Multivariate Differential Dimension Polynomials, their Properties and Invariants

Alexander Levin

The Catholic University of America
Washington, D. C. 20064

Abstract

In this paper we obtain new results on multivariate dimension polynomials of differential field extensions associated with partitions of basic sets of derivations. We prove that the coefficient of the summand of the highest possible degree in the canonical representation of such a polynomial is equal to the differential transcendence degree of the extension. We also give necessary and sufficient conditions under which the multivariate dimension polynomial of a differential field extension of a given differential transcendence degree has the simplest possible form. Furthermore, we describe some relationships between a multivariate dimension polynomial of a differential field extension and dimensional characteristics of subextensions defined by subsets of the basic sets of derivations. In the last part of the paper we show how the invariants of multivariate dimension polynomials can be used for determining the equivalence of systems of algebraic differential equations and discuss the relationship between such polynomials and the concept of Einstein's strength of a system of algebraic partial differential equations.

Key words: differential field extension, differential dimension polynomial, differential transcendence degree, characteristic set
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1. Introduction

Differential dimension polynomials introduced in [5] by E. Kolchin play the same role in differential algebra as Hilbert polynomials play in commutative algebra and algebraic geometry. An important feature of differential dimension polynomials is that they describe in exact terms the freedom degree of a continuous dynamic system as well as the number of arbitrary constants in the general solution of a system of algebraic partial differential equations. The following fundamental result, whose proof can be found in [6,

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Email address: levin@cua.edu (Alexander Levin).

Chapter II], establishes the existence and basic properties of a (univariate) dimension polynomial of a finitely generated differential field extension.

Theorem 1.1. Let K be a differential field of characteristic zero, that is, a field considered together with the action of a set $\Delta = \{\delta_1, \dots, \delta_m\}$ of mutually commuting derivations of K into itself. Let Θ denote the free commutative semigroup of all power products of the form $\theta = \delta_1^{k_1} \dots \delta_m^{k_m}$ $(k_i \geq 0)$, let $\text{ord}\,\theta = \sum_{i=1}^{m} k_i$, and for any $r \geq 0$, let $\Theta(r) = \{\theta \in \Theta \,|\, \text{ord}\,\theta \leq r\}$. Let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a differential field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. (As a field, $L = K(\{\theta\eta_j | \theta \in \Theta, 1 \leq j \leq n\})$.) Then there exists a polynomial $\omega_{\eta|K}(t) \in \mathbb{Q}[t]$ such that

(i) $\omega_{\eta|K}(r) = \text{tr.}\deg_K K(\{\theta\eta_j | \theta \in \Theta(r),\, 1 \leq j \leq n\})$ for all sufficiently large $r \in \mathbb{Z}$;

(ii) $\deg \omega_{\eta|K} \leq m$ and $\omega_{\eta|K}(t)$ can be represented as $\omega_{\eta|K}(t) = \sum_{i=0}^{m} a_i \binom{t+i}{i}$ where $a_0, \dots, a_m \in \mathbb{Z}$;

(iii) $d = \deg \omega_{\eta|K}$, a_m and a_d do not depend on the choice of the system of Δ-generators η of the extension L/K (clearly, $a_d \neq a_m$ if and only if $d < m$, that is, $a_m = 0$). Moreover, a_m is equal to the differential transcendence degree of L over K (denoted by Δ-$\text{tr.}\deg_K L$), that is, to the maximal number of elements $\xi_1, \dots, \xi_k \in L$ such that the set $\{\theta\xi_i | \theta \in \Theta, 1 \leq i \leq k\}$ is algebraically independent over K.

(iv) If the elements $\eta_1, \dots, \eta_n$ are Δ-algebraically independent over K (i. e., the set $\{\theta\eta_i \,|\, \theta \in \Theta, 1 \leq i \leq n\}$ is algebraically independent over K), then $\omega_{\eta|K}(t) = n\binom{t+m}{m}$.

The polynomial $\omega_{\eta|K}$ is called the *differential dimension polynomial* of the differential field extension L/K associated with the system of differential generators η. The invariants $d = \deg \omega_{\eta|K}$ and a_d in part (iii) of the theorem are called the *differential* (or Δ-) *type* and *typical differential* (or Δ-) *transcendence degree* of the extension L/K; they are denoted by Δ-$\text{type}_K L$ and Δ-t. tr. $\deg_K L$, respectively.

Differential dimension polynomials provide a power tool for the study of systems of algebraic differential equations. For a wide class of such systems, the dimension polynomial of the corresponding differential field extension expresses the strength of the system of equations in the sense of A. Einstein. This concept, that was introduced in [1] as an important qualitative characteristic of a system of PDEs, can be expressed as a certain differential dimension polynomial, as it is shown in [14]. Another important application of differential dimension polynomials is based on the fact that if P is a prime (in particular, linear) differential ideal of a finitely generated differential algebra R over a differential field K and L is the quotient field of R/P treated as a differential overfield of K, then the differential dimension polynomial of the extension L/K characterizes the ideal P; assigning such polynomials to prime differential ideals has led to a number of new results on the Krull-type dimension of differential algebras and differential field extensions (see, for example, [3], [4], [13], [7, Chapter 7]), and [17]). It should be also added that the dimension polynomial associated with a finitely generated differential field extension carries certain differential birational invariants, that is, numbers that do not change when we switch to another finite system of generators of the extension. These invariants are closely connected to some other important characteristics; one of them is the differential transcendence degree of the extension. Among recent works on univariate differential dimension polynomials one has to mention the work of O. Sanchez [15] on the evaluation of the coefficients of a differential dimension polynomial, the work of J. Freitag, O. Sanchez and W. Li on the definability of Kolchin polynomials, and works of M. Lange-Hegermann

[8] and [9], where the author introduced a differential dimension polynomial of a characterizable (not necessarily prime) differential ideal and a countable differential polynomial that generalizes the concept of differential dimension polynomial.

In 2001 the author introduced a concept of a multivariate differential dimension polynomial of a finitely generated differential field extension associated with a partition of the set of basic derivations Δ (see [10]). The proof of the corresponding existence theorem that generalizes the first two parts of Theorem 1.1, was based on a special type of reduction in a ring of differential polynomials that takes into account the partition of Δ. It was also shown that a multivariate differential dimension polynomial carries essentially more differential birational invariants of the corresponding differential field extension than its univariate counterpart. As it is demonstrated in Example 4.7, a multivariate dimension polynomial associated with an algebraic differential equation with parameters can carry all this parameters, while the univariate dimension polynomial determines just some relation between the parameters. Therefore, there is a strong motivation for the study of multivariate differential dimension polynomials and their invariants. The main difficulty in this study is due to the fact that a multivariate dimension polynomial of a prime differential polynomial ideal is determined by a characteristic set with respect to several term orderings. Such sets were introduced in [10], but the corresponding theory is in its infancy. Another problem, that is partially solved in this paper, is to characterize invariants of multivariate dimension polynomials and to find relationships between invariants of such polynomials associated with different partitions of the basic set of derivations.

In this paper we obtain new results on multivariate differential dimension polynomials of differential field extensions associated with partitions of the basic sets of derivations. We give necessary and sufficient conditions under which the multivariate dimension polynomial of a differential field extension of a given differential transcendence degree has the simplest possible form. This result (Theorem 4.4) generalizes the corresponding property of univariate differential dimension polynomials proved in [16]. We also prove that the coefficient of the summand of the highest possible degree in the canonical representation of a multivariate dimension polynomial is equal to the differential transcendence degree of the extension (Theorem 4.2). Furthermore, we obtain some relationships between a multivariate dimension polynomial of a differential field extension and dimensional characteristics of subextensions defined by subsets of the basic sets of derivations.

This paper essentially extends the results obtained in [13]: we give a constructive proof of the main theorem on the reduction of differential polynomials with respect to several term orderings (Theorem 3.6 and Corollary 3.7) and present the corresponding algorithm; we also establish the existence of a special differential transcendence basis B of a finitely generated differential field extension L/K such that the multivariate differential dimension polynomial of $L/K\langle B\rangle$ is naturally connected with the corresponding dimension polynomial of L/K (Theorem 4.6); in the last part of the paper we show how the invariants of multivariate dimension polynomials can be used for the study of equivalence of systems of algebraic differential equations and discuss the relationship between such polynomials and the concept of Einstein's strength of a system of algebraic partial differential equations.

2. Preliminaries

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of all integers, all nonnegative integers and all rational numbers, respectively. If M is a finite set, then $\operatorname{Card} M$ will denote the number of elements of M. By a ring we always mean an associative ring with unity. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring, and every algebra over a commutative ring is unitary. Unless otherwise indicated, every field is supposed to have zero characteristic.

A *differential ring* is a commutative ring R considered together with a finite set Δ of mutually commuting derivations of R into itself. The set Δ is called a basic set of the differential ring R that is also called a Δ-ring. A subring (ideal) R_0 of a Δ-ring R is called a differential (or Δ-) subring of R (respectively, a differential (or Δ-) ideal of R) if $\delta(R_0) \subseteq R_0$ for any $\delta \in \Delta$. If a differential (Δ-) ring is a field, it is called a differential (or Δ-) field. In what follows, Θ (or Θ_Δ if we want to indicate the basic set) denotes the free commutative semigroup generated by Δ (that is, if $\Delta = \{\delta_1, \dots, \delta_m\}$, then $\Theta = \{\theta = \delta_1^{k_1} \dots \delta_m^{k_m} \,|\, k_1, \dots, k_m \in \mathbb{N}\}$).

If R is a Δ-ring and $S \subseteq R$, then the smallest Δ-ideal of R containing S is denoted by $[S]$ (as an ideal, it is generated by the set $\{\theta\xi \,|\, \xi \in S\}$). If the set S is finite, $S = \{\xi_1, \dots, \xi_q\}$, we say that the Δ-ideal $I = [S]$ is finitely generated, write $I = [\xi_1, \dots, \xi_q]$) and call $\xi_1, \dots, \xi_q$ differential (or Δ-) generators of I. If a Δ-ideal is prime (in the usual sense), it is called a *prime* differential (or Δ-) *ideal*.

Let R_1 and R_2 be two differential rings with the same basic set $\Delta = \{\delta_1, \dots, \delta_m\}$. (More rigorously, we assume that there exist injective mappings of the set Δ into the sets of mutually commuting derivations of the rings R_1 and R_2. For convenience we will denote the images of elements of Δ under these mappings by the same symbols $\delta_1, \dots, \delta_m$). A ring homomorphism $\phi : R \longrightarrow S$ is called a *differential* (or Δ-) *homomorphism* if $\phi(\delta a) = \delta\phi(a)$ for any $\delta \in \Delta$, $a \in R$.

If K is a Δ-field and K_0 a subfield of K which is also a Δ-subring of K, then K_0 is said to be a differential (or Δ-) subfield of K, and K is called a differential (or Δ-) field extension or a Δ-overfield of K_0. We also say that we have a Δ-field extension K/K_0. In this case, if $S \subseteq K$, then the intersection of all Δ-subfields of K containing K_0 and S is the unique Δ-subfield of K containing K_0 and S and contained in every Δ-subfield of K containing K_0 and S. It is denoted by $K_0\langle S\rangle$ or by $K_0\langle S\rangle_\Delta$ if we want to indicate the set of basic derivations Δ. If $K = K_0\langle S\rangle$ and the set S is finite, $S = \{\eta_1, \dots, \eta_n\}$, then K is said to be a finitely generated Δ-field extension of K_0 with the set of Δ-generators $\{\eta_1, \dots, \eta_n\}$. In this case we write $K = K_0\langle \eta_1, \dots, \eta_n\rangle$. It is easy to see that the field $K_0\langle \eta_1, \dots, \eta_n\rangle$ coincides with the field $K_0(\{\theta\eta_i | \theta \in \Theta, 1 \le i \le n\})$.

Let L be a Δ-field extension of a Δ-field K. We say that a set $U \subseteq L$ is Δ-*algebraically dependent* over K, if the family $\{\theta(u) \,|\, u \in U, \theta \in \Theta\}$ is algebraically dependent over K. Otherwise, the family U is said to be Δ-*algebraically independent* over K. An element $u \in L$ is said to be Δ-algebraic over K if the set $\{u\}$ is Δ-algebraically dependent over K. A maximal Δ-algebraically independent over K subset of L is called a differential (or Δ-) transcendence basis of L over K (or of the extension L/K). It is known (see [6, Chapter II]) that every system of Δ-generators of a Δ-field extension L/K contains a Δ-transcendence basis of L over K and if L/K is finitely generated as a Δ-field extension, then all Δ-transcendence bases have the same number of elements called the *differential (or Δ-) transcendence degree* of L over K; it is denoted by $\Delta\text{-}\operatorname{tr.deg}_K L$.

If K is a Δ-field and $Y = \{y_1, \dots, y_n\}$ is a finite set of symbols, then one can consider the countable set of symbols $\Theta Y = \{\theta y_j | \theta \in \Theta, 1 \le j \le n\}$ and the polynomial ring $R = K[\{\theta y_j | \theta \in \Theta, 1 \le j \le n\}]$ in the set of indeterminates ΘY over K. This polynomial ring is naturally viewed as a Δ-ring where $\delta(\theta y_j) = (\delta\theta) y_j$ ($\delta \in \Delta, \theta \in \Theta, 1 \le j \le n$) and the elements of Δ act on the coefficients of the polynomials of R as they act in the field K. The ring R is called a *ring of differential (or Δ-) polynomials* in the set of differential (Δ-) indeterminates $y_1, \dots, y_n$ over the Δ-field K. This ring is denoted by $K\{y_1, \dots, y_n\}$ and its elements are called differential (or Δ-) polynomials.

MULTIVARIATE NUMERICAL POLYNOMIALS OF SUBSETS OF $\mathbb{N}^m$

Definition 2.1. A polynomial $f(t_1, \dots, t_p)$ in p variables ($p \ge 1$) with rational coefficients is said to be *numerical* if $f(t_1, \dots, t_p) \in \mathbb{Z}$ for all sufficiently large $t_1, \dots, t_p \in \mathbb{Z}$, that is, there exists $(s_1, \dots, s_p) \in \mathbb{Z}^p$ such that $f(r_1, \dots, r_p) \in \mathbb{Z}$ whenever $(r_1, \dots, r_p) \in \mathbb{Z}^p$ and $r_i \ge s_i$ ($1 \le i \le p$).

Clearly, every polynomial with integer coefficients is numerical. As an example of a numerical polynomial in p variables with non-integer coefficients one can consider $\prod_{i=1}^{p} \binom{t_i}{m_i}$ ($m_1, \dots, m_p \in \mathbb{Z}$), where $\binom{t}{k} = \frac{t(t-1)\dots(t-k+1)}{k!}$ for any $k \in \mathbb{Z}, k \ge 1$, $\binom{t}{0} = 1$, and $\binom{t}{k} = 0$ if k is a negative integer.

If f is a numerical polynomial in p variables ($p > 1$), then deg f and $\deg_{t_i} f$ ($1 \le i \le p$) will denote the total degree of f and the degree of f relative to the variable t_i, respectively. The following theorem gives the "canonical" representation of a numerical polynomial in several variables.

Theorem 2.2. Let $f(t_1, \dots, t_p)$ be a numerical polynomial in p variables $t_1, \dots, t_p$, and let $\deg_{t_i} f = m_i$ ($1 \le i \le p$). Then the polynomial $f(t_1, \dots, t_p)$ can be represented as

$$f(t_1, \dots t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \dots \binom{t_p + i_p}{i_p} \tag{2. 1}$$

with integer coefficients $a_{i_1 \dots i_p}$ that are uniquely defined by the numerical polynomial.

In the rest of this section we deal with subsets of $\mathbb{N}^m$ where the positive integer m is represented as a sum of p nonnegative integers $m_1, \dots, m_p$ ($p \ge 1$). In other words, we fix a partition $(m_1, \dots, m_p)$ of m.

If $x = (x_1, \dots, x_m) \in \mathbb{N}^m$, we set $\text{ord}_1 x = \sum_{j=1}^{m_1} x_j$ and $\text{ord}_i = \sum_{j=m_{i-1}+1}^{m_i} x_j$ for $i = 2, \dots, p$. If $A \subseteq \mathbb{N}^m$, then for any $r_1, \dots, r_p \in \mathbb{N}$, $A(r_1, \dots, r_p)$ will denote the subset of A that consists of all m-tuples $a = (a_1, \dots, a_m)$ such that $\text{ord}_i a \le r_i$ ($1 \le i \le p$). Furthermore, we shall associate with the set A a set $V_A \subseteq \mathbb{N}^m$ that consists of all m-tuples $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ that are not greater than or equal to any m-tuple from A with respect to the product order on $\mathbb{N}^m$. (Recall that the product order on the set $\mathbb{N}^k$ ($k \in \mathbb{N}, k \ge 1$) is a partial order $\le_P$ on $\mathbb{N}^k$ such that $c = (c_1, \dots, c_k) \le_P c' = (c'_1, \dots, c'_k)$ if and only if $c_i \le c'_i$ for all $i = 1, \dots, k$. If $c \le_P c'$ and $c \ne c'$, we write $c <_P c'$). Clearly, an element $v = (v_1, \dots, v_m) \in \mathbb{N}^m$ belongs to V_A if and only if for any element $(a_1, \dots, a_m) \in A$ there exists $i \in \mathbb{N}, 1 \le i \le m$, such that $a_i > v_i$.

The following two theorems proved in [7, Chapter 2] generalize the well-known Kolchin's result on the numerical polynomials of subsets of $\mathbb{N}^m$ (see [6, Chapter 0, Lemma 17]) and give an explicit formula for the numerical polynomials in p variables associated with a finite subset of $\mathbb{N}^m$.

Theorem 2.3. Let A be a subset of $\mathbb{N}^m$ where $m = m_1 + \cdots + m_p$ for some nonnegative integers $m_1, \dots, m_p$ $(p \geq 1)$. Then there exists a numerical polynomial $\omega_A(t_1, \dots, t_p)$ in p variables with the following properties:

(i) $\omega_A(r_1, \dots, r_p) = \text{Card}\, V_A(r_1, \dots, r_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$ (i. e., there exist $(s_1, \dots, s_p) \in \mathbb{N}^p$ such that the equality holds for all $(r_1, \dots, r_p) \in \mathbb{N}^p$ such that $(s_1, \dots, s_p) \leq_P (r_1, \dots, r_p)$).

(ii) $\deg \omega_A \leq m$ and $\deg_{t_i} \omega_A \leq m_i$ for $i = 1, \dots, p$.

(iii) $\deg\, \omega_A = m$ if and only if the set A is empty. In this case $\omega_A(t_1, \dots, t_p) = \prod_{i=1}^{p} \binom{t_i + m_i}{m_i}$.

(iv) ω_A is a zero polynomial if and only if $(0, \dots, 0) \in A$.

Definition 2.4. The polynomial $\omega_A(t_1, \dots, t_p)$ is called the dimension polynomial of the set $A \subseteq \mathbb{N}^m$ associated with the partition $(m_1, \dots, m_p)$ of m.

Theorem 2.5. Let $A = \{a_1, \dots, a_n\}$ be a finite subset of $\mathbb{N}^m$ where n is a positive integer and $m = m_1 + \cdots + m_p$ for some nonnegative integers $m_1, \dots, m_p$ $(p \geq 1)$. Let $a_i = (a_{i1}, \dots, a_{im})$ $(1 \leq i \leq n)$ and for any $l \in \mathbb{N}$, $0 \leq l \leq n$, let $\Gamma(l, n)$ denote the set of all l-element subsets of the set $\mathbb{N}_n = \{1, \dots, n\}$. Furthermore, for any $\sigma \in \Gamma(l, p)$, let $\bar{a}_{\sigma j} = \max\{a_{ij} | i \in \sigma\}$ $(1 \leq j \leq m)$ and $b_{\sigma j} = \sum_{h \in \sigma_j} \bar{a}_{\sigma h}$. Then

$$\omega_A(t_1, \dots, t_p) = \sum_{l=0}^{n} (-1)^l \sum_{\sigma \in \Gamma(l,n)} \prod_{j=1}^{p} \binom{t_j + m_j - b_{\sigma j}}{m_j} \qquad (2.\ 2)$$

Remark 2.6. Clearly, if $A \subseteq \mathbb{N}^m$ and A' is the set of all minimal elements of the set A with respect to the product order on $\mathbb{N}^m$, then the set A' is finite and $\omega_A(t_1, \dots, t_p) = \omega_{A'}(t_1, \dots, t_p)$. Thus, Theorem 2.5 gives an algorithm that allows one to find the dimension polynomial of any subset of $\mathbb{N}^m$ (with a given representation of m as a sum of p positive integers): one should first find the set of all minimal points of the subset and then apply Theorem 2.5.

Proposition 2.7. Let

$$f(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} a_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p}$$

and

$$g(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \cdots \sum_{i_p=0}^{m_p} b_{i_1 \dots i_p} \binom{t_1 + i_1}{i_1} \cdots \binom{t_p + i_p}{i_p}$$

be numerical polynomials in p variables $t_1, \dots, t_p$ written in the form (2.1) (all coefficients $a_{i_1 \dots i_p}$ and $b_{i_1 \dots i_p}$ are integers). Suppose that there exists $r^{(0)} = (r_1^{(0)}, \dots, r_p^{(0)}) \in \mathbb{N}^p$ such that for all $r = (r_1, \dots, r_p) \in \mathbb{N}^p$ such that $r^{(0)} \leq_P r$, one has $f(r_1, \dots, r_p) = g(r_1, \dots, r_p)$. Then $f(t_1, \dots, t_p) = g(t_1, \dots, t_p)$ (that is, $a_{i_1 \dots i_p} = b_{i_1 \dots i_p}$ for all $(i_1, \dots, i_p) \in \mathbb{N}^p$, $(i_1, \dots, i_p) \leq_P (m_1, \dots, m_p)$).

Proof. We proceed by induction on p. If $p = 1$, the statement is true because in this case $f - g$ has infinitely many roots and therefore $f = g$. Suppose that $p > 1$ and our statement is true for numerical polynomials with less than p variables. Then $f - g = \sum_{i_1=0}^{m_1} C_{i_1}(t_2, \dots, t_p)\binom{t_1+i_1}{i_1}$ where

$$C_{i_1}(t_2, \dots, t_p) = \sum_{i_2=0}^{m_2} \dots \sum_{i_p=0}^{m_p} [a_{i_1 \dots i_p} - b_{i_1 \dots i_p}]\binom{t_2+i_1}{i_2} \dots \binom{t_p+i_p}{i_p} \quad (0 \le i_1 \le m_1).$$

If we set $t_i = r_i$, where $r_i \ge r_i^{(0)}$ $(i = 2, \dots, p)$, in the above expression for $f - g$, we obtain a polynomial in one variable t_1 that vanishes for all integer values of t_1 that are greater than or equal to $r_1^{(0)}$. Therefore, all coefficients $C_{i_1}(t_2, \dots, t_p)$ $(0 \le i_1 \le m_1)$ vanish at $(r_2, \dots, r_p)$. By the induction hypothesis, $C_{i_1}(t_2, \dots, t_p) = 0$ $(0 \le i_1 \le m_1)$, hence $f - g = 0$, so $f = g$. $\square$

3. Reduction with respect to several term orderings and multivariate differential dimension polynomials

Let K be a differential (Δ-) field whose basic set of derivations Δ is represented as the union of p nonempty disjoint subsets ($p \ge 1$):

$$\Delta = \Delta_1 \bigcup \dots \bigcup \Delta_p \tag{3. 1}$$

where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ for $i = 1, \dots, p$ $(m_1 + \dots + m_p = m$ where $m = \operatorname{Card}\Delta)$. In other words, we fix a partition of the set Δ.

Let Θ_i denote the free commutative semigroup generated by Δ_i $(1 \le i \le p)$ and let Θ be the free commutative semigroup generated by the whole set Δ. For any element $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}} \in \Theta$, the numbers $\operatorname{ord}_i \theta = \sum_{j=1}^{m_i} k_{ij}$ $(i = 1, \dots, p)$ and $\operatorname{ord} \theta = \sum_{i=1}^{p} \operatorname{ord}_i \theta$ will be called the *order of* θ *with respect to* Δ_i and the *order of* θ, respectively. If $\theta, \theta' \in \Theta$, we say that θ' divides θ (or that θ is a multiple of θ') and write $\theta' \,|\, \theta$ if there exists $\theta'' \in \Theta$ such that $\theta = \theta''\theta'$. As usual, the least common multiple of elements $\theta_1 = \prod_{i=1}^{p}\prod_{j=1}^{m_i} \delta_{ij}^{k_{ij1}}, \dots, \theta_q = \prod_{i=1}^{p}\prod_{j=1}^{m_i} \delta_{ij}^{k_{ijq}} \in \Theta$ is the element $\theta = \prod_{i=1}^{p}\prod_{j=1}^{m_i} \delta_{ij}^{k_{ij}}$, where $k_{ij} = \max\{k_{ijl} | 1 \le l \le q\}$ $(1 \le i \le p, 1 \le j \le m_i)$, denoted by $\operatorname{lcm}(\theta_1, \dots, \theta_q)$.

If $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, the set $\{\theta \in \Theta \,|\, \operatorname{ord}_i \theta \le r_i \text{ for } i = 1, \dots, p\}$ will be denoted by $\Theta(r_1, \dots, r_p)$ or $\Theta(r)$. If ξ is an element of a Δ-field K and $\Theta' \subseteq \Theta$, then $\Theta'\xi$, will denote the set $\{\theta(\xi) \,|\, \theta \in \Theta'\}$.

We consider p orderings $<_1, \dots <_p$ of the semigroup Θ defined as follows.
If $\theta = \delta_{11}^{k_{11}} \dots \delta_{1m_1}^{k_{1m_1}} \delta_{21}^{k_{21}} \dots \delta_{pm_p}^{k_{pm_p}}$ and $\theta' = \delta_{11}^{l_{11}} \dots \delta_{1m_1}^{l_{1m_1}} \delta_{21}^{l_{21}} \dots \delta_{pm_p}^{l_{pm_p}}$ are elements of Θ, then $\theta <_i \theta'$ if and only if the vector $(\operatorname{ord}_i \theta, \operatorname{ord}_1 \theta, \dots, \operatorname{ord}_{i-1} \theta, \operatorname{ord}_{i+1} \theta, \dots, \operatorname{ord}_p \theta, k_{i1}, \dots, k_{im}, k_{11}, \dots, k_{1m_1}, k_{21}, \dots, k_{i-1,m_{i-1}}, k_{i+1,1}, \dots, k_{pm_p})$ is less than the vector $(\operatorname{ord}_i \theta', \operatorname{ord}_1 \theta', \dots, \operatorname{ord}_{i-1} \theta', \operatorname{ord}_{i+1} \theta', \dots, \operatorname{ord}_p \theta', l_{i1}, \dots, l_{im}, l_{11}, \dots, l_{1m_1}, l_{21}, \dots, l_{i-1,m_{i-1}}, l_{i+1,1}, \dots, l_{pm_p})$ with respect to the lexicographic order on $\mathbb{N}^{m+p}$.

Let $K\{y_1, \dots, y_n\}$ be the ring of Δ-polynomials in Δ-indeterminates $y_1, \dots, y_n$ over K. Then the elements θy_i $(\theta \in \Theta, 1 \le i \le n)$ will be called terms, and the set of all terms ΘY will be considered together with p orderings that correspond to the orderings of Θ

and are denoted by the same symbols $<_1, \dots, <_p$. These orderings of ΘY are defined as follows. $\theta y_j <_i \theta' y_k$ $(\theta, \theta' \in \Theta, 1 \le j, k \le n, 1 \le i \le p)$ if and only if $\theta <_i \theta'$ or $\theta = \theta'$ and $j < k$. By the ith order of a term $u = \theta y_j$ we mean the number $\text{ord}_i\, u = \text{ord}_i\, \theta$. The number $\text{ord}\, u = \text{ord}\, \theta$ is called the order of u.

We say that a term $u = \theta y_i$ is divisible by a term $v = \theta' y_j$ and write $v \,|\, u$, if $i = j$ and $\theta' \,|\, \theta$. If $v \,|, u$ and $v \neq u$, we say that u is a *proper derivative* 0f v. For any terms $u_1 = \theta_1 y_j, \dots, u_q = \theta_q y_j$ with the same Δ-indeterminate y_j, the term $\text{lcm}(\theta_1, \dots, \theta_q) y_j$ is called the least common multiple of $u_1, \dots, u_q$, it is denoted by $\text{lcm}(u_1, \dots, u_q)$.

If $A \in K\{y_1, \dots, y_n\}$, $A \notin K$, and $1 \le i \le p$, then the highest with respect to the ordering $<_i$ term that appears in A is called the i-leader of the Δ-polynomial A. It is denoted by $u_A^{(i)}$. If A is written as a polynomial in one variable $u_A^{(1)}$, $A = I_d (u_A^{(1)})^d + I_{d-1} (u_A^{(1)})^{d-1} + \dots + I_0$ ($I_d, I_{d-1}, \dots, I_0$ do not contain $u_A^{(1)}$), then I_d is called the *leading coefficient* of the Δ-polynomial A and the partial derivative $\partial A / \partial u_A^{(1)} = d I_d (u_A^{(1)})^{d-1} + (d-1) I_{d-1} (u_A^{(1)})^{d-2} + \dots + I_1$ is called the *separant* of A. The leading coefficient and the separant of A are denoted by I_A and S_A, respectively.

Definition 3.1. Let A and B be two Δ-polynomials from $K\{y_1, \dots, y_n\}$. We say that A has lower rank than B (or that B has higher rank than A) and write $\text{rk}\, A < \text{rk}\, B$ if either $A \in K$, $B \notin K$, or the vector $(u_A^{(1)}, \deg_{u_A^{(1)}} A, \text{ord}_2\, u_A^{(2)}, \dots, \text{ord}_p\, u_A^{(p)})$ is less than the corresponding vector $(u_B^{(1)}, \deg_{u_B^{(1)}} B, \text{ord}_2\, u_B^{(2)}, \dots, \text{ord}_p\, u_B^{(p)})$ with respect to the lexicographic order ($u_A^{(1)}$ and $u_B^{(1)}$ are compared with respect to $<_1$ and all other coordinates are compared with respect to the natural order on $\mathbb{N}$). If the two vectors are equal (or $A \in K$ and $B \in K$) we say that the Δ-polynomials A and B are of the same rank and write $\text{rk}\, A = \text{rk}\, B$.

Definition 3.2. Let A and B be two Δ-polynomials in $K\{y_1, \dots, y_n\}$ and $A \notin K$. We say that B is reduced with respect to A if the following two conditions hold.

(i) B does not contain any term $\theta u_A^{(1)}$ $(\theta \in \Theta, \theta \neq 1)$ such that $\text{ord}_i(\theta u_A^{(i)}) \le \text{ord}_i\, u_B^{(i)}$ for $i = 2, \dots, p$.

(ii) If B contains $u_A^{(1)}$, then either there exists j, $2 \le j \le p$, such that $ord_j u_B^{(j)} < \text{ord}_j\, u_A^{(j)}$ or $ord_j u_A^{(j)} \le \text{ord}_j\, u_B^{(j)}$ for all $j = 2, \dots, p$ and $\deg_{u_A^{(1)}} B < \deg_{u_A^{(1)}} A$.

A Δ-polynomial B is said to be reduced with respect to a set $\mathcal{A} \subseteq K\{y_1, \dots, y_n\}$ if B is reduced with respect to every element of $\mathcal{A}$.

Remark 3.3. It follows from the last definition that a Δ-polynomial B is not reduced with respect to a Δ-polynomial A ($A \notin K$) if either B contains a term $\theta u_A^{(1)}$ $(\theta \in \Theta, \theta \neq 1)$ such that $ord_i(\theta u_A^{(i)}) \le ord_i u_B^{(i)}$ for $i = 2, \dots, p$ or B contains $u_A^{(1)}$ and in this case $ord_j u_A^{(j)} \le ord_j u_B^{(j)}$ for $j = 2, \dots, p$ and $deg_{u_A^{(1)}} A \le deg_{u_A^{(1)}} B$. This observation is helpful if one would like to show that a Δ-polynomial is not reduced with respect to some other Δ-polynomial.

Definition 3.4. A set of Δ-polynomials $\mathcal{A}$ is called autoreduced if $\mathcal{A} \bigcap K = \emptyset$ and every element of $\mathcal{A}$ is reduced with respect to any other element of this set.

The following statement is proved in [10] (see[10, Theorem 4.5]).

Proposition 3.5. Every autoreduced set is finite.

Theorem 3.6. Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be an autoreduced set in the ring of Δ-polynomials $K\{y_1, \dots, y_n\}$ and $B \in K\{y_1, \dots, y_n\}$. Then there exist a Δ-polynomial B_0 and nonnegative integers p_j, q_j $(1 \le j \le r)$ such that B_0 is reduced with respect to $\mathcal{A}$, $rk\, B_0 \le rk\, B$, and

$$\prod_{j=1}^{r} I_{A_i}^{p_j} S_{A_j}^{q_j} B \equiv B_0\, (\mathrm{mod}[\mathcal{A}]). \tag{3. 2}$$

Proof. Suppose that B is not reduced with respect to $\mathcal{A}$ and B contains a term $v = \theta u_{A_j}^{(1)}$, where $\theta \in \Theta$, $\mathrm{ord}\,\theta > 0$, $1 \le j \le r$, such that $\mathrm{ord}_i(\theta u_{A_j}^{(i)}) \le \mathrm{ord}_i\, u_B^{(i)}$ for $i = 2, \dots, p$. Let v be the greatest such a term with respect to $<_1$. We will call it the $\mathcal{A}$-*leader* of B. Let $e = \deg_v B$ and J the coefficient of v^e when B is written as a polynomial in v. Since $\theta A_j = S_{A_j}\theta u_{A_j}^{(1)} + C = S_{A_j} v + C$, where all terms of C are of lower rank than v and $\mathrm{ord}_i\, u_C^{(i)} \le \mathrm{ord}_i(\theta u_{A_j}^{(i)}) \le \mathrm{ord}_i\, u_B^{(i)}$ for $i = 2, \dots, p$, either the $\mathcal{A}$-leader of the Δ-polynomial

$$B_1 = S_{A_j} B - J v^{e-1} \theta A_j$$

has lower rank than v or it is equal to v and in this case $\deg_v B_1 < \deg_w B$. Applying the same procedure to B_1 and continuing this process we obtain a Δ-polynomial B' and some nonnegative integers $q_1, \dots, q_r$ such that $\prod_{i=1}^{r} S_{A_i}^{q_i} B \equiv B'\, (\mathrm{mod}[\mathcal{A}])$ and the $\mathcal{A}$-leader of B' is either one of the 1-leaders of A_j $(1 \le j \le r)$ or has lower rank than any $u_{A_j}^{(1)}$. In the last case, B' is reduced with respect to $\mathcal{A}$ and we are done.

Suppose that the $\mathcal{A}$-leader of B' is $w = u_{A_j}^{(1)}$ for some j $(1 \le j \le r)$, $k = \deg_w B' \ge d_j$, where $d_j = \deg_{u_{A_j}^{(1)}} A_j$, $\mathrm{ord}_i\, u_{A_j}^{(i)} \le \mathrm{ord}_i\, u_{B'}^{(i)}$ for $i = 2, \dots, p$, and F is the coefficient of w^k when B is written as a polynomial in w. Then either the $\mathcal{A}$-leader of the Δ-polynomial

$$B'_1 = I_{A_j} B' - (u_{A_j}^{(i)})^{k-d_j} F A_j$$

has lower rank than w or it is equal to w, $\deg_w B_1 < \deg_w B_1$, and $\mathrm{ord}_i\, u_{A_j}^{(i)} \le \mathrm{ord}_i\, u_{B'_1}^{(i)}$ for $i = 2, \dots, p$. Applying the same procedure to B'_1 and continuing this process we obtain a Δ-polynomial B_0 and some nonnegative integers $p_1, \dots, p_r$ (in addition to $q_1, \dots, q_r$) such that (3. 2) holds. □

The process of reduction described in the proof of the last theorem can be realized by the following algorithm where $R = K\{y_1, \dots, y_n\}$.

Algorithm I. $(B, r, A_1, \dots, A_r;\, B_0;)$
Input: $B \in R$, a positive integer r, $\mathcal{A} = \{A_1, \dots, A_r\} \subseteq R$ where $A_j \ne 0$ for $j = 1, \dots, r$
Output: $p_1, \dots, p_r, q_1, \dots, q_r \in \mathbb{N}$, and $B_0 \in R$such that $B_0 - \prod_{j=1}^{r} I_{A_j}^{p_j} S_{A_j}^{q_j} B \in [\mathcal{A}]$ and B_0 is reduced with respect to $\mathcal{A}$, and $rk\, B_0 \le rk\, B$.
Begin $B_0 := B$
While there exist j, $1 \le j \le r$, and a term v, that appears in B_0 with a nonzero coefficient, such that $u_{A_j}^{(1)} | v$, $u_{A_j}^{(1)} \ne v$, and $\mathrm{ord}_i(\frac{v}{u_{A_j}^{(1)}} u_{A_j}^{(i)}) \le \mathrm{ord}_i\, u_{B_0}^{(i)}$ for $i = 2, \dots, p$

do

z:= the greatest (with respect to $<_1$) of the terms v that satisfy the above conditions.

k:= the smallest number j, $1 \le j \le r$, for which $u^{(1)}_{A_j}$ is the greatest (with respect to $<_1$) 1-leader of an element $A_j \in \mathcal{A}$ such that $u^{(1)}_{A_j} | z$ and $\mathrm{ord}_i(\frac{z}{u^{(1)}_{A_j}} u_{A_j}) \le \mathrm{ord}_i\, u^{(i)}_{B_0}$ for $i = 2, \dots, p$.

$e := \deg_{u^{(1)}_{A_k}} A_k$; J:= the coefficient of z^e when B_0 is written as a polynomial in z;

$\theta := \frac{z}{u^{(1)}_{A_j}}$.

$B_0 := S_{A_k} B_0 - J z^{e-1} \theta A_k$.

While for every $j = 1, \dots, r$, B_0 contains no proper derivatives of $u^{(1)}_{A_j}$ and there exist l, $1 \le l \le r$ such that B_0 contains $(u^{(1)}_{A_l})^t$ with a nonzero coefficient such that $t \ge d_l = \deg_{u^{(1)}_{A_l}} A_l$ and $\mathrm{ord}_i\, u^{(i)}_{A_l} \le \mathrm{ord}_i\, u^{(i)}_{B_0}$ for $i = 2, \dots, p$

do

s:= the smallest number j, $1 \le j \le r$, for which $u^{(1)}_{A_j}$ is the greatest (with respect to $<_1$) 1-leader of an element $A_j \in \mathcal{A}$ such that $(u^{(1)}_{A_j})^t$ appears in B_0 with nonzero coefficient, $t \ge d_j$, and $\mathrm{ord}_i\, u^{(i)}_{A_j} \le \mathrm{ord}_i\, u^{(i)}_{B_0}$ for $i = 2, \dots, p$.

$h := \deg_{u^{(1)}_{A_s}} B_0$; F:= the coefficient of $(u^{(1)}_{A_s})^h$ when B_0 is written as a polynomial in $u^{(1)}_{A_s}$;

$Q_s := Q_s + F (u^{(1)}_{A_s})^{h-1} A_s$; $\quad B_0 := I_{A_s} B_0 - F (u^{(1)}_{A_s})^{h-1} A_s$.

End

Corollary 3.7. Let $\mathcal{A} = \{A_1, \dots, A_r\}$ be an autoreduced set in the ring of Δ-polynomials $K\{y_1, \dots, y_n\}$ and $B_1, \dots, B_s \in K\{y_1, \dots, y_n\}$. Then there exist Δ-polynomials $C_1, \dots, C_s$ and nonnegative integers p_j, q_j $(1 \le j \le r)$ such that each C_i $(1 \le i \le r)$ is reduced with respect to $\mathcal{A}$, the rank of each C_i is no higher than the highest of the ranks of $B_1, \dots, B_s$, and

$$\prod_{j=1}^{r} I^{p_j}_{A_j} S^{q_j}_{A_j} B_i \equiv C_i \,(\mathrm{mod}[\mathcal{A}]) \quad (1 \le i \le s).$$

Proof. Without loss of generality we can assume that $\mathrm{rk}\, B_1 < \dots < \mathrm{rk}\, B_s$. Then we can apply the process of successive reduction described in the proof of the last theorem (and Algorithm I) to obtain the result of our statement. □

In what follows, while considering an autoreduced set $\mathcal{A} = \{A_1, \dots, A_r\}$, we always assume that its elements are arranged in order of increasing rank: $\mathrm{rk}\ A_1 < \dots < \mathrm{rk}\ A_r$.

Definition 3.8. Let $\mathcal{A} = \{A_1, \dots, A_r\}$ and $\mathcal{B} = \{B_1, \dots, B_s\}$ be two autoreduced sets. Then $\mathcal{A}$ is said to have lower rank than $\mathcal{B}$ if one of the following two cases holds:

(i) There exists $k \in \mathbb{N}$ such that $k \le \min\{r, s\}$, $\mathrm{rk}\ A_i = \mathrm{rk}\ B_i$ for $i = 1, \dots, k-1$ and $\mathrm{rk}\ A_k < \mathrm{rk}\ B_k$.

(ii) $r > s$ and $\mathrm{rk}\ A_i = \mathrm{rk}\ B_i$ for $i = 1, \dots, s$.

If $r = s$ and $\mathrm{rk}\ A_i = \mathrm{rk}\ B_i$ for $i = 1, \dots, r$, then $\mathcal{A}$ is said to have the same rank as $\mathcal{B}$.

The statements of Propositions 3.9 and 3.11 below can be obtained by mimicking the proofs of the corresponding statements for classical Ritt-Kolchin autoreduced sets (see [7, Proposition 5.3.10 and Lemma 5.3.12]).

Proposition 3.9. In every nonempty family of autoreduced sets of differential polynomials there exists an autoreduced set of lowest rank.

Definition 3.10. Let J be a Δ-ideal of the ring of Δ-polynomials $K\{y_1, \dots, y_n\}$. Then an autoreduced subset of J of lowest rank is called a characteristic set of the ideal J.

Proposition 3.11. Let $\mathcal{A} = \{A_1, \dots, A_d\}$ be a characteristic set of a Δ-ideal J of the ring of Δ-polynomials $R = K\{y_1, \dots, y_n\}$. Then an element $B \in J$ is reduced with respect to the set $\mathcal{A}$ if and only if $B = 0$. In particular, $I_A \notin J$ and $S_A \notin J$ for every $A \in \mathcal{A}$.

Let K be a Δ-field and $L = K\langle \eta_1, \dots, \eta_n \rangle$ a finitely generated Δ-extension of K with a set of Δ-generators $\eta = \{\eta_1, \dots, \eta_n\}$. Then there exists a natural Δ-homomorphism ϕ_η of the ring of Δ-polynomials $K\{y_1, \dots, y_n\}$ onto the Δ-subring $K\{\eta_1, \dots, \eta_n\}$ of L such that $\phi_\eta(a) = a$ for any $a \in K$ and $\phi_\eta(y_j) = \eta_j$ for $j = 1, \dots, n$. If $A \in K\{y_1, \dots, y_n\}$, then $\phi_\eta(A)$ is called the *value* of A at η and is denoted by $A(\eta)$. Obviously, $P = \operatorname{Ker} \phi_\eta$ is a prime Δ-ideal of $K\{y_1, \dots, y_n\}$. It is called the *defining ideal* of η. If we consider the quotient field Q of $R = K\{y_1, \dots, y_n\}/P$ as a Δ-field (where $\delta(\frac{u}{v}) = \frac{v\delta(u) - u\delta(v)}{v^2}$ for any $u, v \in R$), then this quotient field is naturally Δ-isomorphic to the field L. The Δ-isomorphism of Q onto L is identical on K and maps the images of the Δ-indeterminates $y_1, \dots, y_n$ in the factor ring R onto the elements $\eta_1, \dots, \eta_n$, respectively.

Let K be a differential (Δ-) field, $\operatorname{Card} \Delta = m$, and let a partition (3. 1) of Δ be fixed: $\Delta = \Delta_1 \bigcup \dots \bigcup \Delta_p$ ($p \geq 1$), where $\Delta_i = \{\delta_{i1}, \dots, \delta_{im_i}\}$ ($1 \leq i \leq p$). Furthermore, let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a Δ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$. Let P be the defining ideal of η and $\mathcal{A} = \{A_1, \dots, A_d\}$ a characteristic set of P.

For any $r_1, \dots, r_p \in \mathbb{N}$, let

$U'_{r_1 \dots r_p} = \{u \in \Theta Y \mid \operatorname{ord}_i u \leq r_i$ for $i = 1, \dots, p$ and u is not a derivative of any $u^{(1)}_{A_i}$ (that is, $u \neq \theta u^{(1)}_{A_i}$ for any $\theta \in \Theta; i = 1, \dots, d$) and let

for $i = 1, \dots, p$ and for every $\theta \in \Theta, A \in \mathcal{A}$ such that $u = \theta u^{(1)}_A$, there exists $i \in \{2, \dots, p\}$ such that $\operatorname{ord}_i(\theta u^{(i)}_A) > r_i\}$.

(If $p = 1$, $U'_{r_1} = \{u \in \Theta Y \mid \operatorname{ord}_1 u \leq r_1$ and u is not a derivative of any $u^{(1)}_{A_i}\}$ and $U''_{r_1} = \emptyset$.) Furthermore, for any $(r_1 \dots r_p) \in \mathbb{N}^p$, let $U_{r_1 \dots r_p} = U'_{r_1 \dots r_p} \bigcup U''_{r_1 \dots r_p}$.

The following theorem proved in [10, Section 5] establishes the existence and describes the form of a multivariate dimension polynomial associated with a finite system of Δ-generators of a Δ-field extension and with a partition of the set Δ. We give an extended version of this result that follows from the proof of [10, Theorem 5.1].

Theorem 3.12. With the above notation,

(i) For all sufficiently large $(r_1 \dots r_p) \in \mathbb{N}^p$, the set $U_{r_1 \dots r_p}$ is a transcendence basis of $K(\bigcup_{j=1}^{n} \Theta(r_1, \dots, r_p)\eta_j)$ over K.

(ii) There exist numerical polynomials $\omega_{\eta|K}(t_1,\dots,t_p)$ and $\phi_{\eta|K}(t_1,\dots,t_p)$ in p variables such that $\omega_{\eta|K}(r_1,\dots,r_p) = \operatorname{Card} U'_{r_1\dots r_p}$ and $\phi_{\eta|K}(r_1,\dots,r_p) = \operatorname{Card} U''_{r_1\dots r_p}$ for all sufficiently large $(r_1\dots r_p) \in \mathbb{N}^p$, so that the polynomial $\Phi_{\eta|K}(t_1,\dots,t_p) = \omega_{\eta|K}(t_1,\dots,t_p)+\phi_{\eta|K}(t_1,\dots,t_p)$ has the property that for all sufficiently large $(r_1,\dots,r_p) \in \mathbb{N}^p$, $\Phi_\eta(r_1,\dots,r_p) = \operatorname{tr.deg}_K K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j)$.

(iii) $\deg_{t_i}\Phi_{\eta|K} \le m_i$ $(1 \le i \le p)$, so that $\deg\Phi_{\eta|K} \le m$ and the polynomial $\Phi_{\eta|K}(t_1,\dots,t_p)$ can be represented as

$$\Phi_{\eta|K}(t_1,\dots,t_p) = \sum_{i_1=0}^{m_1}\dots\sum_{i_p=0}^{m_p} a_{i_1\dots i_p}\binom{t_1+i_1}{i_1}\dots\binom{t_p+i_p}{i_p} \tag{3. 3}$$

where $a_{i_1\dots i_p} \in \mathbb{Z}$ for all $i_1\dots i_p$.

(iv) $\phi_{\eta|K}(t_1,\dots,t_p)$ is an alternating sum of polynomials in p variables of the form

$$\phi_{j;k_1,\dots,k_q} = \binom{t_1+m_1-b_{1j}}{m_1}\dots\binom{t_{k_1-1}+m_{k_1-1}-b_{k_1-1,j}}{m_{k_1-1}}\Bigg[\binom{t_{k_1}+m_{k_1}-a_{k_1,j}}{m_{k_1}}-$$

$$\binom{t_{k_1}+m_{k_1}-b_{k_1,j}}{m_{k_1}}\Bigg]\cdot\binom{t_{k_1+1}+m_{k_1+1}-b_{k_1+1,j}}{m_{k_1+1}}\dots\binom{t_{k_q-1}+m_{k_q-1}-b_{k_q-1,j}}{m_{k_q-1}}\cdot$$

$$\Bigg[\binom{t_{k_q}+m_{k_q}-a_{k_q,j}}{m_{k_q}}-\binom{t_{k_q}+m_{k_q}-b_{k_q,j}}{m_{k_q}}\Bigg]\dots\binom{t_p+m_p-b_{pj}}{m_p},$$ so $\deg\phi_{\eta|K} < m$.

Definition 3.13. Numerical polynomial $\Phi_{\eta|K}(t_1,\dots,t_p)$, whose existence is established by the last theorem, is called a differential (or Δ-) dimension polynomial of the differential field extension $L = K\langle\eta_1,\dots,\eta_n\rangle$ associated with the system of Δ-generators $\eta = \{\eta_1,\dots,\eta_n\}$ and with partition (3. 1) of the basic set of derivations Δ.

Remark 3.14. With the notation of the last theorem, if $\eta_1,\dots,\eta_n$ are Δ-algebraically independent over K, then

$$\Phi_{\eta|K}(t_1,\dots,t_p) = n\prod_{i=1}^{p}\binom{t_i+m_i}{m_i}. \tag{3. 4}$$

Indeed, all elements $\delta_{11}^{k_{11}}\dots\delta_{1m_1}^{k_{1m_1}}\delta_{21}^{k_{21}}\dots\delta_{pm_p}^{k_{pm_p}}$ such that $\sum_{j=1}^{m_i}k_{ij} \le r_i$ $(1\le i\le p)$ form a transcendence basis of the field $K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j)$ over K. By Theorem 2.3 (iii), the number of such elements is $n\prod_{i=1}^{p}\binom{r_i+m_i}{m_i}$, so we arrive at formula (3. 5).

Remark 3.15. Theorem 3.12 shows that the main problem in computing the multivariate Δ-dimension polynomial $\Phi_{\eta|K}$ is constructing a characteristic set of the defining Δ-ideal of the Δ-field extension. If this ideal is linear (that is, the defining system of differential equations on the generators of the extension is linear), then this problem was solved in [11] by algorithm for constructing a Gröbner basis with respect to several term orderings (see [11, Algorithm 1] and [11, Theorem 3.10]). In the nonlinear case the problem of generalizing the Ritt-Kolchin algorithm to the case of autoreduced sets with respect to several term orderings defined above is still open.

4. Properties and invariants of multivariate differential dimension polynomials

In this section we determine some invariants of multivariate dimension polynomials associated with a differential field extension, that is, numerical characteristics of the extension that do not depend on the system of its differential generators and that are carried by any its dimension polynomials (associated with a given partition of the basic set of derivations).

For any permutation $(j_1, \dots, j_p)$ of the set $\{1, \dots, p\}$ $(p \geq 1)$, let $\leq_{j_1,\dots,j_p}$ denote the corresponding lexicographic order on $\mathbb{N}^p$ such that $(r_1, \dots, r_p) \leq_{j_1,\dots,j_p} (s_1, \dots, s_p)$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbb{N}$, $1 \leq k \leq p-1$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$. If E is a finite subset of $\mathbb{N}^p$, then E' will denote the set of all p-tuples $e \in E$ that are maximal elements of E with respect to one of the $p!$ orders $\leq_{j_1,\dots,j_p}$. Say, if $E = \{(3,0,2), (2,1,1), (0,1,4), (1,0,3), (1,1,6), (3,1,0), (1,2,0)\} \subseteq \mathbb{N}^3$, then $E' = \{(3,0,2), (3,1,0), (1,1,6), (1,2,0)\}$.

The following result gives differential birational invariants carried by a multivariate dimension polynomial of a differential field extension. In particular, it shows that multivariate differential dimension polynomials carry essentially more such invariants than their univariate counterparts introduced by E. Kolchin.

Theorem 4.1. Let K be a differential field with a basic set of derivations Δ and let partition (3.1) of the set Δ into the union of p disjoint sets $(p \geq 1)$ be fixed. Let $L = K\langle \eta_1, \dots, \eta_n \rangle$ be a Δ-field extension of K with the finite set of Δ-generators $\eta = \{\eta_1, \dots, \eta_n\}$ and let

$$\Phi_{\eta|K}(t_1, \dots, t_p) = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} a_{i_1\dots i_p} \binom{t_1+i_1}{i_1} \dots \binom{t_p+i_p}{i_p} \qquad (4.1)$$

be the corresponding differential dimension polynomial. Let $E_\eta = \{(i_1 \dots i_p) \in \mathbb{N}^p \mid 0 \leq i_k \leq m_k \, (k = 1, \dots, p)$ and $a_{i_1\dots i_p} \neq 0\}$. Then $d = \deg \Phi_{\eta|K}$, $a_{m_1\dots m_p}$, the elements $(k_1, \dots, k_p) \in E'_\eta$, the corresponding coefficients $a_{k_1\dots k_p}$, and the coefficients of the terms of total degree d in $\Phi_{\eta|K}$ do not depend on the system of Δ-generators η.

Proof. The fact that the elements $(k_1, \dots, k_p)$ of the set E'_η and the corresponding coefficients $a_{k_1\dots k_p}$ do not depend on the system of Δ-generators η of L/K is established in the proof of Theorem 5.3 of [10] using the observation that if $\zeta = \{\zeta_1, \dots, \zeta_q\}$ is another system of Δ-generators of L/K, then there exists $(s_1, \dots, s_p) \in \mathbb{N}^p$ such that $\Phi_{\eta|K}(r_1, \dots, r_p) \leq \Phi_{\zeta|K}(r_1 + s_1, \dots, r_p + s_p)$ and $\Phi_{\zeta|K}(r_1, \dots, r_p) \leq \Phi_{\eta|K}(r_1 + s_1, \dots, r_p + s_p)$ for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$. Clearly, these inequalities show that $\deg \Phi_{\zeta|K} = \deg \Phi_{\eta|K}$. Let $d = \deg \Phi_{\eta|K}$. Let us order the terms of total degree d in $\Phi_{\eta|K}$ and $\Phi_{\zeta|K}$ using the lexicographic order $\leq_{p,p-1,\dots,1}$ and for sufficiently large $r \in \mathbb{N}$, set $x_1 = r$, $x_2 = 2^{x_1}$, $x_3 = 2^{x_2}, \dots, x_p = 2^{x_{p-1}}$, $R = 2^{x_p}$ and $r_i = x_i R$ $(1 \leq i \leq p)$. If $r \to \infty$, then the last two inequalities immediately imply that $\Phi_{\eta|K}$ and $\Phi_{\zeta|K}$ have the same coefficients of the corresponding terms of total degree d. □

The next theorem characterizes one of the invariants of polynomial (4.1).

Theorem 4.2. With the notation of the last theorem, $a_{m_1\dots m_p} = \Delta\text{-tr.}\deg_K L$.

Proof. Let $\Delta\text{-tr.}\deg_K L = d$. Then, as it was mentioned in section 2, one can choose a Δ-transcendence basis of L/K from the set η, so we can assume that $\eta_1, \dots, \eta_d$ form such a basis. Since the family $\{\theta\eta_i \,|\, \theta \in \Theta,\ 1 \le i \le d\}$ is algebraically independent over K, it follows from Remark 3.14 that $\text{tr.}\deg_K K(\bigcup\limits_{j=1}^{d}\Theta(r_1, \dots, r_p)\eta_j) = d\prod\limits_{i=1}^{p}\binom{r_i+m_i}{m_i}$ for all $(r_1, \dots, r_p) \in \mathbb{N}^p$. Let $F = K\langle\eta_1, \dots, \eta_d\rangle$. Then every element η_j, $d+1 \le j \le n$, is Δ-algebraic over K. It means that there exists a Δ-polynomial $A_j \in F\{y_j\}$ ($F\{y_j\}$ is the ring of Δ-polynomials in one Δ-indeterminate y_j over F) such that $A_j(\eta_j) = 0$. Taking such a polynomial of the smallest possible degree we can assume that $S_{A_j}(\eta_j) \neq 0$. Let $A_j = \sum_{k=0}^{q_j} I_{jk}(u^{(1)}_{A_j})^k$ where all terms of all I_{jk} are less than $u^{(1)}_{A_j}$ with respect to $<_1$.

If $\delta \in \Delta$, then $\delta A_j(\eta_j) = 0$, so $S_{A_j}(\eta_j)\delta(u^{(1)}_{A_j}(\eta_j)) + \sum_{k=0}^{q_j}\delta(I_{jk}(\eta_j))u^{(1)}_{A_j}(\eta_j) = 0$. The term $\delta u^{(1)}_{A_j}$ has the form $\theta_j y_j$ for some $\theta_j \in \Theta$ and one can easily see that for any term v in any S_{A_j} or $\delta(I_{jk})$, we have $v <_1 \theta_j y_j$ and $\text{ord}_i\, v \le \text{ord}_i\, u^{(i)}_{A_j} + 1$ ($i = 1, \dots, p$). It follows that $\theta_j\eta_j \in F(\{\theta\eta_k \,|\, \theta \in \Theta, \theta y_k <_1 \theta_j y_j$ and $\text{ord}_i\,\theta y_k \le a_{ji}$ for $i = 1, \dots, p\})$ where $a_{ji} = \text{ord}_i\, u^{(i)}_{A_j} + 1$ ($1 \le i \le p$).

Since $F = K\left(\bigcup_{k=1}^{d}\bigcup_{(l_1,\dots,l_p)\in\mathbb{N}^p}\Theta(l_1, \dots, l_p)\eta_k\right)$, there exist $h_1, \dots, h_p \in \mathbb{N}$ such that $\theta_j\eta_j \in K(\bigcup_{k=1}^{d}\Theta(h_1, \dots, h_p)\eta_k \bigcup\{\theta\eta_k \,|\, \theta \in \Theta, \theta y_k <_1 \theta_j y_j, \text{ord}_i\,\theta y_k \le a_{ji}\ (i = 1, \dots, p)\})$.

Let $\theta' \in \Theta$ and $\theta_j \,|\, \theta'$. For any $i = 1, \dots, p$, let $s_i = \text{ord}_i\,\theta'$ (clearly, $s_i \ge a_{ji}$). Then $\theta'\eta_j \in K(\bigcup_{k=1}^{d}\Theta(s_1+h_1, \dots, s_p+h_p)\eta_k \bigcup\{\theta\eta_l \,|\, \theta \in \Theta, \theta y_j <_1 \theta' y_j, \text{ord}_i\,\theta \le \text{ord}_i\,\theta'$ $(1 \le i \le p)$ and $\theta_j \nmid \theta\})$.

Therefore, if $r_i \in \mathbb{N}$, $r_i \ge \max_{d+1\le j\le n}\{a_{ji}\}$ ($i = 1, \dots, p$), then one has $K(\bigcup_{k=1}^{n}\Theta(r_1, \dots, r_p)\eta_k) \subseteq K(\bigcup_{k=1}^{d}\Theta(r_1+h_1, \dots, r_p+h_p)\eta_k \cup \bigcup_{j=d+1}^{n}[\Theta(r_1, \dots, r_p) \setminus \Theta(r_1 - a_{j1}, \dots, r_p - a_{jp})]\eta_j)$. It follows that

$$\Phi_{\eta|K}(r_1, \dots, r_p) \le d\prod_{i=1}^{p}\binom{r_i+m_i}{m_i} + \sum_{j=d+1}^{n}\left[\prod_{i=1}^{p}\binom{r_i+m_i}{m_i} - \prod_{i=1}^{p}\binom{r_i-a_{ji}+m_i}{m_i}\right]$$

for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$. Since the total degree of the polynomial $\sum_{j=d+1}^{n}\left[\prod_{i=1}^{p}\binom{t_i+m_i}{m_i} - \prod_{i=1}^{p}\binom{t_i-a_{ji}+m_i}{m_i}\right]$ is less than m, we obtain that $a_{m_1\dots m_p} \le d$.

On the other hand, for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$, $\Phi_{\eta|K}(r_1, \dots, r_p) = \text{tr.}\deg_K K(\bigcup_{k=1}^{n}\Theta(r_1, \dots, r_p)\eta_k) \ge \text{tr.}\deg_K K(\bigcup_{k=1}^{d}\Theta(r_1, \dots, r_p)\eta_k) = d\prod_{i=1}^{p}\binom{r_i+m_i}{m_i}$, hence $a_{m_1\dots m_p} \ge d$. Thus, $a_{m_1\dots m_p} = d = \Delta\text{-tr.}\deg_K L$. □

With the notation of Theorem 3.15, let $p \ge 2$, $1 \le k < p$, and $\Delta^{(k)} = \Delta_1 \bigcup \dots \bigcup \Delta_k$. For any $r_{k+1}, \dots, r_p \in \mathbb{N}$, let $F_{r_{k+1},\dots,r_p}$ denote the $\Delta^{(k)}$-field extension of K generated by $\bigcup_{j=1}^{n}\Theta(0, \dots, 0, r_{k+1}, \dots, r_p)\eta_j$, so $F_{r_{k+1},\dots,r_p} = K\langle\bigcup_{j=1}^{n}\Theta_{\Delta\setminus\Delta^{(k)}}(r_{k+1}, \dots, r_p)\eta_j\rangle_{\Delta^{(k)}}$. Since $\Theta(r_1, \dots, r_p) = \Theta(r_1, \dots, r_k, 0, \dots, 0)\Theta(0, \dots, 0, r_{k+1}, \dots, r_p)$, we can combine the results of Theorems 3.12 and 4.2 to obtain the following statement.

Corollary 4.3. With the above notation, and the Δ-dimension polynomial (4.1) of the extension $K\langle\eta_1, \dots, \eta_n\rangle/K$, the numerical polynomial in $p-k$ variables

$$\phi(t_{k+1}, \dots, t_p) = \sum_{i_{k+1}=0}^{m_{k+1}}\dots\sum_{i_p=0}^{m_p} a_{m_1\dots m_k i_{k+1}\dots i_p}\binom{t_{k+1}+i_{k+1}}{i_{k+1}}\dots\binom{t_p+i_p}{i_p}$$

describes the growth of $\Delta^{(k)}$-tr. $\deg_K F_{r_{k+1},\dots,r_p}$, that is,

$$\phi(r_{k+1},\dots,r_p) = \Delta^{(k)}\text{- tr.}\deg_K F_{r_{k+1},\dots,r_p}$$

for all sufficiently large $r_{k+1},\dots,r_p \in \mathbb{N}^{p-k}$.

This corollary, in particular, shows that if $L = K\langle\eta_1,\dots,\eta_n\rangle$ is a finitely generated differential field extension of a differential field K with a basic set Δ, $\Delta' \subseteq \Delta$, $\Delta'' = \Delta\backslash\Delta'$, and $m_1 = \text{Card}\,\Delta'$, $m_2 = \text{Card}\,\Delta''$ ($m_1 + m_2 = m$ where $m = \text{Card}\,\Delta$), then there exists a univariate numerical polynomial $\phi(t) = \sum_{i=0}^{m_2} c_i\binom{t+i}{i}$ ($c_i \in \mathbb{Z}$) such that $\phi(r) = \Delta'$-tr. $\deg_K K\langle\bigcup_{k=1}^{n}\Theta_{\Delta''}(r)\eta_k\rangle_{\Delta'}$ and $c_{m_2} = \Delta$-tr. $\deg_K L$. Furthermore, if

$$\Phi_{\eta|K}(t_1,t_2) = \sum_{i=0}^{m_1}\sum_{i_2=0}^{m_2} a_{ij}\binom{t_1+i}{i}\binom{t_2+j}{j}$$

is the bivariate Δ-dimension polynomial of L/K associated with the partition $\Delta = \Delta'\bigcup\Delta''$ and $d = \deg_{t_1}\Phi_{\eta|K} < m_1$, then

$$\Delta'\text{- t. tr.}\deg_K K\langle\bigcup_{k=1}^{n}\Theta_{\Delta''}(r)\eta_k\rangle_{\Delta'} = \sum_{i=0}^{m_2} a_{dj}\binom{r+j}{j}.$$

In this case $d = \Delta'\text{-type}_K\, K\langle\bigcup_{k=1}^{n}\Theta_{\Delta''}(r)\eta_k\rangle_{\Delta'}$.

The following theorem provides necessary and sufficient conditions on generators of a differential field extension of a given differential transcendence degree d under which the corresponding multivariate dimension polynomial has the simplest possible form.

Theorem 4.4. With the notation of Theorem 4.2, the following conditions are equivalent.

(i) $\Phi_{\eta|K}(t_1,\dots,t_p) = d\prod_{i=1}^{p}\binom{t_i+m_i}{m_i}$.

(ii) Δ-tr. $\deg_K K\langle\eta_1,\dots,\eta_n\rangle = \text{tr.}\deg_K(\eta_1,\dots,\eta_n) = d$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 4.2, $d = \Delta$-tr. $\deg_K L$ where $L = K\langle\eta_1,\dots,\eta_n\rangle$. Without loss of generality we can assume that $\eta_1,\dots,\eta_d$ is a Δ-transcendence basis of L over K. Then for all sufficiently large $(r_1,\dots,r_p) \in \mathbb{N}^p$,

$$\Phi_{\eta|K}(r_1,\dots,r_p) = \text{tr.}\deg_K K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j) = \text{tr.}\deg_K K(\bigcup_{j=1}^{d}\Theta(r_1,\dots,r_p)\eta_j),$$

hence

$$\text{tr.}\deg_{K(\bigcup_{j=1}^{d}\Theta(r_1,\dots,r_p)\eta_j)} K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j) = 0.$$

Therefore, every element η_j, $d+1 \le j \le n$, is algebraic over the field $F = K\langle\eta_1,\dots,\eta_d\rangle$, so if η' denotes the $(n-d)$-tuple $(\eta_{d+1},\dots,\eta_n)$, then $\Phi_{\eta'|F}(t_1,\dots,t_p) = 0$.

Let P be the defining Δ-ideal of η' in the ring of Δ-polynomials $F\{y_1,\dots,y_{n-d}\}$ and let $\mathcal{A}$ be a characteristic set of P (we use the terminology and term orderings $<_1,\cdots <_p$ introduced in the beginning of this section). For every $j = 1,\dots,n-d$, let E_j denote the set of all $(k_1,\dots,k_m)\in\mathbb{N}^m$ such that $\delta_1^{k_1}\dots\delta_m^{k_m}y_j$ is a 1-leader of an element of $\mathcal{A}$. Since $\Phi_{\eta'|F}=0$, we also have $\omega_{\eta'|F}=0$ where $\omega_{\eta'|F}$ is the polynomial in p variables defined in Theorem 3.12(ii). Furthermore, it follows from Theorem 2.3(iv) that $\omega_{\eta'|F}=0$ if and only if $E_j=\{(0,\dots,0)\}$ for $j=1,\dots,n-d$.

Since $y_1<_1 y_j$ for $j=2.\dots,n-d$, a Δ-polynomial in $\mathcal{A}$ with leader y_1 is a usual polynomial in y_1 with coefficients in F. Therefore, η_{d+1} and all $\theta\eta_{d+1}$ $(\theta\in\Theta)$ are algebraic over F. If $\eta''=(\eta_{d+2},\dots,\eta_n)$, then $\Phi_{\eta''|F}(r_1,\dots,r_p)\le\Phi_{\eta'|F}(r_1,\dots,r_p)$ for all $(r_1,\dots,r_p)\in\mathbb{N}^p$, so $\Phi_{\eta''|F}=0$ and we can repeat the above arguments and obtain that every $\theta\eta_j$ $(\theta\in\Theta, d+1\le j\le n)$ is algebraic over F.

Since the elements $\eta_{d+1},\dots,\eta_n$ are algebraic over the field $F=K\langle\eta_1,\dots,\eta_d\rangle$, there exist $h_1,\dots,h_p\in\mathbb{N}$ such that $\eta_{d+1},\dots,\eta_n$ are algebraic over $K(\bigcup_{j=1}^{d}\Theta(h_1,\dots,h_p)\eta_j)$. It follows that the field extension $K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j)/K(\bigcup_{j=1}^{d}\Theta(r_1,\dots,r_p)\eta_j)$ is algebraic whenever $(h_1,\dots,h_p)\le_P(r_1,\dots,r_p)$.

Suppose that η_{d+1} is not algebraic over the field $K(\eta_1,\dots,\eta_d)$. Let $(q_1,\dots,q_p)$ be a minimal (with respect to the product order $<_P$) element of $\mathbb{N}^p$ such that η_{d+1} is algebraic over $K(\bigcup_{j=1}^{d}\Theta(q_1,\dots,q_p)\eta_j)$. (By the assumption, $(q_1,\dots,q_p)\neq(0,\dots,0)$). Without loss of generality we can assume that $q_1\ge 1$. Then η_{d+1} is transcendental over the field $K(\bigcup_{j=1}^{d}\Theta(q_1-1,\dots,q_p)\eta_j)$. Then there exists a term v in the ring of Δ-polynomials $K\{y_1,\dots,y_d\}$ such that $\mathrm{ord}_1 v=q_1$, $\mathrm{ord}_i v\le q_i$ for $i=2,\dots,p$, η_{d+1} is transcendental over the field $K'=K(\{\theta\eta_j\,|\,\theta\in\Theta(q_1,\dots,q_p), 1\le j\le d, \theta y_j<_1 v\})$ and algebraic over $K'(v(\eta))$. It follows that $v(\eta)$ is algebraic over $K(\bigcup_{j=1}^{d}\Theta(q_1,\dots,q_p)\eta_j\setminus\{\eta_{d+1}\}\bigcup\{v(\eta)\})$. Therefore, if $\theta'\in\Theta(r_1,\dots,r_p)$ where $(h_1,\dots,h_p)\le_P(r_1,\dots,r_p)$, then $\theta' v(\eta)$ is algebraic over $K(\bigcup_{j=1}^{d}\Theta(r_1+q_1,\dots,r_p+q_p)\eta_j\setminus\{\theta'\eta_{d+1}\}\bigcup\{\theta' v(\eta)\})$.

Since η_{d+1} is algebraic over $K(\bigcup_{j=1}^{d}\Theta(q_1,\dots,q_p)\eta_j)$, the element $\theta'\eta_{d+1}$ is algebraic over $K(\bigcup_{j=1}^{d}\Theta(s_1+q_1,\dots,s_p+q_p)\eta_j)$ where $s_i=\mathrm{ord}_i\,\theta'$, $1\le i\le p$ (clearly, $s_i\le r_i$ for $i=1,\dots,p$). Therefore, $\theta' v(\eta)$ is algebraic over $K(\bigcup_{j=1}^{d}\Theta(s_1+q_1,\dots,s_p+q_p)\eta_j\setminus\{\theta' v(\eta)\})$, hence the set $\bigcup_{j=1}^{d}\Theta(r_1+q_1,\dots,r_p+q_p)\eta_j$ is algebraically dependent over K that contradicts the fact that $\eta_1,\dots,\eta_d$ are Δ-algebraically independent over K.

Thus, η_{d+1} is algebraic over $K(\eta_1,\dots,\eta_d)$ and similarly every η_j, $d+1\le j\le n$, is algebraic over $K(\eta_1,\dots,\eta_d)$, so $d=\Delta\text{-tr.}\deg_K K\langle\eta_1,\dots,\eta_n\rangle=\mathrm{tr.}\deg_K(\eta_1,\dots,\eta_n)$.

(ii) $\Rightarrow$ (i). As in the proof of Theorem 4.2, without loss of generality we can assume that $\eta_1,\dots,\eta_d$ is a Δ-transcendence basis of the Δ-field $L=K\langle\eta_1,\dots,\eta_n\rangle$ over K. Then the elements $\eta_1,\dots,\eta_d$ are algebraically independent over K, so $K(\eta_1,\dots,\eta_n)$ is an algebraic extension of $K(\eta_1,\dots,\eta_d)$. Thus, $K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j)$ is an algebraic extension of the field $K(\bigcup_{j=1}^{d}\Theta(r_1,\dots,r_p)\eta_j)$ for any $(r_1,\dots,r_p)\in\mathbb{N}^p$.

Since $\Phi_{(\eta_1,\dots,\eta_d)|K}(t_1,\dots,t_p)=d\prod_{i=1}^{p}\binom{t_i+m_i}{m_i}$ and the fields $K(\bigcup_{j=1}^{n}\Theta(r_1,\dots,r_p)\eta_j)$ and $K(\bigcup_{j=1}^{d}\Theta(r_1,\dots,r_p)\eta_j)$ have the same transcendence degree over K, we obtain the equality of statement (i). $\square$

Proposition 4.5. Let $L=K\langle\eta_1,\dots,\eta_n\rangle$ be a Δ-field extension generated by a finite set $\eta=\{\eta_1,\dots,\eta_n\}$ and let partition (3.1) of the set Δ be fixed. Suppose that Δ-tr. $\deg_K L=0$ or that Δ-tr. $\deg_K L=d\ge 1$, $\eta_1,\dots,\eta_d$ form a Δ-transcendence basis of

L over K and $\eta' = \{\eta_{d+1}, \dots, \eta_n\}$. Then

$$\Phi_{\eta'|K\langle\eta_1,\dots,\eta_d\rangle}(r_1, \dots, r_p) \leq \Phi_{\eta|K}(r_1, \dots, r_p) - d\prod_{i=1}^{p}\binom{r_i + m_i}{m_i} \qquad (4.\,2)$$

for all sufficiently large $(r_1, \dots, r_p) \in \mathbb{N}^p$.

Proof. If Δ-$\operatorname{tr.deg}_K L = 0$, the statement is obvious. Let $d = \Delta$-$\operatorname{tr.deg}_K L \geq 1$ and $\{\eta_1, \dots, \eta_d\}$ a Δ-transcendence basis of L/K. Let $K' = K\langle\eta_1, \dots, \eta_d\rangle$ and for any $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, $\Lambda_1(r) = \bigcup_{k=1}^{d}\Theta(r_1, \dots, r_p)\eta_k$, $\Lambda_2(r) = \bigcup_{l=d+1}^{n}\Theta(r_1, \dots, r_p)\eta_l$ and $\Lambda_3(r) = \Lambda_1(r)\bigcup\Lambda_2(r)$. Then

$\Phi_{\eta'|K'}(r_1, \dots, r_p) = \operatorname{tr.deg}_{K'} K'(\Lambda_2(r)) \leq \operatorname{tr.deg}_{K(\Lambda_1(r))} K(\Lambda_3(r)) = \operatorname{tr.deg}_K K(\Lambda_3(r))$

$-\operatorname{tr.deg}_K K(\Lambda_1(r)) = \Phi_{\eta|K}(r_1, \dots, r_p) - d\prod_{i=1}^{p}\binom{r_i + m_i}{m_i}$

for all sufficiently large $(r_1, \dots, r_p)$. □

Theorem 4.6. Let $L = K\langle\eta_1, \dots, \eta_n\rangle$ be a Δ-field extension generated by a finite set $\eta = \{\eta_1, \dots, \eta_n\}$ and let $d = \Delta$-$\operatorname{tr.deg}_K L$. Then there exists $c_1, \dots, c_p|in\mathbb{N}$ and a Δ-transcendence basis B of L over K such that $B = \{\eta_{i_1}, \dots \eta_{i_d}\} \subset \eta$ $(1 \leq i_1 < \dots < i_d \leq n)$ and if η' denotes the set $\eta \setminus B$, then

$$\Phi_{\eta'|K\langle B\rangle}(r_1 + c_1, \dots, r_p + c_p) \geq \Phi_{\eta|K}(r_1, \dots, r_p) - d\prod_{i=1}^{p}\binom{t_i + m_i}{m_i} \qquad (4.\,3)$$

for all $(r_1, \dots, r_p)) \in \mathbb{N}^p$.

Proof. Let $\Psi_{\eta|K}(t_1, \dots, t_p) = \Phi_{\eta|K}(t_1, \dots, t_p) - d\prod_{i=1}^{p}\binom{t_i + m_i}{m_i}$. Let P be the defining ideal of η in the ring of Δ-polynomials $K\{y_1, \dots, y_n\}$ and let $\mathcal{A}$ be a characteristic set of P (in the sense of Definition 3.9 and the preceding considerations). For every $j = 1, \dots, n$, let

$$E_j = \{(k_1, \dots, k_m) \in \mathbb{N}^m \mid \delta_1^{k_1} \dots \delta_m^{k_m} y_j \text{ is a 1-leader of an element of } \mathcal{A}\}.$$

Using the notation of Theorem 3.12 (iv), we obtain that $\deg\phi_{\eta|K}(t_1, \dots, t_p) \leq m - 1$. Also, part (ii) of Theorem 3.12 shows that $\omega_{\eta|K}(t_1, \dots, t_p) = \sum_{j=1}^{n}\omega_{E_j}(t_1, \dots, t_p)$ where $\omega_{E_j}(t_1, \dots, t_p)$ is the dimension polynomial of the set $E_j \subseteq \mathbb{N}^m$ associated with the partition $(m_1, \dots, m_p)$ of m corresponding to our partition (3. 1) of the set Δ (see Definition 2.4). Since

$$\Phi_{\eta|K}(t_1, \dots, t_p) = d\prod_{i=1}^{p}\binom{t_i + m_i}{m_i} + \text{ terms of total degree less than } m,$$

there are exactly d indices $j \in \{1, \dots, n\}$ for which $E_j = \emptyset$, that is, $\deg\omega_{E_j} = m$ (see Theorem 2.3(iii)). Without loss of generality , we can assume that these indices are $1, \dots, d$, so the leader of any element of $\mathcal{A}$ is of the form θy_k where $\theta \in \Theta$ and $d + 1 \leq k \leq n$. It follows that $B = \{\eta_1, \dots, \eta_d\}$ is a Δ-transcendence basis of L over K. Indeed, if there are elements $\zeta_1 = \theta_1\eta_{j_1}, \dots, \zeta_s = \theta_s\eta_{j_s}$ with $\theta_i \in \Theta$ and $1 \leq j_i \leq d$ $(i = 1, \dots, s)$ and a polynomial f in s variables with coefficients in K such that $f(\zeta_1, \dots, \zeta_s) = 0$, then $f(\theta_1 y_{j_1}, \dots, \theta_s y_{j_s}) \in P$. This Δ-polynomial is reduced with respect to $\mathcal{A}$, hence all coefficients of f are zeros. Thus, $\eta_1, \dots, \eta_d$ are Δ-algebraically independent over K. Also, every Δ-generator η_i with $d+1 \leq i \leq n$ is Δ-algebraic over $K\langle B\rangle$, since there exists $\theta_0 \in \Theta$ such that $\theta_0 y_i$ is the 1-leader of some element $A \in \mathcal{A}$ for which $A(\eta) = 0$. This equality shows that $\theta_0\eta_i$ is algebraic over the field extension of $K\langle B\rangle$ generated by the set $\{\theta\eta_k \,|\, \theta \in \Theta, d+1 \leq k \leq n \text{ and } \theta y_k <_1 \theta_0 y_i\}$. Using the induction with respect to the well-ordering $<_1$ of the set of terms of $K\{y_1, \dots, y_n\}$ we obtain that η_i is Δ-algebraic over $K\langle B\rangle$.

In what follows, we use the notation from the proof of Proposition 4.5: for any $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, $\Lambda_1(r) = \bigcup_{k=1}^{d} \Theta(r_1, \dots, r_p)\eta_k$, $\Lambda_2(r) = \bigcup_{j=d+1}^{n} \Theta(r_1, \dots, r_p)\eta_j$, and $\Lambda_3(r) = \Lambda_1(r) \bigcup \Lambda_2(r)$. By Proposition 4.5,

$$\Phi_{\eta'|K\langle\eta_1,\dots,\eta_d\rangle}(r_1, \dots, r_p) \leq \Psi_{\eta|K}(r_1, \dots, r_p),$$

that is,

$$\operatorname{tr.deg}_{K\langle\eta_1,\dots,\eta_d\rangle} K\langle\eta_1, \dots, \eta_d\rangle(\Lambda_2(r)) \leq \operatorname{tr.deg}_{K(\Lambda_1(r))} K(\Lambda_3(r)) \qquad (4.\,4)$$

for all sufficiently large $r = (r_1, \dots, r_p) \in \mathbb{N}^p$, that is, there exists a p-tuple $r^{(0)} = (r_1^{(0)}, \dots, r_p^{(0)}) \in \mathbb{N}^p$ such that the last equality holds for all $r \in \mathbb{N}^p$ such that $r \geq_P r^{(0)}$ (as before, $\geq_P$ denotes the product order on $\mathbb{N}^p$).

Let us show that for all $r \in \mathbb{N}^p$ with $r \geq_P r^{(0)}$, we also have

$$\operatorname{tr.deg}_{K(\Lambda_1(r))} K(\Lambda_3(r)) \leq \operatorname{tr.deg}_{K\langle\eta_1,\dots,\eta_d\rangle} K\langle\eta_1, \dots, \eta_d\rangle(\Lambda_2(r)). \qquad (4.\,5)$$

Then the inequalities (4. 4) and (4. 5), together with Proposition 2.7, will imply the desired result.

Assume for contradiction that this is not true. Then there exists $s = (s_1, \dots, s_p) \in \mathbb{N}^p$ with $s \geq_P r^{(0)}$ and a set $W \subseteq \bigcup_{i=d+1}^{n} \Theta(s_1, \dots, s_p)y_i$ such that the set $W(\eta) = \{\theta\eta_i \,|\, \theta \in \Theta(s_1, \dots, s_p), d+1 \leq i \leq n\}$ is algebraically independent over $K(\Lambda_1(s_1, \dots, s_n))$, but algebraically dependent over $K\langle\eta_1, \dots, \eta_d\rangle$. Let $\mathbb{N}(s) = \{r \in \mathbb{N}^m \,|\, s \leq_P r\}$ and let $e = (e_1, \dots, e_p)$ be the smallest element of $\mathbb{N}(s)$ with respect to the lexicographic order $\leq_{lex}$ on $\mathbb{N}^p$ such that $W(\eta)$ is algebraically dependent over $K(\Lambda_1(e))$. Then $s <_{lex} e$ and there exists a nonzero polynomial $f \in K(\Lambda_1(e))[\{w \,|\, w \in W\}]$ such that $f(\eta) = 0$. Clearing the denominators of f, we obtain a nonzero Δ-polynomial $g \in K[\{\theta y_j \,|\, \theta \in \Theta(e), 1 \leq\leq d\} \bigcup W]$ such that $g(\eta) = 0$. Let g be such a Δ-polynomial of the lowest possible rank (in the sense of Definition 3.1). Then the 1-leader of g is of the form $\theta_1 y_k$ where $\operatorname{ord}_i \theta_1 = e_i$ $(1 \leq i \leq p)$ and $1 \leq k \leq d$.

Let us write $g = \sum g_t t$ where t runs over a finite set $\mathcal{M}$ of monomials in the indeterminates w $(w \in W)$ and the coefficients g_t are nonzero Δ-polynomials in $K[\{\theta y_j \,|\, \theta \in \Theta(e), 1 \leq d\}]$. By Corollary 3.7, there exists a Δ-polynomial H, which is a product of initials and separants of elements of $\mathcal{A}$ (so $H \notin P$), and for each $t \in \mathcal{M}$ there exists a Δ-polynomial T such that $Ht \equiv T\,(\operatorname{mod}[\mathcal{A}])$, T is reduced with respect to $\mathcal{A}$, and the rank of T is no higher than the highest of the ranks of monomials t $(t \in \mathcal{M})$ and therefore lower than the rank of $u_g^{(1)}$.

Let $h = \sum_{t\in\mathcal{M}} g_t T$. Clearly, h is reduced with respect to $\mathcal{A}$ and $h \in P$ (because $h \equiv Hg(\operatorname{mod}[\mathcal{A}])$ and $g \in P$). By Proposition 3.11, $h = 0$. Now, since $\partial T/\partial u_g^{(1)} = 0$ and

$\partial T/\partial u_g^{(1)} = 0$, we have

$$HS_g = H\sum_{t\in\mathcal{M}}\left(\frac{\partial g_t}{\partial u_g^{(1)}}\right)t \equiv \sum_{t\in\mathcal{M}}\left(\frac{\partial g_t}{\partial u_g^{(1)}}\right)T(\mathrm{mod}[\mathcal{A}]) \equiv \frac{\partial h}{\partial u_g^{(1)}}\,(\mathrm{mod}[\mathcal{A}]) \equiv 0\,(\mathrm{mod}[\mathcal{A}]).$$

It follows that $HS_g \in P$, hence $S_g \in P$, so $S_g(\eta) = 0$. We have obtained a contradiction with the choice of g as the Δ-polynomial of the lowest rank that vanishes at η. This completes the proof of the theorem. □

The following example illustrates the fact that a multivariate dimension polynomial of a differential field extension carries essentially more information about the extension than its univariate counterpart.

Example 4.7. Let K be a differential field with a basic set of derivations $\Delta = \{\delta_1, \delta_2, \delta_3\}$ and let L be a Δ-field extension of K generated by a single Δ-generator η with the defining equation

$$\delta_1^a\delta_2^b\delta_3^c\eta + \delta_1^a\eta + \delta_2^b\eta + \delta_3^{b+c}\eta = 0 \qquad (4.\,6)$$

where a, b and c are some positive integers. In other words, $L = K\langle\eta\rangle$ is Δ-isomorphic to the quotient field of the factor ring $K\{y\}/P$ where P is the linear (and therefore prime) Δ-ideal of the ring of differential (Δ-) polynomials $K\{y\}$ generated by the Δ-polynomial $f = \delta_1^a\delta_2^b\delta_3^c y + \delta_1^a y + \delta_2^b y + \delta_3^{b+c}y$. ($P$ is the defining ideal of η over K.)

By [6, Chapter II, Theorem 6], the univariate Kolchin differential dimension polynomial $\omega_{\eta/K}(t)$ of L/K is equal to the univariate dimension polynomial of the subset $\{(a, b, c)\}$ of $\mathbb{N}^3$. Using formula (2. 2) for $p = 1$, we obtain that

$$\omega_{\eta/K}(t) = \binom{t+3}{3} - \binom{t+3-(a+b+c)}{3} = \left(\frac{a+b+c}{2}\right)t^2 +$$
$$\left(\frac{(a+b+c)(4-a-b-c)}{2}\right)t + \frac{(a+b+c)[(a+b+c)^2 - 6(a+b+c) + 11]}{6}. \qquad (4.\,7)$$

Now, let us fix a partition $\Delta = \Delta_1 \cup \Delta_2$ with $\Delta_1 = \{\delta_1, \delta_2\}$. Let $\Delta_2 = \{\delta_3\}$, and $\Phi_\eta(t_1, t_2)$ denote the Δ-dimension polynomial of L/K associated with this partition and the Δ-generator η. With the notation of section 3, we obtain that $u_f^{(1)} = \delta_1^a\delta_2^b\delta_3^c y$ and $u_f^{(2)} = \delta_3^{b+c}y$. Using the notation of Theorem 3.12 and formula (2. 2) we obtain that for all sufficiently large $(r_1, r_2) \in \mathbb{N}^2$,

$$\mathrm{Card}\, U'_{r_1,r_2} = \binom{r_1+2}{2}(r_2+1) - \binom{r_1+2-(a+b)}{2}(r_2+1-c).$$

Expanding the last expression and using symbols t_1 and t_2 for the variables representing r_1 and r_2, respectively, we obtain the polynomial $\omega_{\eta|K}(t_1, t_2)$ (see Theorem 3.12) that describes the size of $\mathrm{Card}\, U'_{r_1,r_2}$:

$$\omega_{\eta|K}(t_1, t_2) = \frac{c}{2}t_1^2 + (a+b)t_1t_2 + \frac{2a+2b+3c-2ac-2bc}{2}t_1 + \frac{(a+b)(3-a-b)}{2}t_2 +$$
$$\frac{1}{2}[(a+b-2)(a+b-1)(c-1)+2].$$

Furthermore, for all sufficiently large $(r_1, r_2) \in \mathbb{N}^2$,

$\mathrm{Card}\, U''_{r_1,r_2} = \mathrm{Card}\{\delta_1^{a+k_1}\delta_2^{b+k_2}\delta_3^{c+k_3} \mid k_1, k_2, k_3 \in \mathbb{N}, k_1+k_2 \le r_1-(a+b), r_3-(b+c) < k_3$

$\leq r_3 - c\} = \binom{r_1+2-(a+b)}{2} b.$

Thus, with the notation of Theorem 3.12,

$$\phi_{\eta|K}(t_1, t_2) = \frac{b}{2}t_1^2 + \frac{b(3-2a-2b)}{2}t_1 + \frac{b(a+b-2)(a+b-1)}{2}.$$

It follows that the bivariate differential dimension polynomial of the extension L/K corresponding to the partition $\Delta = \{\delta_1, \delta_2\} \bigcup \{\delta_3\}$ is

$$\Phi_{\eta|K}(t_1, t_2) = \frac{b+c}{2}t_1^2 + (a+b)t_1t_2 + \frac{1}{2}[2a+5b+3c-2ab-2ac-2bc-2b^2]t_1+$$

$$\frac{(a+b)(3-a-b)}{2}t_2 + \frac{1}{2}[(a+b-2)(a+b-1)(b+c-1)+2]. \quad (4.\,8)$$

Finally, let us fix a partition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$ with $\Delta_i = \{\delta_i\}$ $(i = 1, 2, 3)$. Proceeding as before (with the notation of Theorem 3.12), we obtain that

$$\omega_{\eta|K}(t_1, t_2, t_3) = ct_1t_2 + bt_1t_3 + at_2t_3 + (b+c-bc)t_1 + (a+c-ac)t_2 + (a+b-ab)t_3 + a+b+c-ab$$

$$-ac-bc+abc$$

and

$$\phi_{\eta|K} = bt_1t_2 + (b-b^2)t_1 + (b-ab)t_2 + (b-ab-b^2+ab^2),$$

so in this case

$$\Phi_{\eta|K}(t_1, t_2, t_3) = (b+c)t_1t_2 + bt_1t_3 + at_2t_3 + (2b+c-bc-b^2)t_1 + (a+b+c-ab-ac)t_2+$$

$$(a+b-ab)t_3 + (a+2b+c-2ab-ac-bc-b^2+ab^2+abc). \quad (4.\,9)$$

It follows from Theorem 4.1 that the dimension Δ-polynomial in three variables given by (4. 9) carries four invariants of the extension L/K: the total degree 2 and the coefficients $b+c$, b and a of the terms t_1t_2, t_1t_3 and t_2t_3, respectively. The dimensional polynomial (4. 8) carries three invariants, the total degree 2 and the coefficients $b+c$ and $a+b$, while the univariate Kolchin polynomial (4. 7) carries only two invariants of the extension, the total degree 2 and the sum of the parameters $a+b+c$. Therefore, the Δ-dimension polynomial (4. 9) corresponding to the partition of Δ into the union of three disjoint subsets determines all three parameters a, b and c of the defining differential equation (4. 6) while the univariate dimension polynomial gives just the sum of the parameters. Also, in accordance with the above considerations, the dimension polynomial (4. 8) (corresponding to the partition $\Delta = \Delta_1 \bigcup \Delta_2$ with $\Delta_1 = \{\delta_1, \delta_2\}$ and $\Delta_2 = \{\delta_3\}$) shows that

$$\Delta_2\text{- tr.}\deg_K K\langle\{\delta_1^{k_1}\delta_2^{k_2}\eta \,|\, k_1+k_2 \leq r\}\rangle_{\Delta_2} = (a+b)r + \frac{(a+b)(3-a-b)}{2}$$

for all sufficiently large $r \in \mathbb{N}$.

We conclude with an analytical interpretation of multivariate differential dimension polynomials as generalized Einstein's strength of systems of algebraic partial differential equations of certain type.

Let $K\{y_1, \dots, y_n\}$ be the ring of Δ-polynomials over a Δ-field K (we use the above notation and assume that partition (3. 1) of Δ is fixed). If $\Sigma = \{A_\lambda \,|\, \lambda \in \Lambda\}$ is a set of Δ-polynomials in $K\{y_1, \dots, y_n\}$ then the system of equations

$$A_\lambda(y_1, \dots, y_n) = 0 \qquad (\lambda \in \Lambda) \quad (4.\,10)$$

is said to be a system of algebraic (partial if $\operatorname{Card} \Delta > 0$) differential equations. An n-tuple $\eta = (\eta_1, \dots, \eta_n)$ with coordinates in some Δ-overfield of K is said to be a solution of this system if Σ is contained in the kernel of the substitution of $(\eta_1, \dots, \eta_n)$ for $(y_1, \dots, y_n)$ which is a Δ-homomorphism $K\{y_1, \dots, y_n\} \to K\langle \eta_1, \dots, \eta_n \rangle$ sending each y_i to η_i and leaving elements of K fixed. (Note that by the Ritt-Raudenbush basis theorem, see [6, Chapter III, Theorem 1], the solution set of system (4. 10) is the same as the solution set of some its finite subsystem, so we can assume that the set Λ is finite.) A system of the form (4. 10) is said to be *prime* if the differential radical ideal P generated in the ring $K\{y_1, \dots, y_n\}$ by the set Σ is prime. (Since a linear Δ-ideal of the ring $K\{y_1, \dots, y_n\}$ is prime, see [7, Proposition 3.2.28]), every system of linear homogeneous differential equations is prime.) In this case, if L is the field of fractions of the integral domain $K\{y_1, \dots, y_n\}/P$ (which can be naturally treated as a Δ-field extension of K) and η_i is the canonical image of y_i in L $(1 \le i \le n)$, then $L = K\langle \eta_1, \dots, \eta_n \rangle$. The differential dimension polynomial in p variables associated with the system of generators $\{\eta_1, \dots, \eta_n\}$ of the extension L/K (and partition (3. 1) of Δ) is said to be the differential dimension polynomial of the system associated with the given partition of the set Δ.

Let us consider a system of partial differential equations of the form

$$A_i(f_1, \dots, f_n) = 0 \qquad (i = 1, \dots, q) \tag{4. 11}$$

over a field K of infinitely differentiable functions of m real variables $x_1, \dots, x_m$ (f_i are unknown functions of $x_1, \dots, x_m$). Let $\Delta = \{\delta_1, \dots, \delta_m\}$ where δ_i is the partial differentiation $\partial/\partial x_i$, and suppose that $A_i(y_1, \dots, y_s)$ are elements of the ring of Δ-polynomials $K\{y_1, \dots, y_n\}$. We also fix partition (3. 1) of the set of basic derivations Δ (such a partition can be, for example, a natural separation of (all or some) derivations with respect to coordinates and the derivation with respect to time). For any $r_1, \dots, r_p \in \mathbb{N}$, consider the values at some fixed point c of the unknown functions $f_1, \dots, f_n$ and their partial derivatives, whose order with respect to Δ_i does not exceed r_i $(1 \le i \le p)$. If $f_1, \dots, f_n$ should not satisfy any system of equations, these values can be chosen arbitrarily. Because of the system (and equations obtained from the equations of the system by partial differentiations), the number of independent values at c of the functions $f_1, \dots, f_n$ and their partial derivatives whose ith order does not exceed r_i $(1 \le i \le p)$ decreases. This number, which is a function of p variables $r_1, \dots, r_p$, is the "measure of strength" of the system in the sense of A. Einstein (with respect to the given partition of Δ). We denote it by $S_{r_1, \dots, r_p}$.

If the given system is prime, that is, the radical Δ-ideal P of $K\{y_1, \dots, y_n\}$ generated by the Δ-polynomials $A_1, \dots, A_q$ is prime (e. g., the Δ-polynomials are linear), then the Δ-dimension polynomial $\Phi_{\eta|K}(t_1, \dots, t_p)$ of the system (defined by Theorem 3.12 for the Δ-field extension L/K described above) has the property that

$$\Phi_{\eta|K}(r_1, \dots, r_p) = S_{r_1, \dots, r_p}$$

for all sufficiently large $(r_1, \dots, r_p) \in \mathbf{N}^p$, so this dimension polynomial is the measure of Einstein's strength of the system of differential equations (4. 11) with respect to the given partition of the basic set of derivations Δ.

Considering differential dimension polynomials of prime systems of algebraic partial differential equations with basic set of derivations Δ, we say that two such systems with coefficients in a Δ-field K are *equivalent* if there is a Δ-isomorphism between the corresponding Δ-field extensions of K that leaves elements of K fixed (it is called a Δ-K-isomorphism). The following example shows how multivariate differential dimension polynomials can be used for determining the equivalence or non-equivalence of two systems.

Example 4.8. Let us consider two algebraic differential equations over a differential field K with a basic set of derivation $\Delta = \{\delta_1, \delta_2, \delta_3\}$,

$$\delta_1^a \delta_2^b \delta_3^c y + \delta_1^a y + \delta_2^b y + \delta_3^{b+c} y = 0 \tag{4. 12}$$

and

$$\delta_1^a \delta_2^b \delta_3^c y + \delta_3^{a+b+c} y = 0, \tag{4. 13}$$

where a, b, and c are some positive integers. As we have seen in Example 4.7, the univariate Kolchin differential dimension polynomial of equation (4. 12) is given by (4. 7). The univariate differential dimension polynomial of equation (4. 13) is the same (see [11, Example 4.9]). The Δ-dimension polynomials in three variables associated with system (4. 12) and the partition $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3$ with $\Delta_i = \{\delta_i\}$ $(i = 1, 2, 3)$ is given by (4. 9). The corresponding polynomial of equation (4. 13), as it is shown in [11, Example 4.9], is

$$\Phi_{\eta|K}(t_1, t_2, t_3) = (a+b+c+1)t_1t_2 + bt_1t_3 + at_2t_3 + (a+b+c+1-ab-b^2-bc)t_1 + (a+b+c+1-$$

$$ab-a^2-ac)t_2 + (a+b-ab)t_3 + a+b+c+1+ab^2+a^2b-a^2-b^2-2ab-bc-ac+abc. \tag{4. 14}$$

Since the polynomials (4. 9) and (4. 14) have different coefficients of the term t_1t_2, there is no Δ-K-isomorphism between the differential field extensions of K defined by equations (4. 12) and (4. 13).

Our example shows that using a partition of the basic set of derivations and the computation of the corresponding multivariate Δ-dimension polynomials, one can determine that two systems of Δ-equations are not equivalent, even though they have the same univariate differential dimension polynomial.

References

[1] A. Einstein. *The Meaning of Relativity. Appendix II (Generalization of gravitation theory)*, 153–165. Princeton University Press, Princeton, NJ, 1953.

[2] J. Freitag, O. L. Sanchez, O. L.; W. Li. Effective definability of Kolchin polynomials. *Proc.Amer. Math. Soc.*, 148 (2020), 1455–1466.

[3] Joseph L. Johnson A notion on Krull dimension for differential rings. *Comment. Math. Helv.*, 44 (1969), 207–216.

[4] Joseph L. Johnson. Kähler differentials and differential algebra. *Ann. of Math.* (2), 89 (1969), 92–98.

[5] E. R. Kolchin. The notion of dimension in the theory of algebraic differential equations. *Bull. Amer. Math. Soc.*, 70 (1964), 570–573.

[6] E. R. Kolchin. *Differential Algebra and Algebraic Groups.* Academic Press, 1973.

[7] M. V. Kondrateva, A. B. Levin, A. V. Mikhalev, and E. V. Pankratev. *Differential and Difference Dimension Polynomials.* Kluwer Acad. Publ., 1999.

[8] M. Lange-Hegermann. The Differential Dimension Polynomial for Characterizable Differential Ideals. *Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory* (2018), 443–453.

[9] M. Lange-Hegermann. M. The Differential Counting Polynomial. *Foundations of Computational Mathematics*, 18, no. 2, (2018), 291–308.

[10] A. B. Levin. Multivariable dimension polynomials and new invariants of differential field extensions. *Internat. J. Math. and Math. Sci.*, 27 (2001), no. 4, 201–214.

[11] A. B. Levin. Gröbner bases with respect to several orderings and multivariable dimension polynomials. *J. Symb. Comput.*, 42 (2007), no. 5, 561–578.

[13] A. B. Levin. Dimension polynomials of intermediate fields and Krull-type dimension of finitely generated differential field extensions. *Mathematics in Computer Science*, 4 (2010), no. 2–3, 143–150.

[13] A. B. Levin. Some properties of multivariate differential dimension polynomials and their invariants. ISSAC'20. Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation (2020), 320–327.

[14] A. V. Mikhalev and E. V. Pankratev. Differential dimension polynomial of a system of differential equations. *Algebra. Collection of papers.*, 57–67. Moscow State University Press, 1980.

[15] O. L. Sanchez. Estimates for the coefficients of differential dimension polynomials. *Mathematics of Computation*, 88 (2019), 2959–2985.

[16] W. Sit. On the differential transcendence polynomials of finitely generated differential field extensions. *Amer. J. Math.*, 101 (1979), no. 6, 1249–1263.

[17] I. Smirnov. Differential Krull dimension in differential polynomial extensions. *J. Algebra*, 344 (2011), no. 1, 354–372.