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Random sampling and reconstruction of concentrated signals in a reproducing kernel space

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ABSTRACT

In this paper, we consider (random) sampling of signals concentrated on a bounded Corkscrew domain Ω of a metric measure space, and reconstructing concentrated signals approximately from their (un)corrupted sampling data taken on a sampling set contained in Ω. We establish a weighted stability of bi-Lipschitz type for a (random) sampling scheme on the set of concentrated signals in a reproducing kernel space. The weighted stability of bi-Lipschitz type provides a weak robustness to the sampling scheme, however due to the nonconvexity of the set of concentrated signals, it does not imply the unique signal reconstruction. From (un)corrupted samples taken on a finite sampling set contained in Ω, we propose an algorithm to find approximations to signals concentrated on a bounded Corkscrew domain Ω. Random sampling is a sampling scheme where sampling positions are randomly taken according to a probability distribution. Next we show that, with high probability, signals concentrated on a bounded Corkscrew domain Ω can be reconstructed approximately from their uncorrupted (or randomly corrupted) samples taken at i.i.d. random positions drawn on Ω, provided that the sampling size is at least of the order $\mu(\Omega)\ln(\mu(\Omega))$, where $\mu(\Omega)$ is the measure of the concentrated domain Ω. Finally, we demonstrate the performance of proposed approximations to the original concentrated signals when the sampling procedure is taken either with large density or randomly with large size.

1. Introduction

Sampling signals of interest in a stable way and reconstructing the original signals exactly or approximately from their (un)corrupted sampling data are fundamental problems in sampling theory. A common assumption is that signals of interest have some additional properties, such as residing in a linear space, or having sparse representation in a dictionary, or having finite rate of innovation [5,16,19,26,40,46,50,53,56].

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In this paper, we consider (random) sampling and reconstruction of signals in a reproducing kernel space concentrated on a bounded Corkscrew domain.

Let (X, ρ, μ) be a metric measure space and $L^p := L^p(X, \rho, \mu), 1 \le p \le \infty$, be the linear space of all p-integrable functions on the metric measure space (X, ρ, μ) with the standard p-norm denoted by $\|\cdot\|_p$. In this paper, we use the range space

$$V_p := \{Tf,\ f \in L^p\} = \{f \in L^p,\ Tf = f\},\ 1 \le p \le \infty, \tag{1.1}$$

of an idempotent integral operator

$$Tf(x) = \int_X K(x,y) f(y) d\mu(y),\ f \in L^p, \tag{1.2}$$

as the reproducing kernel space for signals to reside in, where the integral kernel K having certain off-diagonal decay and Hölder continuity, see Assumption 2.5. The above range space $V_p, 1 \le p \le \infty$, was introduced by Nashed and Sun in the Euclidean setting [40], and it has rich geometric structure and lots of flexibility to approximate real data set in signal processing and learning theory. Our illustrative examples are spaces of p-integrable (non-)uniform splines [4,48,54], shift-invariant spaces with their generators having certain regularity and decay at infinity [5,8,53], and spaces of signals with finite rate of innovation [20,23,50,56]. Sampling and reconstruction of signals in the range spaces of integral operators in the Euclidean space has been well studied, see [20,34,40] and references therein. For signals in $V_p, 1 \le p \le \infty$, as shown in Proposition 2.7, they can be recovered exactly via an exponentially convergent algorithm from their samples taken on a sampling set with large density.

For some engineering applications, signals of interest are concentrated on a bounded domain Ω and only finitely many sampling data taken inside the domain Ω are available [1,11,12,20,31,33]. This motivates us to consider sampling and reconstruction of signals in the space $V_p, 1 \le p \le \infty$, concentrated on a bounded domain Ω,

$$V_{p,\Omega,\varepsilon} := \big\{f \in V_p,\ \|f\|_{p,\Omega^c} \le \varepsilon \|f\|_p\big\}, \tag{1.3}$$

where $\varepsilon \in (0,1)$, $\Omega^c \subseteq X$ is the complement of the domain Ω, and $\|f\|_{p,\Omega^c}$ is the standard p-norm on the complement Ω^c. The set $V_{p,\Omega,\varepsilon}$ of ε-concentrated signals has been introduced for time-frequency analysis [28,55], phase retrieval [2,3,30], and (random) sampling of bandlimited and wavelet signals [11,12,20,27,59]. As signals in $V_{p,\Omega,\varepsilon}$ are essentially supported on the domain Ω, it is more natural to consider a sampling procedure taken on a finite sampling set Γ_Ω contained inside the domain Ω only. In Section 3, we show that the sampling procedure $f \mapsto (f(\gamma))_{\gamma\in\Gamma_\Omega}$ for ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ has weighted stability of bi-Lipschitz type when the Hausdorff distance

$$d_H(\Gamma_\Omega, \Omega) := \sup_{x\in\Omega} \rho(x, \Gamma_\Omega)$$

between the sampling set $\Gamma_\Omega \subset \Omega$ and the bounded Corkscrew domain Ω is small, see Theorem 3.1 and Corollary 3.2. For signals in a linear space, stability of a sampling scheme guarantees robustness and uniqueness of reconstructing signals from their (noisy) samples, see [5,20,40,51]. However, the weighted stability in Theorem 3.1 does not imply the unique reconstruction even it provides a weak robustness for the sampling scheme on $V_{p,\Omega,\varepsilon}$, see [11] and Remark 3.1. Therefore we should consider reconstructing ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ approximately, instead of exactly, from their samples inside the domain. A challenge to derive such good approximations to ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ is that the set $V_{p,\Omega,\varepsilon}$ is a nonconvex subset of the reproducing kernel space V_p (and hence it is not a linear space), which prevents the direct application of reconstruction algorithms used for signals in a linear space [4,5,9,20,25,40,50,54,58]. In Theorems 3.3 and 3.4, we propose an algorithm to construct suboptimal approximations to ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ from their (un)corrupted samples taken on a finite sampling set $\Gamma_\Omega \subset \Omega$.

Random sampling is a sampling scheme where sampling positions are randomly taken according to a probability distribution [15,39,57]. It has received a lot of attention in the communities of signal processing, compressive sensing, learning theory and sampling theory, see [13,16,17,21,36,38,41,43,45,46,60] and references therein. Random sampling of concentrated signals was first discussed by Bass and Gröchenig, and they proved the following result for bandlimited signals concentrated on the cube $C_R := [-R/2, R/2]^d$, see [12, Theorem 3.1].

Theorem 1.1. *Let $R \geq 2$, $\varepsilon \in (0,1)$ and $\mu \in (0, 1-\varepsilon)$. If sampling positions $\gamma \in \Gamma_R$ are i.i.d. random variables that are uniformly distributed over the cube C_R, then there exist absolute positive constants A and B such that the sampling inequalities*

$$\frac{N}{R^d}(1-\varepsilon-\mu)\|f\|_2^2 \leq \sum_{\gamma\in\Gamma_R} |f(\gamma)|^2 \leq \frac{N}{R^d}(1+\mu)\|f\|_2^2, \; f \in \mathcal{B}_{\varepsilon,R} \tag{1.4}$$

hold with probability at least $1 - A\exp(-B\mu^2 N/R^d)$, where $N = \#\Gamma_R$ is the size of the sampling set Γ_R, $\mathcal{B}$ is the space of signals bandlimited to $[-1/2, 1/2]^d$, and $\mathcal{B}_{\varepsilon,R} = \{f \in \mathcal{B}, \; \|f\|_{2,\mathbb{R}^d\setminus C_R} \leq \sqrt{\varepsilon}\|f\|_2\}$ is the set of bandlimited signals concentrated on C_R.

The sampling inequality of the form (1.4) has been extended to signals in a shift-invariant space, with finite rate of innovation and in a reproducing kernel space on the Euclidean space $\mathbb{R}^d$, see [11,27,34,37,42,59]. In this paper, we introduce a completely different approach to obtain a weighted sampling inequality, see Theorem 4.1 and Remarks 4.1 and 4.2. The sampling inequality of the form (1.4) provides an estimate to the signal energy with high probability, however it does not yield a stable reconstruction of ε-concentrated signals in a reproducing kernel space. To the best of our knowledge, there is no algorithm available to perform the reconstruction of ε-concentrated signals in a reproducing kernel space approximately from their random samples in the considered domain. In Theorems 4.3 and 4.6, we show that the algorithm proposed in Theorem 3.3 provides good approximations to the ε-concentrated signal from its uncorrupted (or randomly corrupted) random samples, with high probability, when the sampling size is large enough.

The main contributions of this paper are as follows: (i) We consider sampling and reconstruction of signals concentrated on a bounded Corkscrew domain Ω of a metric measure space, instead of signals concentrated on a cube $[-R/2, R/2]^d$ of the d-dimensional Euclidean space in the literature [11,12,27,34,37,59]. (ii) For a (random) sampling scheme for signals concentrated on a bounded Corkscrew domain, we establish a weighted stability of bi-Lipschitz type instead of the sampling inequality of the form (1.4), which provides weak robustness of the sampling scheme. (iii) The set of ε-concentrated signals is nonconvex and the (random) sampling operator is not one-to-one in general. We propose an algorithm to construct suboptimal approximations to the original ε-concentrated signals from their (random) samples on the considered domain. (iv) We show that, with high probability, signals concentrated on a bounded Corkscrew domain Ω can be reconstructed approximately from their samples taken at i.i.d. random positions drawn on Ω, provided that the sampling size is at least of the order $\mu(\Omega)\ln(\mu(\Omega))$, where $\mu(\Omega)$ is the measure of the concentrated domain Ω. (v) We show that with high probability, an original ε-concentrated signal can be reconstructed approximately from its random samples corrupted by i.i.d. random noises, when the random sampling size is large enough.

The paper is organized as follows. In Section 2, we present some preliminaries on Corkscrew domains Ω of a metric measure space on which sampling is taken, and reproducing kernel spaces $V_p, 1 \leq p \leq \infty$, in which ε-concentrated signals on a Corkscrew domain Ω reside. In Section 3, we consider the sampling procedure $f \mapsto (f(\gamma))_{\gamma\in\Gamma_\Omega}$ taken on a finite sampling set Γ_Ω contained in a Corkscrew domain Ω for ε-concentrated signals f in the reproducing kernel space V_p. We establish the stability of bi-Lipschitz type for the above sampling procedure in Theorem 3.1, and we construct suboptimal approximations to the original ε-concentrated signals from their (un)corrupted samples, see Theorems 3.3 and 3.4. In Section 4, we consider random sampling of ε-concentrated signals in the reproducing kernel space V_p with large sampling size, and we show that, with high probability, any ε-concentrated signal can be reconstructed approximately from its (un)corrupted samples taken randomly on the Corkscrew domain Ω, see Theorems 4.3 and 4.6. In Section 5, we demonstrate the performance of the proposed approximations to the original ε-concentrated signals when the sampling procedure is taken either with sufficient density or randomly with large size. In Section 6, we include the proofs of all theorems and propositions.

2. Preliminaries on Corkscrew domains and reproducing kernel spaces

In this section, we present some preliminaries on Corkscrew domains Ω of a metric measure space (X,ρ,μ) and the range space V_p of an idempotent integral operator T for ε-concentrated signals on Ω to reside in. Our illustrative model of Corkscrew domains is Lipschitz domains in $\mathbb{R}^d$, such as rectangular regions $[-R/2,R/2]^d$ with side length $R\ge 1$ or balls $B(0,R)$ with center at the origin and radius $R\ge 1$. For a bounded Corkscrew domain Ω with $\mathrm{diam}(\partial\Omega)\ge 1$, we observe that for any $\delta\in(0,1)$ there exist a finite set Ω_δ and a disjoint partition $I_\gamma, \gamma\in\Omega_\delta$, of the domain Ω with the property that

$$B(\gamma,c\delta)\subset I_\gamma\subset B(\gamma,\delta)\ \text{ for all } \gamma\in\Omega_\delta, \tag{2.1}$$

where $c\in(0,1)$ is an absolute constant, see Proposition 2.4. For the range space V_p with the integral kernel K having certain off-diagonal decay and Hölder continuity, we show that it is a reproducing kernel space and signals in V_p can be reconstructed from their samples by an exponentially convergent iterative algorithm, see Propositions 2.6 and 2.7.

2.1. Corkscrew domains in a metric measure space

A *metric* ρ on a set X is a function $\rho: X\times X\longmapsto[0,\infty)$ such that (i) $\rho(x,y)=0$ if and only if $x=y$; (ii) $\rho(x,y)=\rho(y,x)$ for all $x,y\in X$; and (iii) $\rho(x,y)\le\rho(x,z)+\rho(z,y)$ for all $x,y,z\in X$. A *metric measure space* (X,ρ,μ) is a metric space (X,ρ) with a non-negative Borel measure μ compatible with the topology generated by open balls $\{y\in X,\ \rho(x,y)<r\}$ with center $x\in X$ and radius $r>0$. For a metric measure space (X,ρ,μ), we denote the diameter of a set $Y\subset X$ by $\mathrm{diam}(Y)$, and define the closed ball with center $x\in X$ and radius $r\ge 0$ by

$$B(x,r):=\big\{y\in X,\ \rho(x,y)\le r\big\}.$$

In this paper, we always assume the following:

Assumption 2.1. The metric measure space (X,ρ,μ) has dimension $d>0$ in the sense that

$$D_1 r^d\le\mu\big(B(x,r)\big)\le D_2 r^d\quad\text{for all } x\in X \text{ and } 0\le r\le\mathrm{diam}(X), \tag{2.2}$$

where D_1 and D_2 are positive constants.

We call a Borel measure μ satisfying (2.2) to be *Ahlfors d-regular*, and denote the maximal lower bound and minimal upper bound in (2.2) by $D_1(\mu)$ and $D_2(\mu)$ respectively [22,52]. Our models of metric measure spaces are the Euclidean space $\mathbb{R}^d$, the sphere $S^d\subset\mathbb{R}^{d+1}$ and the torus $\mathbb{T}^d$.

For the metric measure space (X, ρ, μ), we can find a finite overlapping cover by balls $B(x_i, \delta), x_i \in X_\delta$, for all $\delta > 0$ such that $B(x_i, \delta/2), x_i \in X_\delta$, are mutually disjoint. In particular, given a dense subset $Y \subset X$, let X_δ be a maximal subset of Y such that $B(x_i, \delta/2), x_i \in X_\delta$, are mutually disjoint, i.e.,

$$B(x_i, \delta/2) \cap B(x_j, \delta/2) = \emptyset \ \text{ for all distinct } x_i, x_j \in X_\delta, \tag{2.3}$$

and

$$B(y, \delta/2) \cap \big(\cup_{x_i \in X_\delta} B(x_i, \delta/2) \big) \neq \emptyset \ \text{ for all } y \in Y. \tag{2.4}$$

Then one may verify that the above family of closed balls $\{B(x_i, \delta), x_i \in X_\delta\}$ covers the whole space X with finite overlapping,

$$1 \leq \sum_{x_i \in X_\delta} \chi_{B(x_i,\delta)}(x) \leq \frac{3^d D_2(\mu)}{D_1(\mu)}, \ x \in X, \tag{2.5}$$

where the first inequality holds as X_δ is closed and

$$\rho(x, X_\delta) = \inf_{y \in X_\delta} \rho(x, y) \leq \delta, \ x \in X$$

by (2.4), and the second one follows since

$$\begin{aligned} \sum_{x_i \in X_\delta} \chi_{B(x_i,\delta)}(x) &\leq \sum_{x_i \in B(x,\delta)} \frac{\mu(B(x_i, \delta/2))}{D_1(\mu)(\delta/2)^d} = \frac{\mu\big(\cup_{x_i \in B(x,\delta)} B(x_i, \delta/2)\big)}{D_1(\mu)(\delta/2)^d} \\ &\leq \frac{\mu(B(x, 3\delta/2))}{D_1(\mu)(\delta/2)^d} \leq \frac{3^d D_2(\mu)}{D_1(\mu)} \end{aligned}$$

by (2.2) and (2.3).

We say that a domain D of the metric measure space (X, ρ, μ) is a *Corkscrew domain* if any ball $B(x, r)$ with center at the boundary $x \in \partial D$ and radius $0 < r \leq$ diam ∂D contains one ball inside the domain with a fraction of radius,

$$B(y, cr) \subset D \cap B(x, r) \text{ for some } y \in D, \tag{2.6}$$

where $c \in (0, 1)$ is an absolute constant. Our illustrative model is Lipschitz domains in $\mathbb{R}^d$, such as rectangular regions $[-R/2, R/2]^d$ with side length $R \geq 1$ or balls $B(0, R)$ with center at the origin and radius $R \geq 1$. In this paper, we consider bounded Corkscrew domain Ω satisfying the following:

Assumption 2.2. The bounded domain Ω and its complement Ω^c satisfy the Corkscrew condition (2.6) and

$$\text{diam}(\partial\Omega) \geq 1.$$

For a bounded domain Ω satisfying the above assumption, we find a nice covering in the following proposition, see Section 6.1 for the proof, which plays a crucial role in our consideration of sampling and reconstruction of signals concentrated on the domain Ω.

Proposition 2.3. *Let (X, ρ, μ) be a d-dimensional metric measure space and Ω be a bounded Corkscrew domain satisfying Assumption 2.2. Then for any $\delta \in (0, 1)$ there exists a discrete set $\Omega_\delta \subset \Omega$ such that*

$$B(x_i, c\delta) \subset \Omega \quad \text{for all } x_i \in \Omega_\delta, \tag{2.7a}$$

$$B(x_i, \delta/2) \cap B(x_j, \delta/2) = \emptyset \quad \text{for all distinct } x_i, x_j \in \Omega_\delta, \tag{2.7b}$$

and

$$1 \le \sum_{x_i \in \Omega_\delta} \chi_{B(x_i, 5\delta)}(x) \le \frac{11^d D_2(\mu)}{D_1(\mu)} \quad \text{for all } x \in \Omega, \tag{2.7c}$$

where $D_1(\mu)$ and $D_2(\mu)$ are the maximal lower bound and minimal upper bound in (2.2) respectively, and c is the ratio in the Corkscrew condition (2.6) for the domain Ω.

Given a discrete set $\Gamma_\Omega \subset \Omega$, we say that $I_\gamma, \gamma \in \Gamma_\Omega$, is a *Voronoi partition* of the domain Ω if

$$\cup_{\gamma \in \Gamma_\Omega} I_\gamma = \Omega, \quad I_\gamma \cap I_{\gamma'} = \emptyset \quad \text{for all distinct } \gamma, \gamma' \in \Gamma_\Omega, \tag{2.8a}$$

and

$$I_\gamma \subset \left\{x \in \Omega,\ \rho(x, \gamma) = \rho(x, \Gamma_\Omega)\right\} \quad \text{for all } \gamma \in \Gamma_\Omega. \tag{2.8b}$$

By Proposition 2.3, we have the following unit partition of the Corkscrew domain Ω.

Proposition 2.4. *Let (X, ρ, μ) be a metric measure space and Ω be a bounded Corkscrew domain satisfying Assumption 2.2. Then for any $\delta \in (0, 1)$ there exists a discrete set $\Omega_\delta \subset \Omega$ such that the corresponding Voronoi partition $I_\gamma, \gamma \in \Omega_\delta$, of the domain Ω satisfies (2.1), where c is the ratio in the Corkscrew condition (2.6) for the domain Ω.*

2.2. Sampling and reconstruction of signals in a reproducing kernel space

For a kernel function K on $X \times X$, we define its *Schur norm* $\|K\|_{\mathcal{S}}$ and *modulus of continuity* $\omega_\delta(K)$ by

$$\|K\|_{\mathcal{S}} = \max\Big(\sup_{x \in X} \|K(x, \cdot)\|_1,\ \sup_{y \in X} \|K(\cdot, y)\|_1\Big)$$

and

$$\omega_\delta(K)(x, y) = \sup_{\rho(x', x) \le \delta,\ \rho(y', y) \le \delta} |K(x', y') - K(x, y)|,\ \ x, y \in X,$$

respectively. To consider sampling and reconstruction of ε-concentrated signals in $V_{p,\Omega,\varepsilon}$, we always assume the following:

Assumption 2.5. The integral kernel K of the idempotent operator T in (1.2) has certain off-diagonal decay and Hölder continuity,

$$\|K\|_{\mathcal{S},\theta} := \|K\|_{\mathcal{S}} + \sup_{0 < \delta \le 1} \delta^{-\theta} \|\omega_\delta(K)\|_{\mathcal{S}} < \infty \tag{2.9}$$

for some $0 < \theta \le 1$.

One may verify that Assumption 2.5 is met for kernels K being Hölder continuous,

$$|K(x,y) - K(x',y')| \le C(\rho(x,x') + \rho(y,y'))^\theta (1 + \rho(x,y) + \rho(x',y'))^{-\alpha}$$

for all $x, x', y, y' \in X$, and having polynomial decay,

$$|K(x,y)| \le C(1 + \rho(x,y))^{-\alpha}$$

for all $x, y \in X$, where $\alpha > d$ and C is a positive constant. It is well known that the integral operator with its kernel having finite Schur norm is bounded operator on L^p. Hence the range space V_p in (1.1) is a closed subspace of L^p. In the following proposition, we show that it is a reproducing kernel space of L^p, which is established in [40] for the Euclidean space setting, see Section 6.2 for a sketch of the proof.

Proposition 2.6. *Let (X, ρ, μ) be a metric measure space, T be an idempotent operator whose kernel K satisfies Assumption 2.5, and $V_p, 1 \le p \le \infty$, be the range space of the operator T defined by (1.1). Then V_p is a reproducing kernel space of L^p, and for any $f \in V_p$*

$$\|f\|_q \le (D_1(\mu))^{-1/p+1/q} \|K\|_{\mathcal{S},\theta}^{1-p/q} \|f\|_p,\ p \le q \le \infty, \tag{2.10}$$

where $D_1(\mu)$ is the maximal lower bound in (2.2) and $\|K\|_{\mathcal{S},\theta}$ is given in (2.9).

To consider sampling and reconstruction of signals $f \in V_p$ concentrated on a bounded domain Ω, we recall the iterative algorithm

$$f_0 = S_\Gamma f \quad \text{and} \quad f_n = f_0 + f_{n-1} - S_\Gamma f_{n-1},\ n \ge 1, \tag{2.11}$$

to reconstruct signals $f \in V_p$ from their samples $f(\gamma), \gamma \in \Gamma$, taken on the sampling set $\Gamma \cap \Omega$ in the domain Ω and the sampling set $\Gamma \cap \Omega^c$ outside the domain Ω, where $\{I_\gamma, \gamma \in \Gamma \cap \Omega\}$ and $\{I_\gamma, \gamma \in \Gamma \cap \Omega^c\}$ are Voronoi partitions of the domain Ω and its complement Ω^c respectively, and the preconstruction operator S_Γ on L^p is defined by

$$S_\Gamma g(x) = \sum_{\gamma \in \Gamma} \mu(I_\gamma)(Tg)(\gamma) K(x,\gamma),\ g \in L^p. \tag{2.12}$$

The above iterative algorithm (2.11) has been widely used in reconstructing signals in various linear spaces, see for instance [4,5,9,20,25,40,50]. In the following proposition, we show that the above algorithm converges exponentially, see Section 6.3 for the proof.

Proposition 2.7. *Let (X, ρ, μ) be a metric measure space, T be an idempotent operator whose kernel K satisfies Assumption 2.5, $V_p, 1 \le p \le \infty$, be the range space (1.1) of the operator T, and Ω be a bounded domain. If Γ is a sampling set satisfying*

$$\delta(\Gamma) := \max\Big(\sup_{x \in \Omega} \rho(x, \Gamma \cap \Omega), \sup_{y \in \Omega^c} \rho(y, \Gamma \cap \Omega^c)\Big) < \|K\|_{\mathcal{S},\theta}^{-2/\theta},$$

then for any $f \in V_p$, the sequence $f_n, n \ge 0$, in the iterative algorithm (2.11) converges to f exponentially,

$$\|f_n - f\|_p \le \frac{1 + \|K\|_{\mathcal{S},\theta}^2 (\delta(\Gamma))^\theta}{1 - \|K\|_{\mathcal{S},\theta}^2 (\delta(\Gamma))^\theta} \big(\|K\|_{\mathcal{S},\theta}^2 (\delta(\Gamma))^\theta\big)^{n+1} \|f\|_p.$$

3. Sampling and reconstruction of concentrated signals

Stability of a sampling scheme is an important concept for the robustness and uniqueness for sampling and reconstruction of signals in a linear space, see [5,8,20,40,50,51,53]. In this section, we first consider weighted stability of bi-Lipschitz type for the sampling procedure on a sampling set $\Gamma_\Omega \subset \Omega$ for ε-concentrated signals on the domain Ω.

Theorem 3.1. *Let $1 \le p \le \infty$, $\varepsilon \in (0,1)$, (X, ρ, μ) be a metric measure space, T be an idempotent integral operator whose kernel K satisfies Assumption 2.5, V_p be the range space of the operator T defined by (1.1), Ω be a bounded domain, and $V_{p,\Omega,\varepsilon}$ be the set of ε-concentrated signals given in (1.3). If $\Gamma_\Omega \subset \Omega$ is a discrete sampling set satisfying*

$$d_H(\Gamma_\Omega, \Omega) < \Big(\frac{1-\varepsilon}{\|K\|_{\mathcal{S},\theta}}\Big)^{1/\theta}, \tag{3.1}$$

then for all $f, g \in V_{p,\Omega,\varepsilon}$,

$$\begin{aligned} &\big(1-\varepsilon-\|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega,\Omega))^\theta\big)\|f-g\|_p - 2\varepsilon\min(\|f\|_p,\|g\|_p) \\ &\le \big\|\big(f(\gamma)-g(\gamma)\big)_{\gamma\in\Gamma_\Omega}\big\|_{p,\mu(\Gamma_\Omega)} \le \big(1+\|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega,\Omega))^\theta\big)\|f-g\|_p, \end{aligned} \tag{3.2}$$

where $d_H(\Gamma_\Omega, \Omega)$ is the Hausdorff distance between Γ_Ω and Ω, $I_\gamma, \gamma \in \Gamma_\Omega$, is a Voronoi partition of the domain Ω, and for any $h \in V_p$

$$\big\|\big(h(\gamma)\big)_{\gamma\in\Gamma_\Omega}\big\|_{p,\mu(\Gamma_\Omega)} = \begin{cases} \big(\sum_{\gamma\in\Gamma_\Omega}|h(\gamma)|^p\mu(I_\gamma)\big)^{1/p} & \text{if } 1 \le p < \infty \\ \sup_{\gamma\in\Gamma_\Omega}|h(\gamma)| & \text{if } p = \infty. \end{cases} \tag{3.3}$$

By (2.2) and (2.8b), we have

$$\mu(I_\gamma) \le D_2(\mu)(d_H(\Gamma_\Omega,\Omega))^d,\ \gamma \in \Gamma_\Omega,$$

where $D_2(\mu)$ is the minimal upper bound in (2.2). Therefore we have the following unweighted inequalities for the sampling scheme $f \longmapsto (f(\gamma))_{\gamma\in\Gamma_\Omega}$, cf. (1.4).

Corollary 3.2. *Let the metric measure space (X, ρ, μ) and the set $V_{p,\Omega,\varepsilon}$ of ε-concentrated signals be as in Theorem 3.1. If $\Gamma_\Omega \subset \Omega$ is a discrete sampling set of the domain Ω satisfying (3.1), then*

$$\Big(\sum_{\gamma\in\Gamma_\Omega}|f(\gamma)|^p\Big)^{1/p} \ge \frac{1-\varepsilon-\|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega,\Omega))^\theta}{(D_2(\mu))^{1/p}(d_H(\Gamma_\Omega,\Omega))^{d/p}}\|f\|_p$$

hold for all $f \in V_{p,\Omega,\varepsilon}, 1 \le p < \infty$.

Given a sampling set Γ_Ω and noiseless samples $h(\gamma), \gamma \in \Gamma_\Omega$, of $h \in V_p$, we define

$$h_I = \sum_{\gamma\in\Gamma_\Omega} h(\gamma)\chi_{I_\gamma}, \tag{3.4}$$

where χ_E is the indicator function on a set E. One may verify easily that

$$h_I(\gamma) = h(\gamma),\ \gamma \in \Gamma_\Omega, \tag{3.5}$$

and

$$\|h_I\|_{p,\Omega} = \big\|(h(\gamma))_{\gamma\in\Gamma_\Omega}\big\|_{p,\mu(\Gamma_\Omega)}. \tag{3.6}$$

By (3.6) with h replaced by $f-g$, the proof of Theorem 3.1 reduces to showing

$$\|h_I - h\|_{p,\Omega} \le \|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega,\Omega))^\theta \|h\|_p,\ \ h\in V_p, \tag{3.7}$$

see Section 6.4 for the detailed argument.

Given noiseless samples $f(\gamma), \gamma\in\Gamma_\Omega$, of an ε-concentrated signal $f\in V_{p,\Omega,\varepsilon}\subset V_p$, the signal f_I in (3.4) coincides with the original signal f on the sampling set Γ_Ω by (3.5). However the signal f_I does not reside in the reproducing kernel space V_p and it does not provide an approximation to the original signal f, except that the Hausdorff distance $d_H(\Gamma_\Omega,\Omega)$ between Γ_Ω and Ω is small, since in that case

$$\|f_I - f\|_p \le \|f_I - f\|_{p,\Omega} + \|f\|_{p,\Omega^c} \le \big(\|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega,\Omega))^\theta + \varepsilon\big)\|f\|_p$$

by (1.3) and (3.7).

Based on the iterative algorithm (2.11), for any $f\in V_p$ we define

$$g_0 = \sum_{\gamma\in\Gamma_\Omega} \mu(I_\gamma) f(\gamma) K(\cdot,\gamma) \in V_p, \tag{3.8a}$$

and $g_n\in V_p, n\ge 1$, inductively by

$$g_n = g_0 + g_{n-1} - S_\Gamma g_{n-1},\ \ n\ge 1, \tag{3.8b}$$

where the preconstruction operator S_Γ on L^p is given in (2.12) with $\Gamma=\Gamma_\Omega\cup\Gamma_{\Omega^c}$, Γ_{Ω^c} is a discrete sampling set of the complement Ω^c of the domain Ω satisfying

$$d_H(\Gamma_{\Omega^c},\Omega^c) \le \min\big(\varepsilon^{1/\theta}\|K\|_{\mathcal{S},\theta}^{-1/\theta}, (2\|K\|_{\mathcal{S},\theta}^2)^{-1/\theta}\big), \tag{3.9}$$

and $\{I_\gamma, \gamma\in\Gamma_\Omega\}$ and $\{I_\gamma,\gamma\in\Gamma_{\Omega^c}\}$ are Voronoi partitions of the domain Ω and its complement Ω^c respectively. In the following theorem, we show that $g_n, n\ge 0$, reconstructed from samples $f(\gamma),\gamma\in\Gamma_\Omega$, inside the domain Ω provide good approximations to the original ε-concentrated signal $f\in V_{p,\Omega,\varepsilon}$, see Section 6.5 for the detailed argument.

Theorem 3.3. *Let $1\le p\le\infty$, $\varepsilon\in(0,1)$, (X,ρ,μ) be a metric measure space, T be an idempotent operator whose kernel K satisfies Assumption 2.5, V_p be the range space of the operator T defined by (1.1), Ω be a bounded Corkscrew domain satisfying Assumption 2.2, Γ_Ω be a discrete sampling set of the domain Ω, and $V_{p,\Omega,\varepsilon}$ be the set of ε-concentrated signals given in (1.3). If the Hausdorff distance between the sampling set Γ_Ω and the domain Ω satisfies*

$$d_H(\Gamma_\Omega,\Omega) < \|K\|_{\mathcal{S},\theta}^{-2/\theta}, \tag{3.10}$$

then for any ε-concentrated signal $f\in V_{p,\Omega,\varepsilon}$, the reconstructed signals g_n in (3.8) with

$$n+1 \ge \max\Big(\frac{\ln(1/\varepsilon) - \ln\|K\|_{\mathcal{S},\theta}}{\theta\ln(1/d_H(\Gamma_\Omega,\Omega)) - 2\ln\|K\|_{\mathcal{S},\theta}}, \frac{\ln(1/\varepsilon)-\ln\|K\|_{\mathcal{S},\theta}}{\ln 2}\Big) \tag{3.11}$$

are $(9C_0\varepsilon)$-concentrated signals in V_p, i.e.,

$$g_n \in V_{p,\Omega,9C_0\varepsilon}, \tag{3.12}$$

and they provide good approximations to the original signal f,

$$\|g_n - f\|_p \le 4C_0\varepsilon\|f\|_p, \tag{3.13}$$

where

$$C_0 = \|K\|_{\mathcal{S},\theta} \max\Big(2, \big(1 - \|K\|_{\mathcal{S},\theta}^2 (d_H(\Gamma_\Omega, \Omega))^\theta\big)^{-1}\Big). \tag{3.14}$$

The reconstructed signals $g_n, n \ge 0$, in (3.8) do not interpolate the sampling data $f(\gamma), \gamma \in \Gamma_\Omega$, however they have small sampling difference to the original signal f on the sampling set Γ_Ω, since it follows from (3.7) and (3.13) that

$$\begin{aligned} \big\|\big(g_n(\gamma) - f(\gamma)\big)_{\gamma\in\Gamma_\Omega}\big\|_{p,\mu(\Gamma_\Omega)} &\le \big(1 + \|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega, \Omega))^\theta\big)\|g_n - f\|_p \\ &\le 4C_0\big(1 + \|K\|_{\mathcal{S},\theta}(d_H(\Gamma_\Omega, \Omega))^\theta\big)\varepsilon\|f\|_p \end{aligned} \tag{3.15}$$

for all $f \in V_{p,\Omega,\varepsilon}, 1 \le p \le \infty$.

Remark 3.1. Take the hat function $h(x) = \max(1-|x|, 0)$, the concentration domain $\Omega_R = [-R, R]$ for some integer $R \ge 2$, and signals $f_\pm(x) = h(x) \pm \delta h(x - R - 1), \delta \in (0, 1)$, in the shift-invariant space

$$V_p(h) = \Big\{\sum_{k\in\mathbb{Z}} c(k)h(x - k),\ (c(k))_{k\in\mathbb{Z}} \in \ell^p\Big\},\ 1 \le p \le \infty,$$

generated by the integer shifts of the hat function h. One may verify that the shift-invariant space $V_p(h)$ is the range space of some idempotent integral operator with kernel satisfying Assumption 2.5, and $f_\pm$ are ε-concentrated signals onto Ω_R with $\varepsilon = \delta$ for $p = \infty$ and $\varepsilon = \delta(1 + \delta^p)^{-1/p}$ for $1 \le p < \infty$, since $\|f_\pm\|_{p,\mathbb{R}\setminus\Omega_R} = \delta\|h\|_p$ and $\|f_\pm\|_p = \delta\|h\|_p/\varepsilon$. As signals $f_\pm$ coincide on the domain Ω_R, the signals $g_{n,\pm}$ constructed in (3.8) from their samples inside the domain Ω_R are the same, which implies that

$$\max\big(\|g_{n,+} - f_+\|_p, \|g_{n,-} - f_-\|_p\big) \ge \frac{1}{2}\|f_+ - f_-\|_p = \delta\|h\|_p = \varepsilon\|f_\pm\|_p.$$

Therefore the error estimate in (3.13) is suboptimal in the sense that the constant $4C_0$ cannot be replaced by a positive constant strictly less than one in general.

Reconstructing a signal from noisy data and estimating the reconstruction error are important problems in sampling theory [1,5,7,9,40,50,53]. In this paper, we propose the following algorithm $\tilde g_n, n \ge 0$, for signal reconstruction when samples $f(\gamma), \gamma \in \Gamma_\Omega$, of $f \in V_p$ are corrupted by some deterministic noise $\boldsymbol{\xi} = (\xi(\gamma))_{\gamma\in\Gamma_\Omega}$:

$$\tilde g_n = \tilde g_0 + \tilde g_{n-1} - S_\Gamma \tilde g_{n-1},\ n \ge 1, \tag{3.16a}$$

where

$$\tilde g_0 = \sum_{\gamma\in\Gamma_\Omega} \mu(I_\gamma)(f(\gamma) + \xi(\gamma))K(\cdot, \gamma) \in V_p, \tag{3.16b}$$

and the preconstruction operator S_Γ on L^p is given in (2.12) with $\Gamma = \Gamma_\Omega \cup \Gamma_{\Omega^c}$, and Γ_{Ω^c} is a discrete sampling set of the complement Ω^c satisfying (3.9). In the following theorem, we show that the reconstructed signals

$\tilde{g}_n$ with large n provide good approximations to the original ε-concentrated signal f, see Section 6.6 for the proof.

Theorem 3.4. *Let $1 \le p \le \infty$, $\varepsilon \in (0,1)$, (X,ρ,μ) be a metric measure space, T be an idempotent operator whose kernel K satisfies Assumption 2.5, V_p be the range space of the operator T defined by (1.1), Ω be a bounded Corkscrew domain satisfying Assumption 2.2, Γ_Ω be a discrete sampling set of the domain Ω satisfying (3.10), $V_{p,\Omega,\varepsilon}$ be the set of ε-concentrated signals given in (1.3), and $\boldsymbol{\xi} = (\xi(\gamma))_{\gamma\in\Gamma_\Omega}$ be a deterministic noise vector with $\|\boldsymbol{\xi}\|_{p,\mu(\Gamma_\Omega)} < \infty$. Then for any ε-concentrated signal $f \in V_{p,\Omega,\varepsilon}$, signals $\tilde{g}_n$ in (3.16) with n satisfying (3.11) provide approximations to the original signal f,*

$$\|\tilde{g}_n - f\|_p \le 4C_0\varepsilon\|f\|_p + C_0\|\boldsymbol{\xi}\|_{p,\mu(\Gamma_\Omega)}, \tag{3.17}$$

where C_0 is given in (3.14) and the norm $\|\cdot\|_{p,\mu(\Gamma_\Omega)}$ is defined by (3.3).

4. Random sampling and reconstruction of concentrated signals

In this section, we consider sampling ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ at i.i.d. random positions drawn on Ω, and reconstructing the original ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ from their samples taken on these random positions. We establish a weighted stability inequality of bi-Lipschitz type for the random sampling procedure in Theorem 4.1. In Theorem 4.3 and Corollary 4.4, we show that, with high probability, signals concentrated on a bounded Corkscrew domain Ω can be reconstructed approximately from their samples taken at i.i.d. random positions drawn on Ω, provided that the sampling size is at least of the order $\mu(\Omega)\ln(\mu(\Omega))$. Finally in Theorem 4.6 we prove that with high probability, an original ε-concentrated signal can be reconstructed approximately from its random samples corrupted by i.i.d. random noises, when the random sampling size is large enough.

Theorem 4.1. *Let (X,ρ,μ) be a metric measure space, $V_p, 1 \le p \le \infty$, be the range space of an idempotent integral operator T whose kernel K satisfies Assumption 2.5, Ω be a bounded Corkscrew domain satisfying Assumption 2.2, and let $V_{p,\Omega,\varepsilon}, \varepsilon \in (0,1)$, be the set of ε-concentrated signals given in (1.3). If $\{\gamma, \gamma \in \Gamma_\Omega\}$ are i.i.d. random positions drawn on Ω with respect to the probability measure $(\mu(\Omega))^{-1}d\mu$, then for any $\tilde{\varepsilon} \in (0, 1-\varepsilon)$, the following weighted stability inequalities of bi-Lipschitz type*

$$\begin{aligned} &\big(1-\varepsilon-\tilde{\varepsilon}\big)\|f-g\|_p - 2\varepsilon\min(\|f\|_p, \|g\|_p) \\ &\le \big\|\big(f(\gamma)-g(\gamma)\big)_{\gamma\in\Gamma_\Omega}\big\|_{p,\mu(\Gamma_\Omega)} \le \big(1+\tilde{\varepsilon}\big)\|f-g\|_p,\ \ f,g \in V_{p,\Omega,\varepsilon} \end{aligned} \tag{4.1}$$

hold with probability at least

$$1 - \frac{10^d\mu(\Omega)}{c^d D_1(\mu)(\tilde{\varepsilon}/\|K\|_{\mathcal{S},\theta})^{d/\theta}}\Big(1 - \frac{c^d D_1(\mu)(\tilde{\varepsilon}/\|K\|_{\mathcal{S},\theta})^{d/\theta}}{10^d\mu(\Omega)}\Big)^N,$$

where N is the size of the sampling set Γ_Ω and the norm $\|\cdot\|_{p,\mu(\Gamma_\Omega)}$ is defined by (3.3).

By Theorem 3.1, the proof of Theorem 4.1 reduces to the following crucial estimate on the probability on the Hausdorff distance $d_H(\Gamma_\Omega, \Omega) > \delta_1$ where $0 < \delta_1 < 1$, see Section 6.7 for the proof.

Proposition 4.2. *Let (X,ρ,μ) be a d-dimensional metric measure space and Ω be a bounded Corkscrew domain satisfying Assumption 2.2. Suppose that $\{\gamma, \gamma \in \Gamma_\Omega\}$ are i.i.d. random positions drawn on Ω with respect to the probability measure $(\mu(\Omega))^{-1}d\mu$. Then for $0 < \delta_1 \le 1$,*

$$\mathbb{P}\{d_H(\Gamma_\Omega,\Omega)>\delta_1\}\le \frac{10^d\mu(\Omega)}{c^dD_1(\mu)\delta_1^d}\Big(1-\frac{c^dD_1(\mu)\delta_1^d}{10^d\mu(\Omega)}\Big)^N. \tag{4.2}$$

Remark 4.1. Applying Corollary 3.2 and Proposition 4.2 with δ_1 replaced by $(2\|K\|_{\mathcal{S},\theta})^{-1/\theta}$, we obtain that the following sampling inequalities

$$\Big(\sum_{\gamma\in\Gamma_\Omega}|f(\gamma)|^p\Big)^{1/p}\ge (1/2-\varepsilon)(D_2(\mu))^{-1/p}(2\|K\|_{\mathcal{S},\theta})^{d/(p\theta)}\|f\|_p,\ f\in V_{p,\Omega,\varepsilon} \tag{4.3}$$

hold with probability at least

$$1-\frac{10^d(2\|K\|_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}{c^dD_1(\mu)}\Big(1-\frac{c^dD_1(\mu)}{10^d(2\|K\|_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}\Big)^N,$$

where $\varepsilon\in(0,1/2)$ and $1\le p<\infty$. We remark that the above sampling inequalities (4.3) for random sampling of ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ can be considered as a weak version of the corresponding sampling inequalities for bandlimited/wavelet signals concentrated on $[-R/2,R/2]^d$ in [11,12,27,34,37,42,59].

Remark 4.2. Let $\tau\in(0,1/2], 1\le p<\infty$, and

$$N\ge N_0(\mu(\Omega),\tau):=\frac{5^d2^{d+1+d/\theta}\|K\|_{\mathcal{S},\theta}^{d/\theta}\mu(\Omega)}{c^dD_1(\mu)}\ln\Big(\frac{10^d(2\|K\|_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}{c^dD_1(\mu)\tau}\Big). \tag{4.4}$$

Applying Corollary 3.2 and Proposition 4.2 with

$$\delta_1=\Big(\frac{10^d}{c^dD_1(\mu)}\frac{\mu(\Omega)}{N}\ln\Big(\frac{N}{\tau}\Big)\Big)^{1/d},$$

we conclude that the following sampling inequalities

$$\sum_{\gamma\in\Gamma_\Omega}|f(\gamma)|^p\ge\frac{(1/2-\varepsilon)^pc^dD_1(\mu)}{10^dD_2(\mu)}\Big(\ln\frac{N}{\tau}\Big)^{-1}\frac{N}{\mu(\Omega)}\|f\|_p^p,\ f\in V_{p,\Omega,\varepsilon} \tag{4.5}$$

hold with probability at least $1-\tau$. We remark that the sampling inequalities (4.5) for random sampling of ε-concentrated signals in $V_{p,\Omega,\varepsilon}$ can be considered as a weak version of the corresponding sampling inequalities for bandlimited/wavelet signals concentrated on $[-R/2,R/2]^d$ in [11,12,27,37], where the lower bound in (4.5) is replaced by a multiple of $N\|f\|_p^p/\mu(\Omega)$.

To the best of our knowledge, there is no algorithm available to find good approximations to ε-concentrated signals from their random samples inside the domain Ω. By Theorem 3.3 and Proposition 4.2 with δ_1 replaced by $(2\|K\|_{\mathcal{S},\theta}^2)^{-1/\theta}$, such approximations are constructed explicitly.

Theorem 4.3. *Let the metric measure space (X,ρ,μ), the domain Ω, the set $V_{p,\Omega,\varepsilon}$ of ε-concentrated signals, and the sequence $g_n\in V_p, n\ge 0$, be as in Theorem 3.3. Suppose that $\{\gamma,\gamma\in\Gamma_\Omega\}$ are i.i.d. random positions drawn on Ω with respect to probability measure $(\mu(\Omega))^{-1}d\mu$, and denote the size of Γ_Ω by N. Then for*

$$n+1\ge\frac{\ln(1/\varepsilon)-\ln\|K\|_{\mathcal{S},\theta}}{\ln 2}, \tag{4.6}$$

the following reconstruction error estimates

$$\|g_n - f\|_p \le 8\|K\|_{\mathcal{S},\theta}\varepsilon\|f\|_p,\ \ f \in V_{p,\Omega,\varepsilon} \tag{4.7}$$

hold with probability at least

$$1-\tau(\mu(\Omega), N) := 1 - \frac{10^d(2\|K\|^2_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}{c^d D_1(\mu)}\Big(1 - \frac{c^d D_1(\mu)}{10^d(2\|K\|^2_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}\Big)^N. \tag{4.8}$$

For any $0 < \tau < 1$, one may verify that

$$\tau(\mu(\Omega), N) \le \tau$$

when

$$N \ge N_1(\mu(\Omega), \tau) := \frac{10^d(2\|K\|^2_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}{c^d D_1(\mu)} \ln \frac{10^d(2\|K\|^2_{\mathcal{S},\theta})^{d/\theta}\mu(\Omega)}{c^d D_1(\mu)\tau}. \tag{4.9}$$

Therefore by Proposition 2.6 and Theorem 4.3, we have the following corollary.

Corollary 4.4. *Let $\varepsilon, \tau \in (0,1)$, and let the metric measure space (X, ρ, μ), the domain Ω, the set $V_{p,\Omega,\varepsilon}$ of ε-concentrated signals, the random sampling set Γ_Ω, and the reconstructed signals $g_n, n \ge 0$, be as in Theorem 4.3. If the size N of the random sampling set Γ_Ω satisfies (4.9), then for any integer n satisfying (4.6) and $p \le q \le \infty$,*

$$\|g_n - f\|_q \le 8(D_1(\mu))^{-1/p+1/q}\|K\|^{2-p/q}_{\mathcal{S},\theta}\varepsilon\|f\|_p,\ \ f \in V_{p,\Omega,\varepsilon} \tag{4.10}$$

hold with probability at least $1-\tau$.

Next, we consider signal reconstruction when random samples $f(\gamma), \gamma \in \Gamma_\Omega$, of a signal $f \in V_p$ are corrupted by some bounded noise $\boldsymbol{\xi} = (\xi(\gamma))_{\gamma\in\Gamma_\Omega}$,

$$\tilde{f}_\gamma = f(\gamma) + \xi(\gamma),\ \ \gamma \in \Gamma_\Omega. \tag{4.11}$$

Following the argument used in the proofs of Theorem 3.4 and Corollary 4.4, we have the following result when random samples are corrupted by bounded deterministic noises.

Corollary 4.5. *Let $\varepsilon, \tau \in (0,1)$, and let the metric measure space (X, ρ, μ), the domain Ω, the set $V_{p,\Omega,\varepsilon}$ of ε-concentrated signals and the random sampling set Γ_Ω be as in Theorem 4.3, $\boldsymbol{\xi} = (\xi(\gamma))_{\gamma\in\Gamma_\Omega}$ be bounded noise vector with bound $\|\boldsymbol{\xi}\|_\infty = \sup_{\gamma\in\Gamma_\Omega}|\xi(\gamma)|$, and the reconstructed signals $\tilde{g}_n, n \ge 0$, be as in Theorem 3.4. If the size N of the random sampling set Γ_Ω satisfies (4.9), then for any integer n satisfying (4.6),*

$$\|\tilde{g}_n - f\|_\infty \le 8(D_1(\mu))^{-1/p}\|K\|^2_{\mathcal{S},\theta}\varepsilon\|f\|_p + 2\|K\|_{\mathcal{S},\theta}\|\boldsymbol{\xi}\|_\infty,\ \ f \in V_{p,\Omega,\varepsilon} \tag{4.12}$$

hold with probability at least $1-\tau$.

Remark 4.3. Let $0 \ne h_0 \in V_{p,\Omega,\varepsilon_0}$ satisfy

$$32(D_1(\mu))^{-1/p}\|K\|^2_{\mathcal{S},\theta}\|h_0\|_p\varepsilon_0 \le \|h_0\|_\infty. \tag{4.13}$$

Such a signal exists for sufficiently small ε_0 when V_p is the shift-invariant space generated by the integer shifts of the hat function and $\Omega = [-R/2, R/2]$, $R \ge 2$, see Remark 3.1. Take $x \in \Omega$ with $|h_0(x)| \ge \|h_0\|_\infty/2$

and let $\tilde{g}_n, n \geq 0$, be as in Theorem 3.4 reconstructed from noisy sampling data (4.11) with $f = 0$ and $\xi(\gamma) = h_0(\gamma), \gamma \in \Gamma_\Omega$. Then we obtain from Corollary 4.4 that

$$|\tilde{g}_n(x) - h_0(x)| \leq 8(D_1(\mu))^{-1/p} \|K\|^2_{\mathcal{S},\theta} \varepsilon_0 \|h_0\|_p \tag{4.14}$$

hold with probability at least $1 - \tau$ for large n. Therefore for large n,

$$|\tilde{g}_n(x) - f(x)| = |\tilde{g}_n(x)| \geq \|\boldsymbol{\xi}\|_\infty / 4$$

hold with probability at least $1 - \tau$, since $\|\boldsymbol{\xi}\|_\infty \leq \|h_0\|_\infty$ by the definition of the noise vector $\boldsymbol{\xi}$, and

$$|\tilde{g}_n(x)| \geq |h_0(x)| - 8(D_1(\mu))^{-1/p} \|K\|^2_{\mathcal{S},\theta} \varepsilon_0 \|h_0\|_p \geq \|h_0\|_\infty / 4$$

by (4.13) and (4.14). This demonstrates that the error estimate in (4.12) is suboptimal in the sense that the second part of the bound estimate $2\|K\|_{\mathcal{S},\theta} \|\boldsymbol{\xi}\|_\infty$ cannot be replaced by $A\|\boldsymbol{\xi}\|_\infty$ for some small constant A.

By Remark 4.3, the term $2\|K\|_{\mathcal{S},\theta} \|\boldsymbol{\xi}\|_\infty$ related to the noise vector $\boldsymbol{\xi}$ can not be ignored in the error estimate (4.12) of Corollary 4.5, no matter how large the sampling size N is. In the following theorem, we show that the scenario will be *completely different* if the noise vector $\boldsymbol{\xi}$ has its components being i.i.d. random variables, see [6,7,18,24] and references therein for reconstruction of signals in various linear spaces from their samples corrupted by random noises.

Theorem 4.6. *Let the metric measure space (X, ρ, μ), the domain Ω, the set $V_{p,\Omega,\varepsilon}$ of ε-concentrated signals, the random sampling set Γ_Ω, and the sequence $\tilde{g}_n, n \geq 0$, be as in Theorem 4.3. Suppose that $\tau \in (0, 1/2)$ and $\xi(\gamma), \gamma \in \Gamma_\Omega$, are i.i.d. random variables with mean zero and variance σ^2,*

$$\mathbb{E}(\xi(\gamma)) = 0, \quad \mathrm{Var}(\xi(\gamma)) = \sigma^2, \ \gamma \in \Gamma_\Omega. \tag{4.15}$$

Let $f \in V_{p,\Omega,\varepsilon}$ and set

$$\tilde{\delta}_1 = \min\left((2\|K\|^2_{\mathcal{S},\theta})^{-1/\theta}, \Big(\frac{\tau \varepsilon^2 \sigma^{-2} \|f\|_p^2}{D_2(\mu)(D_1(\mu))^{2/p-1}} \Big)^{1/d} \right). \tag{4.16}$$

If the size N of the random sampling set Γ_Ω satisfies

$$N \geq \frac{10^d \mu(\Omega)}{c^d D_1(\mu) \tilde{\delta}_1^d} \ln \frac{10^d \mu(\Omega)}{c^d D_1(\mu) \tau \tilde{\delta}_1^d}, \tag{4.17}$$

then for any integer n satisfying (4.6), the approximation error estimates

$$\|\tilde{g}_n - f\|_\infty \leq 10(D_1(\mu))^{-1/p} \|K\|^2_{\mathcal{S},\theta} \varepsilon \|f\|_p \tag{4.18}$$

hold with probability at least $1 - 2\tau$, where $D_1(\mu)$ and $D_2(\mu)$ are the maximal lower bound and minimal upper bound in (2.2) respectively, and c is the ratio in the Corkscrew condition (2.6) for the domain Ω.

5. Numerical demonstrations

In this section, we demonstrate effectiveness of the algorithms (3.8) and (3.16) to approximate concentrated signals in the reproducing kernel space

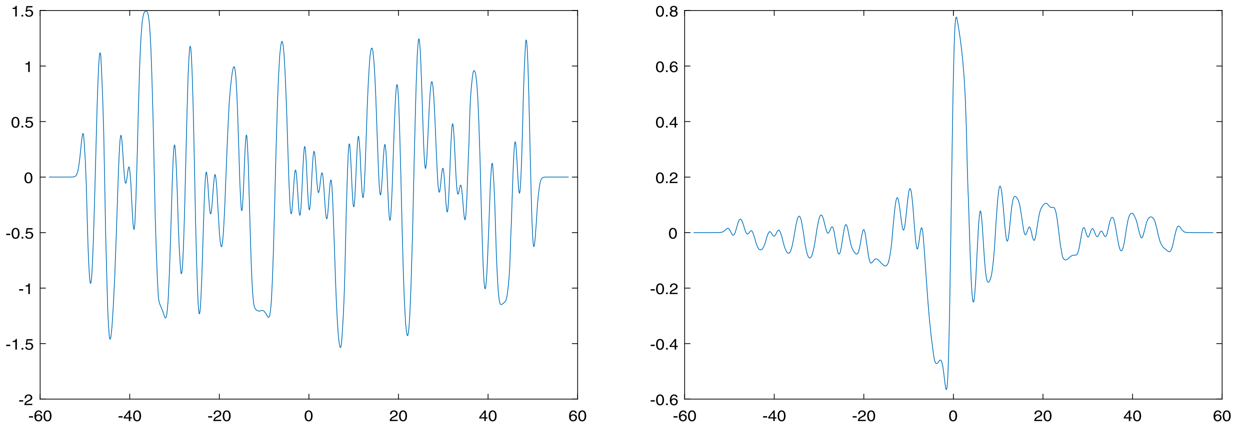

Fig. 1. Plotted on the left is a concentrated signal $f_{L,\alpha}$ in (5.1) with $L = 50$ and $\alpha = 0$, while on the right is another concentrated signal $f_{L,\alpha}$ in (5.1) with $L = 50$ and $\alpha = 0.8$, where the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c}/\|f_{L,\alpha}\|_2 = 0.6433/7.2628 = 0.0886$ for the signal in the left figure and $0.0244/1.6869 = 0.0145$ for the signal in the right figure.

$$V_2(\Phi) = \Big\{ \sum_{i\in\mathbb{Z}} c_i\phi_i : \ \sum_{i\in\mathbb{Z}} |c(i)|^2 < \infty \Big\}$$

generated by (non-)uniform shifts of the Gaussian function $\exp(-x^2)$, where $\Phi := \{\phi_i(x) = \exp(-(x - i - \theta_i)^2),\ i \in \mathbb{Z}\}$ and $\theta_i \in [-1/10, 1/10], i \in \mathbb{Z}$, are randomly selected [10,20,35,48]. Our numerical simulations indicate that the correlation matrix $A_\Phi := (\langle \phi_i, \phi_j\rangle)_{i,j\in\mathbb{Z}}$ has bounded inverse on ℓ^2, and hence the inverse $A_\Phi^{-1} = (b_{ij})_{i,j\in\mathbb{Z}}$ has polynomial off-diagonal decay of any order by Wiener's lemma for infinite matrices [29,32,44,47,49]. Therefore the linear space $V_2(\Phi)$ is the range space of an idempotent integral operator with integral kernel function

$$K_\Phi(x, y) = \sum_{i,j\in\mathbb{Z}} b_{ji}\phi_i(x)\phi_j(y)$$

satisfying Assumption 2.5 with $\theta = 1$.

In our simulations, we consider the following family of signals

$$f_{L,\alpha} = \sum_{i=-L}^{L} r_i(1 + |i|)^{-\alpha}\phi_i \in V_2(\Phi) \tag{5.1}$$

concentrated on the interval $\Omega_L = [-L, L]$, where $L \geq 1, \alpha \geq 0$, and random variables $r_i, -L \leq i \leq L$, are independently selected in $[-1, 1]\backslash(-1/2, 1/2)$ with uniform distribution, see Fig. 1 for two examples of concentrated signals $f_{L,\alpha}$ with $L = 50$ and $\alpha = 0, 0.8$ respectively.

Due to the Riesz basis property for the generator Φ and randomness of $r_i, -L \leq i \leq L$, we have

$$\frac{\|f_{L,\alpha}\|_{2,\Omega_L^c}}{\|f_{L,\alpha}\|_2} \lesssim \frac{L^{-\alpha}}{\big(\sum_{|i|\leq L}(1+|i|)^{-2\alpha}\big)^{1/2}} \lesssim \begin{cases} L^{-1/2} & \text{if } \alpha < 1/2 \\ (L\ln L)^{-1/2} & \text{if } \alpha = 1/2 \\ L^{-\alpha} & \text{if } \alpha > 1/2, \end{cases}$$

and

$$\frac{\sqrt{\mathbb{E}\|f_{L,\alpha}\|^2_{2,\Omega_L^c}}}{\sqrt{\mathbb{E}\|f_{L,\alpha}\|_2^2}} = \frac{\big(\sum_{|i|\leq L}(1+|i|)^{-2\alpha}\int_{\mathbb{R}\backslash[-L,L]}|\phi_i(x)|^2dx\big)^{1/2}}{\big(\sum_{|i|\leq L}(1+|i|)^{-2\alpha}\int_{\mathbb{R}}|\phi_i(x)|^2dx\big)^{1/2}}$$

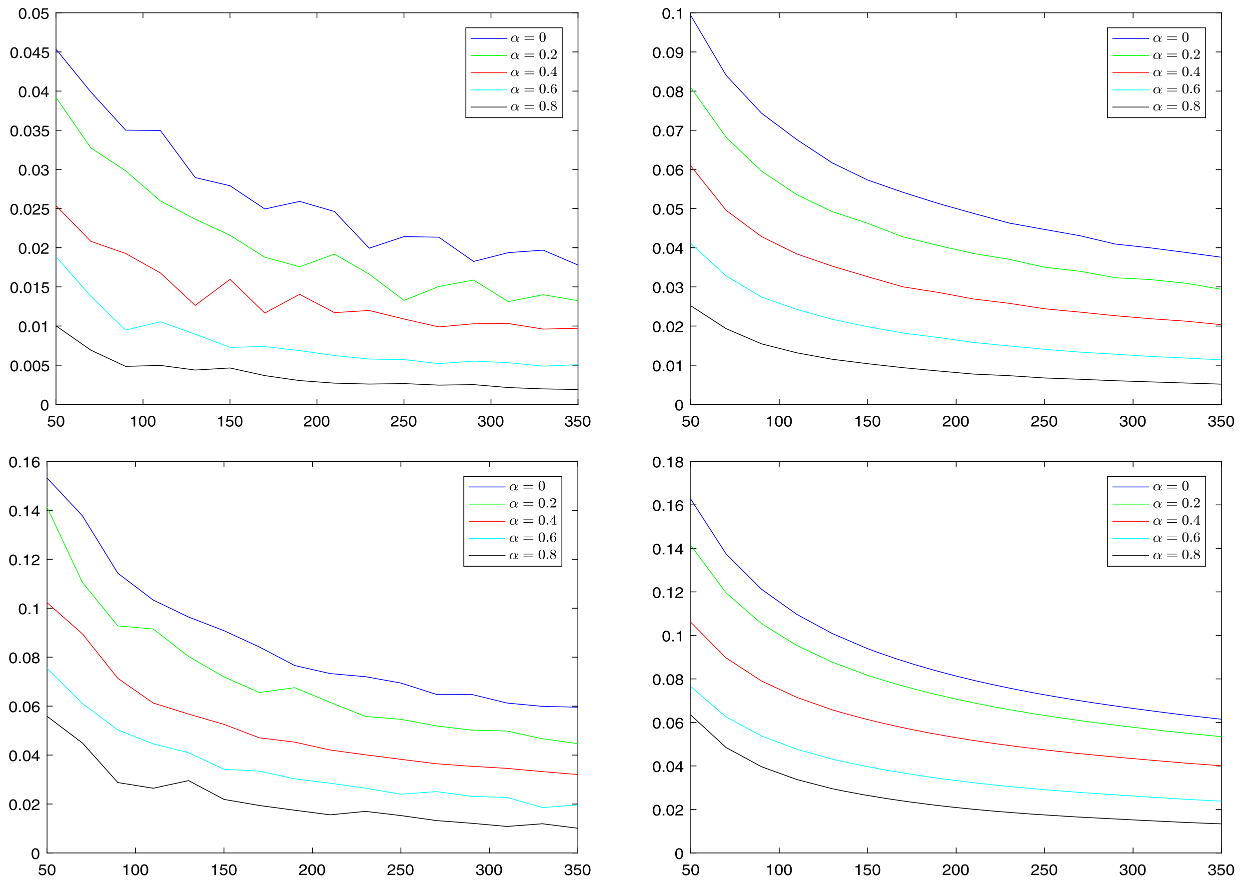

Fig. 2. Plotted on the top left/top right/bottom left are the minimal/average/maximal concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c}/\|f_{L,\alpha}\|_2$ over 1000 trials for $\alpha = 0, 0.2, 0.4, 0.6, 0.8$ respectively. On the bottom right is the concentration ratio $\varepsilon_{L,\alpha}$ selected in (5.2) for the family of concentrated signals $f_{L,\alpha}, 50 \le L \le 350$, which is approximately the maximal concentration ratio plotted on the bottom left figure. It is observed that the average concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c}/\|f_{L,\alpha}\|_2$ is almost proportional to the selected concentration ratio $\varepsilon_{L,\alpha}$ for $\alpha = 0, 0.2, 0.4, 0.6, 0.8$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\approx \begin{cases} L^{-1/2} & \text{if } \alpha < 1/2 \\ (L \ln L)^{-1/2} & \text{if } \alpha = 1/2 \\ L^{-\alpha} & \text{if } \alpha > 1/2. \end{cases}$$

Here for two positive items A and B, $A \lesssim B$ means that A/B is bounded by an absolute constant, and $A \approx B$ if both A/B and B/A are bounded by an absolute constant. Therefore signals $f_{L,\alpha}$ in (5.1) are concentrated on Ω_L with concentration ratio being about a multiple of $L^{-\max(\alpha,1/2)}$ for $\alpha \neq 1/2$. The above estimate on the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c}/\|f_{L,\alpha}\|_2$ is confirmed by our numerical simulations, see Fig. 2. So in our simulations, we consider that the family of concentrated signals $f_{L,\alpha}$ in (5.1) have concentration ratio

$$\varepsilon_{L,\alpha} = C_\alpha L^{-\max(\alpha,1/2)}, \tag{5.2}$$

where $C_\alpha = 1.15, 1, 0.75, 0.80, 1.45$ for $\alpha = 0, 0.2, 0.4, 0.6, 0.8$ respectively.

In the first part of our numerical simulations, we consider the sampling set $\Gamma_L = \{\gamma_k, 1 \le k \le N\}$ with $\gamma_{i+1} - \gamma_i, 0 \le i \le N-1$, being independently selected on $[1/4, 3/4]$ with uniform distribution, where $\gamma_0 = -L, \gamma_{N+1} = L$ and N is chosen so that $\gamma_{N+1} - \gamma_N \in [0, 1/4]$. The size of the sampling set Γ_L is between $8L/3$ and $8L$, while most of them have their sizes around $4L$. To construct the preconstruction operator in (2.12) associated with the above sampling set Γ_L, we take uniformly sampling set on the complement

Table 1
Average of the relative approximation error $E_{L,\alpha}(n)$ in (5.3) over 500 trials for $n = 0$ and 3.

$n = 0$

RAE \ α / L	0	0.2	0.4	0.6	0.8
50	0.1021	0.0891	0.0777	0.0674	0.0617
70	0.0926	0.0814	0.0722	0.0643	0.0608
90	0.0857	0.0766	0.0683	0.0615	0.0593
110	0.0811	0.0736	0.0665	0.0605	0.0582
170	0.0741	0.0682	0.0631	0.0589	0.0584
230	0.0701	0.0657	0.0623	0.0593	0.0580
290	0.0677	0.0639	0.0609	0.0586	0.0579
350	0.0665	0.0630	0.0604	0.0581	0.0580

$n = 3$

RAE \ α / L	0	0.2	0.4	0.6	0.8
50	0.0846	0.0679	0.0515	0.0343	0.0213
70	0.0725	0.0574	0.0427	0.0279	0.0165
90	0.0634	0.0501	0.0368	0.0235	0.0133
110	0.0567	0.0453	0.0330	0.0207	0.0111
170	0.0461	0.0363	0.0256	0.0152	0.0077
230	0.0397	0.0315	0.0223	0.0128	0.0063
290	0.0352	0.0275	0.0190	0.0108	0.0053
350	0.0328	0.0248	0.0174	0.0097	0.0044

$\Omega_L^c = \mathbb{R}\backslash[-L, L]$ with gap $\delta_{L,\alpha} := C_\alpha L^{-\max(\alpha,1/2)}/2$, where C_α is given in (5.2). Under the above setting, the preconstruction operator in (2.12) becomes

$$S_{L,\alpha}f(x) = \sum_{k=1}^{N} |I_{\gamma_k}| f(\gamma_k) K_\Phi(x, \gamma_k) + \delta_{L,\alpha} \sum_{m=0}^{\infty} f(\gamma_m^+) K_\Phi(x, \gamma_m^+)$$
$$+\delta_{L,\alpha} \sum_{m=0}^{\infty} f(\gamma_m^-) K_\Phi(x, \gamma_m^-), \;\; f \in V_2(\Phi),$$

where $\gamma_m^\pm = \pm(L+(m+1/2)\delta_{L,\alpha})$, $m \ge 0$, $|I_{\gamma_1}| = L+\frac{\gamma_2+\gamma_1}{2}$, $|I_{\gamma_N}| = L-\frac{\gamma_N+\gamma_{N-1}}{2}$, and $|I_{\gamma_k}| = \frac{\gamma_{k+1}-\gamma_{k-1}}{2}, 2 \le k \le N-1$. Let $g_{n,L,\alpha}, n \ge 0$, be the n-th term in the iterative algorithm (3.8) with the original concentrated signal being $f_{L,\alpha}$ and the above sampling set Γ_L with the Hausdorff distance $d_H(\Gamma_L, [-L, L]) \le 3/8$. Shown in Table 1 is the average of the relative approximation error (RAE)

$$E_{L,\alpha}(n) = \|g_{n,L,\alpha} - f_{L,\alpha}\|_2 / \|f_{L,\alpha}\|_2 \tag{5.3}$$

over 500 trials for $n = 0, 3$. Our numerical simulations show that

$$E_{L,\alpha}(n) \le \varepsilon_{L,\alpha}, \;\; n \ge 3,$$

for all $50 \le L \le 350$ and $\alpha = i/5, 0 \le i \le 4$, see Table 1 and Fig. 3. This demonstrates the conclusion in Theorem 3.3 on the approximation property of $g_{n,L,\alpha}, n \ge 0$, to the original concentrated signal $f_{L,\alpha}$ for large n. We observe from Fig. 3 that $g_{n,L,\alpha}, n \ge 1$, in the iterative algorithm (3.8) provide better approximations to the original signal $f_{L,\alpha}$ than the preconstructed signal $g_{0,L,\alpha}$ does, and that $g_{n,L,\alpha}, n \ge 3$, have almost perfect approximations to the original signal $f_{L,\alpha}$ inside the domain far from the boundary.

In the second part of our numerical simulations, we consider the sampling set $\Gamma_{N,L} = \{\gamma_k, 1 \le k \le N\}$ with $\gamma_k, 1 \le k \le N$, being independently selected on $[-L, L]$ with uniform distribution. We order these random sampling positions in increasing order and denote by $-L \le \mu_1 \le \ldots \le \mu_N \le L$. Similar to our first simulation, we take uniformly sampling set on the complement $[-L, L]^c$ with gap $\delta_{L,\alpha} := C_\alpha L^{-\max(\alpha,1/2)}/2$ and C_α given in (5.2). Under the above setting, the preconstruction operator in (2.12) becomes

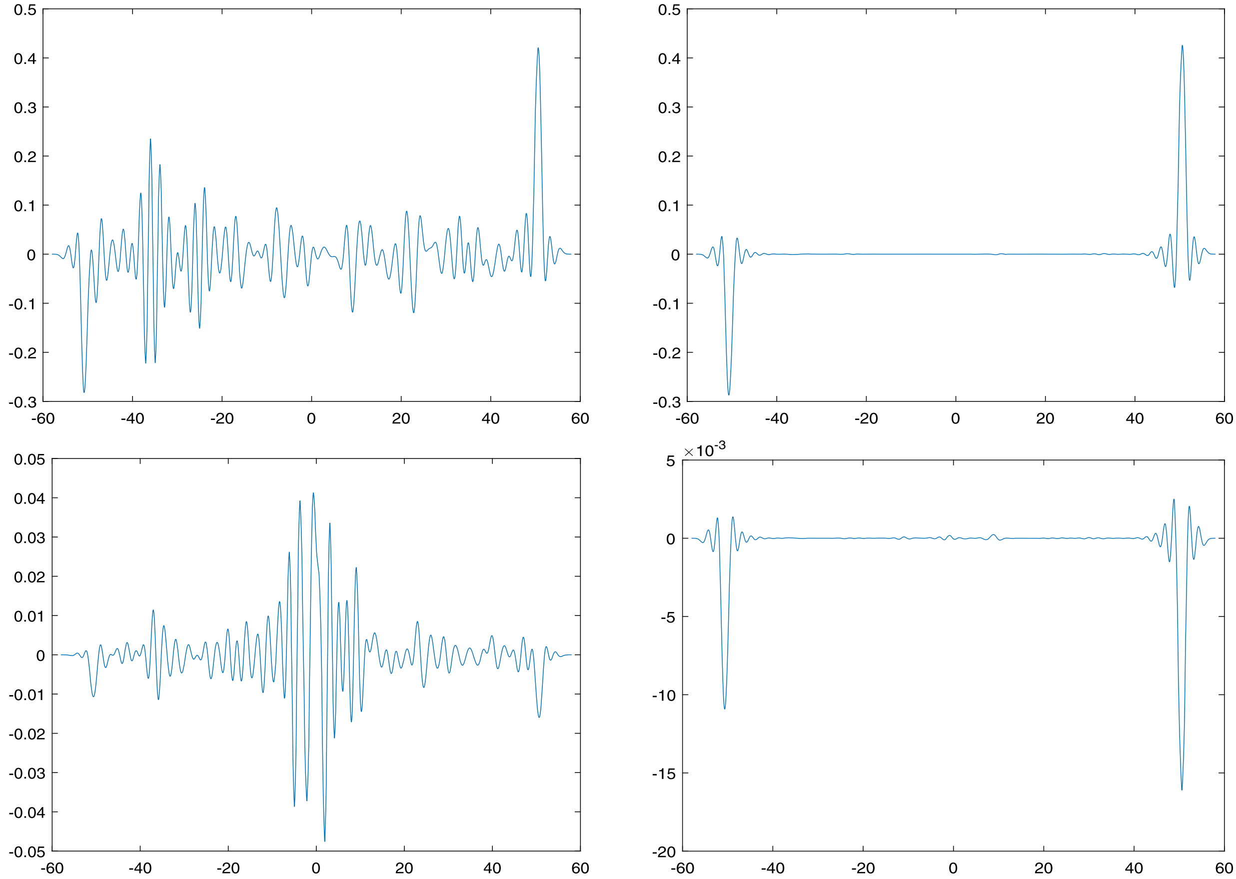

Fig. 3. Plotted on the top left and bottom left are the difference $g_{0,L,\alpha} - f_{L,\alpha}$ between the preconstructed signal $g_{0,L,\alpha}$ and the original signal $f_{L,\alpha}$ given in Fig. 1 with $\alpha = 0$ (top) and $\alpha = 0.8$ (bottom), while on the top right and bottom right are the difference $g_{n,L,\alpha} - f_{L,\alpha}$ between the constructed signal $g_{n,L,\alpha}$ at the third iteration ($n = 3$) and the original signal $f_{L,\alpha}$ with $\alpha = 0$ (top) and $\alpha = 0.8$ (bottom). Here the number of samples in the reconstruction procedure is $4L + 3 = 203$, and the relative preconstruction error $\|g_{0,L,\alpha} - f_{L,\alpha}\|_2/\|f_{L,\alpha}\|_2$, the relative approximation error $\|g_{n,L,\alpha} - f_{L,\alpha}\|_2/\|f_{L,\alpha}\|_2$ and the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c}/\|f_{L,\alpha}\|_2$ of the original signal $f_{L,\alpha}$ are $0.1050, 0.0772$ and 0.0886 respectively for the top figures, and 0.0598, 0.0126 and 0.0145 respectively for the bottom figures.

$$\tilde{S}_{N,L,\alpha} f(x) = \sum_{k=1}^{N} |I_{\mu_k}| f(\mu_k) K_\Phi(x, \mu_k) + \delta_{L,\alpha} \sum_{m=0}^{\infty} f(\gamma_m^+) K_\Phi(x, \gamma_m^+)$$
$$+ \delta_{L,\alpha} \sum_{m=0}^{\infty} f(\gamma_m^-) K_\Phi(x, \gamma_m^-), \ \ f \in V_2(\Phi),$$

where $\gamma_m^\pm = \pm(L + (m+1/2)\delta_{L,\alpha}), m \ge 0, |I_{\mu_1}| = L + \frac{\mu_2+\mu_1}{2}, |I_{\mu_N}| = L - \frac{\mu_N + \mu_{N-1}}{2}$ and $|I_{\mu_k}| = \frac{\mu_{k+1} - \mu_{k-1}}{2}, 2 \le k \le N-1$. Let $g_{N,L,\alpha}^{(n)}, n \ge 0$, be the n-th term in the iterative algorithm (3.8) with the original signal being $f_{L,\alpha}$ and the random sampling set being $\Gamma_{N,L}$ of size N. Our simulations indicate that most of signals $g_{N,L,\alpha}^{(n)}, n \ge 6$, reconstructed from the iterative algorithm (3.8) provide good approximations to the original signal $f_{L,\alpha}$ when $N \ge 12L$, see Fig. 4 for the average of the relative approximation error

$$E_{N,L,\alpha}(n) = \|g_{N,L,\alpha}^{(n)} - f_{L,\alpha}\|_2/\|f_{L,\alpha}\|_2$$

to two concentrated signals in Fig. 1 over 500 trials. Shown in Table 2 is the success rate of the iterative algorithm (3.8) to approximate the original signal $f_{L,\alpha}$ over 500 trials, where a trial is considered as successful if the relative approximation error satisfies $E_{N,L,\alpha}(n) \le \varepsilon_{L,\alpha}$ for $n = 6$. We observe that the success rate is higher as the random sampling size N increases. Recall from Fig. 2 that $\varepsilon_{L,\alpha} = C_\alpha L^{-\max(\alpha,1/2)}$

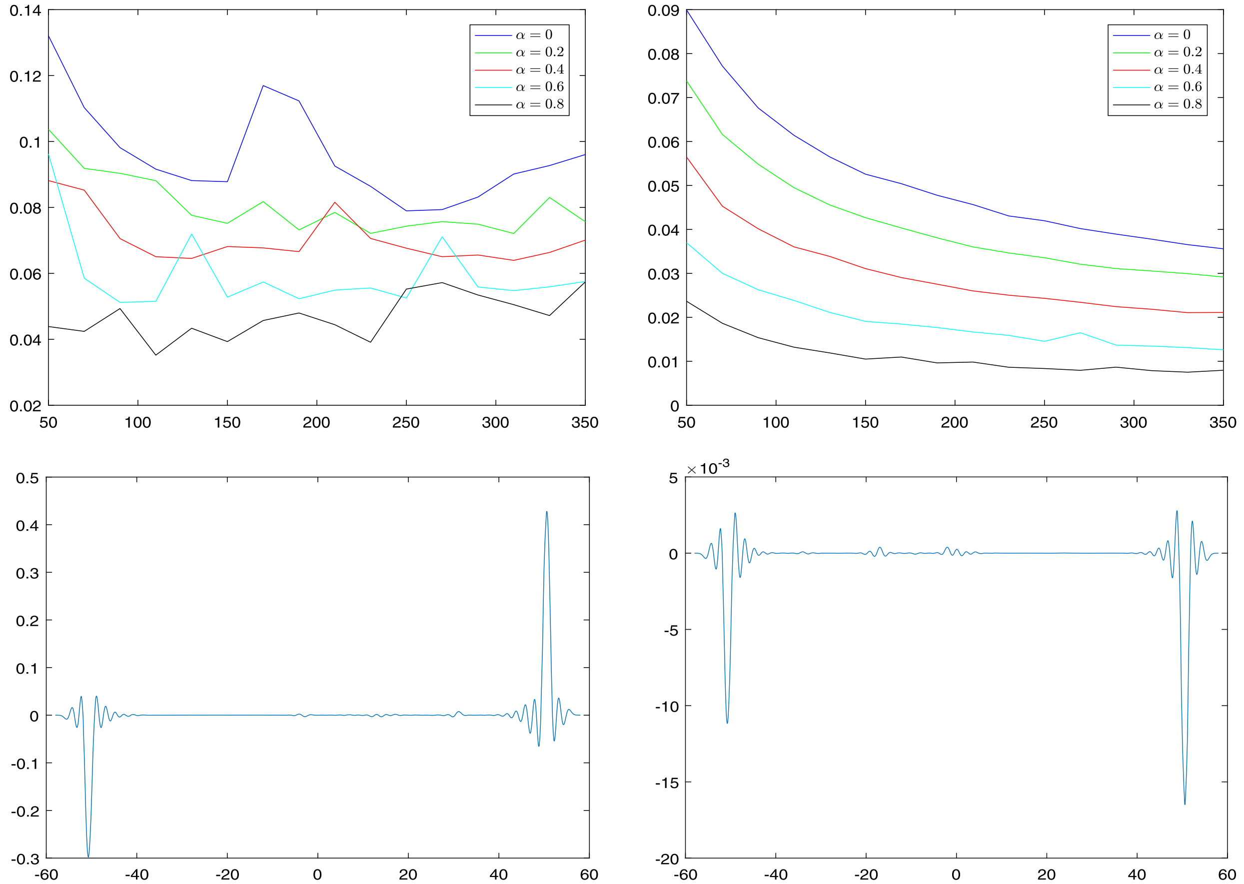

Fig. 4. Shown on the top left and right are the average of the relative approximation error $E_{N,L,\alpha}(n)$ over 500 trials with $n = 6$ and $N = 8L$ (left) and $N = 12L$ (right). Plotted on the bottom left and right are the difference $g^{(n)}_{N,L,\alpha} - f_{L,\alpha}$ between the reconstructed signal $g^{(n)}_{N,L,\alpha}$ at the sixth iteration ($n = 6$) and the original signal $f_{L,\alpha}$ given in Fig. 1, where $L = 50, \alpha = 0$ (left) and $\alpha = 0.8$ (right), cf. Fig. 3. Here the number of random samples in the reconstruction procedure is $N = 8L = 400$, and the relative approximation error $\|g^{(n)}_{N,L,\alpha} - f_{L,\alpha}\|_2/\|f_{L,\alpha}\|_2$ is 0.0783 for the bottom left figure and 0.0128 for the bottom right figure.

Table 2
Success rate to reconstruct the concentrated signals $f_{L,\alpha}$ from random samples of size $N = 8L, 12L$ over 500 trials.

N	$8L$					$12L$				
SR \ α / L	0	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.8
50	94.0	92.6	86.8	83.0	87.2	99.8	100.0	99.6	99.2	98.8
70	90.8	88.0	84.2	78.6	85.0	99.8	100.0	99.0	99.0	98.2
90	87.4	85.4	81.4	75.8	72.8	99.8	100.0	98.8	98.8	98.4
110	85.0	80.4	76.0	68.8	68.4	99.6	99.2	99.0	98.6	98.4
170	72.4	73.0	56.8	49.4	51.8	99.2	99.4	98.8	97.0	96.4
230	60.4	57.4	51.2	35.8	36.8	98.8	99.6	98.2	96.8	96.6
290	52.2	48.6	39.4	30.6	24.4	99.4	99.8	98.4	96.8	93.2
350	39.0	36.2	31.0	15.6	19.4	99.2	98.2	96.8	95.6	91.6

decreases as L and α increase. This together with Table 2 demonstrates the conclusion in Theorem 4.3 that with high probability, $g^{(n)}_{N,L,\alpha}, n \geq O(\ln L)$, provide good approximations to the original signal $f_{L,\alpha}$ when $N \geq O(L \ln L)$.

In the third part of our numerical simulations, we follow numerical simulations in the second part, except that the sampling data of a concentrated signal f on the sampling set $\Gamma_{N,L} = \{\gamma_k, 1 \leq k \leq N\}$ being corrupted by i.i.d. random noises $\xi(\gamma_k) \in [-\delta, \delta], \gamma_k \in \Gamma_{N,L}$, with uniform distribution,

$$\tilde{f}_{\gamma_k} = f(\gamma_k) + \xi(\gamma_k),\ \gamma_k \in \Gamma_{N,L}. \tag{5.4}$$

Let $\tilde{g}^{(n)}_{N,L,\alpha}, n \ge 0$, be the n-th term in the algorithm (3.16) with the original concentrated signal $f_{L,\alpha}$ in Fig. 1 and the noisy data given in (5.4). By Theorem 4.6, the reconstructed signals $\tilde{g}^{(n)}_{N,L,\alpha}, n \ge O(\ln L)$, provide good approximations to the original signal $f_{L,\alpha}$ when

$$N \ge O(L\delta^2/\varepsilon^2_{L,\alpha} \ln(L\delta^2/\varepsilon^2_{L,\alpha})) = O(L^{\max(2\alpha+1,2)}\delta^2 \ln(L^{\max(2\alpha+1,2)}\delta^2)).$$

The above conclusion is observed from Fig. 5, where $L = 50, \delta = L^{\min(1/2-\alpha,0)}/2$ and $\tilde{E}_{N,L,\alpha} = \|\tilde{g}^{(n)}_{N,L,\alpha} - f_{L,\alpha}\|/\|f_{L,\alpha}\|_2$ is the relative approximation error between the reconstructed signal $\tilde{g}^{(n)}_{N,L,\alpha}$ at the sixth iteration ($n = 6$) from noisy data and the original signal $f_{L,\alpha}$ given in Fig. 1.

6. Proofs

In this section, we collect the proofs of Propositions 2.3, 2.6, 2.7 and 4.2, and Theorems 3.1, 3.3, 3.4 and 4.6.

6.1. Proof of Proposition 2.3

Let X_δ be a discrete set of X such that (2.3) and (2.5) hold, and set

$$\tilde{X}_\delta = \{x_i \in X_\delta \cap \Omega,\ \rho(x_i, \partial\Omega) > \delta\}. \tag{6.1}$$

Let $Y_\delta = \{y_k\} \subset \partial\Omega$ and $\tilde{Y}_\delta = \{z_k\} \subset \Omega$ be chosen so that

$$\partial\Omega \subset \cup_{y_k \in Y_\delta} B(y_k, \delta) \tag{6.2}$$

and

$$B(z_k, c\delta) \subset B(y_k, \delta) \cap \Omega,\ y_k \in Y_\delta,\ z_k \in \tilde{Y}_\delta. \tag{6.3}$$

The existence of the set $\tilde{Y}_\delta$ follows from the Corkscrew condition (2.6) for the domain Ω.

Let $\Omega_\delta \subset \tilde{X}_\delta \cup \tilde{Y}_\delta$ be a maximal set such that

$$\tilde{X}_\delta \subset \Omega_\delta, \tag{6.4}$$

$$B(x_i, \delta/2) \cap B(x_j, \delta/2) = \emptyset \text{ for all distinct } x_i, x_j \in \Omega_\delta, \tag{6.5}$$

and

$$B(y, \delta/2) \cap \left(\cup_{x_i \in \Omega_\delta} B(x_i, \delta/2)\right) \ne \emptyset \ \text{ for all } y \in \tilde{X}_\delta \cup \tilde{Y}_\delta. \tag{6.6}$$

Now we show that the above maximal set Ω_δ satisfies (2.7). By (6.1), (6.3) and (6.5), we see that the maximal set Ω_δ satisfies (2.7a) and (2.7b).

Next we establish the first inequality in (2.7c). Take $x \in \Omega$. For the case that $\rho(x, \partial\Omega) > 2\delta$, there exists $x_i \in X_\delta$ by (2.5) such that $\rho(x, x_i) \le \delta$, which implies that $\rho(x_i, \partial\Omega) > \delta$ and hence $x_i \in \tilde{X}_\delta$. Then for the case that $\rho(x, \partial\Omega) > 2\delta$,

$$\rho(x, \Omega_\delta) \le \rho(x, \tilde{X}_\delta) \le \delta \tag{6.7}$$

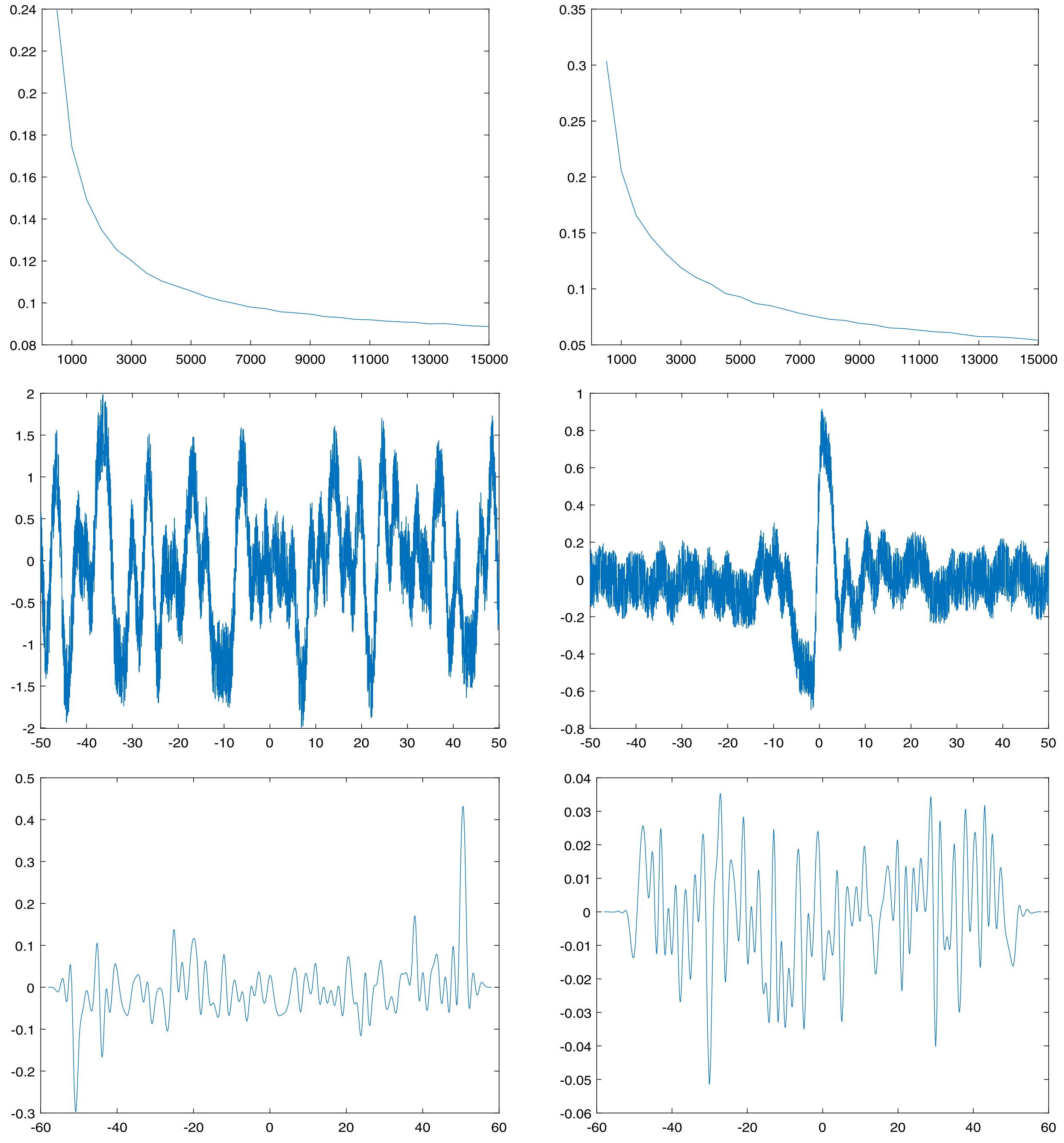

Fig. 5. Presented on the top left and right are the average of the relative approximation error $\tilde{E}_{N,L,0}$ and $\tilde{E}_{N,L,0.8}$, $L^2/5 \le N \le 6L^2$, over 100 trials respectively. Plotted on the middle left and right are noisy sampling data in (5.4) with $N = 2L^2 = 5000$ and $\alpha = 0, 0.8$ respectively. Shown on the bottom left and right are the difference $\tilde{g}^{(n)}_{N,L,\alpha} - f_{L,\alpha}$ between the reconstructed signal $\tilde{g}^{(n)}_{N,L,\alpha}$ at the sixth iteration ($n = 6$) from noisy sampling data in the middle figures and the original signal $f_{L,\alpha}$, where $\alpha = 0, 0.8$ for left/right figures respectively, cf. Figs. 3 and 4. The relative approximation error $\|g^{(n)}_{N,L,\alpha} - f_{L,\alpha}\|_2 / \|f_{L,\alpha}\|_2$ and the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega_L^c} / \|f_{L,\alpha}\|_2$ are 0.1036 and 0.0886 for the bottom left figure and 0.0948 and 0.0145 for the bottom right figure.

by (6.4). For the case that $\rho(x, \partial\Omega) \le 2\delta$, there exists $y_k \in Y_\delta$ such that $\rho(y_k, x) \le 3\delta$ by the covering property (6.2), which together with (6.3) implies that

$$\rho(x, \tilde{Y}_\delta) \le 4\delta. \tag{6.8}$$

By the maximal property (6.6), we have

$$\rho(y, \Omega_\delta) \le \delta \text{ for all } y \in \tilde{X}_\delta \cup \tilde{Y}_\delta. \tag{6.9}$$

Combining (6.8) and (6.9), we show that

$$\rho(x, \Omega_\delta) \le 5\delta \tag{6.10}$$

for the case that $\rho(x, \partial\Omega) \le 2\delta$. Therefore the first inequality in (2.7c) follows from (6.7) and (6.10).

Finally we establish the second inequality in (2.7c). Take $x \in \Omega$. We obtain from (2.2) and (6.5) that

$$\begin{aligned}\sum_{x_i \in \Omega_\delta} \chi_{B(x_i,5\delta)}(x) &\le \sum_{x_i \in B(x,5\delta)\cap\Omega_\delta} \frac{\mu(B(x_i, \delta/2))}{D_1(\mu)(\delta/2)^d} = \frac{\mu\big(\cup_{x_i \in B(x,5\delta)\cap\Omega_\delta} B(x_i, \delta/2)\big)}{D_1(\mu)(\delta/2)^d} \\ &\le \frac{\mu(B(x, 11\delta/2))}{D_1(\mu)(\delta/2)^d} \le 11^d \frac{D_2(\mu)}{D_1(\mu)}.\end{aligned}$$

This completes the proof.

6.2. Proof of Proposition 2.6

Following the argument used in [40], we have

$$\|Tf\|_p \le \|K\|_{\mathcal{S}} \|f\|_p \text{ for all } f \in L^p. \tag{6.11}$$

Then V is a closed subspace of L^p.

By the interpolation property between L^p and L^∞ [14], it suffices to prove

$$\|f\|_\infty \le (D_1(\mu))^{-1/p} \|K\|_{\mathcal{S},\theta} \|f\|_p, \;\; f \in V_p. \tag{6.12}$$

Take $x \in X$. By (1.1), we obtain

$$|f(x)| = \Big| \int_X K(x, y) f(y) d\mu(y) \Big| \le \|K(x, \cdot)\|_{p'} \|f\|_p, \tag{6.13}$$

where $1/p + 1/p' = 1$.

By the definition of the modulus of continuity, we have

$$|K(x, y')| \le \omega_1(K)(x, y) + |K(x, y)|, \;\; y \in B(y', 1).$$

This together with (2.2) and (2.9) implies that

$$\begin{aligned}\|K(x, \cdot)\|_\infty &\le \sup_{y' \in X} \frac{1}{\mu(B(y', 1))} \int_{B(y',1)} \big(\omega_1(K)(x, y) + |K(x, y)|\big) d\mu(y) \\ &\le (D_1(\mu))^{-1} \|K\|_{\mathcal{S},\theta}.\end{aligned} \tag{6.14}$$

Obviously, $\|K(x, \cdot)\|_1 \le \|K\|_{\mathcal{S},\theta}$. Interpolating the L^1 and L^∞ norms of $K(x, \cdot)$ yields

$$\sup_{x \in X} \|K(x, \cdot)\|_{p'} \le (D_1(\mu))^{-1/p} \|K\|_{\mathcal{S},\theta}. \tag{6.15}$$

Combining (6.13) and (6.15) proves (6.12) and hence (2.10).

6.3. Proof of Proposition 2.7

We follow the argument used in [40], where a similar result is established for a reproducing kernel space on the Euclidean space $\mathbb{R}^d$. Set $\delta = \delta(\Gamma), \Gamma_\Omega = \Gamma \cap \Omega$ and $\Gamma_{\Omega^c} = \Gamma \cap \Omega^c$. Then it follows from (2.8b) that $\rho(y,\gamma) = \rho(y,\Gamma_\Omega) \le \delta$ for every $\gamma \in \Gamma_\Omega$ and $y \in I_\gamma$, and that $\rho(y,\gamma) = \rho(y,\Gamma_{\Omega^c}) \le \delta$ for every $\gamma \in \Gamma_{\Omega^c}$ and $y \in I_\gamma$. Therefore $\rho(y,\gamma) \le \delta$ for all $\gamma \in \Gamma$ and $y \in I_\gamma$. Hence for all $x \in X$, we obtain

$$\begin{aligned}|S_\Gamma f(x) - f(x)| &\le \sum_{\gamma\in\Gamma} \int_{I_\gamma} |K(x,\gamma)f(\gamma) - K(x,y)f(y)| d\mu(y)\\ &\le \sum_{\gamma\in\Gamma} \int_{I_\gamma} |K(x,\gamma) - K(x,y)||f(y)| d\mu(y)\\ &\quad + \sum_{\gamma\in\Gamma} \int_{I_\gamma} |K(x,\gamma)|\Big(\int_X |K(\gamma,z) - K(y,z)||f(z)|d\mu(z)\Big) d\mu(y)\\ &\le \int_X \omega_\delta(K)(x,y)|f(y)|d\mu(y)\\ &\quad + \int_X\int_X \big(|K(x,y)| + \omega_\delta(K)(x,y)\big)\omega_\delta(K)(y,z)|f(z)|d\mu(z)d\mu(y)\\ &=: \int_X K_\Gamma(x,y)|f(y)|d\mu(y),\ \ f \in V_p. \end{aligned} \tag{6.16}$$

Observe that

$$\|K_\Gamma\|_{\mathcal{S}} \le \|\omega_\delta(K)\|_{\mathcal{S}}(1 + \|K\|_{\mathcal{S}} + \|\omega_\delta(K)\|_{\mathcal{S}}) \le \|K\|^2_{\mathcal{S},\theta}\delta^\theta. \tag{6.17}$$

By (6.16) and (6.17), we obtain the following crucial estimate in the proof,

$$\|S_\Gamma f - f\|_p \le \|\omega_\delta(K)\|_{\mathcal{S}}(1 + \|K\|_{\mathcal{S}} + \|\omega_\delta(K)\|_{\mathcal{S}})\|f\|_p \le \|K\|^2_{\mathcal{S},\theta}\delta^\theta\|f\|_p \tag{6.18}$$

for $f \in V_p$.

By (2.11), we can prove by induction on $n \ge 1$ that

$$f_n - f_{n-1} = (I - S_\Gamma)^{n-1}(f_1 - f_0) = (I - S_\Gamma)^n S_\Gamma f \tag{6.19}$$

and

$$f_n = \sum_{k=0}^{n} (I - S_\Gamma)^k f_0 = \Big(T + \sum_{k=1}^{n} (T - S_\Gamma)^k\Big) S_\Gamma f,\ \ n \ge 1. \tag{6.20}$$

Define

$$R := T + \sum_{k=1}^{\infty} (T - S_\Gamma)^k. \tag{6.21}$$

Then one may verify that R is a bounded operator on L^p by (6.18),

$$\|Rf\|_p \le \sum_{k=0}^{\infty} \|(I - S_\Gamma)^k T f\|_p \le \big(1 - \|K\|_{\mathcal{S},\theta}^2 \delta^\theta\big)^{-1} \|Tf\|_p, \; f \in L^p, \tag{6.22}$$

and R is a pseudo-inverse of the preconstruction operator S_Γ,

$$RS_\Gamma f = S_\Gamma R f = f, \; f \in V_p. \tag{6.23}$$

By (6.18), (6.20), (6.21) and (6.23), we have

$$\begin{aligned}\|f_n - f\|_p &\le \sum_{k=n+1}^{\infty} \|(I - S_\Gamma)^k S_\Gamma f\|_p \\ &\le \frac{1 + \|K\|_{\mathcal{S},\theta}^2 \delta^\theta}{1 - \|K\|_{\mathcal{S},\theta}^2 \delta^\theta} \big(\|K\|_{\mathcal{S},\theta}^2 \delta^\theta\big)^{n+1} \|f\|_p, \; f \in V_p.\end{aligned}$$

This proves that $f_n, n \ge 0$, converge to f exponentially.

6.4. Proof of Theorem 3.1

For $h \in V_p$, let h_I be as in (3.4) and set $\delta = d_H(\Gamma_\Omega, \Omega)$. Following the argument after the statement of Theorem 3.1, it suffices to prove

$$\|h_I - h\|_{p,\Omega} \le \|K\|_{\mathcal{S},\theta} \delta^\theta \|h\|_p, \; h \in V_p. \tag{6.24}$$

For any $\gamma \in \Gamma_\Omega$ and $x \in I_\gamma$, it follows from the Voronoi partition property (2.8) that $\rho(x, \gamma) = \rho(x, \Gamma_\Omega) \le \delta$. This together with (1.1) implies that

$$\begin{aligned}|h_I(x) - h(x)| &= \Big| \int_X \sum_{\gamma \in \Gamma_\Omega} \big(K(\gamma, y) - K(x, y)\big) \chi_{I_\gamma}(x) h(y) d\mu(y) \Big| \\ &\le \int_X \omega_\delta(K)(x, y) |h(y)| d\mu(y), \; x \in \Omega.\end{aligned} \tag{6.25}$$

Combining (6.11) and (6.25) proves (6.24).

For any $f, g \in V_p$, by (6.24) with h_I and h replaced by $f_I - g_I$ and $f - g$ respectively, we have

$$\|f_I - g_I - (f - g)\|_{p,\Omega} \le \|K\|_{\mathcal{S},\theta} \delta^\theta \|f - g\|_p, \tag{6.26}$$

which implies

$$\begin{aligned}\|f_I - g_I\|_{p,\Omega} &\le \|f - g\|_{p,\Omega} + \|K\|_{\mathcal{S},\theta} \delta^\theta \|f - g\|_p \\ &\le \big(1 + \|K\|_{\mathcal{S},\theta} \delta^\theta\big) \|f - g\|_p.\end{aligned} \tag{6.27}$$

Moreover, for any $f, g \in V_{p,\Omega,\varepsilon}$, we obtain by (6.26) that

$$\begin{aligned}\|f_I - g_I\|_{p,\Omega} &\ge \|f - g\|_{p,\Omega} - \|K\|_{\mathcal{S},\theta} \delta^\theta \|f - g\|_p \\ &\ge \|f - g\|_p - \|f - g\|_{p,\Omega^c} - \|K\|_{\mathcal{S},\theta} \delta^\theta \|f - g\|_p \\ &\ge \big(1 - \varepsilon - \|K\|_{\mathcal{S},\theta} \delta^\theta\big) \|f - g\|_p - 2\varepsilon \min(\|f\|_p, \|g\|_p),\end{aligned} \tag{6.28}$$

where we use

$$\begin{aligned}\|f-g\|_{p,\Omega^c} &\le \|f\|_{p,\Omega^c}+\|g\|_{p,\Omega^c}\le \varepsilon(\|f\|_p+\|g\|_p)\\ &= \varepsilon\big|\|f\|_p-\|g\|_p\big| + 2\varepsilon\min(\|f\|_p,\|g\|_p)\\ &\le \varepsilon\|f-g\|_p + 2\varepsilon\min(\|f\|_p,\|g\|_p).\end{aligned}$$

Combining (3.6), (6.27) and (6.28) completes the proof.

6.5. Proof of Theorem 3.3

For $f\in V_{p,\Omega,\varepsilon}$ and a sampling set Γ_{Ω^c} outside the domain Ω, define

$$f_0^c=\sum_{\gamma\in\Gamma_{\Omega^c}}\mu(I_\gamma)f(\gamma)K(\cdot,\gamma)\in V_p \ \text{ and } \ f_I^c=\sum_{\gamma\in\Gamma_{\Omega^c}} f(\gamma)\chi_{I_\gamma}.$$

One may verify easily that

$$\|f_0^c\|_p\le \|K\|_{\mathcal{S},\theta}\|f_I^c\|_p=\|K\|_{\mathcal{S},\theta}\|f_I^c\|_{p,\Omega^c}. \tag{6.29}$$

Set $\tilde\delta=d_H(\Gamma_{\Omega^c},\Omega^c)$. Applying similar argument used to prove (3.7) and using (3.9), we obtain

$$\|f_I^c-f\|_{p,\Omega^c}\le \|K\|_{\mathcal{S},\theta}\tilde\delta^\theta\|f\|_p\le\varepsilon\|f\|_p.$$

This together with (1.3) and (6.29) implies that

$$\|f_0^c\|_p\le \|K\|_{\mathcal{S},\theta}\|f_I^c\|_{p,\Omega^c}\le 2\|K\|_{\mathcal{S},\theta}\varepsilon\|f\|_p. \tag{6.30}$$

Set $\delta=\max(d_H(\Gamma_\Omega,\Omega),d_H(\Gamma_{\Omega^c},\Omega^c))$. Define $f_n, n\ge 0$, as in (2.11) with $\Gamma=\Gamma_\Omega\cup\Gamma_{\Omega^c}$. Then it follows from (3.9), (3.10) and Proposition 2.7 that $f_n, n\ge 0$, converge to f exponentially,

$$\|f_n-f\|_p\le \frac{2}{1-\|K\|^2_{\mathcal{S},\theta}\delta^\theta}\big(\|K\|^2_{\mathcal{S},\theta}\delta^\theta\big)^{n+1}\|f\|_p,\ n\ge 0. \tag{6.31}$$

Observe that $f_0=g_0+f_0^c$ and

$$f_n=g_n+\Big(\sum_{k=0}^n(I-S_\Gamma)^k\Big)f_0^c, n\ge 1.$$

Therefore

$$\|f_n-g_n\|_p\le\sum_{k=0}^n\big(\|K\|^2_{\mathcal{S},\theta}\delta^\theta\big)^k\|f_0^c\|_p\le\frac{\|f_0^c\|_p}{1-\|K\|^2_{\mathcal{S},\theta}\delta^\theta}\le 2C_0\varepsilon\|f\|_p \tag{6.32}$$

by (3.9), (3.10), (6.18) and (6.30).

Combining (6.31), (6.32) and then using (3.11), we have

$$\|g_n-f\|_p\le\|f_n-f\|_p+\|g_n-f_n\|_p\le 4C_0\varepsilon\|f\|_p,\ n\ge 0.$$

This proves (3.13).

The conclusion (3.12) is trivial for $\varepsilon \geq 1/(9C_0)$. Now we consider the case that $\varepsilon < 1/(9C_0)$. By (3.13) and the assumption $f \in V_{p,\Omega,\varepsilon}$, we have

$$\|g_n\|_{p,\Omega^c} \leq \|f - g_n\|_p + \|f\|_{p,\Omega^c} \leq 5C_0\varepsilon\|f\|_p \tag{6.33}$$

and

$$\|g_n\|_p \geq \|f\|_p - \|g_n - f\|_p \geq (1 - 4C_0\varepsilon)\|f\|_p \geq \frac{5}{9}\|f\|_p. \tag{6.34}$$

Combining (6.33) and (6.34) proves (3.12), and hence the reconstructed signals g_n in (3.8) are $(9C_0\varepsilon)$-concentrated signals in V_p.

6.6. Proof of Theorem 3.4

Let $g_n, n \geq 0$, be as in (3.8). By Theorem 3.3, it suffices to prove that

$$\|g_n - \tilde{g}_n\|_p \leq C_0\|\boldsymbol{\xi}\|_{p,\mu(\Gamma_\Omega)}. \tag{6.35}$$

Following similar argument used to establish (6.32), we obtain

$$\|g_n - \tilde{g}_n\|_p \leq \big(1 - \|K\|^2_{\mathcal{S},\theta}\delta^\theta\big)^{-1}\Big\|\sum_{\gamma\in\Gamma_\Omega}\mu(I_\gamma)\xi(\gamma)K(\cdot,\gamma)\Big\|_p,$$

where δ is in (6.31). This together with (6.29) proves (6.35) and hence completes the proof.

6.7. Proof of Proposition 4.2

Let $\Omega_{\delta_1/10}$ be the discrete set in Proposition 2.3 with δ replaced by $\delta_1/10$. By (2.7c), we have

$$\begin{aligned}\mathbb{P}\big\{d_H(\Gamma_\Omega,\Omega) > \delta_1\big\} &\leq \mathbb{P}\big\{B(x_i,\delta_1/2)\cap\Gamma_\Omega = \emptyset \text{ for some } x_i \in \Omega_{\delta_1/10}\big\}\\ &\leq \sum_{x_i\in\Omega_{\delta_1/10}}\mathbb{P}\big\{B(x_i,\delta_1/2)\cap\Gamma_\Omega = \emptyset\big\}.\end{aligned} \tag{6.36}$$

We observe that

$$\mathbb{P}\big\{B(x_i,\delta_1/2)\cap\Gamma_\Omega = \emptyset\big\} \leq \Big(1 - \frac{\mu(B(x_i,\delta_1/2)\cap\Omega)}{\mu(\Omega)}\Big)^N \leq \Big(1 - \frac{D_1(\mu)(c\delta_1/10)^d}{\mu(\Omega)}\Big)^N \tag{6.37}$$

by (2.7a) and the assumption on the random sampling, and also that

$$\#\Omega_{\delta_1/10} \leq \sum_{x_i\in\Omega_{\delta_1/10}}\frac{\mu(B(x_i,c\delta_1/10))}{D_1(\mu)(c\delta_1/10)^d} = \frac{\mu(\cup_{x_i\in\Omega_{\delta_1/10}}B(x_i,c\delta_1/10))}{D_1(\mu)(c\delta/10)^d} \leq \frac{10^d\mu(\Omega)}{c^dD_1(\mu)\delta_1^d} \tag{6.38}$$

by Assumption 2.1 and Proposition 2.3. Combining (6.36), (6.37) and (6.38) completes the proof.

6.8. Proof of Theorem 4.6

By (4.17) and Proposition 4.2, we have

$$\mathbb{P}\{d_H(\Gamma_\Omega, \Omega) > \tilde{\delta}_1\} \le \frac{10^d \mu(\Omega)}{c^d D_1(\mu)\tilde{\delta}_1^d}\Big(1 - \frac{c^d D_1(\mu)\tilde{\delta}_1^d}{10^d \mu(\Omega)}\Big)^N \le \tau. \tag{6.39}$$

Therefore it suffices to establish the conclusion under the hypothesis that

$$d_H(\Gamma_\Omega, \Omega) \le \tilde{\delta}_1. \tag{6.40}$$

Take $x \in X$ and let g_n and $\tilde{g}_n, n \ge 0$, be defined by (3.8) and (3.16) respectively. For a sampling set Γ_Ω with $d_H(\Gamma_\Omega, \Omega) \le \tilde{\delta}_1$, we obtain from Proposition 2.6 and Theorem 3.3 that

$$|g_n(x) - f(x)| \le 8(D_1(\mu))^{-1/p}\|K\|^2_{\mathcal{S},\theta}\varepsilon\|f\|_p \tag{6.41}$$

for all integers n satisfying (4.6).

Set $h_n = \tilde{g}_n - g_n, n \ge 0$. Following the argument used in the proof of Proposition 2.7, we can show that

$$h_n(x) = \sum_{\gamma\in\Gamma_\Omega} \xi(\gamma)\mu(I_\gamma)\int_X K_{n,\Gamma_\Omega}(x, y)K(y, \gamma)d\mu(y), \tag{6.42}$$

and

$$\|K_{n,\Gamma_\Omega}\|_{\mathcal{S}} \le \sum_{k=0}^{n}\Big(\|K\|^2_{\mathcal{S},\theta}\big(\max(d_H(\Gamma_\Omega, \Omega), d_H(\Gamma_{\Omega^c}, \Omega^c))\big)^\theta\Big)^k \le 2, \tag{6.43}$$

where the last inequality follows from (3.9), (4.16) and (6.40).

By (4.15) and (6.42), we have

$$\mathbb{E}_{\boldsymbol{\xi}}\{h_n(x)|d_H(\Gamma_\Omega, \Omega) \le \tilde{\delta}_1\} = \sum_{\gamma\in\Gamma_\Omega} \mathbb{E}_{\boldsymbol{\xi}}(\xi(\gamma))\mu(I_\gamma)\int_X K_{n,\Gamma_\Omega}(x, y)K(y, \gamma)d\mu(y) = 0, \tag{6.44}$$

and

$$\mathrm{Var}_{\boldsymbol{\xi}}\{h_n(x)|d_H(\Gamma_\Omega, \Omega) \le \tilde{\delta}_1\} = \sigma^2 \sum_{\gamma\in\Gamma_\Omega} |\mu(I_\gamma)|^2\Big|\int_X K_{n,\Gamma_\Omega}(x, y)K(y, \gamma)d\mu(y)\Big|^2. \tag{6.45}$$

For a sampling set Γ_Ω satisfying (6.40), we obtain from (2.2), (2.8), (6.14) and (6.43) that

$$\mu(I_\gamma) \le D_2(\mu)\tilde{\delta}_1^d \tag{6.46}$$

and

$$\Big|\int_X K_{n,\Gamma_\Omega}(x, y)K(y, \gamma)d\mu(y)\Big| \le \|K_{n,\Gamma_\Omega}\|_{\mathcal{S}}\|K(\cdot, \gamma)\|_\infty \le 2(D_1(\mu))^{-1}\|K\|_{\mathcal{S},\theta} \tag{6.47}$$

for all $\gamma \in \Gamma_\Omega$. Similarly, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_\Omega} \mu(I_\gamma)\Big|\int_X K_{n,\Gamma_\Omega}(x,y)K(y,\gamma)d\mu(y)\Big| \\
&\le \int_\Omega\int_X |K_{n,\Gamma_\Omega}(x,y)|(|K(y,z)|+\omega_\delta(K)(y,z))d\mu(y)d\mu(z) \\
&\le \|K_{n,\Gamma_\Omega}\|_{\mathcal{S}}\big(\|K\|_{\mathcal{S}}+\|\omega_\delta(K)\|_{\mathcal{S}}\big) \le 2\|K\|_{\mathcal{S},\theta}, \qquad (6.48)
\end{aligned}$$

where $\delta = \max(d_H(\Gamma_\Omega,\Omega), d_H(\Gamma_{\Omega^c},\Omega^c))$. Combining (6.44)–(6.48), we get

$$\mathrm{Var}_{\boldsymbol{\xi}}\big\{h_n(x)|d_H(\Gamma_\Omega,\Omega)\le\tilde\delta_1\big\} \le 4\sigma^2(D_1(\mu))^{-1}D_2(\mu)\|K\|^2_{\mathcal{S},\theta}\tilde\delta_1^d.$$

Then applying Chebyshev inequality yields

$$\mathbb{P}_{\boldsymbol{\xi}}\big\{|h_n(x)|\ge 2(D_1(\mu))^{-1/p}\|K\|_{\mathcal{S},\theta}\varepsilon\|f\|_p\ |d_H(\Gamma_\Omega,\Omega)\le\tilde\delta_1\big\} \le \frac{\sigma^2 D_2(\mu)\tilde\delta_1^d}{(D_1(\mu))^{1-2/p}\varepsilon^2\|f\|_p^2} \le \tau, \qquad (6.49)$$

where the second inequality holds by (4.16). Combining (6.39), (6.41) and (6.49) completes the proof.

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References

[1] B. Adcock, M. Gataric, A. Hansen, On stable reconstructions from nonuniform Fourier measurements, SIAM J. Imaging Sci. 7 (2014) 1690–1723.

[2] R. Alaifari, I. Daubechies, P. Grohs, R. Yin, Stable phase retrieval in infinite dimensions, Found. Comput. Math. 19 (2019) 869–900.

[3] R. Alaifari, P. Grohs, Gabor phase retrieval is severely ill-posed, Appl. Comput. Harmon. Anal. 50 (2021) 401–419.

[4] A. Aldroubi, H. Feichtinger, Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: the L^p theory, Proc. Am. Math. Soc. 126 (1998) 2677–2686.

[5] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (2001) 585–620.

[6] A. Aldroubi, L. Huang, I. Krishtal, A. Ledeczi, R.R. Lederman, P. Volgyesi, Dynamical sampling with additive random noise, Sampl. Theory Signal Image Process. 17 (2018) 153–182.

[7] A. Aldroubi, C. Leonetti, Q. Sun, Error analysis of frame reconstruction from noisy samples, IEEE Trans. Signal Process. 56 (2008) 2311–2325.

[8] A. Aldroubi, Q. Sun, W.-S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, J. Fourier Anal. Appl. 11 (2005) 215–244.

[9] A. Aldroubi, Q. Sun, W.-S. Tang, Nonuniform average sampling and reconstruction in multiply generated shift-invariant spaces, Constr. Approx. 20 (2004) 173–189.

[10] N.D. Atreas, On a class of non-uniform average sampling expansions and partial reconstruction in subspaces of $L_2(\mathbb{R})$, Adv. Comput. Math. 36 (2012) 21–38.

[11] R.F. Bass, K. Gröchenig, Relevant sampling of band-limited functions, Ill. J. Math. 57 (2013) 43–58.

[12] R.F. Bass, K. Gröchenig, Random sampling of bandlimited functions, Isr. J. Math. 177 (2010) 1–28.

[13] R.F. Bass, K. Gröchenig, Random sampling of multivariate trigonometric polynomials, SIAM J. Math. Anal. 36 (2005) 773–795.

[14] J. Bergh, J. Löfström, Interpolation Spaces, an Introduction, Grundlehren der Mathematischen Wissenschaften, vol. 223, Springer-Verlag, Berlin-New York, 1976.

[15] F.J. Beutler, O.A.Z. Leneman, Random sampling of random processes: stationary point processes, Inf. Control 9 (1966) 325–346.

[16] E.J. Candès, M.B. Wakin, An introduction to compressive sampling, IEEE Signal Process. Mag. 25 (2008) 21–30.
[17] S.H. Chan, T. Zickler, Y.M. Lu, Monte Carlo non-local means: random sampling for large-scale image filtering, IEEE Trans. Image Process. 23 (2014) 3711–3725.
[18] Y. Chen, S. Fomel, Random noise attenuation using local signal-and-noise orthogonalization, Geophysics 80 (2015) WD1–WD9.
[19] C. Cheng, Y. Jiang, Q. Sun, Spatially distributed sampling and reconstruction, Appl. Comput. Harmon. Anal. 47 (2019) 109–148.
[20] C. Cheng, Y. Jiang, Q. Sun, Sampling and Galerkin reconstruction in reproducing kernel spaces, Appl. Comput. Harmon. Anal. 41 (2016) 638–659.
[21] F. Cucker, S. Smale, On the mathematical foundations of learning, Bull. Am. Math. Soc. (N.S.) 39 (2002) 1–49.
[22] D. Deng, Y. Han, Harmonic Analysis on Spaces of Homogeneous Type, Springer, Berlin, 2009.
[23] P.L. Dragotti, M. Vetterli, T. Blu, Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets Strang-Fix, IEEE Trans. Signal Process. 55 (2007) 1741–1757.
[24] K.R. Duffy, J. Li, M. Médard, Capacity-achieving guessing random additive noise decoding, IEEE Trans. Inf. Theory 65 (2019) 4023–4040.
[25] H. Feichtinger, K. Gröchenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, SIAM J. Math. Anal. 23 (1992) 244–261.
[26] S. Foucart, H. Rauhut, A Mathematical Introduction to Compressive Sensing, Birkhäuser, New York, 2013.
[27] H. Führ, J. Xian, Relevant sampling in finitely generated shift-invariant spaces, J. Approx. Theory 240 (2019) 1–15.
[28] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.
[29] K. Gröchenig, M. Leinert, Wiener's lemma for twisted convolution and Gabor frames, J. Am. Math. Soc. 17 (2004) 1–18.
[30] P. Grohs, M. Rathmair, Stable Gabor phase retrieval and spectral clustering, Commun. Pure Appl. Math. 72 (2019) 981–1043.
[31] J.A. Hogan, J.D. Lakey, Duration and Bandwidth Limiting: Prolate Functions, Sampling, and Applications, Birkhäuser, New York, 2012.
[32] S. Jaffard, Propriétés des matrices bien localisées près de leur diagonale et quelques applications, Ann. Inst. Henri Poincaré 7 (1990) 461–476.
[33] P. Jaming, A. Karoui, S. Spektor, The approximation of almost time- and band-limited functions by their expansion in some orthogonal polynomials bases, J. Approx. Theory 212 (2016) 41–65.
[34] Y. Jiang, W. Li, Random sampling in weighted reproducing kernel subspaces of $L^p_\nu(\mathbb{R}^d)$, arXiv:2003.02993.
[35] A. Kumar, S. Sampath, Sampling and average sampling in quasi shift-invariant spaces, Numer. Funct. Anal. Optim. 41 (2020) 1246–1271.
[36] Y. Li, J. Wen, J. Xian, Reconstruction from convolution random sampling in local shift invariant spaces, Inverse Probl. 35 (2019) 125008.
[37] Y. Lu, J. Xian, Non-uniform random sampling and reconstruction in signal spaces with finite rate of innovation, Acta Appl. Math. 169 (2020) 247–277.
[38] F.A. Marvasti, Spectral analysis of random sampling and error free recovery by an iterative method, Trans. Inst. Electron. Commun. Eng. Jpn., Sect. E 69 (1986) 79–82.
[39] E. Masry, Random sampling and reconstruction of spectra, Inf. Control 19 (1971) 275–288.
[40] M.Z. Nashed, Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$, J. Funct. Anal. 258 (2010) 2422–2452.
[41] F. Particke, A. Schmidt, C. Rugheimer, T. Mahr, Additive random sampling for radar signal processing, in: 2017 IEEE-APS Topical Conference on Antennas and Propagation in Wireless Communications (APWC), Verona, 2017, pp. 93–96.
[42] D. Patel, S. Sampath, Random sampling on reproducing kernel subspaces of $L^p(\mathbb{R}^n)$, J. Math. Anal. Appl. 491 (2020) 124270.
[43] H. Rauhut, Random sampling of sparse trigonometric polynomials, Appl. Comput. Harmon. Anal. 22 (2007) 16–42.
[44] C.E. Shin, Q. Sun, Polynomial control on stability, inversion and powers of matrices on simple graphs, J. Funct. Anal. 276 (2019) 148–182.
[45] S. Smale, D.X. Zhou, Shannon sampling II: connections to learning theory, Appl. Comput. Harmon. Anal. 19 (2005) 285–302.
[46] S. Smale, D.X. Zhou, Shannon sampling and function reconstruction from point values, Bull. Am. Math. Soc. (N.S.) 41 (2004) 279–305.
[47] Q. Sun, Wiener's lemma for infinite matrices II, Constr. Approx. 34 (2011) 209–235.
[48] Q. Sun, Frames in spaces with finite rate of innovation, Adv. Comput. Math. 28 (2008) 301–329.
[49] Q. Sun, Wiener's lemma for infinite matrices, Trans. Am. Math. Soc. 359 (2007) 3099–3123.
[50] Q. Sun, Nonuniform average sampling and reconstruction of signals with finite rate of innovation, SIAM J. Math. Anal. 38 (2006) 1389–1422.
[51] Q. Sun, J. Xian, Rate of innovation for (non-)periodic signals and optimal lower stability bound for filtering, J. Fourier Anal. Appl. 20 (2014) 119–134.
[52] J.T. Tyson, Metric and geometric quasiconformality in Ahlfors regular Loewner spaces, Conform. Geom. Dyn. 5 (2001) 21–73.
[53] M. Unser, Sampling—50 years after Shannon, Proc. IEEE 88 (2000) 569–587.
[54] M. Unser, Splines: a perfect fit for signal and image processing, IEEE Signal Process. Mag. 16 (1999) 22–38.
[55] G.A. Velasco, Relevant sampling of the short-time Fourier transform of time-frequency localized functions, arXiv:1707.09634.
[56] M. Vetterli, P. Marziliano, T. Blu, Sampling signals with finite rate of innovation, IEEE Trans. Signal Process. 50 (2002) 1417–1428.

[57] J.S. Vitter, Random sampling with a reservoir, ACM Trans. Math. Softw. 11 (1985) 37–57.
[58] J. Xian, S. Li, Sampling set conditions in weighted multiply generated shift-invariant spaces and their applications, Appl. Comput. Harmon. Anal. 23 (2007) 171–180.
[59] J. Yang, W. Wei, Random sampling in shift invariant spaces, J. Math. Anal. Appl. 398 (2013) 26–34.
[60] N. Zarmehi, S. Shahsavari, F. Marvasti, Comparison of uniform and random sampling for speech and music signals, in: 2017 International Conference on Sampling Theory and Applications (SampTA), Tallinn, 2017, pp. 552–555.