension polynomials by relative Grobner bases in difference-differential modules. *J. Symb. Comput.*, 43 (2008), no. 10, 726–745.