Hilbert-type dimension polynomials of intermediate difference-differential field extensions*

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Abstract. Let K be an inversive difference-differential field and L a (not necessarily inversive) finitely generated difference-differential field extension of K. We consider the natural filtration of the extension L/K associated with a finite system η of its difference-differential generators and prove that for any intermediate difference-differential field F, the transcendence degrees of the components of the induced filtration of F are expressed by a certain numerical polynomial $\chi_{K,F,\eta}(t)$. This polynomial is closely connected with the dimension Hilbert-type polynomial of a submodule of the module of Kähler differentials $\Omega_{L^{*}|K}$ where L^{*} is the inversive closure of L. We prove some properties of polynomials $\chi_{K,F,\eta}(t)$ and use them for the study of the Krull-type dimension of the extension L/K. In the last part of the paper, we present a generalization of the obtained results to multidimensional filtrations of L/K associated with partitions of the sets of basic derivations and translations.

Keywords: Difference-differential field · difference-differential module · Kähler differentials · dimension polynomial.

1 Introduction

Dimension polynomials associated with finitely generated differential field extensions were introduced by E. Kolchin in [4]; their properties and various applications can be found in his fundamental monograph [5, Chapter 2]. A similar technique for difference and inversive difference field extensions was developed in [7], [8], [12], [13] and some other works of the author. Almost all known results on differential and difference dimension polynomials can be found in [6] and [10]. One can say that the role of dimension polynomials in differential and difference algebra is similar to the role of Hilbert polynomials in commutative algebra and algebraic geometry. The same can be said about dimension polynomials associated with difference-differential algebraic structures. They appear as generalizations of their differential and difference counterparts and play a key role in the study of dimension of difference-differential modules and extensions

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of difference-differential fields. Existence theorems, properties and methods of computation of univariate and multivariate difference-differential dimension polynomials can be found in [15], [6, Chapters 6 and 7], [14], [19] and [20].
In this paper we prove the existence and obtain some properties of a univariate dimension polynomial associated with an intermediate difference-differential field of a finitely generated difference-differential field extension (see Theorem 2 that can be considered as the main result of the paper). Then we use the obtained results for the study of the Krull-type dimension of such an extension. In particular, we establish relationships between invariants of dimension polynomials and characteristics of difference-differential field extensions that can be expressed in terms of chains of intermediate fields. In the last part of the paper we generalize our results on univariate dimension polynomials and obtain multivariate dimension polynomials associated with multidimensional filtrations induced on intermediate difference-differential fields. (Such filtrations naturally arise when one considers partitions of the sets of basic derivations and translations.) Note that we consider arbitrary (not necessarily inversive) difference-differential extensions of an inversive difference-differential field. In the particular case of purely differential extensions and in the case of inversive difference field extensions, the existence and properties of dimension polynomials were obtained in [11] and [13]. The main problem one runs into while working with a non-inversive difference (or difference-differential) field extension is that the translations are not invertible and there is no natural difference (respectively, difference-differential) structure on the associated module of Kähler differentials. We overcome this obstacle by considering such a structure on the module of Kähler differentials associated with the inversive closure of the extension. Finally, the results of this paper allow one to assign a dimension polynomial to a system of algebraic difference-differential equations of the form $f_i = 0$, $i \in I$ (f_i lie in the algebra of difference-differential polynomials $K\{y_1, \dots, y_n\}$ over a ground field K) such that the difference-differential ideal P generated by the left-hand sides is prime and the solutions of the system should be invariant with respect to the action of a group G that commutes with basic derivations and translations. As in the case of systems of differential or difference equations, the dimension polynomial of such a system is defined as the dimension polynomial of the subfield of the difference-differential quotient field $K\{y_1, \dots, y_n\}/P$ whose elements remain fixed under the action of G. Using the correspondence between dimension polynomials and Einstein's strength of a system of algebraic differential or difference equations established in [16] and [6, Chapter 6] (this characteristic of a system of PDEs governing a physical field was introduced in [1]), one can consider this dimension polynomial as an expression of the Einstein's strength of a system of difference-differential equations with group action.

2 Preliminaries

Throughout the paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ denote the sets of all integers, all non-negative integers and all rational numbers, respectively. As usual, $\mathbb{Q}[t]$ will denote the ring of polynomials in one variable t with rational coefficients. By a ring we always mean an associative ring with a unity. Every ring homomorphism is unitary (maps unit onto unit), every subring of a ring contains the unity of the ring. Every module is unitary and every algebra over a commutative ring is unitary as well. Every field is supposed to have characteristic zero.

A *difference-differential ring* is a commutative ring R considered together with finite sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivations and injective endomorphisms of R, respectively, such that any two mappings of the set $\Delta \bigcup \sigma$ commute. The elements of the set σ are called *translations* and the set $\Delta \bigcup \sigma$ will be referred to as a *basic set* of the difference-differential ring R, which is also called a Δ-σ-ring. We will often use prefix Δ-σ- instead of the adjective "difference-differential". If all elements of σ are automorphisms of R, we say that the Δ-σ-ring R is *inversive*. In this case we set $\sigma^* = \{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$ and call R a Δ-σ^*-ring.

If a Δ-σ-ring R is a field, it is called a *difference-differential field* or a Δ-σ-field. If R is inversive, we say that R is a Δ-σ^*-field.

In what follows, Λ will denote the free commutative semigroup of all power products $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ where $k_i, l_j \in \mathbb{N}$ $(1 \le i \le m,\ 1 \le j \le n)$. Furthermore, Θ and T will denote the commutative semigroups of power products $\delta_1^{k_1} \dots \delta_m^{k_m}$ and $\alpha_1^{l_1} \dots \alpha_n^{l_n}$ $(k_i, l_j \in \mathbb{N})$, respectively. If $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \in \Lambda$, we define the order of λ as $\operatorname{ord} \lambda = \sum_{i=1}^{m} k_i + \sum_{j=1}^{n} l_j$ and set $\Lambda(r) = \{\lambda \in \Lambda \mid \operatorname{ord} \lambda \le r\}$ for any $r \in \mathbb{N}$.

If the elements of σ are automorphisms, then Λ^* and Γ will denote the free commutative semigroup of all power products $\mu = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n}$ with $k_i \in \mathbb{N}$, $l_j \in \mathbb{Z}$ and the free commutative group of power products $\gamma = \alpha_1^{l_1} \dots \alpha_n^{l_n}$ with $l_1, \dots, l_n \in \mathbb{Z}$, respectively. The order of such elements μ and γ are defined as $\operatorname{ord} \lambda = \sum_{i=1}^{m} k_i + \sum_{j=1}^{n} |l_j|$ and $\operatorname{ord} \gamma = \sum_{j=1}^{n} |l_j|$, respectively. We also set $\Lambda^*(r) = \{\mu \in \Lambda^* \mid \operatorname{ord} \mu \le r\}$ $(r \in \mathbb{N})$.

A subring (ideal) S of a Δ-σ-ring R is said to be a difference-differential (or Δ-σ-) subring of R (respectively, difference-differential (or Δ-σ-) ideal of R) if S is closed with respect to the action of any operator of $\Delta \bigcup \sigma$. In this case the restriction of a mapping from $\Delta \bigcup \sigma$ on S is denoted by same symbol. If S is a Δ-σ-subring R, we also say that R is a Δ-σ-overring of S. If S is a Δ-σ-ideal of R and for any $\tau \in T$, the inclusion $\tau(a) \in S$ implies that $a \in S$, we say that the Δ-σ-ideal S is *reflexive* or that S is a Δ-σ^*-ideal of R.

If L is a Δ-σ-field and K a subfield of L which is also a Δ-σ-subring of L, then K is said to be a Δ-σ-subfield of L; L, in turn, is called a difference-differential (or Δ-σ-) field extension or a Δ-σ-overfield of K. In this case we also say that we have a Δ-σ-field extension L/K.

If R is a Δ-σ-ring and $S \subseteq R$, then the intersection of all Δ-σ-ideals of R containing the set S is, obviously, the smallest Δ-σ-ideal of R containing S. This ideal is denoted by $[S]$; as an ideal, it is generated by the set $\{\lambda(x) \mid x \in S, \lambda \in \Lambda\}$. If S is finite, $S = \{x_1, \dots, x_k\}$, we say that the Δ-σ-ideal $I = [S]$ is finitely generated, write $I = [x_1, \dots, x_k]$ and call $x_1, \dots, x_k$ Δ-σ-generators of I.

If K is a Δ-σ-subfield of the Δ-σ-field L and $S \subseteq L$, then the intersection of all Δ-σ-subfields of L containing K and S is the unique Δ-σ-subfield of L containing K and S and contained in every Δ-σ-subfield of L with this property. It is denoted by $K\langle S\rangle$. If S is finite, $S = \{\eta_1, \dots, \eta_s\}$ we write $K\langle \eta_1, \dots, \eta_s\rangle$ for $K\langle S\rangle$ and say that this is a finitely generated Δ-σ-extension of K with the set of Δ-σ-generators $\{\eta_1, \dots, \eta_s\}$. It is easy to see that $K\langle \eta_1, \dots, \eta_s\rangle$ coincides with the field $K(\{\lambda\eta_i \mid \lambda \in \Lambda, 1 \leq i \leq s\})$. (If there might be no confusion, we often write $\lambda\eta$ for $\lambda(\eta)$ where $\lambda \in \Lambda$ and η is an element of a Δ-σ-ring.)

Let R_1 and R_2 be two difference-differential rings with the same basic set $\Delta \bigcup \sigma$. (More rigorously, we assume that there exist injective mappings of the sets Δ and σ into the sets of derivations and automorphisms of the rings R_1 and R_2, respectively, such that the images of any two elements of $\Delta \bigcup \sigma$ commute. We will denote the images of elements of $\Delta \bigcup \sigma$ under these mappings by the same symbols $\delta_1, \dots, \delta_m, \alpha_1, \dots, \alpha_n$). A ring homomorphism $\phi : R_1 \longrightarrow R_2$ is called a *difference-differential* (or Δ-σ-) *homomorphism* if $\phi(\tau a) = \tau\phi(a)$ for any $\tau \in \Delta \bigcup \sigma$, $a \in R$. It is easy to see that the kernel of such a mapping is a Δ-σ^*-ideal of R_1.

If R is a Δ-σ-subring of a Δ-σ-ring R^* such that the elements of σ act as automorphisms of R^* and for every $a \in R^*$ there exists $\tau \in T$ such that $\tau(a) \in R$, then the Δ-σ^*-ring R^* is called the *inversive closure* of R.

The proof of the following result can be obtained by mimicking the proof of the corresponding statement about inversive closures of difference rings, see [10, Proposition 2.1.7].

Proposition 1. (i) *Every Δ-σ-ring has an inversive closure.*
(ii) *If R_1^* and R_2^* are two inversive closures of a Δ-σ-ring R, then there exists a Δ-σ-isomorphism of R_1^* onto R_2^* that leaves elements of R fixed. .*
(iii) *If a Δ-σ-ring R is a Δ-σ-subring of a Δ-σ^*-ring U, then U contains an inversive closure of R.*
(iv) *If a Δ-σ-ring R is a field, then its inversive closure is also a field.*

If K is an inversive difference-differential field and $L = K\langle \eta_1, \dots, \eta_s\rangle$, then the inversive closure of L is denoted by $K\langle \eta_1, \dots, \eta_s\rangle^*$. Clearly, this Δ-σ^*-field coincides with the field $K(\{\mu\eta_i | \mu \in \Lambda^*, 1 \leq i \leq s\})$.

Let R be a Δ-σ-ring and $U = \{u_i \mid i \in I\}$ a family of elements of some Δ-σ-overring of R. We say that the family U is Δ-σ-*algebraically dependent* over R, if the family $\{\lambda u_i \mid \lambda \in \Lambda,\ i \in I\}$ is algebraically dependent over R. Otherwise, the family U is said to be Δ-σ-*algebraically independent* over R.

If K is a Δ-σ-field and L a Δ-σ-field extension of K, then a set $B \subseteq L$ is said to be a Δ-σ-*transcendence basis* of L over K if B is Δ-σ-algebraically independent over K and every element $a \in L$ is Δ-σ-algebraic over $K\langle B\rangle$ (that is, the set $\{\lambda a \mid \lambda \in \Lambda\}$ is algebraically dependent over $K\langle B\rangle$). If L is a finitely generated Δ-σ-field extension of K, then all Δ-σ-transcendence bases of L over K are finite and have the same number of elements (the proof of this fact can be obtained by mimicking the proof of the corresponding properties of difference transcendence bases, see [10, Section 4.1]). In this case, the number of elements of any Δ-σ-transcendence basis is called the *difference-differential* (or Δ-σ-) *transcendence* *degree* of L over K (or the Δ-σ-transcendence degree of the extension L/K); it is denoted by Δ-σ-$\text{trdeg}_K L$.

The following theorem proved in [15] generalizes the Kolchin's theorem on differential dimension polynomial (see [5, Chapter II, Theorem 6]) and also the author's theorems on dimension polynomials of difference and inversive difference field extensions (see [10, Theorems 4.2.1 and 4.2.5]).

Theorem 1. *With the above notation, let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a Δ-σ-field extension of a Δ-σ-field K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. Then there exists a polynomial $\chi_{\eta|K}(t) \in \mathbb{Q}[t]$ such that*
(i) $\chi_{\eta|K}(r) = \text{trdeg}_K K(\{\lambda\eta_j \mid \lambda \in \Lambda(r), 1 \leq j \leq s\})$ *for all sufficiently large $r \in \mathbb{Z}$ (that is, there exists $r_0 \in \mathbb{Z}$ such that the equality holds for all $r > r_0$).*
(ii) $\deg \chi_{\eta|K} \leq m + n$ *and $\chi_{\eta|K}(t)$ can be written as* $\chi_{\eta|K}(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$, *where $a_i \in \mathbb{Z}$.*
(iii) $d = \deg \chi_{\eta|K}$, a_{m+n} *and a_d do not depend on the set of Δ-σ-generators η of L/K ($a_{m+n} = 0$ if $d < m + n$). Moreover, $a_{m+n} = \Delta$-σ-$\text{trdeg}_K L$.*

The polynomial $\chi_{\eta|K}(t)$ is called the Δ-σ-*dimension polynomial* of the Δ-σ-field extension L/K associated with the system of Δ-σ-generators η . We see that $\chi_{\eta|K}(t)$ is a polynomial with rational coefficients that takes integer values for all sufficiently large values of the argument. Such polynomials are called *numerical*; their properties are thoroughly described in [6, Chapter 2]. The invariants $d = \deg \chi_{\eta|K}$ and a_d (if $d < m + n$) are called the Δ-σ-*type* and *typical Δ-σ-transcendence degree* of L/K; they are denoted by Δ-σ-$\text{type}_K L$ and Δ-σ-$t.\text{trdeg}_K L$, respectively.

3 Dimension polynomials of intermediate difference-differential fields. The main theorem

The following result is an essential generalization of Theorem 1. This generalization allows one to assign certain numerical polynomial to an intermediate Δ-σ-field of a Δ-σ-field extension L/K where K is an inversive Δ-σ-field. (We use the notation introduced in the previous section.)

Theorem 2. *Let K be an inversive Δ-σ-field with basic set $\Delta \bigcup \sigma$ where $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ are the sets of derivations and automorphisms of K, respectively. Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a Δ-σ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. Let F be an intermediate Δ-σ-field of the extension L/K and for any $r \in \mathbb{N}$, let $F_r = F \bigcap K(\{\lambda\eta_j \mid \lambda \in \Lambda(r), 1 \leq j \leq s\})$. Then there exists a numerical polynomial $\chi_{K,F,\eta}(t) \in \mathbb{Q}[t]$ such that*
(i) $\chi_{K,F,\eta}(r) = \text{trdeg}_K F_r$ *for all sufficiently large $r \in \mathbb{N}$;*
(ii) $\deg \chi_{K,F,\eta} \leq m+n$ *and $\chi_{K,F,\eta}(t)$ can be written as* $\chi_{K,F,\eta}(t) = \sum_{i=0}^{m+n} c_i \binom{t+i}{i}$
where $c_i \in \mathbb{Z}$ ($1 \leq i \leq m + n$).

(iii) $d = \deg \chi_{K,F,\eta}(t)$, c_{m+n} *and* c_d *do not depend on the set of* Δ-σ-*generators* η *of the extension* L/K. *Furthermore,* $c_{m+n} = \Delta$-σ-$\mathrm{trdeg}_K F$.

The polynomial $\chi_{K,F,\eta}(t)$ is called a *Δ-σ-dimension polynomial of the intermediate field F associated with the set of Δ-σ-generators η of L/K.*

The proof of Theorem 2 is based on properties of difference-differential modules and the difference-differential structure on the module of Kähler differentials considered below. Similar properties in differential and difference cases can be found in [2] and [10, Section 4.2], respectively.

Let K be a Δ-σ-field and Λ the semigroup of power products of basic operators introduced in section 2. Let $\mathcal{D}$ denote the set of all finite sums of the form $\sum_{\lambda\in\Lambda} a_\lambda\lambda$ where $a_\lambda \in K$ (such a sum is called a *Δ-σ-operator* over K; two Δ-σ-operators are equal if and only if their corresponding coefficients are equal). The set $\mathcal{D}$ can be treated as a ring with respect to its natural structure of a left K-module and the relationships $\delta a = a\delta + \delta(a)$, $\alpha a = \alpha(a)\alpha$ for any $a \in K$, $\delta \in \Delta$, $\alpha \in \sigma$ extended by distributivity. The ring $\mathcal{D}$ is said to be the *ring of Δ-σ-operators* over K.

If $A = \sum_{\lambda\in\Lambda} a_\lambda\lambda \in \mathcal{D}$, then the number $\mathrm{ord}\, A = \max\{\mathrm{ord}\,\lambda \,|\, a_\lambda \neq 0\}$ is called the *order* of the Δ-σ-operator A. In what follows, we treat $\mathcal{D}$ as a filtered ring with the ascending filtration $(\mathcal{D}_r)_{r\in\mathbb{Z}}$ where $\mathcal{D}_r = 0$ if $r < 0$ and $\mathcal{D}_r = \{A \in \mathcal{D} \,|\, \mathrm{ord}\, A \leq r\}$ if $r \geq 0$.

Similarly, if a Δ-σ-field K is inversive and Λ^* is the semigroup defined in section 2, then $\mathcal{E}$ will denote the set of all finite sums $\sum_{\mu\in\Lambda^*} a_\mu\mu$ where $a_\mu \in K$. Such a sum is called a Δ-σ^*-operator over K; two Δ-σ^*-operators are equal if and only if their corresponding coefficients are equal. Clearly, the ring $\mathcal{D}$ of Δ-σ-operators over K is a subset of $\mathcal{E}$. Moreover, $\mathcal{E}$ can be treated as an overring of $\mathcal{D}$ such that $\alpha^{-1}a = \alpha^{-1}(a)\alpha^{-1}$ for every $\alpha \in \sigma$, $a \in K$. This ring is called the *ring of Δ-σ^*-operators* over K.

The order of a Δ-σ^*-operator $B = \sum_{\mu\in\Lambda^*} a_\mu\mu$ is defined in the same way as the order of a Δ-σ-operator: $\mathrm{ord}\, B = \max\{\mathrm{ord}\,\mu \,|\, a_\mu \neq 0\}$. In what follows the ring $\mathcal{E}$ is treated as a filtered ring with the ascending filtration $(\mathcal{E}_r)_{r\in\mathbb{Z}}$ such that $\mathcal{E}_r = 0$ if $r < 0$ and $\mathcal{E}_r = \{B \in \mathcal{E} \,|\, \mathrm{ord}\, B \leq r\}$ if $r \geq 0$.

If K is a Δ-σ-field, then a *difference-differential module* over K (also called a Δ-σ-K-module) is a left $\mathcal{D}$-module M, that is, a vector K-space where elements of $\Delta \bigcup \sigma$ act as additive mutually commuting operators such that $\delta(ax) = a(\delta x) + \delta(a)x$ and $\alpha(ax) = \alpha(a)\alpha x$ for any $\delta \in \Delta$, $\alpha \in \sigma$, $x \in M$, $a \in K$. We say that M is a finitely generated Δ-σ-K-module if M is finitely generated as a left $\mathcal{D}$-module.

Similarly, if K is a Δ-σ^*-field, then an *inversive difference-differential module* over K (also called a Δ-σ^*-K-module) is a left $\mathcal{E}$-module (that is, a Δ-σ-K-module M with the action of elements of σ^* such that $\alpha^{-1}(ax) = \alpha^{-1}(a)\alpha^{-1}x$ for every $\alpha \in \sigma$). A Δ-σ^*-K-module M is said to be finitely generated if it is generated as a left $\mathcal{E}$-module by a finite set whose elements are called Δ-σ^*-generators of M.

If M is a Δ-σ-K-module (respectively, a Δ-σ^*-module, if K is a Δ-σ^*-field), then by a filtration of M we mean an exhaustive and separated filtration of

M as a $\mathcal{D}$- (respectively, $\mathcal{E}$-) module, that is, an ascending chain $(M_r)_{r\in\mathbb{Z}}$ of vector K-subspaces of M such that $\mathcal{D}_r M_s \subseteq M_{r+s}$ (respectively, $\mathcal{E}_r M_s \subseteq M_{r+s}$) for all $r, s \in \mathbb{Z}$, $M_r = 0$ for all sufficiently small $r \in \mathbb{Z}$, and $\bigcup_{r\in\mathbb{Z}} M_r = M$. A filtration $(M_r)_{r\in\mathbb{Z}}$ of a Δ-σ-K- (respectively, Δ-σ^*-K) module M is said to be *excellent* if every M_r is a finite dimensional vector K-space and there exists $r_0 \in \mathbb{Z}$ such that $M_r = \mathcal{D}_{r-r_0} M_{r_0}$ (respectively, $M_r = \mathcal{E}_{r-r_0} M_{r_0}$) for any $r \geq r_0$. Clearly, if M is generated as a $\mathcal{D}$- (respectively, $\mathcal{E}$-) module by elements $x_1, \dots x_s$, then $(\sum_{i=1}^{s} \mathcal{D}_r x_i)_{r\in\mathbb{Z}}$ (respectively, $(\sum_{i=1}^{s} \mathcal{E}_r x_i)_{r\in\mathbb{Z}}$) is an excellent filtration of M; it is said to be the natural filtration associated with the set of generators $\{x_1, \dots, x_s\}$.

If M' and M'' are Δ-σ-K- (respectively, Δ-σ^*-K-) modules, then a mapping $f : M' \to M''$ is said to be a Δ-σ-homomorphism if it is a homomorphism of $\mathcal{D}$- (respectively, $\mathcal{E}$-) modules. If M' and M'' are equipped with filtrations $(M'_r)_{r\in\mathbb{Z}}$ and $(M''_r)_{r\in\mathbb{Z}}$, respectively, and $f(M'_r) \subseteq M''_r$ for every $r \in \mathbb{Z}$, then f is said to be a Δ-σ-homomorphism of filtered Δ-σ-K- (respectively, Δ-σ^*-K-) modules.

The following two statements are direct consequences of [6, Theorem 6.7.3] and [6, Theorem 6.7.10], respectively.

Theorem 3. *With the above notation, let K be a Δ-σ-field, M a finitely generated Δ-σ-K-module, and $(M_r)_{r\in\mathbb{Z}}$ the natural filtration associated with some finite system of generators of M over the ring of Δ-σ-operators $\mathcal{D}$. Then there is a numerical polynomial $\phi(t) \in \mathbb{Q}[t]$ such that:*

(i) $\phi(r) = \dim_K M_r$ *for all sufficiently large* $r \in \mathbb{Z}$.

(ii) $\deg \phi \leq m + n$ *and* $\phi(t)$ *can be written as* $\phi(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$ *where* $a_0, \dots, a_{m+n} \in \mathbb{Z}$.

(iii) $d = \deg \phi(t)$, a_n *and* a_d *do not depend on the finite set of generators of the $\mathcal{D}$-module M the filtration $(M_r)_{r\in\mathbb{Z}}$ is associated with. Furthermore, a_{m+n} is equal to the Δ-σ-dimension of M over K (denoted by Δ-σ-$\dim_K M$), that is, to the maximal number of elements $x_1, \dots, x_k \in M$ such that the family $\{\lambda x_i \mid \lambda \in \Lambda, 1 \leq i \leq k\}$ is linearly independent over K.*

Theorem 4. *Let $f : M' \to M''$ be an injective homomorphism of filtered Δ-σ-K-modules M' and M'' with filtrations $(M'_r)_{r\in\mathbb{Z}}$ and $(M''_r)_{r\in\mathbb{Z}}$, respectively. If the filtration of M'' is excellent, then the filtration of M' is excellent as well.*

PROOF OF THEOREM 2

Let $L = K\langle \eta_1, \dots, \eta_s \rangle$ be a Δ-σ-field extension of a Δ-σ^*-field K. Let L^* be the inversive closure of L, that is, $L^* = K\langle \eta_1, \dots, \eta_s \rangle^*$. Let $M = \Omega_{L^*|K}$, the module of Kähler differentials associated with the extension L^*/K. Then M can be treated as a Δ-σ^*-L^*-module where the action of the elements of $\Delta \bigcup \sigma^*$ is defined in such a way that $\delta(d\zeta) = d\delta(\zeta)$ and $\alpha(d\zeta) = d\alpha(\zeta)$ for any $\zeta \in L^*$, $\delta \in \Delta$, $\alpha \in \sigma^*$ (see [2] and [12, Lemma 4.2.8]).

For every $r \in \mathbb{N}$, let M_r denote the vector L^*-subspace of M generated by all elements $d\zeta$ where $\zeta \in K(\bigcup_{i=1}^{s} \Lambda^*(r)\eta_i)$. It is easy to check that $(M_r)_{r\in\mathbb{Z}}$ ($M_r = 0$ if $r < 0$) is the natural filtration of the Δ-σ^*-L^*-module M associated with the system of Δ-σ^*-generators $\{d\eta_1, \dots, d\eta_s\}$.

Let F be any intermediate Δ-σ-field of L/K, $F_r = F \bigcap K(\{\lambda\eta_j \mid \lambda \in \Lambda(r), 1 \leq j \leq s\})$ ($r \in \mathbb{N}$) and $F_r = 0$ if $r < 0$. Let $\mathcal{E}$ and $\mathcal{D}$ denote the ring of Δ-σ^*-operators over L^* and the ring of Δ-σ-operators over L, respectively. Let N be the $\mathcal{D}$-submodule of M generated by all elements of the form $d\zeta$ with $\zeta \in F$ (by $d\zeta$ we always mean $d_{L^*|K}\zeta$). Furthermore, for any $r \in \mathbb{N}$, let N_r be the vector L-space generated by all elements $d\zeta$ with $\zeta \in F_r$ and $N_r = 0$ if $r < 0$.

It is easy to see that $(N_r)_{r\in\mathbb{Z}}$ is a filtration of the Δ-σ-L-module N, and if $M' = \sum_{i=1}^{s} \mathcal{D}d\eta_i$, then the embedding $N \to M'$ is a homomorphism of filtered $\mathcal{D}$-modules. (M' is considered as a filtered $\mathcal{D}$-module with the excellent filtration $(\sum_{i=1}^{s} \mathcal{D}_r d\eta_i)_{r\in\mathbb{Z}}$.) By Theorem 4, $(N_r)_{r\in\mathbb{Z}}$ is an excellent filtration of the $\mathcal{D}$-module N. Applying Theorem 3 we obtain that there exists a polynomial $\chi_{K,F,\eta}(t) \in \mathbb{Q}[t]$ such that $\chi_{K,F,\eta}(t)(r) = \dim_K N_r$ for all sufficiently large $r \in \mathbb{Z}$.

As it is shown in [17, Chapter V, Section 23], elements $\zeta_1, \dots, \zeta_k \in L^*$ are algebraically independent over K if and only if the elements $d\zeta_1, \dots, d\zeta_k$ are linearly independent over L^*. Thus, if $\zeta_1, \dots, \zeta_k \in F_r$ ($r \in \mathbb{Z}$) are algebraically independent over K, then the elements $d\zeta_1, \dots, d\zeta_k \in N_r$ are linearly independent over L^* and therefore over L. Conversely, if elements $dx_1, \dots, dx_h$ ($x_i \in F_r$ for $i = 1, \dots, h$) are linearly independent over L, then $x_1, \dots, x_h$ are algebraically independent over K. Otherwise, we would have a polynomial $f(X_1, \dots, X_h) \in K[X_1, \dots, X_h]$ of the smallest possible degree such that $f(x_1, \dots, x_h) = 0$. Then $df(x_1, \dots, x_h) = \sum_{i=1}^{h} \frac{\partial f}{\partial X_i}(x_1, \dots, x_h)dx_i = 0$ where not all coefficients of dx_i are zeros (they are expressed by polynomials of degree less than $\deg f$). Since all the coefficients lie in L, we would have a contradiction with the linear independence of $dx_1, \dots, dx_h$ over L.
It follows that $\dim_L N_r = \operatorname{trdeg}_K F_r$ for all $r \in \mathbb{N}$. Applying Theorem 3 we obtain the statement of Theorem 2. $\square$

Clearly, if $F = L$, then Theorem 2 implies Theorem 1. Note also that if an intermediate field F of a finitely generated Δ-σ-field extension L/K is not a Δ-σ-subfield of L, there might be no numerical polynomial whose values for sufficiently large integers r are equal to $\operatorname{trdeg}_K(F \bigcap K(\{\lambda\eta_j \mid \lambda \in \Lambda(r), 1 \leq j \leq s\}))$. Indeed, let $\Delta = \{\delta\}$ and $\sigma = \emptyset$. Let $L = K\langle y\rangle$, where the Δ-σ-generator y is Δ-σ-independent over K, and let $F = K(\delta^2 y, \dots, \delta^{2k} y, \dots)$. Then $\Lambda = \{\delta^i \mid i \in \mathbb{N}\}$, $\Lambda(r) = \{1, \delta, \dots, \delta^r\}$, $F_r = F \bigcap K(\lambda y \mid \lambda \in \Lambda(r))$ and $\operatorname{trdeg}_K F_r = [\frac{r}{2}]$ (the integer part of $\frac{r}{2}$), which is not a polynomial of r. In this case, the function $\phi(r) = \operatorname{trdeg}_K F_r$ is a quasi-polynomial, but if one takes $F = K(\delta^2 y, \dots, \delta^{2^k} y, \dots)$, then $\operatorname{trdeg}_K F_r = [\log_2 r]$.

4 Type and dimension of difference-differential field extensions

Let K be an inversive difference-differential (Δ-σ-) field with a basic set $\Delta\bigcup\sigma$ where $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ are the sets of derivations and automorphisms of K, respectively. Let $L = K\langle\eta_1, \dots, \eta_s\rangle$ be a Δ-σ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. (We keep the notation introduced in section 2.)

Let $\mathfrak{U}$ denote the set of all intermediate Δ-σ-fields of the extension L/K and

$$\mathfrak{B}_{\mathfrak{U}} = \{(F, E) \in \mathfrak{U} \times \mathfrak{U} \,|\, F \supseteq E\}.$$

Furthermore, let $\overline{\mathbb{Z}}$ denote the ordered set $\mathbb{Z}\bigcup\{\infty\}$ (where the natural order on $\mathbb{Z}$ is extended by the condition $a < \infty$ for any $a \in \mathbb{Z}$).

Proposition 2. *With the above notation, there exists a unique mapping* $\mu_{\mathfrak{U}} : \mathfrak{B}_{\mathfrak{U}} \to \overline{\mathbb{Z}}$ *such that*

(i) $\mu_{\mathfrak{U}}(F, E) \geq -1$ *for any pair* $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$.

(ii) *If* $d \in \mathbb{N}$*, then* $\mu_{\mathfrak{U}}(F, E) \geq d$ *if and only if* $\operatorname{trdeg}_E F > 0$ *and there exists an infinite descending chain of intermediate* Δ-σ*-fields*

$$F = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_r \supseteq \cdots \supseteq E \tag{1}$$

such that

$$\mu_{\mathfrak{U}}(F_i, F_{i+1}) \geq d - 1 \quad (i = 0, 1, \dots). \tag{2}$$

Proof. In order to show the existence and uniqueness of the desired mapping $\mu_{\mathfrak{U}}$, one can just mimic the proof of the corresponding statement for chains of prime differential ideals given in [3, Section 1] (see also [11, Proposition 4.1] and [13, Section 4] where similar arguments were applied to differential and inversive difference field extensions, respectively). Namely, let us set $\mu_{\mathfrak{U}}(F, E) = -1$ if $F = E$ or the field extension F/E is algebraic. If $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$, $\operatorname{trdeg}_E F > 0$ and for every $d \in \mathbb{N}$, there exists a chain of intermediate Δ-σ-fields (1) with condition (2), we set $\mu_{\mathfrak{U}}(F, E) = \infty$. Otherwise, we define $\mu_{\mathfrak{U}}(F, E)$ as the maximal integer d for which condition (ii) holds (that is, $\mu_{\mathfrak{U}}(F, E) \geq d$). It is clear that the mapping $\mu_{\mathfrak{U}}$ defined in this way is unique. $\square$

With the notation of the last proposition, we define the *type* of a Δ-σ-field extension L/K as the integer

$$\operatorname{type}(L/K) = \sup\{\mu_{\mathfrak{U}}(F, E) \,|\, (F, E) \in \mathfrak{B}_{\mathfrak{U}}\}. \tag{3}$$

and the *dimension* of the Δ-σ-extension L/K as the number

$\dim(L/K) = \sup\{q \in \mathbb{N} \,|\,$ there exists a chain $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_q$ such that $F_i \in \mathfrak{U}$ and

$$\mu_{\mathfrak{U}}(F_{i-1}, F_i) = \operatorname{type}(L/K) \qquad (i = 1, \dots, q)\}. \tag{4}$$

It is easy to see that for any pair of intermediate Δ-σ-fields of L/K such that $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$, $\mu_{\mathfrak{U}}(F, E) = -1$ if and only if the field extension E/F is algebraic. It is also clear that if $\operatorname{type}(L/K) < \infty$, then $\dim(L/K) > 0$.

Proposition 3. *With the above notation, let F and E be intermediate Δ-σ-fields of a Δ-σ-field extension $L = K\langle \eta_1, \dots, \eta_s\rangle$ generated by a finite set $\eta = \{\eta_1, \dots, \eta_s\}$. Let $F \supseteq E$, so that $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$. Then for any integer $d \geq -1$, the inequality $\mu_{\mathfrak{U}}(F, E) \geq d$ implies the inequality $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) \geq d$. ($\chi_{K,F,\eta}(t)$ and $\chi_{K,E,\eta}(t)$ are the Δ-σ-dimensions polynomials of the fields F and E associated with the set of Δ-σ-generators η of L/K.)*

Proof. We proceed by induction on d. Since $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) \geq -1$ for any pair $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$ and $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) \geq 0$ if $\operatorname{trdeg}_E F > 0$, our statement is true for $d = -1$ and $d = 0$. (As usual we assume that the degree of the zero polynomial is -1.)

Let $d > 0$ and let the statement be true for all nonnegative integers less than d. Let $\mu_{\mathfrak{U}}(F, E) \geq d$ for some pair $(F, E) \in \mathfrak{B}_{\mathfrak{U}}$, so that there exists a chain of intermediate Δ-σ-fields (1) such that $\mu_{\mathfrak{U}}(F_i, F_{i+1}) \geq d - 1$ $(i = 0, 1, \dots)$. If $\deg(\chi_{K,F_i,\eta}(t) - \chi_{K,F_{i+1},\eta}(t)) \geq d$ for some $i \in \mathbb{N}$, then $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) \geq \deg(\chi_{K,F_i,\eta}(t) - \chi_{K,F_{i+1},\eta}(t)) \geq d$, so the statement of the proposition is true.

Suppose that $\deg(\chi_{K,F_i,\eta}(t) - \chi_{K,F_{i+1},\eta}(t)) = d - 1$ for every $i \in \mathbb{N}$, that is, $\chi_{K,F_i,\eta}(t) - \chi_{K,F_{i+1},\eta}(t) = \sum_{j=0}^{d-1} a_j^{(i)} \binom{t+j}{j}$ where $a_0^{(1)}, \dots, a_{d-1}^{(i)} \in \mathbb{Z}$, $a_{d-1}^{(i)} > 0$. Then

$$\chi_{K,F,\eta}(t) - \chi_{K,F_{i+1},\eta}(t) = \sum_{k=0}^{i} (\chi_{K,F_k,\eta}(t) - \chi_{K,F_{k+1},\eta}(t)) = \sum_{j=0}^{d-1} b_j^{(i)} \binom{t+j}{j}$$

where $b_0^{(i)}, \dots, b_{d-1}^{(i)} \in \mathbb{Z}$ and $b_{d-1}^{(i)} = \sum_{k=0}^{i} a_{d-1}^{(k)}$. Therefore, $b_{d-1}^{(0)} < b_{d-1}^{(1)} < \dots$ and $\lim_{i\to\infty} b_{d-1}^{(i)} = \infty$. On the other hand, $\deg(\chi_{K,F,\eta}(t) - \chi_{K,F_{i+1},\eta}(t)) \leq \deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t))$. If $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) = d-1$, that is, $\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t) = \sum_{j=0}^{d-1} c_j \binom{t+j}{j}$ for some $c_0, \dots, c_{d-1} \in \mathbb{Z}$, then we would have $b_{d-1}^{(i)} < c_{d-1}$ for all $i \in \mathbb{N}$ contrary to the fact that $\lim_{i\to\infty} b_{d-1}^{(i)} = \infty$. Thus, $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) \geq d$, so the proposition is proved. $\square$

The following theorem provides a relationship between the introduced characteristics of a finitely generated Δ-σ-extension and the invariants of its Δ-σ-dimension polynomial introduced by Theorem 2.

Theorem 5. *Let K be an inversive difference-differential (Δ-σ-) field with basic set $\Delta \bigcup \sigma$ where $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ are the sets of derivations and automorphisms of K, respectively. Let L be a finitely generated Δ-σ-field extension of K. Then*

(i) $\operatorname{type}(L/K) \leq \Delta\text{-}\sigma\text{-}\operatorname{type}_K L \leq m + n$.

(ii) *If* $\Delta\text{-}\sigma\text{-}\operatorname{trdeg}_K L > 0$, *then* $\operatorname{type}(L/K) = m+n$, $\dim(L/K) = \Delta\text{-}\sigma\text{-}\operatorname{trdeg}_K L$.

(iii) *If* $\Delta\text{-}\sigma\text{-}\operatorname{trdeg}_K L = 0$, *then* $\operatorname{type}(L/K) < m + n$.

Proof. Let $\eta = \{\eta_1, \dots, \eta_s\}$ be a system of Δ-σ-generators of L over K and for every $r \in \mathbb{N}$, let $L_r = K(\{\lambda\eta_i \mid \lambda \in \Lambda(r), 1 \leq i \leq s\})$. Furthermore, if F is any intermediate Δ-σ-field of the extension L/K, then F_r ($r \in \mathbb{N}$) will denote the field $F \bigcap L_r$. By Theorem 2, there is a polynomial $\chi_{K,F,\eta}(t) \in \mathbb{Q}[t]$ such that $\chi_{K,F,\eta}(r) = \operatorname{trdeg}_K F_r$ for all sufficiently large $r \in \mathbb{N}$, deg $\chi_{K,F,\eta} \leq m+n$, and this polynomial can be written as $\chi_{K,F,\eta}(t) = \sum_{i=1}^{m+n} a_i \binom{t+i}{i}$ where $a_0, \dots, a_{m+n} \in \mathbb{Z}$ and $a_{m+n} = \Delta\text{-}\sigma\text{-}\operatorname{trdeg}_K F$. Clearly, if E and F are two intermediate Δ-σ-fields of L/K and $F \supseteq E$, then $\chi_{K,F,\eta}(t) \geq \chi_{K,E,\eta}(t)$. (This inequality means that $\chi_F(r) \geq \chi_E(r)$ for all sufficiently large $r \in \mathbb{N}$. As it is first shown in [18], the set W of all differential dimension polynomials of finitely generated differential field extensions is well ordered with respect to this ordering. At the same time, as it is proved in [6, Chapter 2], W is also the set of all Δ-σ-dimension polynomials associated with finitely generated Δ-σ-field extensions).

Note that if $F \supseteq E$ and $\chi_{K,F,\eta}(t) = \chi_{K,E,\eta}(t)$, then the field extension F/E is algebraic. Indeed, if $x \in F$ is transcendental over E, then there exists $r_0 \in \mathbb{N}$ such that $x \in F_r$ for all $r \geq r_0$. Therefore, $\operatorname{trdeg}_K F_r = \operatorname{trdeg}_K E_r + \operatorname{trdeg}_{E_r} F_r > \operatorname{trdeg}_K E_r$ for all $r \geq r_0$ hence $\chi_{K,F,\eta}(t) > \chi_{K,E,\eta}(t)$ contrary to our assumption.

Since $\deg(\chi_{K,F,\eta}(t) - \chi_{K,E,\eta}(t)) \leq m+n$ for any pair $(F,E) \in \mathfrak{B}_{\mathfrak{U}}$, the last proposition implies that $\operatorname{type}(L/K) \leq \Delta\text{-}\sigma\text{-}\operatorname{type}_K L \leq m+n$. If $\Delta\text{-}\sigma\text{-}\operatorname{trdeg}_K L = 0$, then $\operatorname{type}(L/K) \leq \Delta\text{-}\sigma\text{-}\operatorname{type}_K L < m+n$. Thus, it remains to prove statement (ii) of the theorem.

Let $\Delta\text{-}\sigma\text{-}\operatorname{trdeg}_K L > 0$, let element $x \in L$ be Δ-σ-transcendental over K and let $F = K\langle x \rangle$. Clearly, in order to prove that $\operatorname{type}(L/K) = m+n$ it is sufficient to show that $\mu_{\mathfrak{U}}(F,K) \geq m+n$. This inequality, in turn, immediately follows from the consideration of the following $m+n$ strictly descending chains of intermediate Δ-σ-fields of F/K.

$$F = K\langle x \rangle \supset K\langle \delta_1 x \rangle \supset K\langle \delta_1^2 x \rangle \supset \dots \supset K\langle \delta_1^{i_1} x \rangle \supset K\langle \delta_1^{i_1+1} x \rangle \supset \dots \supset K\,,$$

$$K\langle \delta_1^{i_1} x \rangle \supset K\langle \delta_1^{i_1+1} x, \delta_1^{i_1}\delta_2 x \rangle \supset K\langle \delta_1^{i_1+1} x, \delta_1^{i_1}\delta_2^2 x \rangle \supset \dots K\langle \delta_1^{i_1+1} x, \delta_1^{i_1}\delta_2^{i_2} x \rangle \supset K\langle \delta_1^{i_1+1} x, \delta_1^{i_1}\delta_2^{i_2+1} x \rangle \supset \dots \supset K\langle \delta_1^{i_1+1} x \rangle\,,$$

$$\dots$$

$$K\langle \delta_1^{i_1+1} x, \delta_1^{i_1+1}\delta_2^{i_2+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m} x \rangle \supset K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m}(\alpha_1-1) x \rangle \supset$$
$$\supset \dots \supset K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m}(\alpha_1-1)^2 x \rangle \supset$$
$$\supset \dots \supset K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m}(\alpha_1-1)^{i_{m+1}} x \rangle \supset$$
$$\dots \supset K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m}(\alpha_1-1)^{i_{m+1}+1} x \rangle \supset$$
$$\dots \supset K\langle \delta_1^{i_1+1} x, \delta_1^{i_1+1}\delta_2^{i_2+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m+1} x \rangle\,,$$

$$\dots$$

$$K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m+1}(\alpha_1-1)^{i_{m+1}+1} \dots (\alpha_{n-1}-1)^{i_{m+n-1}} x \rangle \supset$$
$$K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m}(\alpha_1-1)^{i_{m+1}+1} \dots (\alpha_{n-1}-1)^{i_{m+n-1}+1}(\alpha_n-1) x \rangle \supset \dots \supset K\langle \delta_1^{i_1+1} x, \dots, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \dots \delta_{m-1}^{i_{m-1}+1}\delta_m^{i_m}(\alpha_1-1)^{i_{m+1}+1} \dots (\alpha_{n-1}-1)^{i_{m+n-1}+1}(\alpha_n-1)^{i_{m+n}} x \rangle \supset \dots \supset K\langle \delta_1^{i_1+1} x,$$

$\ldots, \delta_1^{i_1+1} \ldots \delta_{m-1}^{i_{m-1}+1} x, \delta_1^{i_1+1} \ldots \delta_{m-1}^{i_{m-1}+1} \delta_m^{i_m} (\alpha_1 - 1)^{i_{m+1}+1} \ldots (\alpha_{n-1} - 1)^{i_{m+n-1}+1} x \rangle$.

These $m+n$ chains show that $\mu_{\mathfrak{U}}(F, K) \geq m+n$, hence $\operatorname{type}(L/K) = m+n$. Furthermore, if Δ-σ-$\operatorname{trdeg}_K L = k > 0$ and $x_1, \ldots, x_k$ is a Δ-σ-transcendence basic of L over K, then every x_i ($2 \leq i \leq k$) is Δ-σ-independent over $K\langle x_1, \ldots, x_{i-1} \rangle$. Therefore, the above chains show that $\mu_{\mathfrak{U}}(K\langle x_1 \rangle, K) = \mu_{\mathfrak{U}}(K\langle x_1, x_2 \rangle, K\langle x_1 \rangle) = \cdots = \mu_{\mathfrak{U}}(K\langle x_1, \ldots, x_k \rangle, K\langle x_1, \ldots, x_{k-1} \rangle) = m+n$, hence $\dim(L/K) \geq k = \Delta$-$\sigma$-$\operatorname{trdeg}_K L$.

In order to prove the opposite inequality, suppose that $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_p$ is an ascending chain of intermediate Δ-σ-fields of the extension L/K such that $\mu_{\mathfrak{U}}(F_i, F_{i+1}) = \operatorname{type}(L/K) = m+n$ for $i = 0, \ldots, p-1$. Clearly, in order to prove our inequality, it is sufficient to show that $p \leq k$.

For every $i = 0, \ldots, p$, the Δ-σ-dimension polynomial $\chi_{K,F_i,\eta}(t)$, whose existence is established by Theorem 2, can be written as $\chi_{K,F_i,\eta}(t) = \sum_{j=0}^{m+n} a_j^{(i)} \binom{t+j}{j}$ where $a_j^{(i)} \in \mathbb{Z}$ ($0 \leq i \leq p-1$, $0 \leq j \leq m+n$). Then $\chi_{K,F_0,\eta}(t) - \chi_{K,F_p,\eta}(t) = \sum_{i=1}^{p} (\chi_{K,F_{i-1},\eta}(t) - \chi_{K,F_i,\eta}(t)) = \sum_{i=1}^{p} \sum_{j=0}^{m+n} (a_j^{(i-1)} - a_j^{(i)}) \binom{t+j}{j} =$ $(a_{m+n}^{(0)} - a_{m+n}^{(p)}) \binom{t+m+n}{m+n} + o(t^{m+n})$ where $o(t^{m+n})$ denotes a polynomial of degree at most $m+n-1$.

Since $\mu_{\mathfrak{U}}(F_i, F_{i+1}) = m+n$ ($0 \leq i \leq p-1$), we have $\deg(\chi_{K,F_i,\eta}(t) - \chi_{K,F_{i+1},\eta}(t)) = m+n$ (see Proposition 3). Therefore, $a_{m+n}^{(0)} > a_{m+n}^{(1)} > \cdots > a_{m+n}^{(p)}$, hence

$$a_{m+n}^{(0)} - a_{m+n}^{(q)} = \sum_{i=1}^{p} (a_{m+n}^{(i-1)} - a_{m+n}^{(i)}) \geq p.$$

On the other hand, $\chi_{K,F_0,\eta}(t) - \chi_{K,F_p,\eta}(t) \leq \chi_{K,L,\eta}(t) = \sum_{i=0}^{m+n} a_i \binom{t+i}{i}$ where $a_{m+n} = \Delta$-σ-$\operatorname{trdeg}_K L$. Therefore, $p \leq a_{m+n}^{(0)} - a_{m+n}^{(p)} \leq k = \sigma$-$\operatorname{trdeg}_K L$. This completes the proof of the theorem. □

5 Multivariate dimension polynomials of intermediate difference-differential field extensions

In this section we present a result that generalizes both Theorem 2 and the theorem on multivariate dimension polynomial of a finitely generated differential field extension associated with a partition of the basic set of derivations, see [9, Theorem 4.6].

Let K be a difference-differential (Δ-σ-) field with basic sets $\Delta = \{\delta_1, \ldots, \delta_m\}$ and $\sigma = \{\alpha_1, \ldots, \alpha_n\}$ of derivations and automorphisms, respectively. Suppose that these sets are represented as the unions of p and q nonempty disjoint subsets, respectively ($p, q \geq 1$):

$$\Delta = \Delta_1 \bigcup \cdots \bigcup \Delta_p, \;\; \sigma = \sigma_1 \bigcup \cdots \bigcup \sigma_q, \tag{5}$$

$\Delta_1 = \{\delta_1, \dots, \delta_{m_1}\}$, $\Delta_2 = \{\delta_{m_1+1}, \dots, \delta_{m_1+m_2}\}$, $\dots$, $\Delta_p = \{\delta_{m_1+\cdots+m_{p-1}+1}, \dots, \delta_m\}$, $\sigma_1 = \{\alpha_1, \dots, \alpha_{n_1}\}$, $\sigma_2 = \{\alpha_{n_1+1}, \dots, \alpha_{n_1+n_2}\}$, $\dots$, $\sigma_q = \{\alpha_{n_1+\cdots+n_{q-1}+1}, \dots, \alpha_n\}$; $(m_1 + \cdots + m_p = m;\ n_1 + \cdots + n_q = n)$.

For any element $\lambda = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \in \Lambda$ ($k_i, l_j \in \mathbb{N}$; we use the notation of section 2), the *order of λ with respect to a set Δ_i ($1 \leq i \leq p$)* is defined as $\sum\limits_{\mu=m_1+\cdots+m_{i-1}+1}^{m_1+\cdots+m_i} k_\mu$; it is denoted by $\operatorname{ord}_i \lambda$. (If $i = 1$, the last sum is replaced by $\sum\limits_{\mu=1}^{m_1} k_\mu$.) Similarly, the order of λ with respect to a set σ_j ($1 \leq j \leq q$), denoted by $\operatorname{ord}'_j \lambda$, is defined as $\sum\limits_{\nu=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j} l_\nu$. (If $j = 1$, the last sum is $\sum\limits_{\nu=1}^{n_1} l_\nu$.)

If $r_1, \dots, r_{p+q} \in \mathbb{N}$, we set

$$\Lambda(r_1, \dots, r_{p+q}) = \{\lambda \in \Lambda \mid \operatorname{ord}_i \lambda \leq r_i \ (1 \leq i \leq p) \text{ and } \operatorname{ord}'_j \lambda \leq r_{p+j} (1 \leq j \leq q)\}.$$

Furthermore, for any permutation $(j_1, \dots, j_{p+q})$ of the set $\{1, \dots, p+q\}$, let $<_{j_1,\dots,j_{p+q}}$ be the lexicographic order on $\mathbb{N}^{p+q}$ such that $(r_1, \dots, r_{p+q}) <_{j_1,\dots,j_{p+q}} (s_1, \dots, s_{p+q})$ if and only if either $r_{j_1} < s_{j_1}$ or there exists $k \in \mathbb{N}$, $1 \leq k \leq p+q$, such that $r_{j_\nu} = s_{j_\nu}$ for $\nu = 1, \dots, k$ and $r_{j_{k+1}} < s_{j_{k+1}}$.

If $A \subseteq \mathbb{N}^{p+q}$, then A' will denote the set of all $(p+q)$-tuples $a \in A$ that are maximal elements of this set with respect to one of the $(p+q)!$ orders $<_{j_1,\dots,j_{p+q}}$. Say, if $A = \{(1,1,1), (2,3,0), (0,2,3), (2,0,5), (3,3,1), (4,1,1), (2,3,3)\} \subseteq \mathbb{N}^3$, then $A' = \{(2,0,5), (3,3,1), (4,1,1), (2,3,3)\}$.

Theorem 6. *With the above notation, let F be an intermediate Δ-σ-field of a Δ-σ-field extension $L = K\langle \eta_1, \dots, \eta_s \rangle$ generated by a finite family $\eta = \{\eta_1, \dots, \eta_s\}$. Let partitions (5) be fixed and for any $r_1, \dots, r_{p+q} \in \mathbb{N}^{p+q}$, let*

$$F_{r_1,\dots,r_{p+q}} = F \bigcap K(\bigcup_{j=1}^{s} \Lambda(r_1, \dots, r_{p+q})\eta_j).$$

Then there exists a polynomial in $p+q$ variables $\Phi_{K,F,\eta} \in \mathbb{Q}[t_1, \dots, t_{p+q}]$ such that

(i) $\Phi_{K,F,\eta}(r_1, \dots, r_{p+q}) = \operatorname{trdeg}_K K(\bigcup\limits_{j=1}^{s} \Lambda(r_1, \dots, r_{p+q})\eta_j)$

for all sufficiently large $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$. (That is, there exist $r_1^{(0)}, \dots, r_{p+q}^{(0)} \in \mathbb{N}$ such that the equality holds for all $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$ with $r_i \geq r_i^{(0)}$, $1 \leq i \leq p+q$.) ;

(ii) *$\deg_{t_i} \Phi_\eta \leq m_i$ $(1 \leq i \leq p)$, $\deg_{t_{p+j}} \Phi_\eta \leq n_j$ $(1 \leq j \leq q)$ and $\Phi_\eta(t_1, \dots, t_{p+q})$ can be represented as*

$$\Phi_\eta = \sum_{i_1=0}^{m_1} \dots \sum_{i_p=0}^{m_p} \sum_{i_{p+1}=0}^{n_1} \dots \sum_{i_{p+q}=0}^{n_q} a_{i_1 \dots i_{p+q}} \binom{t_1 + i_1}{i_1} \dots \binom{t_{p+q} + i_{p+q}}{i_{p+q}} \tag{6}$$

where $a_{i_1 \dots i_{p+q}} \in \mathbb{Z}$.

(iii) *Let $E_\eta = \{(i_1, \dots, i_{p+q}) \in \mathbb{N}^{p+q} \,|\, 0 \leq i_k \leq m_k$ for $k = 1, \dots, p$, $0 \leq i_{p+j} \leq n_j$ for $j = 1, \dots, q$, and $a_{i_1 \dots i_{p+q}} \neq 0\}$. Then $d = \deg \Phi_\eta$, $a_{m_1 \dots m_p n_1 \dots n_q}$, elements $(k_1, \dots, k_{p+q}) \in E'_\eta$, the corresponding coefficients $a_{k_1 \dots k_{p+q}}$, and the coefficients of the terms of total degree d do not depend on the choice of the set of Δ-σ-generators η. Furthermore, $a_{m_1 \dots m_p n_1 \dots n_q} = \Delta$-$\sigma$-$\mathrm{trdeg}_K L$.*

Proof. We will mimic the method of the proof of Theorem 2 using the results on multivariate dimension polynomials of Δ-σ-L-modules. Let $\mathcal{D}$ be the ring of Δ-σ-operators over L considered as a filtered ring with $(p+q)$-dimensional filtration $\{\mathcal{D}_{r_1,\dots,r_{p+q}} \,|\, (r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q}\}$ where for any $r_1, \dots, r_{p+q} \in \mathbb{N}^{p+q}$, $\mathcal{D}_{r_1,\dots,r_{p+q}}$ is the vector L-subspace of $\mathcal{D}$ generated by $\Lambda(r_1, \dots, r_{p+q})$, and $\mathcal{D}_{r_1,\dots,r_{p+q}} = 0$ if at least one r_i is negative. If M is a Δ-σ-L-module, then a family $\{M_{r_1,\dots,r_{p+q}} | (r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q}\}$ of vector K-subspaces of M is said to be a $(p+q)$-dimensional filtration of M if

(i) $M_{r_1,\dots,r_{p+q}} \subseteq M_{s_1,\dots,s_{p+q}}$ whenever $r_i \leq s_i$ for $i = 1, \dots, p+q$.

(ii) $\bigcup_{(r_1,\dots,r_{p+q}) \in \mathbb{Z}^{p+q}} M_{r_1,\dots,r_{p+q}} = M$.

(iii) There exists $(r_1^{(0)}, \dots, r_{p+q}^{(0)}) \in \mathbb{Z}^p$ such that $M_{r_1,\dots,r_{p+q}} = 0$ if $r_i < r_i^{(0)}$ for at least one index i.

(iv) $\mathcal{D}_{r_1,\dots,r_{p+q}} M_{s_1,\dots,s_{p+q}} \subseteq M_{r_1+s_1,\dots,r_{p+q}+s_{p+q}}$ for any $(p+q)$-tuples $(r_1, \dots, r_{p+q})$, $(s_1, \dots, s_{p+q}) \in \mathbb{Z}^{p+q}$,

If every vector L-space $M_{r_1,\dots,r_{p+q}}$ is finite-dimensional and there exists an element $(h_1, \dots, h_p) \in \mathbb{Z}^p$ such that $\mathcal{D}_{r_1,\dots,r_{p+q}} M_{h_1,\dots,h_{p+q}} = M_{r_1+h_1,\dots,r_{p+q}+h_{p+q}}$ for any $(r_1, \dots, r_{p+q}) \in \mathbb{N}^{p+q}$, the filtration $\{M_{r_1,\dots,r_{p+q}} | (r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q}\}$ is called *excellent*. Clearly, if $z_1, \dots, z_k$ is a finite system of generators of a Δ-σ-L-module M, then $\{\sum_{i=1}^{k} \mathcal{D}_{r_1,\dots,r_{p+q}} z_i | (r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q}\}$ is an excellent $(p+q)$-dimensional filtration of M.

Let L^* be the inversive closure of L. As we have seen, the module of Kähler differentials $\Omega_{L^*|K}$ can be equipped with a structure of a Δ-σ^*-L-module such that $\beta(d\zeta) = d\beta(\zeta)$ for any $\zeta \in L^*$, $\beta \in \Delta \bigcup \sigma$ $(d = d_{L^*|K})$. Let M' denote a $\mathcal{D}$-submodule $\sum_{i=1}^{s} \mathcal{D} d\eta_i$ of M treated as a filtered $\mathcal{D}$-module with the natural $(p+q)$-dimensional filtration $\{M'_{r_1,\dots,r_{p+q}} | (r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q}\}$ where $M'_{r_1,\dots,r_{p+q}} = \sum_{i=1}^{s} \mathcal{D}_{r_1,\dots,r_{p+q}} d\eta_i$. Let N be a $\mathcal{D}$-submodule of M' generated by all elements $d\zeta$ where $\zeta \in F$ and for any $r_1, \dots, r_{p+q} \in \mathbb{N}$, let $N_{r_1,\dots,r_{p+q}}$ be the vector L-space generated by all elements $d\zeta$ where $\zeta \in F_{r_1,\dots,r_{p+q}}$. Setting $N_{r_1,\dots,r_{p+q}} = 0$ if $(r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q} \setminus \mathbb{N}^{p+q}$, we get a $(p+q)$-dimensional filtration of the Δ-σ-L-module N, and the embedding $N \to M'$ becomes a homomorphism of $(p+q)$-filtered Δ-σ-L-modules. Now, one can mimic the proof of Theorem 3.2.8 of [12] to show that the filtration $\{N_{r_1,\dots,r_{p+q}} | (r_1, \dots, r_{p+q}) \in \mathbb{Z}^{p+q}\}$

is excellent. The result of Theorem 6 immediately follows from the fact that $\dim_L N_{r_1,\dots,r_{p+q}} = \operatorname{trdeg}_K F_{r_1,\dots,r_{p+q}}$ for all $(r_1,\dots,r_{p+q}) \in \mathbb{N}^{p+q}$ (as it is mentioned in the proof of Theorem 2, a family $(\zeta_i)_{i\in I}$ of elements of L (in particular, of $F_{r_1,\dots,r_{p+q}}$) is algebraically independent over K if and only if the family $(d\zeta_i)_{i\in I}$ is linearly independent over L) and the result of [12, Theorem 3.5.8] (it states that under the above conditions, there exists a polynomial $\Phi_{K,F,\eta}(t_1,\dots,t_{p+q}) \in \mathbb{Q}[t_1,\dots,t_{p+q}]$ such that $\Phi_\eta(r_1,\dots,r_{p+q}) = \dim_L N_{r_1,\dots,r_{p+q}}$ for all sufficiently large $(r_1,\dots,r_{p+q}) \in \mathbb{Z}^{p+q}$ and $\Phi_{K,F,\eta}(t_1,\dots,t_{p+q})$ satisfies conditions (ii) of Theorem 6. Statement (iii) of Theorem 6 can be obtained in the same way as statement (iii) of Theorem 2 of [13].)

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