Editors' Suggestion

Viscous Maxwell-Chern-Simons theory for topological electromagnetic phases of matter

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Chern-Simons theories have been very successful in explaining integer and fractional quantum Hall phases of matter, topological insulators, and Weyl semimetals. However, it remains an open question as to whether Chern-Simons theories can be adapted to topological photonics. We develop a viscous Maxwell-Chern-Simons theory to capture the fundamental physics of a topological electromagnetic phase of matter. We show the existence of a unique spin-1 skyrmion in the viscous Hall fluid arising from a photonic Zeeman interaction in momentum space. Our work bridges the gap between electromagnetic and condensed matter topological physics while also demonstrating the central role of photon spin-1 quantization in identifying new phases of matter.

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I. INTRODUCTION

Chern-Simons theory has been studied in condensed matter and high-energy physics for over three decades [1,2]. In a two-dimensional (2D) quantum Hall fluid, it describes the transverse current generated by an applied electric field, which manifests in the Hall conductivity σ_{xy}. Interestingly, 2D Chern-Simons theory also provides an elegant explanation of Hall quantization as well as the chiral edge currents, with no need to invoke electronic band structure. In addition, it has successfully described the fractional quantum Hall effect in many-body systems and even captures the physics of anyons [3]. On the other hand, three-dimensional (3D) Chern-Simons theory, also known as axion electrodynamics, emerges as a residual magnetoelectric response in topological insulators [4]. Lattice gauge theories are also of significant interest in quantum simulation [5].

However, in both 2D and 3D, Chern-Simons theory only elucidates the topological properties of the electron. The topology of the electromagnetic field in these quantum materials has remained largely unexplored. Here we mean quantities such as the *photonic* Chern number and the topological invariants associated with the electromagnetic field coupled to condensed matter. To characterize these topological properties, it is fundamentally necessary to define the photon wave function and understand the dynamical $\omega \neq 0$ and subwavelength $k \neq 0$ behavior of the material response [6,7]. In solids, the topology of the photon wave function is encapsulated in the spatiotemporal dispersion of optical coefficients such as the conductivity tensor $\sigma_{ij}(\omega, \mathbf{k})$. This insight has led to a new electromagnetic classification of topological matter [8] and intriguing phenomena such as unidirectional electromagnetic spin waves [9] that are fundamentally different than magnetoplasmons. These so-called topological electromagnetic phases of matter are intrinsically bosonic (spin-1) and are fundamentally different from fermionic (spin-1/2) phases as they obey differing symmetries, e.g., time-reversal: $\mathcal{T}^2 = +1$ for bosons vs $\mathcal{T}^2 = -1$ for fermions. The prototypical model of a gapped topological electromagnetic phase, with nontrivial photonic Chern number $C_{em} \neq 0$, was first connected to nonlocality (momentum dependence) of the Hall conductivity $\sigma_{xy}(k) = \lambda(\kappa - \xi k^2)$ [6,7]. These observations necessarily require a formalism beyond conventional Chern-Simons theory.

In this paper we lay the foundations for a field theory approach to topological photonic phases. The specific class of systems we focus on are quantum fluids with Hall viscosity. Hall viscosity η_H [10,11], also known as odd viscosity [12] in fluid dynamics, is a fundamental property of quantum Hall fluids and can exhibit topological quantization analogous to the Hall conductivity [13–15]. Like conventional viscosity, it is related to the stress response of the system under deformations and governs the diffusive flow of the electron fluid. However, Hall viscosity is unique because it is dissipationless, inducing diffusive flow in a direction perpendicular to a pressure (force) gradient and therefore does no work. We show that Hall viscosity, intriguingly, defines a topological electromagnetic phase of matter with spin-1 photonic skyrmions. We further describe the central idea of a viscous photon mass arising in viscous Chern-Simons theories—fundamentally different from the Proca mass which breaks gauge invariance [16]. Our viscous Maxwell-Chern-Simons (MCS) Lagrangian also reveals topologically protected chiral (unidirectional) edge states that minimize the surface variation and correspond to massless excitations costing an infinitesimal amount of energy.

An overview of the problem is depicted in Fig. 1. The low-energy $\omega \approx 0$ and long-wavelength $k \approx 0$ quantized Hall conductivity is well understood as a topological phase of electrons. At terahertz frequencies $\omega \neq 0$ but low momentum $k \approx 0$, plateaus and quantized Faraday rotation have been observed in the integer quantum Hall regime [17]. However, at finite $\omega \neq 0$ and $k \neq 0$, the Hall conductivity becomes dynamical and viscous, paving the route for the first-known topological phase for photons in condensed matter. We argue that our low-energy theory applies to graphene's electron fluid

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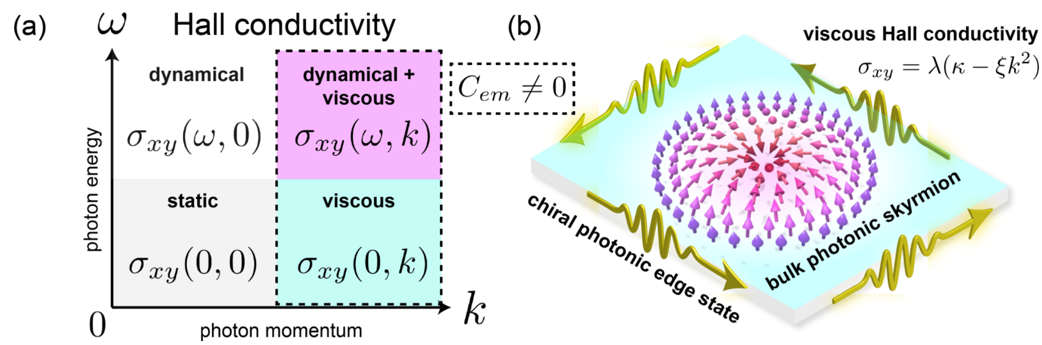

FIG. 1. (a) Summary of the four quantum Hall regimes. Hall quantization and plateauing behavior has been demonstrated in both static and dynamical regimes. However, topological electromagnetic phases $C_{em} \neq 0$ are only realized in the dynamical + viscous (nonlocal) regimes. (b) Overview of viscous Maxwell-Chern-Simons theory. The bulk topology is governed by a spin-1 photonic skyrmion in momentum space which arises from viscous Hall conductivity $\sigma_{xy}(k) = \lambda(\kappa - \xi k^2)$. The arrows represent the direction of the effective spin $\hat{d}$ of the photon. The boundary of the nontrivial phase $\kappa\xi > 0$ hosts topologically protected chiral photons which are linearly dispersing (massless).

[18], where appreciable Hall viscosity [19] was experimentally demonstrated, even under weak magnetic fields. The viscous MCS theory possesses a few limitations as it neglects Coulomb interactions and the high-frequency screening of the magnetic field, which require more sophisticated hydrodynamic models [20]. The main goal here is to formulate a field-theoretic approach to topological photonic phases, make the connection with Chern-Simons theories, and illustrate the importance of Hall viscosity in realizing nontrivial phases. To guide experimentalists in the search for such new topological electromagnetic phases of matter, we have included a summary of a few physical systems exhibiting Hall viscosity along with their characteristic parameters in Table I.

We note that our work is closely related to ideas in topological photonics, but the physical platforms are fundamentally different. We are concerned with condensed matter systems such as viscous Hall fluids. Topological wave phenomena [21–23] have transcended all of photonics: From plasmonics [24,25], metamaterials [26–28], and photonic crystals [29,30]. Nevertheless, it remains an open question whether topological photonic phases can be expressed in terms of an effective gauge theory, i.e., a field-theoretic approach. The advantage of our viscous MCS theory is the proof that the topological edge wave minimizes the action on the boundary. Furthermore, the boundary conditions we derive are fundamentally different from those used for conventional nanoscale systems such as photonic crystals and plasmonics. This difference arises from the presence of Hall viscosity, which is a necessary physical property for defining topological electromagnetic phases of matter.

II. LAGRANGIAN FORMULATION FOR TOPOLOGICAL ELECTROMAGNETIC PHASES

A. Maxwell-Chern-Simons theory

In 2+1 dimensions, the MCS Lagrangian is defined as

$$\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}. \tag{1}$$

$A_\mu = (\phi, A_x, A_y)$ are the 2D gauge fields, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor [1,2]. We have set the dielectric constant to unity $\varepsilon = 1$, but the case with $\varepsilon > 1$ is easily handled and does not alter the topological physics—it simply scales the electric field and the effective speed of light. The first term in $\mathcal{L}_A$ is the familiar Maxwell Lagrangian. The second term is the Chern-Simons Lagrangian, and κ is the coupling constant. Alternately, the MCS theory can be formulated in the more aesthetically pleasing "self-dual"

TABLE I. Summary of two physical systems exhibiting significant Hall viscosity and topologically nontrivial electromagnetic phases $C_{em} \neq 0$. In general, Hall viscosity is always present if the system breaks both parity and time-reversal symmetry. When viscosity repels the magnetic field $\mathcal{C}_2 > 0$, the electromagnetic phase is nontrivial [Eq. (9)], which occurs in both quantum Hall $\nu \in \mathbb{Z}$ [14] and graphene Hall fluids [18]. Hall viscosity is also appreciable in the semiclassical graphene fluid [19] around room temperature 100–300 K and for weak magnetic fields $B_0 \approx 10$ mT.

	Quantum Hall fluid [14]	Graphene Hall fluid ($\nu = 1$) [18]
Biasing magnetic field, B_0	Quantizing	10 T
DC Hall conductivity, $\sigma_{xy}(0)$	$\nu e^2/(2\pi\hbar)$	$2e^2/(2\pi\hbar) \approx 6.97 \times 10^5$ m/s
MCS mass, $\kappa/2\pi$	$\sigma_{xy}(0)/(2\pi\lambda)$	$\sigma_{xy}(0)/(2\pi\lambda) \approx 4.43$ THz
Magnetic length, l	$\sqrt{\hbar c/(eB_0)}$	$\sqrt{\hbar c/(eB_0)} \approx 81$ Å
Cyclotron frequency, $\omega_c/2\pi$	$eB_0/(2\pi cm)$	$eB_0/(2\pi cm^*) \approx 22.6$ THz
Hall viscosity, $\eta_H/(\hbar n)$	$\nu/4$	$\nu/4 = 1/4$
Energy density, $\epsilon(B_0)$	$\nu^2\hbar\omega_c/(4\pi l^2)$	$\sqrt{2}\hbar v_F\zeta(3/2)/(8\pi^2 l^3) \approx 403\ \mu$J/m^2
Topological phase? $\mathcal{C}_2 = \xi/(\kappa l^2)$	yes: $\mathcal{C}_2 = 3\nu/4 > 0$	yes: $\mathcal{C}_2 \approx 1/2 > 0$

picture [31],

$$\mathcal{L}_F = \frac{\kappa}{2}\tilde{F}^\mu \tilde{F}_\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{F}_\mu \partial_\nu \tilde{F}_\rho, \quad (2)$$

which is equivalent to Eq. (1) up to a Legendre transformation [32]. In this case, the field theory is described in terms of the electromagnetic dual $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$, which satisfies the Bianchi identity (Faraday equation) $\partial_\mu \tilde{F}^\mu = 0$ upon variation of the action. In 2D, the dual field $\tilde{F}_\mu$ is a covariant vector,

$$\tilde{F}_\mu = (B_z, E_y, -E_x), \quad (3)$$

with the same number of components as the gauge fields $A_\mu = (\phi, A_x, A_y)$ and therefore is an equally valid description of the field theory.

B. Viscous Maxwell-Chern-Simons theory

Although traditional MCS theory has been studied extensively, we analyze the role of viscosity (nondissipative nonlocality) [33] that leads to topological implications on the electromagnetic field [6–9]. Originally, Hall viscosity was conceived from a geometric perspective, associated with deformations of the underlying metric of the quantum fluid [13]. An equivalent but alternative point of view is to include nonlocal terms that account for the stress-strain response of the quantum Hall fluid. To this end, we introduce the viscous MCS Lagrangian,

$$\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} - \frac{\xi}{4}\epsilon^{\mu\nu\rho}A_\mu \nabla^2 F_{\nu\rho}, \quad (4)$$

which will elucidate these topological electromagnetic phases of matter. An effective action to describe a medium with Hall viscosity was first proposed by Hoyos and Son [14], which was motivated by Galilean invariance as opposed to relativistic invariance. Similarly, our viscous Lagrangian Eq. (4) is Galilean invariant. The one significant difference is that the Hoyos and Son Lagrangian was limited to longitudinal fields $\mathbf{E} = -\nabla\phi$. Our theory is a slight generalization in flat space-time that includes the response of the transverse field $\nabla \times \mathbf{E} \neq 0$. A proof is provided in the Supplemental Material [34]. ξ is the nonlocal Chern-Simons coupling and accounts for viscosity in the MCS Lagrangian.

As before, we can transform to the self-dual picture to obtain an intuitive interpretation:

$$\mathcal{L}_F = \frac{\kappa}{2}\tilde{F}^\mu \tilde{F}_\mu - \frac{\xi}{2}\nabla\tilde{F}^\mu \cdot \nabla\tilde{F}_\mu + \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{F}_\mu \partial_\nu \tilde{F}_\rho. \quad (5)$$

We note there is a striking one-to-one correspondence between Eq. (5), which we derived, and the minimal topological Dirac model [35],

$$\mathcal{L}_\psi = m\bar{\psi}\psi - b\nabla\bar{\psi}\cdot\nabla\psi - i\bar{\psi}\gamma^\mu\partial_\mu\psi, \quad (6)$$

where γ^μ are the 2+1D gamma matrices and ψ is a two-component spinor. Equations (5) and (6) are, in fact, supersymmetric partners [1], describing spin-1 bosons and spin-1/2 fermions, respectively. By direct comparison, we see that κ plays the role of photonic mass in the same way as m for the electron. Likewise, ξ and b dictate the kinetic (viscous) terms, which are essential to realize nontrivial phases. In the long-wavelength (continuum) limit $k \approx 0$, the viscous term regularizes the field at $k \to \infty$ such that the momentum space is effectively a sphere $\mathbb{R}^2 \simeq S^2$. This means topological invariants of the electromagnetic field, like the Chern number C_{em}, can be defined [36]. We also show in the lattice regularized theory (Sec. IV) that nontrivial photonic phases $C_{em} \neq 0$ are only possible when viscosity is nonzero $\xi \neq 0$.

C. Viscous Hall conductivity

Physically, the Chern-Simons coupling is interpreted as a dissipationless Hall conductivity, as the induced current density is

$$J^\mu_{\text{ind}} = -\lambda\frac{\partial\mathcal{L}_A}{\partial A_\mu} = \lambda(\kappa + \xi\nabla^2)\tilde{F}^\mu. \quad (7)$$

Since the induced current J^μ_{ind} is proportional to the dual field $\tilde{F}^\mu$, the nonlocal conductivity tensor $\sigma_{ij}(k)$ is purely antisymmetric,

$$\sigma_{xy}(k) = -\sigma_{yx}(k) = \lambda(\kappa - \xi k^2), \quad (8)$$

with vanishing symmetric components $\sigma_{xx} = \sigma_{yy} = 0$. The prefactor λ is a characteristic length scale of the problem and ensures correct units of the conductivity. For simplicity, we assume λ is the Thomas-Fermi screening length, which is approximately $\lambda \approx 25$ nm in graphene [37]. The viscous Chern-Simons coupling ξ therefore describes the quadratic correction to the Hall response [14],

$$\frac{\sigma_{xy}(k)}{\sigma_{xy}(0)} = 1 - \frac{\xi}{\kappa}k^2 = 1 - \mathcal{C}_2(kl)^2, \quad (9)$$

where $\sigma_{xy}(0) = \lambda\kappa = \nu e^2/(2\pi\hbar)$ is the intrinsic DC Hall response, ν is the filling factor, and $l = \sqrt{\hbar c/(eB_0)}$ is the magnetic length. $\hbar$ is the reduced Planck constant, c is the speed of light, e is the elementary charge, and B_0 is the biasing magnetic field. The coefficient $\mathcal{C}_2 = \xi/(\kappa l^2)$ depends on the Hall viscosity η_H and the energy density $\epsilon(B_0)$ of the Hall fluid,

$$\mathcal{C}_2 = \frac{2\pi}{\nu}\frac{l^2}{\hbar\omega_c}B_0^2\epsilon''(B_0) - \frac{\eta_H}{\hbar n}, \quad (10)$$

where $\omega_c = eB_0/(cm)$ is the cyclotron frequency and n is the density of electrons. The first term involving $B_0^2\epsilon''(B_0)$ is a thermodynamic property related to the internal compressibility [38], while the second term involving η_H is universal. Depending on the material platform, $\mathcal{C}_2$ can be either positive $\mathcal{C}_2 > 0$ or negative $\mathcal{C}_2 < 0$, which either inhibits or enhances the total Hall response. We argue that the *inhibiting* regime $\mathcal{C}_2 > 0$, i.e., when $\kappa\xi > 0$, corresponds to a topologically nontrivial electromagnetic phase [6–9].

D. Equations of motion

$\mathcal{L}_A$ and $\mathcal{L}_F$ generate the same equations of motion when one varies the action with respect to the gauge fields (A_μ) or the dual fields ($\tilde{F}_\mu$). However, to ensure the action does not break gauge invariance on a boundary, it is more convenient to work with the self-dual theory $\mathcal{L}_F$. Varying the dual field $\tilde{F}_\mu \to \tilde{F}_\mu + \delta\tilde{F}_\mu$, we naturally obtain a bulk and surface term $\delta\mathcal{S} = \delta\mathcal{S}_b + \delta\mathcal{S}_s$:

$$\delta\mathcal{S}_b = \int dV[\partial_\mu F^{\mu\nu} + (\kappa + \xi\nabla^2)\tilde{F}^\nu]\delta\tilde{F}_\nu \quad (11)$$

and

$$\delta\mathcal{S}_s = \int_{\partial V} dtdy\left[\left(\frac{1}{2}F^{x\mu} - \xi\partial_x\tilde{F}^{\mu}\right)\delta\tilde{F}_\mu\right]_{x=0}. \tag{12}$$

$dV = dtdxdy$ is the differential space-time volume, and we have taken the boundary at $x = 0$. The minimization principle states that a physical system tends to its lowest energy state, which requires that the fields satisfy the equations of motion within V and the boundary conditions on ∂V. Here we consider an isolated system with no external fields or sources. By requiring a vanishing bulk term $\delta\mathcal{S}_b = 0$, we arrive at the viscous wave equation in the quantum fluid,

$$\partial_\mu F^{\mu\nu} + (\kappa + \xi\nabla^2)\tilde{F}^{\nu} = 0, \tag{13}$$

where $F^{\mu\nu} = \epsilon^{\mu\nu\rho}\tilde{F}_\rho$ is the field strength. Equation (13) represents the equations of motion of the viscous MCS theory. On the other hand, the surface term $\delta\mathcal{S}_s = 0$ vanishes for two distinct boundary conditions. The first is a Dirichlet condition,

$$\delta\tilde{F}^{\mu}|_{x=0} = 0, \tag{14}$$

where the value of the field is fixed on $x = 0$, usually to zero $\tilde{F}^{\mu}|_{x=0} = 0$, corresponding to an open boundary. The second possibility is slightly more interesting and represents the natural (mixed) boundary condition:

$$j_s^{\mu} = \frac{\delta\mathcal{S}_s}{\delta\tilde{F}_\mu|_{x=0}} = [F^{x\mu} - 2\xi\partial_x\tilde{F}^{\mu}]_{x=0} = 0. \tag{15}$$

Equation (15) has a particularly nice explanation—it implies the induced surface current j_s^{μ} vanishes on the boundary. We emphasize that this boundary condition is formally identical to its fermionic counterpart derived from the Dirac equation [Eq. (6)]. Together, the above equations define the bulk and edge physics of photons propagating in the viscous Hall fluid.

III. VISCOUS PHOTON MASS

A. Photonic Zeeman interaction

Our first goal is to study the bulk photonic physics of the viscous Hall fluid by exploiting the equations of motions derived above [Eq. (13)]. For that, we introduce a Hamiltonian formalism of the electromagnetic field coupled to a medium described by its macroscopic response (complete conductivity tensor). To construct the electromagnetic "Dirac equation" it is convenient to utilize the Riemann-Silberstein (RS) vector $\vec{F}$ [39], which is often called the photon wave function. In 2+1D, the RS vector is defined as

$$\vec{F} = [E_x \quad E_y \quad iB_z]. \tag{16}$$

In this Maxwell Hamiltonian picture, $\vec{F}$ is a 3D vector propagating in the 2D plane while the dual field $(\tilde{F}^{\mu})$ is a covariant vector, but the two are equivalent up to a unitary transformation. We now combine Eq. (13) with the Bianchi identity (Faraday equation) to obtain a first-order (in time) wave equation:

$$i\partial_t\vec{F} = i\vec{d}\times\vec{F} = H\vec{F}. \tag{17}$$

We call $\vec{d}$ the effective magnetic field of the photon which is a 3D vector operator,

$$\vec{d} = [p_x \quad p_y \quad \kappa - \xi p^2], \tag{18}$$

and $p_j = -i\nabla_j$ are the corresponding momentum operators. H is the "Maxwell Hamiltonian" and is the projection of the effective magnetic field $\vec{d}$ onto the vector spin operators $\vec{S} = [S_x \quad S_y \quad S_z]$:

$$H = \vec{d}\cdot\vec{S} = p_xS_x + p_yS_y + (\kappa - \xi p^2)S_z. \tag{19}$$

The Maxwell Hamiltonian $H = \vec{d}\cdot\vec{S}$ resembles the Zeeman interaction but for photons [40]. The essential difference is that $\vec{S}$ are spin-1 operators for the photon, as opposed to the Pauli matrices $\vec{\sigma}$ which are spin-1/2 operators for the electron. In the RS basis, $[S_j, S_k] = i\epsilon_{jkl}S^l$ are antisymmetric SO(3) matrices that generate the spin-1 algebra [41,42]. Note that the photon propagating within the viscous Hall fluid experiences a net magnetic field that depends on its momentum, Hall conductivity, as well as the Hall viscosity. The dielectric constant ε simply scales the momentum operators and does not effect the behavior of the net magnetic field $\vec{d}$. This "photonic Zeeman interaction" in a viscous quantum Hall fluid leads to a remarkable spin-1 skyrmion in momentum space (Sec. IV).

B. Difference from Proca mass

The topological physics of the electron is tied to the quantization of Hall conductivity. Our intriguing result is that topological properties for the photon arise from the *viscous* nature of the Hall conductivity. The Chern-Simons coupling (Hall conductivity) $\kappa \neq 0$ behaves as a gauge-invariant photonic mass Λ that opens a low-energy band gap for electromagnetic waves at $\omega = 0$ in the quantum fluid. We note that the MCS mass is fundamentally different from the Proca mass that is often encountered in superconductivity [16]. By choosing a Lorenz gauge $\partial_\mu A^{\mu} = 0$, the London penetration depth λ_L of a superconductor is identified with the Proca mass $\lambda_L^{-1} = m$. Conversely, the MCS mass does not require the specification of a gauge. The Stueckelberg [43] mechanism is an alternative way of generating mass for the photon, but in the quantum Hall effect, parity and time-reversal symmetry breaking is captured specifically by the Chern-Simons coupling term. Since the MCS mass does not preserve parity or time-reversal symmetry, it admits the possibility of nontrivial Chern phases $C_{em} \neq 0$. The Hall viscosity is crucial to realize the nontrivial topological electromagnetic phase $\xi \neq 0$ and makes this photonic mass spatially dispersive:

$$\Lambda(p) = \lambda^{-1}\sigma_{xy}(p) = \kappa - \xi p^2. \tag{20}$$

To appreciate its significance, we translate the system to the energy-momentum space and place the MCS theory on a square lattice $x = n_xa$ and $y = n_ya$. Here $n_{x,y} \in \mathbb{Z}$ is an integer and a is the lattice constant. The lattice regularizes the field theory at high k and ensures quantization of topological invariants like the photonic Chern number. Due to discretization of space [44], the momentum is only unique up to $|k_{x,y}| \leqslant \pi/a$, which defines a torus $\mathbb{T}^2$ in two dimensions. That is, $\mathbf{k}$ is defined within the first Brillouin zone (BZ). The dispersion relation of the dynamical $\omega \neq 0$ modes is found straightforwardly,

$$\omega^2(\mathbf{k}) = \vec{d}^2(\mathbf{k}), \tag{21}$$

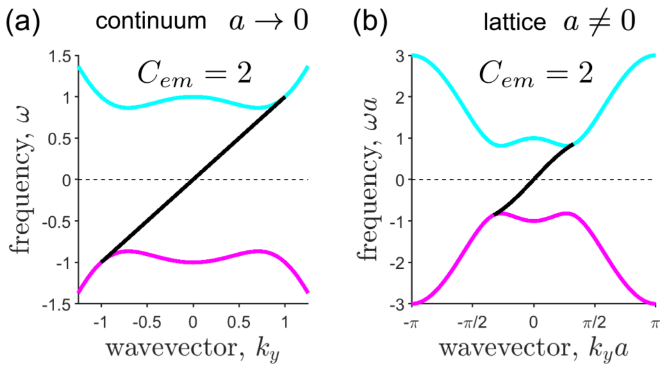

FIG. 2. Bulk and edge dispersion of (a) continuum and (b) lattice models of viscous Maxwell-Chern-Simons theory. Cyan and magnetic lines are positive- and negative-energy topological bands while the black line is the chiral edge state. (a) Parameters are $\kappa = \xi = 1$ in the continuum theory $a \to 0$. (b) Parameters are $\kappa a = \xi/a = 1$ in the lattice theory $a \neq 0$.

where $\vec{d}(\mathbf{k}+\mathbf{g}) = \vec{d}(\mathbf{k})$ is periodic in the reciprocal lattice and $g_{x,y} = N_{x,y}2\pi/a$ is an arbitrary reciprocal vector $N_{x,y} \in \mathbb{Z}$:

$$\vec{d}(\mathbf{k}) = [a^{-1}\sin(k_x a) \quad a^{-1}\sin(k_y a) \quad \Lambda(\mathbf{k})]. \tag{22}$$

$\Lambda(\mathbf{k})$ is the viscous photon mass in the lattice theory and is quadratic in the momentum:

$$\Lambda(\mathbf{k}) = \kappa - \xi\left(\frac{2}{a}\right)^2\left[\sin^2\left(\frac{k_x a}{2}\right) + \sin^2\left(\frac{k_y a}{2}\right)\right]. \tag{23}$$

It is easy to check that the continuum limit is recovered when $a \to 0$. The dispersion relation is depicted in Fig. 2, which shows the bulk bands and the gapless edge states within the band gap. The positive energy $\omega = d > 0$ bulk eigenstate is then derived as

$$\vec{F}_{\mathbf{k}} = \frac{1}{\sqrt{2}}\left[\frac{\vec{d}\times\hat{z}}{|\vec{d}\times\hat{z}|} + i\frac{\vec{d}\times(\vec{d}\times\hat{z})}{d|\vec{d}\times\hat{z}|}\right], \tag{24}$$

which has been normalized to unit energy $|\vec{F}_{\mathbf{k}}|^2 = |\mathbf{E}|^2 + |B_z|^2 = 1$.

IV. SPIN-1 PHOTONIC SKYRMIONS

We now show that a spin-1 photonic skyrmion emerges within the viscous Hall fluid. Our momentum space skyrmion is analogous to those predicted in p-wave superconductors [45]. The reason the skyrmion is spin-1 is because the MCS mass $\Lambda(\mathbf{k})$ also defines the representation theory of the 2+1D Poincaré algebra [2],

$$j_m = \frac{\Lambda(\mathbf{k})}{|\Lambda(\mathbf{k})|} = \mathrm{sgn}[\Lambda(\mathbf{k})], \tag{25}$$

which is a massive spin-1 excitation $j_m = \pm 1$. The representation j_m indicates whether the wave is right $(+1)$ or left (-1) circularly polarized in the x-y plane. The topology is intimately tied to the spin-1 representation of the electromagnetic field. The Berry curvature Ω is precisely [7]

$$\begin{aligned}\Omega &= -i(\partial_x\vec{F}^*_{\mathbf{k}}\cdot\partial_y\vec{F}_{\mathbf{k}} - \partial_y\vec{F}^*_{\mathbf{k}}\cdot\partial_x\vec{F}_{\mathbf{k}})\\ &= \hat{d}\cdot(\partial_x\hat{d}\times\partial_y\hat{d}),\end{aligned} \tag{26}$$

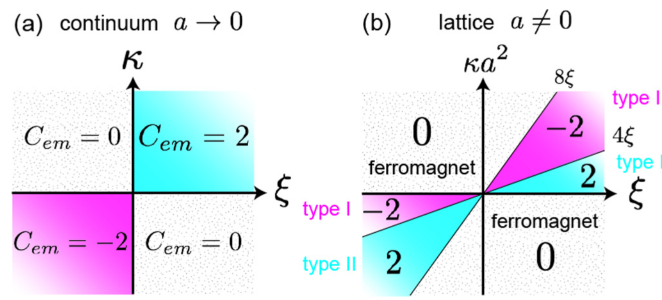

FIG. 3. Topological phase diagrams for (a) continuum and (b) lattice models of viscous Maxwell-Chern-Simons theory. $C_{em} = \pm 2, 0$ is the photonic Chern number of the *positive* energy band $\omega > 0$ for different parameters. κ and ξ are the Chern-Simons and viscous Chern-Simons coupling, respectively. a is the lattice constant of a square grid. $\kappa a^2 = 0, 4\xi, 8\xi$ denote the phase transition lines in the lattice model. These correspond to points of accidental degeneracy, where the band gap closes at $\mathbf{k} = \Gamma, X/Y, M$, respectively. Importantly, conventional MCS theory $\xi = 0$ always corresponds to a topologically trivial phase $C_{em} = 0$ in the lattice regularization.

where $\hat{d} = \vec{d}/d$ is a unit vector. Note that the photonic Chern number for the viscous Hall fluid is always an even integer $C_{em} \in 2\mathbb{Z}$:

$$C_{em} = \frac{1}{2\pi}\int_{\mathrm{BZ}} d\mathbf{k}\,\hat{d}\cdot(\partial_x\hat{d}\times\partial_y\hat{d}) = 2N. \tag{27}$$

$N \in \mathbb{Z}$ is the skyrmion winding number [46,47] that counts the number of times $\hat{d}(\mathbf{k})$ wraps around the unit sphere $\mathbb{T}^2 \to S^2$. We define the skyrmion number N and Chern number C_{em} through the photon wave function $\vec{F}$. The topological invariant is a property of the U(1) gauge field coupled to the viscous quantum Hall fluid. This is in stark contrast to electronic topological materials where the electron wave function ψ plays the central role.

Importantly, at high-symmetry points $\mathbf{k} = \Gamma, X/Y, M$, the spin-1 representation [48,49] can only change if $\kappa\xi > 0$, which requires the Hall coefficient $\mathcal{C}_2 > 0$. After a bit of work, it can be shown that the Chern number is [50]

$$\begin{aligned}C_{em} &= \mathrm{sgn}[\Lambda(\Gamma)] + \mathrm{sgn}[\Lambda(M)] - 2\mathrm{sgn}[\Lambda(X)]\\ &= \mathrm{sgn}(\kappa) + \mathrm{sgn}\left(\kappa - \frac{8\xi}{a^2}\right) - 2\mathrm{sgn}\left(\kappa - \frac{4\xi}{a^2}\right).\end{aligned} \tag{28}$$

The eigenvalues at $\Lambda(X) = \Lambda(Y)$ are identical and thus appear twice in Eq. (28). The topological phase diagram is shown in Fig. 3. For standard MCS theory, the Hall viscosity is zero $\xi = 0$ and the photonic Chern number is identically zero $C_{em} = 0$ in the lattice regularization. This is due to the inherent field doubling that occurs in a periodic system [51], which cancels any parity anomalies that may arise in the continuum limit. Hence, the Hall conductivity κ alone cannot describe a photonic skyrmion or a topological electromagnetic phase. A nontrivial phase with spin-1 photonic skyrmions $|C_{em}| = 2$ is only possible when Hall viscosity is nonzero, $\xi \neq 0$. Note that the continuum limit is recovered when the Hall viscosity is sufficiently large $\sqrt{\xi/\kappa} \gg a$, such that the Chern number reduces to $C_{em} = \mathrm{sgn}(\kappa) + \mathrm{sgn}(\xi)$. The continuum theory predicts the existence of a spin-1 photonic skyrmion. Our work builds on the continuum theory to include lattice

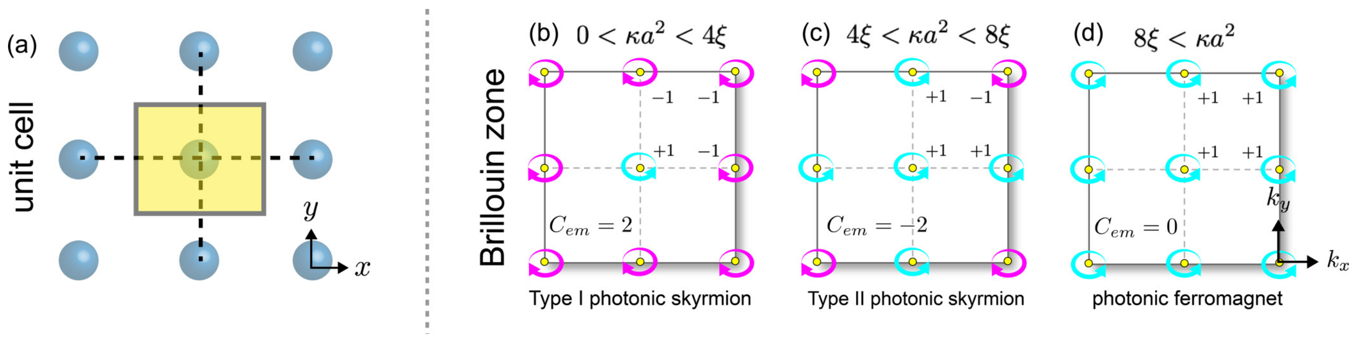

FIG. 4. (a) Unit cell of a square lattice with the primitive Wigner-Seitz cell shown in yellow. (b), (c), (d) The Brillouin zone of the three phases $C_{em} = \pm 2, 0$ in the lattice regularized theory. $\kappa > 0$ and $\xi > 0$ are chosen positive such that (b) and (c) label type-I and type-II photonic skyrmions, respectively. (d) The photonic ferromagnet. The eigenvalue at high-symmetry points denotes the sign of the Maxwell-Chern-Simons mass $j_m = \mathrm{sgn}(\Lambda) = \pm 1$, which determines the spin-1 representation—if the field is right $(+1)$ or left (-1) circularly polarized. The two nontrivial phases possess skyrmion numbers of $N = \pm 1$ corresponding to a spin-1 Chern number of $C_{em} = 2N = \pm 2$.

symmetries, which delineates these skyrmions into type I and type II. As a visualization, examples of type-I and type-II photonic skyrmions are displayed in Fig. 4.

V. TOPOLOGICAL BOUNDARY CONDITIONS

We now analyze the edge physics for the viscous Hall fluid using the MCS theory. We emphasize that the topological boundary conditions are derived through a minimization principle [Eq. (12)]. This is in stark contrast to the conventional approach to solving for topological photonic waves. Thus the edge wave solutions of the viscous Hall fluid satisfy fundamentally different boundary conditions than photonic crystal edge waves or edge magnetoplasmons [25]. These Maxwellian waves are not only unidirectional but are also eigenstates of the photon spin operator [9]. The most striking property is that the contacting medium has no influence and cannot introduce a gap in the edge wave dispersion—the edge wave always exists. This is also a fundamentally unique property of the viscous Hall fluid, as conventional edge magnetoplasmons simply disappear if the contacting medium is a metal (e.g., gold-InSb interface).

The topological boundary conditions have an intuitive interpretation in the RS basis. The open (Dirichlet) boundary condition [Eq. (14)] implies all components of the field vanish at the boundary $\vec{F}|_{x=0} = 0$. This is similar to the no-slip boundary condition in fluid mechanics. On the other hand, the natural boundary condition [Eq. (15)] guarantees that the induced surface current vanishes $v_x\vec{F}|_{x=0} = 0$. Figure 5 shows the truncated lattice corresponding to a viscous Hall fluid and the unidirectional Maxwellian spin waves for two different boundary conditions. The detailed derivation of the bulk-boundary correspondence is appended to the Supplemental Material [34].

VI. CONCLUSIONS

We have presented viscous Maxwell-Chern-Simons theory—the fundamental (exactly solvable) model of a topological electromagnetic phase, the topological physics of which is ultimately governed by viscous (nonlocal) Hall conductivity. To rigorously analyze the problem, we

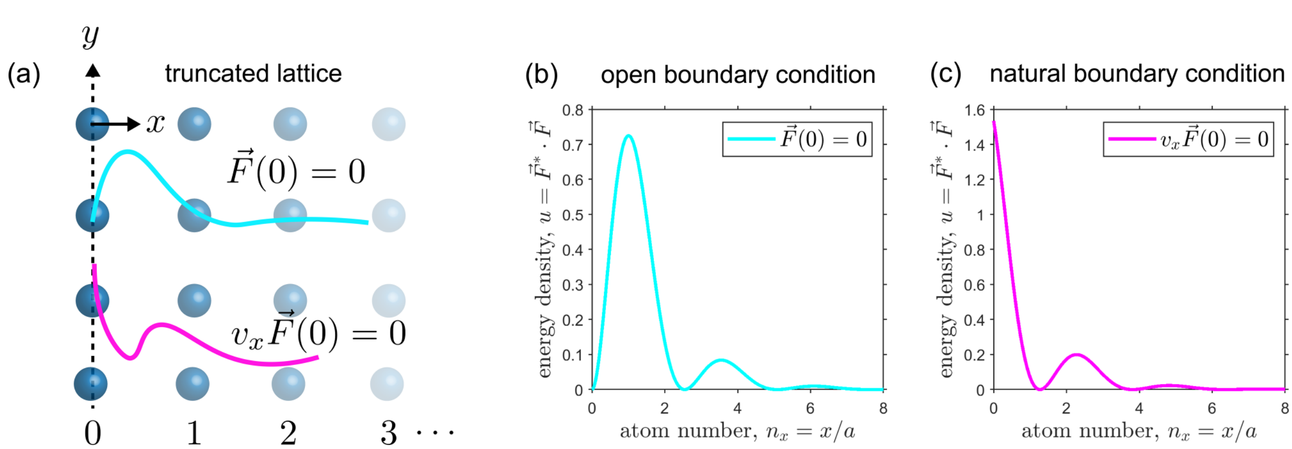

FIG. 5. The two boundary conditions for the viscous Hall fluid that minimize the surface variation $\delta\mathcal{S}_s = 0$ at $x = 0$. (a) Schematic of the truncated atomic lattice at $x = 0$. (b), (c) Plots of the normalized energy density $u = |\vec{F}|^2 = |\mathbf{E}|^2 + |B_z|^2$ of the chiral photonic edge state. The parameters are $\kappa a = 0.1$, $\xi/a = 0.2$, and $k_y a = 0.1$ as a demonstration. (b) The Dirichlet (open) boundary condition $\vec{F}(0) = 0$ has zero measure at $x = 0$. (c) The natural boundary condition $v_x\vec{F}(0) = 0$ is more localized at the surface and resembles an evanescent wave.

introduced the viscous Maxwell-Chern-Simons Lagrangian and derived the equations of motion, as well as the boundary conditions, from the principle of least action. Our work puts forth a fundamentally new field-theoretic approach to merge the fields of topological photonics and quantum Hall fluids.

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[1] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, Ann. Phys. **140**, 372 (1982).
[2] G. V. Dunne, Aspects of Chern-Simons theory, in *Aspects Topologiques de la Physique en Basse Dimension. Topological Aspects of Low Dimensional Systems*, edited by A. Comtet, T. Jolicßur, S. Ouvry, and F. David (Springer, Berlin, Heidelberg, 1999), pp. 177–263.
[3] F. Wilczek, Quantum Mechanics of Fractional-Spin Particles, Phys. Rev. Lett. **49**, 957 (1982).
[4] L. Wu, M. Salehi, N. Koirala, J. Moon, S. Oh, and N. P. Armitage, Quantized Faraday and Kerr rotation and axion electrodynamics of a 3D topological insulator, Science **354**, 1124 (2016).
[5] E. A. Martinez, C. A. Muschik, P. Schindler, D. Nigg, A. Erhard, M. Heyl, P. Hauke, M. Dalmonte, T. Monz, P. Zoller, and R. Blatt, Real-time dynamics of lattice gauge theories with a few-qubit quantum computer, Nature (London) **534**, 516 (2016).
[6] T. Van Mechelen and Z. Jacob, Quantum gyroelectric effect: Photon spin-1 quantization in continuum topological bosonic phases, Phys. Rev. A **98**, 023842 (2018).
[7] T. V. Mechelen and Z. Jacob, Photonic Dirac monopoles and skyrmions: Spin-1 quantization [invited], Opt. Mater. Express **9**, 95 (2019).
[8] T. Van Mechelen and Z. Jacob, Nonlocal topological electromagnetic phases of matter, Phys. Rev. B **99**, 205146 (2019).
[9] T. V. Mechelen and Z. Jacob, Unidirectional Maxwellian spin waves, Nanophotonics **8**, 1399 (2019).
[10] T. I. Tuegel and T. L. Hughes, Hall viscosity and momentum transport in lattice and continuum models of the integer quantum Hall effect in strong magnetic fields, Phys. Rev. B **92**, 165127 (2015).
[11] T. Scaffidi, N. Nandi, B. Schmidt, A. P. Mackenzie, and J. E. Moore, Hydrodynamic Electron Flow and Hall Viscosity, Phys. Rev. Lett. **118**, 226601 (2017).
[12] A. Souslov, K. Dasbiswas, M. Fruchart, S. Vaikuntanathan, and V. Vitelli, Topological Waves in Fluids with Odd Viscosity, Phys. Rev. Lett. **122**, 128001 (2019).
[13] J. E. Avron, R. Seiler, and P. G. Zograf, Viscosity of Quantum Hall Fluids, Phys. Rev. Lett. **75**, 697 (1995).
[14] C. Hoyos and D. T. Son, Hall Viscosity and Electromagnetic Response, Phys. Rev. Lett. **108**, 066805 (2012).
[15] N. Read and E. H. Rezayi, Hall viscosity, orbital spin, and geometry: Paired superfluids and quantum Hall systems, Phys. Rev. B **84**, 085316 (2011).
[16] M. Tajmar, Electrodynamics in superconductors explained by Proca equations, Phys. Lett. A **372**, 3289 (2008).
[17] Y. Ikebe, T. Morimoto, R. Masutomi, T. Okamoto, H. Aoki, and R. Shimano, Optical Hall Effect in the Integer Quantum Hall Regime, Phys. Rev. Lett. **104**, 256802 (2010).
[18] M. Sherafati, A. Principi, and G. Vignale, Hall viscosity and electromagnetic response of electrons in graphene, Phys. Rev. B **94**, 125427 (2016).
[19] A. I. Berdyugin, S. G. Xu, F. M. D. Pellegrino, R. Krishna Kumar, A. Principi, I. Torre, M. Ben Shalom, T. Taniguchi, K. Watanabe, I. V. Grigorieva, M. Polini, A. K. Geim, and D. A. Bandurin, Measuring Hall viscosity of graphene's electron fluid, Science **364**, 162 (2019).
[20] R. Cohen and M. Goldstein, Hall and dissipative viscosity effects on edge magnetoplasmons, Phys. Rev. B **98**, 235103 (2018).
[21] L. Lu, J. D. Joannopoulos, and M. Soljacic, Topological photonics, Nat. Photon. **8**, 821 (2014).
[22] M. Stone, Topology, spin, and light, Science **348**, 1432 (2015).
[23] S. A. R. Horsley, Topology and the optical Dirac equation, Phys. Rev. A **98**, 043837 (2018).
[24] L.-kun Shi and J. C. W. Song, Plasmon Geometric Phase and Plasmon Hall Shift, Phys. Rev. X **8**, 021020 (2018).
[25] S. A. Hassani Gangaraj, G. W. Hanson, M. G. Silveirinha, K. Shastri, M. Antezza, and F. Monticone, Unidirectional and diffractionless surface plasmon polaritons on three-dimensional nonreciprocal plasmonic platforms, Phys. Rev. B **99**, 245414 (2019).
[26] Q. Lin, X.-Q. Sun, M. Xiao, S.-C. Zhang, and S. Fan, A three-dimensional photonic topological insulator using a two-dimensional ring resonator lattice with a synthetic frequency dimension, Sci. Adv. **4**, eaat2774 (2018).
[27] Q. Guo, O. You, B. Yang, J. B. Sellman, E. Blythe, H. Liu, Y. Xiang, J. Li, D. Fan, J. Chen, C. T. Chan, and S. Zhang, Observation of Three-Dimensional Photonic Dirac Points and Spin-Polarized Surface Arcs, Phys. Rev. Lett. **122**, 203903 (2019).
[28] E. Lustig, S. Weimann, Y. Plotnik, Y. Lumer, M. A. Bandres, A. Szameit, and M. Segev, Photonic topological insulator in synthetic dimensions, Nature (London) **567**, 356 (2019).
[29] J. Noh, W. A. Benalcazar, S. Huang, M. J. Collins, K. P. Chen, T. L. Hughes, and M. C. Rechtsman, Topological protection of photonic mid-gap defect modes, Nat. Photon. **12**, 408 (2018).
[30] Y. Yang, Z. Gao, H. Xue, L. Zhang, M. He, Z. Yang, R. Singh, Y. Chong, B. Zhang, and H. Chen, Realization of a three-dimensional photonic topological insulator, Nature (London) **565**, 622 (2019).
[31] P. K. Townsend, K. Pilch, and P. V. Nieuwenhuizen, Self-duality in odd dimensions, Phys. Lett. B **136**, 38 (1984).
[32] E. S. Fradkin and A. A. Tseytlin, Quantum equivalence of dual field theories, Ann. Phys. **162**, 31 (1985).
[33] S. Zhong, J. Orenstein, and J. E. Moore, Optical Gyrotropy from Axion Electrodynamics in Momentum Space, Phys. Rev. Lett. **115**, 117403 (2015).

[34] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.102.155425 for contains full derivations of equations in the main text, including a proof of bulk-boundary correspondence.

[35] S.-Q. Shen, W.-Y. Shan, and H.-Z. Lu, Topological insulator and the Dirac equation, SPIN **01**, 33 (2011).

[36] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, Topological insulators and superconductors: Tenfold way and dimensional hierarchy, New J. Phys. **12**, 065010 (2010).

[37] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. **81**, 109 (2009).

[38] B. Bradlyn, M. Goldstein, and N. Read, Kubo formulas for viscosity: Hall viscosity, Ward identities, and the relation with conductivity, Phys. Rev. B **86**, 245309 (2012).

[39] I. Bialynicki-Birula and Z. Bialynicka-Birula, The role of the Riemann–Silberstein vector in classical and quantum theories of electromagnetism, J. Phys. A: Math. Theor. **46**, 053001 (2013).

[40] J. H. Han, *Skyrmions in Condensed Matter* (Springer, New York, 2017), Vol. 278.

[41] Y.-Q. Zhu, D.-W. Zhang, H. Yan, D.-Y. Xing, and S.-L. Zhu, Emergent pseudospin-1 Maxwell fermions with a threefold degeneracy in optical lattices, Phys. Rev. A **96**, 033634 (2017).

[42] I. C. Fulga, L. Fallani, and M. Burrello, Geometrically protected triple-point crossings in an optical lattice, Phys. Rev. B **97**, 121402(R) (2018).

[43] B. Körs and P. Nath, Aspects of the Stueckelberg extension, J. High Energy Phys. 07 (2005) 069.

[44] F. L. Teixeira and W. C. Chew, Lattice electromagnetic theory from a topological viewpoint, J. Math. Phys. **40**, 169 (1999).

[45] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).

[46] N. Nagaosa and Y. Tokura, Topological properties and dynamics of magnetic skyrmions, Nat. Nanotechnol. **8**, 899 (2013).

[47] S. Tsesses, E. Ostrovsky, K. Cohen, B. Gjonaj, N. H. Lindner, and G. Bartal, Optical skyrmion lattice in evanescent electromagnetic fields, Science **361**, 993 (2018).

[48] H. C. Po, A. Vishwanath, and H. Watanabe, Symmetry-based indicators of band topology in the 230 space groups, Nat. Commun. **8**, 50 (2017).

[49] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, Topological Classification of Crystalline Insulators through Band Structure Combinatorics, Phys. Rev. X **7**, 041069 (2017).

[50] L. O. Nascimento, Introduction to topological phases and electronic interactions in (2+1) dimensions, Braz. J. Phys. **47**, 215 (2017).

[51] T. Z. Olesen, N. D. Vlasii, and U.-J. Wiese, From doubled Chern-Simons-Maxwell lattice gauge theory to extensions of the toric code, Ann. Phys. **361**, 303 (2015).