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On an inverse boundary value problem for a nonlinear elastic wave equation

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We consider an inverse boundary value problem for a nonlinear elastic wave equation which was studied in [1]. We show that all the parameters appearing in the equation can be uniquely determined from boundary measurements under certain geometric assumptions. The proof is based on second order linearization and Gaussian beams.

R É S U M É

Nous considérons un problème inverse avec données au bord, pour une équation des ondes élastiques non-linéaire étudiée précédemment dans l'article [1]. Sous certaines hypothèses géométriques, nous prouvons que tous les paramètres constitutifs de l'équation peuvent être déterminés de manière unique a partir de données au bord. La preuve fait appel à une linéarisation au deuxième ordre ainsi qu'à des faisceaux gaussiens.

1. Introduction

Consider the initial boundary value problem for the quasilinear elastic wave equation

$$\begin{aligned}
&\rho\frac{\partial^2 u}{\partial t^2}-\nabla\cdot S(x,u)=0,\quad (t,x)\in(0,T)\times\Omega,\\
&u(t,x)=f(t,x),\quad (t,x)\in(0,T)\times\partial\Omega,\\
&u(0,x)=\frac{\partial}{\partial t}u(0,x)=0,\quad x\in\Omega.
\end{aligned}\tag{1}$$

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Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. We denote $x = (x_1, x_2, x_3)$ to be the Cartesian coordinates. Then we can write the displacement vector as $u = (u_1, u_2, u_3)$ under the Cartesian coordinates. The stress tensor S has the form

$$\begin{aligned} S_{ij} =& \lambda\varepsilon_{mm}\delta_{ij} + \lambda\widetilde{\varepsilon}_{mm}\frac{\partial u_i}{\partial x_j} + 2\mu\left(\varepsilon_{ij} + \widetilde{\varepsilon}_{jn}\frac{\partial u_i}{\partial x_n}\right) \\ &+ \mathscr{A}\widetilde{\varepsilon}_{in}\widetilde{\varepsilon}_{jn} + \mathscr{B}(2\widetilde{\varepsilon}_{nn}\widetilde{\varepsilon}_{ij} + \widetilde{\varepsilon}_{mn}\widetilde{\varepsilon}_{mn}\delta_{ij}) + \mathscr{C}\widetilde{\varepsilon}_{mm}\widetilde{\varepsilon}_{nn}\delta_{ij} + \mathcal{O}(u^3), \end{aligned} \tag{2}$$

where ε is the strain tensor defined as

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right),$$

and $\widetilde{\varepsilon}$ is the linearized strain tensor

$$\widetilde{\varepsilon}_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

By using the notation $\mathcal{O}(u^3)$ we are considering the small displacement asymptotics. The functions

$$\lambda(x), \mu(x), \rho(x), \mathscr{A}(x), \mathscr{B}(x), \mathscr{C}(x)$$

are all smooth on $\overline{\Omega}$. The parameters λ and μ are called Lamé moduli and ρ is the density. This model is widely used and can be found in [2,3,1].

In this article, we study the inverse problem of recovering the elastic parameters λ, μ, ρ, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ from *displacement-to-traction map*

$$\Lambda : f \to \nu \cdot S(x,u)|_{(0,T)\times\partial\Omega},$$

where ν is the exterior normal unit vector to $\partial\Omega$.

The well-definedness of Λ for small f is guaranteed by the well-posedness of (1) with small boundary data:

Proposition 1 *([1, Theorem 2]). Assume $f \in C^m([0,T] \times \partial\Omega), m \geq 3$ is supported away from $t = 0$. Then there exists $\epsilon_0 > 0$ such that for $\|f\|_{C^m} < \epsilon_0$ there exists a unique solution*

$$u \in \bigcap_{k=0}^{m} C^k([0,T]; W^{m-k,2})([0,T] \times \Omega)).$$

Denote $S = S^L + S^N$, where S^L is the linearized stress

$$S^L_{ij}(x,u) = \lambda\widetilde{\varepsilon}_{mm}\delta_{ij} + 2\mu\widetilde{\varepsilon}_{ij},$$

and

$$\begin{aligned} S^N_{ij}(x,u) =& \frac{\lambda + \mathscr{B}}{2}\frac{\partial u_m}{\partial x_n}\frac{\partial u_m}{\partial x_n}\delta_{ij} + \mathscr{C}\frac{\partial u_m}{\partial x_m}\frac{\partial u_n}{\partial x_n}\delta_{ij} + \frac{\mathscr{B}}{2}\frac{\partial u_m}{\partial x_n}\frac{\partial u_n}{\partial x_m}\delta_{ij} + \mathscr{B}\frac{\partial u_m}{\partial x_m}\frac{\partial u_j}{\partial x_i} \\ &+ \frac{\mathscr{A}}{4}\frac{\partial u_j}{\partial x_m}\frac{\partial u_m}{\partial x_i} + (\lambda + \mathscr{B})\frac{\partial u_m}{\partial x_m}\frac{\partial u_i}{\partial x_j} \\ &+ \left(\mu + \frac{\mathscr{A}}{4}\right)\left(\frac{\partial u_m}{\partial x_i}\frac{\partial u_m}{\partial x_j} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_j}{\partial x_m} + \frac{\partial u_i}{\partial x_m}\frac{\partial u_m}{\partial x_j}\right) + \mathcal{O}(u^3). \end{aligned}$$

The linear elastic wave equation reads

$$\begin{aligned} &\rho\frac{\partial^2 u}{\partial t^2} - \nabla\cdot S^L(x,u) = 0, \quad (t,x)\in(0,T)\times\Omega, \\ &u(t,x) = f(t,x), \quad (t,x)\in(0,T)\times\partial\Omega, \\ &u(0,x) = \frac{\partial}{\partial t}u(0,x) = 0, \quad x\in\Omega. \end{aligned} \tag{3}$$

We denote the Dirichlet-to-Neumann map for the above linear elastic wave equation as

$$\Lambda^{lin} : f \to \nu\cdot S^L(x,u)|_{(0,T)\times\partial\Omega}.$$

The S- and P-wavespeeds are related with the Lamé moduli λ, μ and the density ρ in the following way

$$c_S = \sqrt{\frac{\mu}{\rho}}, \quad c_P = \sqrt{\frac{\lambda+2\mu}{\rho}}.$$

We will assume

$$\mu > 0, \quad 3\lambda + 2\mu > 0 \text{ on } \overline{\Omega}.$$

Then $c_P > c_S$ in $\overline{\Omega}$. Denote the Riemannian metrics associated with P/S- wave speeds to be

$$g_{P/S} = c_{P/S}^{-2}\mathrm{d}s^2,$$

where $\mathrm{d}s^2$ is the Euclidean metric. Then P- and S-waves travel along geodesics in the Riemannian manifolds (Ω, g_P) and (Ω, g_S) respectively. Let $\mathrm{diam}_{P/S}(\Omega)$ be the diameter of Ω with respect to $g_{P/S}$. More precisely,

$$\mathrm{diam}_{P/S}(\Omega) = \sup\{\text{lengths of all geodesics in } (\Omega, g_{P/S})\}$$

For the inverse problem, one can first recover the Dirchlet-to-Neumann map Λ^{lin} for the linear elastic wave equation (3) by first order linearization of Λ (cf. [1])

$$\frac{\partial}{\partial\epsilon}\Lambda(\epsilon f)|_{\epsilon=0} = \Lambda^{lin}(f).$$

It was shown in [4] that from Λ^{lin} one can recover the scattering relation associated to the wave speeds c_P and c_S. Using the result of [5] one can determine c_S and c_P if the foliation condition is satisfied for both metrics $g_{P/S}$ and T is larger than $\mathrm{diam}_S(\Omega)$. For a more precise statement see [6, Theorem 1.4]. Recall that a Riemannian manifold (M, g) satisfies the foliation condition if it can be foliated by strictly convex hypersurfaces [7]. The foliation condition is satisfied for $(\Omega, g_{P/S})$, for instance, if $\partial\Omega$ is strictly convex (with respect to $g_{P/S}$) and the wave speeds $c_{P/S}$ increase with depth. They are also satisfied under some additional conditions on the curvature (see [8,9]), if Ω is simply connected. As pointed out in [6] the foliation condition is a natural generalization of the Herglotz [10] and the Wieckert-Zoeppritz [11] conditions. The foliation condition allows for conjugate points. If the boundary is strictly convex for $g_{P/S}$ and there are no conjugate points for $g_{P/S}$, the uniqueness of c_P and c_S was shown by Rachele in [12]. One can in fact determine the three parameters λ, μ, ρ, assuming further $c_P \neq 2c_S$ except at isolated points in $\overline{\Omega}$, under the foliation condition [13] or the no conjugate points condition (an extra curvature condition is needed) [14]. We summarize the results for the linear elastic wave equation (3) in the following.

Proposition 2. *Assume $T > \max\{\mathrm{diam}_S(\Omega), \mathrm{diam}_P(\Omega)\}$, $\partial\Omega$ is strictly convex with respect to $g_{P/S}$, and either of the following conditions holds*

1. *$(\Omega, g_{P/S})$ has no conjugate points;*
2. *$(\Omega, g_{P/S})$ satisfies the foliation condition.*

Then Λ^{lin} uniquely determines $\frac{\mu}{\rho}$ and $\frac{\lambda}{\rho}$ in $\overline{\Omega}$. Assume further that $\lambda = 2\mu$ only at isolated points in $\overline{\Omega}$. When condition (1) *is satisfied, assume also that (Ω, g_P) is negatively curved. Then ρ is uniquely determined.*

Remark 1. Notice that when $(\Omega, g_{P/S})$ has no conjugate points, an extra curvature condition on (Ω, g_P) is needed for the unique determination of ρ. This is due to the fact that the injectivity (up to natural obstructions) of the related tensor tomography problem is established only under extra curvature conditions. The curvature condition can be relaxed using the results of [15–17].

In this paper we mainly focus on the determination of the nonlinear elastic parameters $\mathscr{A}, \mathscr{B}, \mathscr{C}$. In [1], the authors proved the uniqueness of $\mathscr{A}$ and $\mathscr{B}$, by analyzing the nonlinear interactions of distorted plane waves. The approach originated from [18], and has been successfully used to study inverse problems for nonlinear hyperbolic equations [19–23]. We will present an alternative approach to the proof of the uniqueness of $\mathscr{A}$ and $\mathscr{B}$, and further extend to the uniqueness of $\mathscr{C}$. Our work is still based on the higher order linearization utilized in aforementioned work, but instead of distorted plane waves we will use Gaussian beams. We note here that Gaussian beams have been used to study various inverse problems [24–30]. We emphasize here that Gaussian beams can be constructed allowing conjugate points.

We summarize the main theorem of this article here:

Theorem 1. *Assume $T > 2\max\{\mathrm{diam}_S(\Omega), \mathrm{diam}_P(\Omega)\}$, $\partial\Omega$ is strictly convex with respect to $g_{P/S}$, and either of the following conditions holds*

1. *$(\Omega, g_{P/S})$ has no conjugate points;*
2. *$(\Omega, g_{P/S})$ satisfies the foliation condition.*

Assume λ, μ, ρ can be recovered from Λ^{lin}. Then Λ determines $\lambda, \mu, \rho, \mathscr{A}, \mathscr{B}, \mathscr{C}$ in $\overline{\Omega}$ uniquely.

The rest of this paper is organized as follows. In Section 2, we carry out the second order linearization of the displacement-to-traction map and derive an integral identity, from which we can recover the parameters of interest. In Section 3, we construct Gaussian beam solutions to linear elastic wave equation, for both P- and S-waves. Finally, Section 4 is devoted to the proof of the main theorem.

2. Second-order linearization of displacement-to-traction map

We will apply the higher order linearization technique introduced in [18] to the displacement-to-traction map Λ, and arrive at an integral identity which could be used for the recovery of the parameters. The linearization of Λ itself has already been used in [1]. Higher order linearization of Dirichlet-to-Neumann map and the resulted integral identities for semilinear and quasilinear elliptic equations are used [31–36,28,37]. Assume u solves (1) with Dirichlet boundary value

$$f = \epsilon_1 f^{(1)} + \epsilon_2 f^{(2)}.$$

Denote $u^{(j)}$, $j = 1, 2$ to be the solution to the linearized elastic wave equation with boundary value $f^{(j)}$, i.e.,

$$
\begin{aligned}
&\rho\frac{\partial^2 u^{(j)}}{\partial t^2} - \nabla\cdot S^L(x,u^{(j)}) = 0, \quad (t,x)\in(0,T)\times\Omega,\\
&u^{(j)} = f^{(j)}, \quad \text{on } (0,T)\times\partial\Omega,\\
&u^{(j)}(0,x) = \frac{\partial}{\partial t}u^{(j)}(0,x) = 0, \quad x\in\Omega.
\end{aligned}
\tag{4}
$$

Applying $\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}$ to (1), we obtain the equation for $\mathcal{U}^{(12)} = \frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}u|_{\epsilon_1=\epsilon_2=0}$.

$$
\begin{aligned}
&\rho\frac{\partial^2}{\partial t^2}\mathcal{U}^{(12)} - \nabla\cdot S^L(x,\mathcal{U}^{(12)}) = \nabla\cdot G(u^{(1)},u^{(2)}), \quad (t,x)\in(0,T)\times\Omega,\\
&\mathcal{U}^{(12)}(t,x) = 0, \quad (t,x)\in(0,T)\times\partial\Omega,\\
&\mathcal{U}^{(12)}(0,x) = \frac{\partial}{\partial t}\mathcal{U}^{(12)}(0,x) = 0, \quad x\in\Omega.
\end{aligned}
\tag{5}
$$

Here

$$
\begin{aligned}
G_{ij}(u^{(1)},u^{(2)}) =&(\lambda+\mathscr{B})\frac{\partial u_m^{(1)}}{\partial x_n}\frac{\partial u_m^{(2)}}{\partial x_n}\delta_{ij} + 2\mathscr{C}\frac{\partial u_m^{(1)}}{\partial x_m}\frac{\partial u_n^{(2)}}{\partial x_n}\delta_{ij} + \mathscr{B}\frac{\partial u_m^{(1)}}{\partial x_n}\frac{\partial u_n^{(2)}}{\partial x_m}\delta_{ij}\\
&+\mathscr{B}\left(\frac{\partial u_m^{(1)}}{\partial x_m}\frac{\partial u_j^{(2)}}{\partial x_i} + \frac{\partial u_m^{(2)}}{\partial x_m}\frac{\partial u_j^{(1)}}{\partial x_i}\right) + \frac{\mathscr{A}}{4}\left(\frac{\partial u_j^{(1)}}{\partial x_m}\frac{\partial u_m^{(2)}}{\partial x_i} + \frac{\partial u_j^{(2)}}{\partial x_m}\frac{\partial u_m^{(1)}}{\partial x_i}\right)\\
&+(\lambda+\mathscr{B})\left(\frac{\partial u_m^{(1)}}{\partial x_m}\frac{\partial u_i^{(2)}}{\partial x_j} + \frac{\partial u_m^{(2)}}{\partial x_m}\frac{\partial u_i^{(1)}}{\partial x_j}\right)\\
&+\left(\mu+\frac{\mathscr{A}}{4}\right)\left(\frac{\partial u_m^{(1)}}{\partial x_i}\frac{\partial u_m^{(2)}}{\partial x_j} + \frac{\partial u_m^{(2)}}{\partial x_i}\frac{\partial u_m^{(1)}}{\partial x_j} + \frac{\partial u_i^{(1)}}{\partial x_m}\frac{\partial u_j^{(2)}}{\partial x_m} + \frac{\partial u_i^{(2)}}{\partial x_m}\frac{\partial u_j^{(1)}}{\partial x_m}\right.\\
&\qquad\left.+\frac{\partial u_i^{(1)}}{\partial x_m}\frac{\partial u_m^{(2)}}{\partial x_j} + \frac{\partial u_i^{(2)}}{\partial x_m}\frac{\partial u_m^{(1)}}{\partial x_j}\right).
\end{aligned}
\tag{6}
$$

We note that

$$
\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Lambda(\epsilon_1 f^{(1)} + \epsilon_2 f^{(2)})|_{\epsilon_1=\epsilon_2=0} = \nu\cdot S^L(\mathcal{U}^{(12)}) + \nu\cdot G(u^{(1)},u^{(2)}).
$$

Assume v solves the initial boundary value problem for the backward elastic wave equation

$$
\begin{aligned}
&\rho\frac{\partial^2}{\partial t^2}v - \nabla\cdot S^L(v) = 0, \quad (t,x)\in(0,T)\times\Omega,\\
&v(t,x) = g, \quad (t,x)\in(0,T)\times\partial\Omega,\\
&v(T,x) = \frac{\partial}{\partial t}v(T,x) = 0, \quad x\in\Omega.
\end{aligned}
\tag{7}
$$

Using integration by parts, we get

$$
\begin{aligned}
&\int_0^T\int_{\partial\Omega}\left(\frac{\partial^2}{\partial\epsilon_1\partial\epsilon_2}\Lambda(\epsilon_1 f_1 + \epsilon_2 f_2)|_{\epsilon_1=\epsilon_2=0} - \nu\cdot G(u^{(1)},u^{(2)})\right) g\,\mathrm{d}S\mathrm{d}t\\
=&\int_0^T\int_{\partial\Omega}\nu\cdot S^L(\mathcal{U}^{(12)})g\,\mathrm{d}S\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \left(\nabla \cdot S^L(\mathcal{U}^{(12)}) v + \mathbf{C}\nabla \mathcal{U}^{(12)} : \nabla v \right) \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \left(\rho \frac{\partial^2}{\partial t^2} \mathcal{U}^{(12)} - \nabla \cdot G(u^{(1)}, u^{(2)}) \right) v + \mathbf{C}\nabla \mathcal{U}^{(12)} : \nabla v \, \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \rho \mathcal{U}^{(12)} \frac{\partial^2}{\partial t^2} v - \nabla \cdot G(u^{(1)}, u^{(2)}) v + \mathbf{C}\nabla \mathcal{U}^{(12)} : \nabla v \, \mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_\Omega \mathcal{U}^{(12)} \rho \frac{\partial^2}{\partial t^2} v - \nabla \cdot G(u^{(1)}, u^{(2)}) v - \mathcal{U}^{(12)} \nabla \cdot S^L(v) \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\partial\Omega} \nu \cdot S^L(v) \mathcal{U}^{(12)} \, \mathrm{d}S\mathrm{d}t \\
&= - \int_0^T \int_\Omega \nabla \cdot G(u^{(1)}, u^{(2)}) v \, \mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Here we use the notation $A : B = \sum_{i,j=1}^3 A_{ij} B_{ij}$ for matrices A and B.

Therefore, the displacement-to-traction map determines

$$
\begin{aligned}
&\int_0^T \int_{\partial\Omega} \left(\frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} \Lambda(\epsilon_1 f^{(1)} + \epsilon_2 f^{(2)})|_{\epsilon_1 = \epsilon_2 = 0} \right) g \, \mathrm{d}S\mathrm{d}t \\
&= - \int_0^T \int_\Omega \nabla \cdot G(u^{(1)}, u^{(2)}) v \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\partial\Omega} \nu \cdot G(u^{(1)}, u^{(2)}) g \, \mathrm{d}S\mathrm{d}t \\
&= \int_0^T \int_\Omega \mathcal{G}(\nabla u^{(1)}, \nabla u^{(2)}, \nabla v) \, \mathrm{d}x\mathrm{d}t,
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
&\mathcal{G}(\nabla u^{(1)}, \nabla u^{(2)}, \nabla v) \\
=&(\lambda + \mathscr{B})(\nabla u^{(1)} : \nabla u^{(2)})(\nabla \cdot v) + 2\mathscr{C}(\nabla \cdot u^{(1)})(\nabla \cdot u^{(2)})(\nabla \cdot v) + \mathscr{B}(\nabla u^{(1)} : \nabla^T u^{(2)})(\nabla \cdot v) \\
&+ \mathscr{B} \left((\nabla \cdot u^{(1)})(\nabla u^{(2)} : \nabla^T v) + (\nabla \cdot u^{(2)})(\nabla u^{(1)} : \nabla^T v) \right) \\
&+ \frac{\mathscr{A}}{4} \left(\frac{\partial u_j^{(1)}}{\partial x_m} \frac{\partial u_m^{(2)}}{\partial x_i} + \frac{\partial u_j^{(2)}}{\partial x_m} \frac{\partial u_m^{(1)}}{\partial x_i} \right) \frac{\partial v_i}{\partial x_j} \\
&+ (\lambda + \mathscr{B}) \left((\nabla \cdot u^{(1)})(\nabla u^{(2)} : \nabla v) + (\nabla \cdot u^{(2)})(\nabla u^{(1)} : \nabla v) \right) \\
&+ \left(\mu + \frac{\mathscr{A}}{4} \right) \left(\frac{\partial u_m^{(1)}}{\partial x_i} \frac{\partial u_m^{(2)}}{\partial x_j} + \frac{\partial u_m^{(2)}}{\partial x_i} \frac{\partial u_m^{(1)}}{\partial x_j} + \frac{\partial u_i^{(1)}}{\partial x_m} \frac{\partial u_j^{(2)}}{\partial x_m} + \frac{\partial u_i^{(2)}}{\partial x_m} \frac{\partial u_j^{(1)}}{\partial x_m} \right. \\
&\qquad \left. + \frac{\partial u_i^{(1)}}{\partial x_m} \frac{\partial u_m^{(2)}}{\partial x_j} + \frac{\partial u_i^{(2)}}{\partial x_m} \frac{\partial u_m^{(1)}}{\partial x_j} \right) \frac{\partial v_i}{\partial x_j}.
\end{aligned}
\tag{9}
$$

Here we use the notation $\nabla^T u = (\nabla u)^T$.

We will construct special solutions $u^{(1)}, u^{(2)}, v$ for the linear elastic wave equation and recover the parameters $\mathscr{A}, \mathscr{B}, \mathscr{C}$ from the integral (8). We emphasize here that the solutions will be constructed with known coefficients λ, μ, ρ in the linearized equation.

3. Gaussian beam solutions

Denote

$$M = [0, T] \times \Omega.$$

We note that M can be viewed as a Lorentzian manifold with metric $-\mathrm{d}t^2 + g_P$ or $-\mathrm{d}t^2 + g_S$.

In this section, we will construct Gaussian beam solutions u to the linear elastic wave equation

$$\begin{aligned} &\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot S^L(x, u) = 0, \quad (t, x) \in (0, T) \times \Omega, \\ &u(0, x) = \frac{\partial}{\partial t} u(0, x) = 0, \quad x \in \Omega, \end{aligned} \tag{10}$$

of the form

$$u(t, x) = e^{\mathrm{i}\varrho\varphi(t,x)} \mathfrak{a}(t, x) + R_\varrho(t, x),$$

with a large parameter ϱ. The phase function φ is complex-valued. The principal term $e^{\mathrm{i}\varrho\varphi(t,x)}\mathfrak{a}(t, x)$ is concentrated near a null geodesic ϑ in $(M, -\mathrm{d}t^2 + g_{P/S})$. The remainder term R_ϱ will vanish as $\varrho \to +\infty$.

For the construction of term $e^{\mathrm{i}\varrho\varphi(t,x)}\mathfrak{a}(t, x)$, we consider the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \nabla \cdot S^L(x, u) = 0$$

in an extended domain $\widetilde{\Omega}$, such that $\Omega \subset\subset \widetilde{\Omega}$. The parameters λ, μ, ρ are extended smoothly to $\widetilde{\Omega}$. Also denote $\widetilde{M} = [0, T] \times \widetilde{\Omega}$.

We want to mention here that by assuming $T > 2\max\{\mathrm{diam}_S(\Omega), \mathrm{diam}_P(\Omega)\}$, we have implicitly assumed that $(\Omega, g_{P/S})$ is non-trapping, i.e., every geodesic hits the boundary in finite time. Meanwhile, if $(\Omega, g_{P/S})$ has no conjugate points and is simply connected, or satisfies the foliation condition, then $(\Omega, g_{P/S})$ is non-trapping (cf. [38, Proposition 3.31] and [8, Lemma 2.1]).

Fermi coordinates. We introduce Fermi coordinates in a neighborhood of the null geodesic ϑ. Assume $\vartheta(t) = (t, \gamma(t))$, where γ is a unit-speed geodesic in the Riemannian manifold $(\widetilde{\Omega}, g)$, where $g = g_{P/S}$ is of interest to us. Assume ϑ passes through a point $(t_0, x_0) \in M$, i.e. $t_0 \in (0, T)$ and $\gamma(t_0) = x_0 \in \Omega$, and ϑ joins two points $(t_-, \gamma(t_-))$ and $(t_+, \gamma(t_+))$ where $t_-, t_+ \in (0, T)$ and $\gamma(t_-), \gamma(t_+) \in \partial\Omega$. Extend ϑ to $\widetilde{M}$ such that $\gamma(t)$ is well defined on $[t_- - \epsilon, t_+ + \epsilon] \subset (0, T)$ with ϵ a small constant. We will follow the construction of the coordinates in [27]. See also [20], [22].

Choose α_2, α_3 such that $\{\dot{\gamma}(t_0), \alpha_2, \alpha_3\}$ forms an orthonormal basis for $T_{x_0}\Omega$. Let s denote the arc length along γ from x_0. We note here that s can be positive or negative, and $(t_0 + s, \gamma(t_0 + s)) = \vartheta(t_0 + s)$. For $k = 2, 3$, let $e_k(s) \in T_{\gamma(t_0+s)}\Omega$ be the parallel transport of α_k along γ to the point $\gamma(t_0 + s)$.

Define the coordinate system $(y^1 = s, y^2, y^3)$ through $\mathcal{F}_1 : \mathbb{R}^3 \to \widetilde{\Omega}$:

$$\mathcal{F}_1(s, y^2, y^3) = \exp_{\gamma(t_0+s)}\left(y^2 e_2(s) + y^3 e_3(s)\right).$$

In the new coordinates, we have

$$g|_\gamma = \sum_{j=1}^{3}(\mathrm{d}y^j)^2, \quad \text{and} \quad \frac{\partial g_{jk}}{\partial y^i}\Big|_\gamma = 0, \ 1 \le i,j,k \le 3.$$

Then the Euclidean metric g_E of $\mathbb{R}^3$ takes the form

$$g_E = \sum_{1\le i,j\le 3} c^2 g_{ij}\mathrm{d}y^i\mathrm{d}y^j.$$

The Christoffel symbols then have the form

$$\begin{aligned}&\Gamma^1_{\alpha\beta} = -c^{-1}\frac{\partial c}{\partial s}g_{\alpha\beta}, \quad \Gamma^\beta_{1\alpha} = \delta^\alpha_\beta c^{-1}\frac{\partial c}{\partial s}, \quad \Gamma^1_{1\alpha} = c^{-1}\frac{\partial c}{\partial y^\alpha},\\ &\Gamma^\alpha_{11} = -c^{-1}g^{\alpha\beta}\frac{\partial c}{\partial y^\beta}, \quad \Gamma^1_{11} = c^{-1}\frac{\partial c}{\partial s}.\end{aligned} \tag{11}$$

Here $\alpha, \beta \in \{2,3\}$ and $c = c_{P/S}$.

On the Lorentzian manifold $(\widetilde{M}, -\mathrm{d}t^2 + g)$, near the null geodesic $\vartheta : (t_- - \frac{\epsilon}{2}, t_+ + \frac{\epsilon}{2}) \to \widetilde{M}$ where $\vartheta(t) = (t, \gamma(t))$, we introduce the Fermi coordinates,

$$z^0 = \tau = \frac{1}{\sqrt{2}}(t - t_0 + s), \quad z^1 = r = \frac{1}{\sqrt{2}}(-t + t_0 + s), \quad z^j = y^j,\ j = 2,3.$$

Denote $\tau_\pm = \sqrt{2}(t_\pm - t_0)$. Then on ϑ we have $\overline{g} = -\mathrm{d}t^2 + g$ satisfying

$$\overline{g}|_\vartheta = 2\mathrm{d}\tau\mathrm{d}r + \sum_{j=2}^{3}(\mathrm{d}z^j)^2 \quad \text{and} \quad \frac{\partial \overline{g}_{jk}}{\partial z^i}\Big|_\vartheta = 0, \ 0 \le i,j,k \le 3.$$

We will use the notations $z = (\tau, z') = (\tau = z^0, r = z^1, z'')$ and $y = (s = y^1, y')$.

Construction of Gaussian beams. We will construct approximate Gaussian beam of order N of the form

$$u_\varrho = \mathfrak{a}e^{\mathrm{i}\varrho\varphi}$$

with

$$\varphi = \sum_{k=0}^{N}\varphi_k(\tau, z'), \quad \mathfrak{a}(\tau, z') = \chi\left(\frac{|z'|}{\delta}\right)\sum_{k=0}^{N+1}\varrho^{-k}\mathbf{a}_k(\tau, z'),$$

in a neighborhood of ϑ,

$$\mathcal{V} = \{(\tau, z') \in \widetilde{M} | \tau \in [\tau_- - \frac{\epsilon}{\sqrt{2}}, \tau_+ + \frac{\epsilon}{\sqrt{2}}],\ |z'| < \delta\}.$$

Here $\delta > 0$ is a small parameter. The smooth function $\chi : \mathbb{R} \to [0, +\infty)$ satisfies $\chi(t) = 1$ for $|t| \le \frac{1}{4}$ and $\chi(t) = 0$ for $|t| \ge \frac{1}{2}$. We refer to [29] for more details. We denote $\mathbf{a}_0 = \mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{a}_1 = \mathbf{b} = (b_1, b_2, b_3)$. We note here that an extra term $\mathbf{a}_{N+1}$ is needed here to achieve order N approximation, in compare with the solution constructed in [29]. The parameter δ is small such that $\mathfrak{a}|_{t=0} = \mathfrak{a}|_{t=T} = 0$.

In a neighborhood of ϑ, we calculate

$$\begin{aligned}\widetilde{\varepsilon}_{k\ell}(u_\varrho) =& \frac{1}{2}(a_{k;\,\ell} + a_{\ell;\,k})e^{\mathrm{i}\varrho\varphi} + \frac{1}{2}\varrho^{-1}(b_{k;\,\ell} + b_{\ell;\,k})e^{\mathrm{i}\varrho\varphi}\\ &+ \frac{1}{2}\mathrm{i}\varrho(a_k\varphi_{;\,\ell} + a_\ell\varphi_{;\,k})e^{\mathrm{i}\varrho\varphi} + \frac{1}{2}\mathrm{i}(b_k\varphi_{;\,\ell} + b_\ell\varphi_{;\,k})e^{\mathrm{i}\varrho\varphi},\end{aligned}$$

and

$$\begin{aligned}\sigma_{ij}(u_\varrho) := S^L_{ij}(u_\varrho) =& \lambda e^{k\ell}\widetilde{\varepsilon}_{k\ell}e_{ij} + 2\mu\widetilde{\varepsilon}_{ij}\\ =& \lambda c^{-2}g^{k\ell}\widetilde{\varepsilon}_{k\ell}c^2 g_{ij} + 2\mu\widetilde{\varepsilon}_{ij}\\ =& \mathrm{i}\varrho(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ij} + \mu a_i\varphi_{;\,j} + \mu a_j\varphi_{;\,i})e^{\mathrm{i}\varrho\varphi}\\ &+ (\lambda a_{k;\ell}g^{k\ell}g_{ij} + \mathrm{i}\lambda b_k\varphi_{;\ell}g^{k\ell}g_{ij} + \mu(a_{i;\,j} + a_{j;\,i}) + \mathrm{i}\mu(b_i\varphi_{;\,j} + b_j\varphi_{;\,i}))e^{\mathrm{i}\varrho\varphi}\\ &+ \mathcal{O}(\varrho^{-1}).\end{aligned}$$

We proceed to calculate

$$\begin{aligned}\sigma_{ij;m} =& \partial_m\sigma_{ij} - \Gamma^n_{im}\sigma_{nj} - \Gamma^n_{jm}\sigma_{ni}\\ =& -\varrho^2(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ij} + \mu a_i\varphi_{;j} + \mu a_j\varphi_{;i})\varphi_{;m}e^{\mathrm{i}\varrho\varphi}\\ &+ \mathrm{i}\varrho\partial_m(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ij} + \mu a_i\varphi_{;j} + \mu a_j\varphi_{;i})e^{\mathrm{i}\varrho\varphi}\\ &+ \mathrm{i}\varrho\left(\lambda a_{k;\ell}g^{k\ell}g_{ij} + \mu(a_{i;j} + a_{j;i}) + \mathrm{i}\lambda b_k\varphi_{;\ell}g^{k\ell}g_{ij} + \mathrm{i}\mu(b_i\varphi_{;j} + b_j\varphi_{;i})\right)\varphi_{;m}e^{\mathrm{i}\varrho\varphi}\\ &- \mathrm{i}\varrho\Gamma^n_{im}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{nj} + \mu a_n\varphi_{;j} + \mu a_j\varphi_{;n})e^{\mathrm{i}\varrho\varphi}\\ &- \mathrm{i}\varrho\Gamma^n_{jm}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ni} + \mu a_n\varphi_{;i} + \mu a_i\varphi_{;n})e^{\mathrm{i}\varrho\varphi} + \mathcal{O}(1),\end{aligned}$$

and

$$\begin{aligned}(\nabla\cdot S^L)_i = \sigma_{ij;}{}^{j} =& \sigma_{ij;m}g^{jm}c^{-2}\\ =& -\varrho^2(\lambda a_k\varphi_{;\ell}g^{k\ell}\varphi_{;i} + \mu a_i\varphi_{;j}\varphi_{;m}g^{jm} + \mu a_j\varphi_{;m}g^{jm}\varphi_{;i})c^{-2}e^{\mathrm{i}\varrho\varphi}\\ &+ \mathrm{i}\varrho c^{-2}\left(\partial_i(\lambda a_k\varphi_{;\ell}g^{k\ell}) + \lambda a_k\varphi_{;\ell}g^{k\ell}\partial_m g_{ij}g^{jm} + \partial_m(\mu a_i\varphi_{;j} + \mu a_j\varphi_{;i})g^{jm}\right)e^{\mathrm{i}\varrho\varphi}\\ &+ \mathrm{i}\varrho c^{-2}\Big(\lambda a_{k;\ell}g^{k\ell}\varphi_{;i} + \mu(a_{i;j} + a_{j;i})\varphi_{;m}g^{jm} + \mathrm{i}\lambda b_k\varphi_{;\ell}g^{k\ell}\varphi_{;i}\\ &\qquad + \mathrm{i}\mu(b_i\varphi_{;j}\varphi_{;m}g^{jm} + \varphi_{;i}b_j\varphi_{;m}g^{jm})\Big)e^{\mathrm{i}\varrho\varphi}\\ &- \mathrm{i}\varrho c^{-2}\Gamma^n_{im}g^{mj}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{nj} + \mu a_n\varphi_{;j} + \mu a_j\varphi_{;n})e^{\mathrm{i}\varrho\varphi}\\ &- \mathrm{i}\varrho c^{-2}\Gamma^n_{jm}g^{mj}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ni} + \mu a_n\varphi_{;i} + \mu a_i\varphi_{;n})e^{\mathrm{i}\varrho\varphi} + \mathcal{O}(1).\end{aligned}$$

We also calculate

$$\partial_t^2(u_\varrho)_i = -\varrho^2(\partial_t\varphi)^2a_ie^{\mathrm{i}\varrho\varphi} - \varrho(\partial_t\varphi)^2b_ie^{\mathrm{i}\varrho\varphi} + \mathrm{i}\varrho\partial_t^2\varphi a_ie^{\mathrm{i}\varrho\varphi} + 2\mathrm{i}\varrho(\partial_t\varphi)\partial_ta_ie^{\mathrm{i}\varrho\varphi} + \mathcal{O}(1).$$

In a neighborhood of ϑ, we can write

$$\rho\partial_t^2u_\varrho - \nabla\cdot S^L(u_\varrho) = e^{\mathrm{i}\varrho\varphi}\left(\varrho^2\mathcal{I}_1 + \sum_{k=0}^{N}\varrho^{1-k}\mathcal{I}_{k+2} + \mathcal{O}(\varrho^{-N})\right), \tag{12}$$

where

$$\mathcal{I}_1 = -\rho(\partial_t\varphi)^2\mathbf{a} + (\lambda+\mu)\langle\mathbf{a},\nabla\varphi\rangle\nabla\varphi + \mu|\nabla\varphi|^2\mathbf{a},$$

or component-wisely

$$(\mathcal{I}_1)_i = -\rho(\partial_t\varphi)^2a_i + \lambda a_j\varphi_{;\ell}g^{j\ell}c^{-2}\varphi_{;i} + \mu a_i\varphi_{;j}\varphi_{;\ell}c^{-2}g^{j\ell} + \mu a_j\varphi_{;\ell}g^{j\ell}c_P^{-2}\varphi_{;i},$$

and

$$
\begin{aligned}
(\mathcal{I}_2)_i =& \rho(\partial_t^2\varphi)a_i + 2\rho\partial_t\varphi\partial_t a_i + \mathrm{i}\rho(\partial_t\varphi)^2 b_i \\
&- c_P^{-2}\left(\partial_i(\lambda a_k\varphi_{;\ell}g^{k\ell}) + \lambda a_k\varphi_{;\ell}g^{k\ell}\partial_m g_{ij}g^{jm} + \partial_m(\mu a_i\varphi_{;j} + \mu a_j\varphi_{;i})g^{jm}\right) \\
&- c_P^{-2}\left(\lambda a_{k;\ell}g^{k\ell}\varphi_{;i} + \mu(a_{i;j} + a_{j;i})\varphi_{;m}g^{jm} + \mathrm{i}\lambda b_k\varphi_{;\ell}g^{k\ell}\varphi_{;i} + \mathrm{i}\mu(b_i\varphi_{;j}\varphi_{;m}g^{jm} + \varphi_{;i}b_j\varphi_{;m}g^{jm})\right) \\
&+ \Gamma^n_{im}g^{mj}c_P^{-2}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{nj} + \mu a_n\varphi_{;j} + \mu a_j\varphi_{;n}) \\
&+ \Gamma^n_{jm}g^{mj}c_P^{-2}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ni} + \mu a_n\varphi_{;i} + \mu a_i\varphi_{;n}).
\end{aligned}
$$

We will construct the phase function φ and the amplitude $\mathfrak{a}$ such that

$$
\frac{\partial^\Theta}{\partial z^\Theta}\mathcal{I}_k = 0 \text{ on } \vartheta \tag{13}
$$

for $\Theta = (0, \Theta_1, \Theta_2, \Theta_3)$ with $|\Theta| \leq N$ and $k = 1, 2, \cdots, N+2$. The detailed construction will be given later.

Assume that (13) is satisfied, we construct the remainder term R_ϱ. We let R_ϱ be the solution to the following initial boundary value problem

$$
\begin{aligned}
&\rho\frac{\partial^2 R_\varrho}{\partial t^2} - \nabla\cdot S^L(x, R_\varrho) = F_\varrho, \quad (t,x)\in(0,T)\times\Omega, \\
&R_\varrho = 0, \quad \text{on } (0,T)\times\partial\Omega, \\
&R_\varrho(0,x) = \frac{\partial}{\partial t}R_\varrho(0,x) = 0, \quad x\in\Omega.
\end{aligned} \tag{14}
$$

Here

$$
F_\varrho = -\rho\partial_t^2 u_\varrho + \nabla\cdot S^L(u_\varrho),
$$

in a neighborhood of ϑ. By (13) and [29, Lemma 2], we have

$$
\|F_\varrho\|_{H^k(M)} \leq C\varrho^{-K},
$$

where $K = \frac{N+1-k}{2} + 1$.

By L_2 estimates for second order hyperbolic equation, we have

$$
\|R_\varrho\|_{H^{k+1}(M)} \leq C\|F_\varrho\|_{H^k(M)}.
$$

We can take N large enough and use Sobolev imbedding to obtain

$$
\|R_\varrho\|_{W^{1,3}(M)} = \mathcal{O}(\varrho^{-1/2}). \tag{15}
$$

We remark here that $u = u_\varrho + R_\varrho$ solves the equation (3) with

$$
f = u_\varrho|_{[0,T]\times\partial\Omega}.
$$

3.1. Construction of the phase

We will construct phase function $\varphi = \varphi_{P/S}$ such that

$$
\mathcal{S}\varphi_P = (\lambda + 2\mu)|\nabla\varphi_P|^2 - \rho(\partial_t\varphi_P)^2,
$$

or

$$\mathcal{S}\varphi_S = \mu|\nabla\varphi_S|^2 - \rho(\partial_t\varphi_S)^2$$

vanishes on ϑ up to order N. In terms of Fermi coordinates $z = (z^0 = \tau, z^1, z^2, z^3)$ for $\overline{g}_{P/S} = -\mathrm{d}t^2 + g_{P/S}$ we need

$$\frac{\partial^\Theta}{\partial z^\Theta}(\mathcal{S}\varphi_{P/S})(\tau, 0) = 0 \tag{16}$$

for $\Theta = (0, \Theta_1, \Theta_2, \Theta_3)$ with $|\Theta| \le N$.

Notice that (16) is equivalent to

$$\frac{\partial^\Theta}{\partial z^\Theta}\langle \mathrm{d}\varphi_\bullet, \mathrm{d}\varphi_\bullet\rangle_{\overline{g}_\bullet}\Big|_\vartheta = 0.$$

Thus the phase function φ can be constructed as in [29] of the form

$$\varphi = \sum_{k=0}^{N} \varphi_k(\tau, z').$$

Here for each k, φ_k is a complex valued homogeneous polynomial of degree k with respect to the variables z^i, $i = 1, 2, 3$. In this paper, we will use the explicit forms of $\varphi_0, \varphi_1, \varphi_2$, which will be constructed below. Following the lines in [27], one can take

$$\varphi_0 = 0, \quad \varphi_1 = r = \frac{-t + t_0 + s}{\sqrt{2}},$$

and

$$\varphi_2(\tau, z') = \sum_{1 \le i,j \le 3} H_{ij}(\tau) z^i z^j.$$

Here H is a symmetric matrix with $\Im H(\tau) > 0$.

The matrix H satisfies a Ricatti type ODE,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}H + HCH + D = 0, \tau \in (\tau_- - \frac{\epsilon}{2}, \tau_+ + \frac{\epsilon}{2}), \quad H(0) = H_0, \text{ with } \Im H_0 > 0, \tag{17}$$

where C, D are matrices with $C_{11} = 0$, $C_{ii} = 2$, $i = 2, 3$, $C_{ij} = 0$, $i \ne j$ and $D_{ij} = \frac{1}{4}(\partial^2_{ij} g^{11})$.

Lemma 1 *([27, Lemma 3.2]). The Ricatti equation (17) has a unique solution. Moreover the solution H is symmetric and $\Im(H(\tau)) > 0$ for all $\tau \in (\tau_- - \frac{\delta}{2}, \tau_+ + \frac{\delta}{2})$. For solving the above Ricatti equation, one has*

$$H(\tau) = Z(\tau)Y(\tau)^{-1},$$

where $Y(\tau)$ and $Z(\tau)$ solve the ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau) = CZ(\tau), \quad Y(0) = Y_0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) = -D(\tau)Y(\tau), \quad Z(0) = Y_1 = H_0Y_0.$$

In addition, $Y(\tau)$ is non-degenerate.

Lemma 2 *([27, Lemma 3.3]). The following identity holds:*

$$\det(\Im(H(\tau))|\det(Y(\tau))|^2 = c_0$$

with c_0 independent of τ.

We see that the matrix $Y(\tau)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Y + CDY = 0, \quad Y(0) = Y_0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}Y(0) = CY_1. \tag{18}$$

3.2. Construction of the amplitude for P-waves

We consider the Lorentzian manifold $(M, -\mathrm{d}t^2 + g_P)$ and ϑ is a null-geodesic in it. For P-waves, the polarization vector $\mathbf{a}$ should be in parallel with the wave vector $\nabla\varphi$ on ϑ. Denote $\varphi = \varphi_P$ and take

$$\mathbf{a} = A_P \nabla\varphi. \tag{19}$$

Component-wisely, the gradient of φ has the form

$$\varphi_{\,;\,1}|_\vartheta = \frac{1}{\sqrt{2}}, \quad \varphi_{\,;\,\alpha}|_\vartheta = 0.$$

By (19), we have

$$a_1|_\vartheta = \frac{1}{\sqrt{2}}A_P, \quad a_\alpha|_\vartheta = 0,$$

and

$$\mathcal{I}_1 = (-\rho(\partial_t\varphi)^2 + (\lambda + 2\mu)|\nabla\varphi|^2)A_P\nabla\varphi.$$

By the construction of the phase function φ, we have (13) satisfied for $k = 1$.

Next, we proceed to construct A_P. Let us first consider the equation (13) with $k = 2$, $\Theta = 0$ and $i = 1$. On ϑ, we calculate

$$\begin{aligned}
&\rho(\partial_t^2\varphi)a_1 = \frac{1}{\sqrt{2}}\rho(\partial_t^2\varphi)A_P,\\
&2\rho(\partial_t\varphi)(\partial_t a_1) = -\sqrt{2}\rho\left(\frac{1}{\sqrt{2}}\partial_t A_P + A_P\frac{\partial^2\varphi}{\partial s\partial t}\right),\\
&\mathrm{i}\rho(\partial_t\varphi)^2 b_i = \frac{1}{2}\mathrm{i}\rho b_1,\\
&\partial_1(\lambda a_k\varphi_{;\ell}g^{k\ell}) = \frac{1}{2}\partial_s(\lambda A_P) + \sqrt{2}\lambda A_P\frac{\partial^2\varphi}{\partial s^2},\\
&\lambda a_k\varphi_{;\ell}g^{k\ell}\partial_m g_{1j}g^{jm} = 0,\\
&\partial_m(\mu a_1\varphi_{;j} + \mu a_j\varphi_{;1})g^{jm} = \partial_s(\mu A_P) + 2\sqrt{2}\mu A_P\frac{\partial^2\varphi}{\partial s^2} + \sqrt{2}\mu A_P\sum_{\alpha=2}^{3}\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha},\\
&\lambda a_{k;\ell}g^{k\ell}\varphi_{;1} = \lambda\left(\frac{1}{\sqrt{2}}A_P\frac{\partial^2\varphi}{\partial s^2} + \frac{1}{2}\partial_s A_P + \frac{1}{2}A_P c_P^{-1}\frac{\partial c_P}{\partial s} + \frac{1}{\sqrt{2}}A_P\sum_{\alpha=2}^{3}\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}\right),
\end{aligned}$$

$$\mu(a_{1;j}+a_{j;1})\varphi_{;m}g^{jm}=\mu(\partial_s A_P+\sqrt{2}A_P\frac{\partial^2\varphi}{\partial s^2}-A_P c_P\frac{\partial c_P}{\partial s}),$$
$$\mathrm{i}\lambda b_k\varphi_{;\ell}g^{k\ell}\varphi_{;1}+\mathrm{i}\mu(b_1\varphi_{;j}\varphi_{;m}g^{jm}+\varphi_{;1}b_j\varphi_{;m}g^{jm})=\frac{1}{2}\mathrm{i}(\lambda+2\mu)b_1,$$
$$\Gamma^n_{1m}g^{mj}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{nj}+\mu a_n\varphi_{;j}+\mu a_j\varphi_{;n})=(\frac{3}{2}\lambda+\mu)c_P^{-1}\frac{\partial c_P}{\partial s}A_P,$$
$$\Gamma^n_{jm}g^{mj}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{ni}+\mu a_n\varphi_{;i}+\mu a_i\varphi_{;n})=-(\frac{1}{2}\lambda+\mu)c_P^{-1}\frac{\partial c_P}{\partial s}A_P.$$

Then we obtain the following equation on ϑ:

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left(\rho\partial_t^2\varphi-c_P^{-2}(\lambda+2\mu)\partial_s^2\varphi-c_P^{-2}(\lambda+2\mu)\sum_{\alpha=2}^{3}\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}\right)A_P\\&-\sqrt{2}c_P^{-2}(\lambda+2\mu)A_P\frac{\partial^2\varphi}{\partial s^2}-\sqrt{2}\rho A_P\frac{\partial^2\varphi}{\partial s\partial t}-c_P^{-2}(\lambda+2\mu)\partial_s A_P-\rho\partial_t A_P\\&+\frac{1}{2}(\lambda+2\mu)c_P^{-3}\frac{\partial c_P}{\partial s}A_P-\frac{1}{2}c_P^{-2}\partial_s(\lambda+2\mu)A_P\\&+\mathrm{i}\frac{1}{2}b_1[\rho-c_P^{-2}(\lambda+2\mu)]=0.\end{aligned}$$

Since $c_P^{-2}(\lambda+2\mu)=\rho$, b_1 can not be determined at this step, but will be determined from lower order asymptotics. Notice that on ϑ,

$$\begin{aligned}&\rho\partial_t^2\varphi-c_P^{-2}(\lambda+2\mu)\partial_s^2\varphi-c_P^{-2}(\lambda+2\mu)\sum_{\alpha=2}^{3}\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}\\=&\rho\left(\partial_t^2\varphi-\partial_s^2\varphi-\sum_{\alpha=2}^{3}\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}\right)\\=&\rho\Box_{\overline{g}}\varphi\\=&-\rho\sum_{\alpha=2}^{3}\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}\\=&-\rho\mathrm{Tr}(CH)=-\rho\frac{\partial}{\partial\tau}\log(\det(Y_P(\tau))),\end{aligned}\tag{20}$$

and, using $\frac{\partial}{\partial s}+\frac{\partial}{\partial t}=\sqrt{2}\frac{\partial}{\partial\tau}$,

$$\begin{aligned}&-\sqrt{2}c_P^{-2}(\lambda+2\mu)A_P\frac{\partial^2\varphi}{\partial s^2}-\sqrt{2}\rho A_P\frac{\partial^2\varphi}{\partial s\partial t}\\=&-\sqrt{2}\rho A_P\left(\frac{\partial^2\varphi}{\partial s^2}+\frac{\partial^2\varphi}{\partial s\partial t}\right)\\=&-2\rho A_P\frac{\partial}{\partial\tau}(\partial_s\varphi)\\=&0.\end{aligned}$$

Then we arrive at the transport equation for the amplitude A_P on ϑ,

$$2\frac{\partial A_P}{\partial\tau}+\left[\frac{1}{\lambda+2\mu}\frac{\partial(\lambda+2\mu)}{\partial\tau}-c_P^{-1}\frac{\partial c_P}{\partial\tau}+\frac{1}{\det(Y_P)}\frac{\partial\det(Y_P)}{\partial\tau}\right]A_P=0,\tag{21}$$

or equivalently

$$\frac{\partial}{\partial \tau} \ln \left[A_P^2 \det(Y_P) c_P^{-1}(\lambda+2\mu)\right] = 0.$$

Then we can take

$$A_P(\tau) = c \det(Y_P(\tau))^{-1/2} c_P(\tau,0)^{-1/2} \rho(\tau,0)^{-1/2},$$

with some constant c.

Next we consider (13) for $k=2$ and $i=\alpha=2,3$ and obtain an equation for b_α, $\alpha=2,3$ on the null geodesic ϑ

$$\mathrm{i}(\rho - c_P^{-2}\mu) b_\alpha - c_P^{-2}(\lambda+2\mu)\frac{\partial A_P}{\partial z^\alpha} + \mathfrak{F}(\varphi, A_P|_\vartheta) = 0. \tag{22}$$

We can get an expression for b_α on ϑ.

Substituting the expression for b_α into the equation (13) with $k=2$ and $|\Theta|=1$ and $i=1$, we end up with a transport equation for $\frac{\partial^\Theta}{\partial z^\Theta} A_P$ on ϑ, from which we can determine the value of $\frac{\partial^\Theta}{\partial z^\Theta} A_P$. Then using again (22), we can determine b_α. Finally, the equation (13) with $k=2$ and $|\Theta|=1$ and $\alpha=2,3$ gives us the value of $\frac{\partial^\Theta}{\partial z^\Theta} b_\alpha$ on ϑ. Continuing with this process, we can have (13) satisfied with $k=2$ and $|\Theta| \le N$.

The lower order terms $\mathbf{a}_k$, $k=1,2,\cdots,N+1$ in the amplitude $\mathfrak{a}$ can be constructed as in [29] such that (13) is satisfied for all $k=2,\cdots,N+2$ and $|\Theta|\le N$. We note here that $(\mathbf{a}_{N+1})_1$ can take any value, while $(\mathbf{a}_{N+1})_\alpha$, $\alpha=2,3$ need to be determined.

3.3. Construction of the amplitude for S-waves

Let us now consider a null geodesic ϑ in the Lorentzian manifold $(M, -\mathrm{d}t^2 + g_S)$. Denote $\varphi=\varphi_S$. In the Fermi coordinates,

$$\varphi_{;1}|_\vartheta = \frac{1}{\sqrt{2}}, \quad \varphi_{;\alpha}|_\vartheta = 0, \text{ for } \alpha=2,3.$$

Now

$$\mathcal{I}_1 = (\mathcal{S}\varphi)\mathbf{a} + (\lambda+\mu)\langle \mathbf{a}, \nabla\varphi\rangle \nabla\varphi.$$

For S-waves, the polarization vector $\mathbf{a}$ should be perpendicular to the wave vector $\nabla\varphi$ on ϑ. In order for (13) to hold, we also need

$$\frac{\partial^\Theta}{\partial z^\Theta}\langle \mathbf{a}, \nabla\varphi\rangle|_\vartheta = 0, \tag{23}$$

for $\Theta=(0,\Theta_1,\Theta_2,\Theta_3)$ with $|\Theta|\le N$. For this we take

$$\mathbf{a} = A_S \mathbf{e}$$

with $\mathbf{e}=(e_1,e_2,e_3)$ satisfying $|\mathbf{e}|=1$ on ϑ and

$$\frac{\partial^\Theta}{\partial z^\Theta}\langle \mathbf{e}, \nabla\varphi\rangle|_\vartheta = 0. \tag{24}$$

First we construct such vector $\mathbf{e}$. Without loss of generality, we can fix an $\alpha \in \{2,3\}$ and let

$$e_1|_\vartheta, \quad e_\alpha|_\vartheta = 1, \quad e_{\alpha'}|_\vartheta = 0, \text{ for } \alpha' \neq \alpha.$$

The equation (24) with $|\Theta| = 1$ implies

$$\frac{1}{\sqrt{2}}\frac{\partial e_1}{\partial z^k} + \frac{\partial^2 \varphi}{\partial z^k \partial z^\alpha} = 0, \text{ on } \vartheta \tag{25}$$

for $k \in \{1,2,3\}$. Successively we can obtain, from (24) for $|\Theta| \geq 2$, equations for $\frac{\partial^\Theta e_1}{\partial z^\Theta}$ that will be determined. From now on, we just assume $\mathbf{e}$ has already been chosen.

We first consider (13) for $k = 2$ and $i = \alpha$. We calculate

$$\begin{aligned}
&\rho(\partial_t^2\varphi)a_\alpha = \rho(\partial_t^2\varphi)A_S,\\
&2\rho(\partial_t\varphi)(\partial_t a_\alpha) = -\sqrt{2}\rho\partial_t A_S - \sqrt{2}\rho A_S\frac{\partial e_\alpha}{\partial t},\\
&\mathrm{i}\rho(\partial_t\varphi)^2 b_\alpha = \frac{1}{2}\mathrm{i}\rho b_\alpha,\\
&\partial_\alpha(\lambda a_k \varphi_{;\ell}g^{k\ell}) = \lambda A_S\frac{\partial^2\varphi}{\partial y^\alpha \partial y^\alpha} + \frac{1}{\sqrt{2}}\lambda A_S\frac{\partial e_1}{\partial y^\alpha},\\
&\lambda a_k\varphi_{;\ell}g^{k\ell}\partial_m g_{\alpha j}g^{jm} = 0\\
&\partial_m(\mu a_\alpha\varphi_{;j} + \mu a_j\varphi_{;\alpha})g^{jm} = \frac{1}{\sqrt{2}}\partial_s(\mu A_S) + \frac{1}{\sqrt{2}}\mu A_S\frac{\partial e_\alpha}{\partial s} + \mu A_S\left(\frac{\partial^2\varphi}{\partial s^2} + \frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha} + \sum_{\beta=2}^{3}\frac{\partial^2\varphi}{\partial y^\beta\partial y^\beta}\right),\\
&\lambda a_{k;\ell}g^{k\ell}\varphi_{;\alpha} = 0,\\
&\mu(a_{\alpha;j} + a_{j;\alpha})\varphi_{;m}g^{jm} = \frac{1}{\sqrt{2}}\mu\left(\frac{\partial A_S}{\partial s} + A_S\frac{\partial e_\alpha}{\partial s} + A_S\frac{\partial e_1}{\partial y^\alpha} - 2c_S^{-1}\frac{\partial c_S}{\partial s}A_S\right),\\
&\mathrm{i}\lambda b_k\varphi_{;\ell}g^{k\ell}\varphi_{;\alpha} + \mathrm{i}\mu(b_\alpha\varphi_{;j}\varphi_{;m}g^{jm} + \varphi_{;\alpha}b_j\varphi_{;m}g^{jm}) = \frac{1}{2}\mathrm{i}\mu b_\alpha,\\
&\Gamma^n_{\alpha m}g^{mj}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{nj} + \mu a_n\varphi_{;j} + \mu a_j\varphi_{;n}) = 0,\\
&\Gamma^n_{jm}g^{mj}(\lambda a_k\varphi_{;\ell}g^{k\ell}g_{n\alpha} + \mu a_n\varphi_{;\alpha} + \mu a_\alpha\varphi_{;n}) = -\frac{1}{\sqrt{2}}\mu c_S^{-1}\frac{\partial c_S}{\partial s}A_S.
\end{aligned}$$

Then we obtain the following equation on ϑ

$$\begin{aligned}
&\left(\rho\partial_t^2\varphi - c_S^{-2}\mu\partial_s^2\varphi - c_S^{-2}\mu\sum_{\beta=2}^{3}\frac{\partial^2\varphi}{\partial y^\beta\partial y^\beta}\right)A_S\\
&-\sqrt{2}\left(\rho\frac{\partial e_\alpha}{\partial t} + c_s^{-2}\mu\frac{\partial e_\alpha}{\partial s}\right)A_S - \sqrt{2}(\rho\partial_t A_S + c_S^{-2}\mu\partial_s A_S)\\
&+\frac{1}{\sqrt{2}}\mu c_S^{-3}\frac{\partial c_S}{\partial s}A_S - \frac{1}{\sqrt{2}}c_S^{-2}\frac{\partial\mu}{\partial s}A_S + \frac{1}{2}\mathrm{i}b_\alpha(\rho - c_S^{-2}\mu)\\
&-c_S^{-2}(\lambda+\mu)A_S\left(\frac{1}{\sqrt{2}}\frac{\partial e_1}{\partial y^\alpha} + \frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}\right) = 0.
\end{aligned}$$

Since $\rho - c_S^{-2}\mu = 0$, b_α can not be determined at this step. Notice

$$\frac{1}{\sqrt{2}}\frac{\partial e_1}{\partial y^\alpha}+\frac{\partial^2\varphi}{\partial y^\alpha\partial y^\alpha}=0,$$

on ϑ by setting $k=\alpha$ in (25) and

$$\rho\frac{\partial e_\alpha}{\partial t}+c_s^{-2}\mu\frac{\partial e_\alpha}{\partial s}=\sqrt{2}\rho\frac{\partial e_\alpha}{\partial \tau}=0.$$

Similar to (20), we have

$$\rho\partial_t^2\varphi-c_S^{-2}\mu\partial_s^2\varphi-c_S^{-2}\mu\sum_{\beta=2}^{3}\frac{\partial^2\varphi}{\partial y^\beta\partial y^\beta}=-\rho\frac{\partial}{\partial\tau}\log(\det(Y_S(\tau))).$$

Thus we end up with the following equation for the amplitude A_S on ϑ:

$$2\frac{\partial A_S}{\partial \tau}+\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\tau}-c_S^{-1}\frac{\partial c_S}{\partial\tau}+\frac{1}{\det(Y_S)}\frac{\partial\det(Y_S)}{\partial\tau}\right]A_S=0. \tag{26}$$

The above equation is similar to the equation for A_P (21). Therefore we can take

$$A_S(\tau)=c\det(Y_S(\tau))^{-1/2}c_S(\tau,0)^{-1/2}\rho(\tau,0)^{-1/2},$$

with some constant c.

By calculation, we find that the equation $\mathcal{I}_2=0$ for $i=\alpha'$ always holds on ϑ, with arbitrary choice of $b_{\alpha'}$.

Next we consider the equation $(\mathcal{I}_2)_i=0$ for $i=1$. We obtain the following equation on the null geodesic ϑ

$$\frac{1}{2}\mathrm{i}(\rho-c_S^{-2}(\lambda+2\mu))b_1-c_S^{-2}(\lambda+\mu)\frac{\partial A_S}{\partial z^\alpha}+\mathfrak{F}(\varphi,A_S|_\vartheta)=0. \tag{27}$$

Substitute the expression for b_1 into the equation (13) with $k=2$, $|\Theta|=1$ and $i=1$. We will end up with a transport equation for $\frac{\partial^\Theta}{\partial z^\Theta}A_S$ on ϑ, from which we can determine the value of $\frac{\partial^\Theta}{\partial z^\Theta}A_S$. Then using again (27), we can determine b_1. Finally, the equation (13) with $k=2$ and $|\Theta|=1$ gives us the value of $\frac{\partial^\Theta}{\partial z^\Theta}b_1$ on ϑ.

Similar as for P-waves, we can construct lower order terms in the amplitude $\mathfrak{a}$ such that (13) is satisfied.

4. Proof of the main theorem

We will prove the uniqueness of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ from the displacement-to-traction Λ in this section. For the determination of $\mathscr{A}$, $\mathscr{B}$, we will have a pointwise recovery in the interior Ω. With $\mathscr{A}$ and $\mathscr{B}$ determined, certain type of weighted ray transform of $\mathscr{C}$ (along any geodesic in (Ω,g_P)) can be obtained from Λ. Under some geometric conditions, the weighted ray transform is invertible.

4.1. Determination of $\mathscr{A}$ and $\mathscr{B}$

We introduce the notations

$$L_p^{S,*}M=\{(\tau,\xi)\in T_p^*M,\,\tau^2=c_S^2|\xi|^2\},$$
$$L_p^{P,*}M=\{(\tau,\xi)\in T_p^*M,\,\tau^2=c_P^2|\xi|^2\},$$

for the S-wave and P-wave light cones at a point $p\in M$. Let us start with a lemma used in [1]:

Lemma 3. *There exist nonzero $\zeta^{(2)}, \zeta^{(0)} \in L_p^{S,*}M$, $\zeta^{(1)} \in L_p^{P,*}M$ such that $\zeta^{(0)}$, $\zeta^{(1)}$ and $\zeta^{(2)}$ are linearly dependent, while $\zeta^{(2)}$ and $\zeta^{(0)}$ are linearly independent.*

For readers' convenience, we still include the proof here.

Proof. Assume $\zeta^{(k)} = (\tau^{(k)}, \xi^{(k)})$, $k = 1, 2$. We have

$$(\tau^{(1)})^2 = c_P^2 |\xi^{(1)}|^2, \quad (\tau^{(2)})^2 = c_S^2 |\xi^{(2)}|^2.$$

Now we consider the vector $\zeta^{(0)} = a\zeta^{(1)} + b\zeta^{(2)}$. Without loss of generality, we can assume $a = 1$, $|\xi^{(k)}| = 1$ for $k = 1, 2$. In order for $\zeta^{(0)} \in L_p^{S,*}M$, we need

$$(\tau^{(1)} + b\tau^{(2)})^2 = c_S^2 |\xi^{(1)} + b\xi^{(2)}|^2.$$

Then we must have

$$2b\sqrt{\mu(\lambda + 2\mu)} - 2\mu b \xi^{(1)} \cdot \xi^{(2)} + (\lambda + 2\mu) = 0.$$

The above equation (with b as the unknown) always has a nonzero solution. This finishes the proof of the lemma. $\square$

Fix a point $x_0 \in \Omega$. Let $p = (\frac{T}{2}, x_0) \in M$, $\xi^{(0)}, \xi^{(1)}, \xi^{(2)} \in T^*_{x_0}\Omega$, $|\xi^{(k)}| = 1$. By Lemma 3, we can choose $\xi^{(k)}$, $k = 0, 1, 2$, such that

$$\begin{aligned} \zeta^{(1)} &= (c_P, \xi^{(1)}) \in L_p^{P,*}M, \\ \zeta^{(2)} &= (c_S, \xi^{(2)}) \in L_p^{S,*}M, \\ \zeta^{(0)} &= (c_S, \xi^{(0)}) \in L_p^{S,*}M, \end{aligned}$$

satisfying

$$\kappa_0 \zeta^{(0)} + \kappa_1 \zeta^{(1)} + \kappa_2 \zeta^{(2)} = 0. \tag{28}$$

Remark 2. The only term with parameter $\mathscr{C}$ in (9) is $\mathscr{C}(\nabla \cdot u^{(1)})(\nabla \cdot u^{(2)})(\nabla \cdot v)$. If any of the three solutions $u^{(1)}, u^{(2)}, v$ represents S-wave, this term will essentially vanish. Thus for the recovery of $\mathscr{C}$, we need to take $u^{(1)}, u^{(2)}, v$ all representing P-waves. However, one can not choose three vectors in $L_p^{P,*}M$, such that they are linearly dependent but pairwise linearly independent. This explains why we need to recover $\mathscr{C}$ in a different way.

Denote $\vartheta^{(0)}, \vartheta^{(2)}$ to be the null geodesics in Lorentzian manifold $(M, -\mathrm{d}t^2 + g_S)$ with cotangent vector $\zeta^{(0)}$, $\zeta^{(2)}$ at point p, and $\vartheta^{(1)}$ the null geodesic in Lorentzian manifold $(M, -\mathrm{d}t^2 + g_P)$ with cotangent vector $\zeta^{(1)}$ at point p. We will construct solutions:

- $u_\varrho^{(1),P}$ is the Gaussian beam solution representing P-waves, concentrated near the null geodesic $\vartheta^{(1)}$;
- $u_\varrho^{(2),SV}$ is the Gaussian beam solution representing SV-waves, concentrated near the null geodesic $\vartheta^{(2)}$;
- v_ϱ^{SV} is the Gaussian beam solution representing SV-waves, concentrated near the null geodesic $\vartheta^{(0)}$. (See Fig. 1.)

More specifically, denote

$$u_\varrho^{(1),P} = e^{\mathrm{i}\varrho\kappa_1\varphi^{(1)}} \chi^{(1)} (\mathbf{a}^{(1)} + \mathcal{O}(\varrho^{-1})), \quad u_\varrho^{(2),SV} = e^{\mathrm{i}\varrho\kappa_2\varphi^{(2)}} \chi^{(2)} (\mathbf{a}^{(2)} + \mathcal{O}(\varrho^{-1})),$$

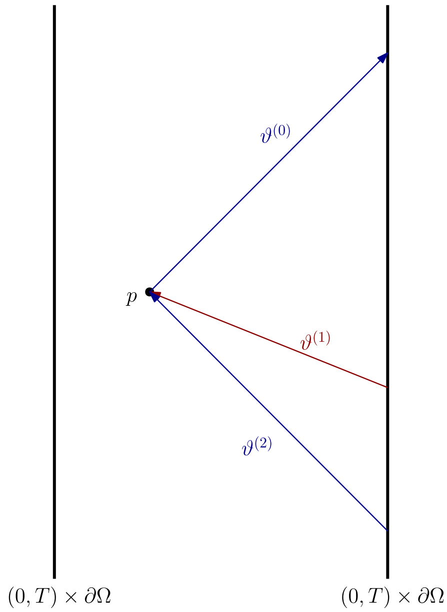

Fig. 1. Illustration of the choices of $\vartheta^{(1)}, \vartheta^{(2)}$ and $\vartheta^{(0)}$.

and

$$v_\varrho^{SV} = e^{\mathrm{i}\varrho\kappa_0\varphi^{(0)}}\chi^{(0)}(\mathbf{a}^{(0)} + \mathcal{O}(\varrho^{-1})).$$

By the construction of the phase functions $\varphi^{(k)}$, we can let

$$\nabla\varphi^{(k)}(p) = \xi^{(k)}, \text{ for } k = 0, 1, 2.$$

The amplitudes can be chosen such that

$$\mathbf{a}^{(1)}(p) = \xi^{(1)}, \quad \mathbf{a}^{(2)}(p) = \mathbf{a}^{(0)}(p) \perp \operatorname{span}\{\xi^{(1)}, \xi^{(2)}\}, \quad |\mathbf{a}^{(2)}(p)| = 1. \tag{29}$$

Similar to [29, Lemma 5], we have

Lemma 4. *The function*

$$S := \kappa_0\varphi^{(0)} + \kappa_1\varphi^{(1)} + \kappa_2\varphi^{(2)}$$

is well-defined in a neighborhood of p *and*

1. $S(p) = 0$*;*
2. $(\partial_t S(p), \nabla S(p)) = 0$*;*
3. $\Im S(q) \geq cd(q,p)^2$ *for* q *in a neighborhood of* p*, where* $c > 0$ *is a constant.*

Proof. The first claim is trivial since each of the three phases $\phi^{(k)}$ vanishes along $\vartheta^{(k)}$. For the second claim, one only needs to notice that

$$(\partial_t\varphi^{(0)}, \nabla\varphi^{(0)}) = (c_S, \xi^{(0)}), \quad (\partial_t\varphi^{(1)}, \nabla\varphi^{(1)}) = (c_P, \xi^{(1)}), \quad (\partial_t\varphi^{(2)}, \nabla\varphi^{(2)}) = (c_S, \xi^{(2)})$$

and use (28).

For the third claim, first notice $\Im\varphi^{(1)}(q) \geq 0$. We will prove $\Im\varphi^{(0)}(q) + \Im\varphi^{(2)}(q) \geq cd(p,q)$ for some constant $c > 0$. Using Fermi coordinates for $(M, -\mathrm{d}t^2 + g_S)$, we see that for $k = 0, 2$,

$$D^2\Im\varphi^{(k)}(X,X) \geq 0, \quad \forall X \in T_pM,$$
$$D^2\Im\varphi^{(k)}(X,X) > 0, \quad \forall X \in T_pM \setminus \operatorname{span}(\xi^{(k),\sharp}).$$

Since $\xi^{(0)}$ and $\xi^{(2)}$ are linearly independent, the claim follows. $\square$

Now let $u^{(k)}$, $k = 1, 2$ to be the solution to (4) with

$$f^{(1)} = u_\varrho^{(1),\,P}|_{[0,T]\times\partial\Omega}, \quad f^{(2)} = u_\varrho^{(2),\,SV}|_{[0,T]\times\partial\Omega}$$

and v to be the solution to (7) with

$$g = v_\varrho^{SV}|_{[0,T]\times\partial\Omega}.$$

We remark here that $u_\varrho^{(k),\bullet}|_{t=0} = \frac{\partial}{\partial t}u_\varrho^{(k),\bullet}|_{t=0} = 0$, and $v_\varrho^{SV}|_{t=T} = \frac{\partial}{\partial t}v_\varrho^{SV}|_{t=T} = 0$ since $T > 2\operatorname{diam}_S(\Omega)$.

We will need the estimate (15) and the following ones

$$\|u_\varrho^{(1),P}\|_{W^{1,3}}, \quad \|u_\varrho^{(2),SV}\|_{W^{1,3}}, \quad \|v_\varrho^{SV}\|_{W^{1,3}} = \mathcal{O}(\varrho^{1/2}).$$

Substituting $f^{(1)}, f^{(2)}, g$ constructed above in (8), we know that the displacement-to-traction map determines

$$\begin{aligned} &\varrho^{-1}\frac{\mathrm{i}}{\kappa_1\kappa_2\kappa_3}\int_0^T\int_\Omega \mathcal{G}(\nabla u^{(1)}, \nabla u^{(2)}, \nabla v)\mathrm{d}x\mathrm{d}t \\ =&\varrho^2\int_Q e^{\mathrm{i}\varrho S}\chi^{(1)}\chi^{(2)}\chi^{(0)}\mathcal{G}(\mathbf{a}^{(1)}\otimes\nabla\varphi^{(1)}, \mathbf{a}^{(2)}\otimes\nabla\varphi^{(2)}, \mathbf{a}^{(0)}\otimes\nabla\varphi^{(0)})\mathrm{d}x\mathrm{d}t + \mathcal{O}(\varrho^{-1/2}). \end{aligned} \tag{30}$$

First let us assume there are no conjugate points in (Ω, g_P)**.** Then the three null-geodesic $\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(0)}$ intersect only at p, and thus the function $\chi^{(1)}\chi^{(2)}\chi^{(0)}$ is supported in a small neighborhood of p. Denote

$$\mathcal{A} := \mathcal{G}(\mathbf{a}^{(1)}\otimes\nabla\varphi^{(1)}, \mathbf{a}^{(2)}\otimes\nabla\varphi^{(2)}, \mathbf{a}^{(0)}\otimes\nabla\varphi^{(0)})$$

with

$$\begin{aligned} &\mathcal{G}(\mathbf{a}^{(1)}\otimes\nabla\varphi^{(1)}, \mathbf{a}^{(2)}\otimes\nabla\varphi^{(2)}, \mathbf{a}^{(0)}\otimes\nabla\varphi^{(0)}) \\ =&\mathscr{B}[(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(1)})(\mathbf{a}^{(2)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(2)}) + (\mathbf{a}^{(2)}\cdot\nabla\varphi^{(2)})(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(2)}) \\ &\quad + (\mathbf{a}^{(2)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(2)})] \\ &+ \frac{\mathscr{A}}{4}\Big((\mathbf{a}^{(2)}\cdot\nabla\varphi^{(1)})(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(2)}) + (\mathbf{a}^{(1)}\cdot\nabla\varphi^{(2)})(\mathbf{a}^{(2)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(1)})\Big) \\ &+ (\lambda+\mathscr{B})[(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(1)})(\mathbf{a}^{(2)}\cdot\mathbf{a}^{(0)})(\nabla\varphi^{(2)}\cdot\nabla\varphi^{(0)}) + (\mathbf{a}^{(2)}\cdot\nabla\varphi^{(2)})(\mathbf{a}^{(1)}\cdot\mathbf{a}^{(0)})(\nabla\varphi^{(1)}\cdot\nabla\varphi^{(0)}) \\ &\quad + (\mathbf{a}^{(1)}\cdot\mathbf{a}^{(2)})(\nabla\varphi^{(1)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(0)})] + 2\mathscr{C}(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(1)})(\mathbf{a}^{(2)}\cdot\nabla\varphi^{(2)})(\mathbf{a}^{(0)}\cdot\nabla\varphi^{(0)}) \\ &+ (\mu+\frac{\mathscr{A}}{4})\Big((\mathbf{a}^{(1)}\cdot\mathbf{a}^{(2)})(\nabla\varphi^{(1)}\cdot\mathbf{a}^{(0)})(\nabla\varphi^{(2)}\cdot\nabla\varphi^{(0)}) + (\nabla\varphi^{(1)}\cdot\nabla\varphi^{(2)})(\mathbf{a}^{(1)}\cdot\mathbf{a}^{(0)})(\mathbf{a}^{(2)}\cdot\nabla\varphi^{(0)}) \\ &\quad + (\mathbf{a}^{(2)}\cdot\mathbf{a}^{(1)})(\nabla\varphi^{(2)}\cdot\mathbf{a}^{(0)})(\nabla\varphi^{(1)}\cdot\nabla\varphi^{(0)}) + (\mathbf{a}^{(2)}\cdot\mathbf{a}^{(0)})(\mathbf{a}^{(1)}\cdot\nabla\varphi^{(0)})(\nabla\varphi^{(1)}\cdot\nabla\varphi^{(2)}) \\ &\quad + (\nabla\varphi^{(1)}\cdot\mathbf{a}^{(2)})(\nabla\varphi^{(2)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(1)}\cdot\mathbf{a}^{(0)}) + (\nabla\varphi^{(2)}\cdot\mathbf{a}^{(1)})(\nabla\varphi^{(1)}\cdot\nabla\varphi^{(0)})(\mathbf{a}^{(2)}\cdot\mathbf{a}^{(0)})\Big). \end{aligned}$$

By the choice of $\alpha^{(k)} = \mathbf{a}^{(k)}(p)$ and $\xi^{(k)}$, $k = 0, 1, 2$ in (29). We have

$$
\begin{aligned}
\mathcal{A}(p) =& \mathscr{B}(x_0)[(\alpha^{(1)}\cdot\xi^{(1)})(\alpha^{(2)}\cdot\xi^{(0)})(\alpha^{(0)}\cdot\xi^{(2)}) + (\alpha^{(2)}\cdot\xi^{(2)})(\alpha^{(1)}\cdot\xi^{(0)})(\alpha^{(0)}\cdot\xi^{(2)}) \\
&\qquad + (\alpha^{(2)}\cdot\xi^{(0)})(\alpha^{(1)}\cdot\xi^{(0)})(\alpha^{(0)}\cdot\xi^{(2)})] \\
&+ \frac{\mathscr{A}}{4}(x_0)\Big((\alpha^{(2)}\cdot\xi^{(1)})(\alpha^{(1)}\cdot\xi^{(0)})(\alpha^{(0)}\cdot\xi^{(2)}) + (\alpha^{(1)}\cdot\xi^{(2)})(\alpha^{(2)}\cdot\xi^{(0)})(\alpha^{(0)}\cdot\xi^{(1)})\Big) \\
&+ (\lambda+\mathscr{B})(x_0)[(\alpha^{(1)}\cdot\xi^{(1)})(\alpha^{(2)}\cdot\alpha^{(0)})(\xi^{(2)}\cdot\xi^{(0)}) + (\alpha^{(2)}\cdot\xi^{(2)})(\alpha^{(1)}\cdot\alpha^{(0)})(\xi^{(1)}\cdot\xi^{(0)}) \\
&\quad + (\alpha^{(1)}\cdot\alpha^{(2)})(\xi^{(1)}\cdot\xi^{(0)})(\alpha^{(0)}\cdot\xi^{(0)})] + 2\mathscr{C}(x_0)(\alpha^{(1)}\cdot\xi^{(1)})(\alpha^{(2)}\cdot\xi^{(2)})(\alpha^{(0)}\cdot\xi^{(0)}) \\
&+ (\mu+\frac{\mathscr{A}}{4})(x_0)\Big((\alpha^{(1)}\cdot\alpha^{(2)})(\xi^{(1)}\cdot\alpha^{(0)})(\xi^{(2)}\cdot\xi^{(0)}) + (\xi^{(1)}\cdot\xi^{(2)})(\alpha^{(1)}\cdot\alpha^{(0)})(\alpha^{(2)}\cdot\xi^{(0)}) \\
&\quad + (\alpha^{(2)}\cdot\alpha^{(1)})(\xi^{(2)}\cdot\alpha^{(0)})(\xi^{(1)}\cdot\xi^{(0)}) + (\alpha^{(2)}\cdot\alpha^{(0)})(\alpha^{(1)}\cdot\xi^{(0)})(\xi^{(1)}\cdot\xi^{(2)}) \\
&\quad + (\xi^{(1)}\cdot\alpha^{(2)})(\xi^{(2)}\cdot\xi^{(0)})(\alpha^{(1)}\cdot\alpha^{(0)}) + (\xi^{(2)}\cdot\alpha^{(1)})(\xi^{(1)}\cdot\xi^{(0)})(\alpha^{(2)}\cdot\alpha^{(0)})\Big) \\
=& (\lambda+\mathscr{B})(x_0)\xi^{(2)}\cdot\xi^{(0)} + (2\mu+\frac{\mathscr{A}}{2})(x_0)(\xi^{(1)}\cdot\xi^{(2)})(\xi^{(1)}\cdot\xi^{(0)}).
\end{aligned}
$$

Apply the method of stationary phase (cf., for example, [39, Theorem 7.7.5]) to (30), we can recover

$$
(\lambda+\mathscr{B})(x_0)\xi^{(2)}\cdot\xi^{(0)} + (2\mu+\frac{\mathscr{A}}{2})(x_0)(\xi^{(1)}\cdot\xi^{(2)})(\xi^{(1)}\cdot\xi^{(0)}),
$$

by taking $\varrho \to +\infty$. This is exactly the same quantity recovered in [1] using the nonlinear interaction of distorted plane P and SV waves. By varying $\xi^{(1)}, \xi^{(2)}$ ($\xi^{(0)}$ will be varying accordingly), we can recover $\lambda+\mathscr{B}$ and $2\mu + \frac{\mathscr{A}}{2}$ separately at the point x_0. Since x_0 can be any point in Ω, this completes the determination of $\mathscr{A}$ and $\mathscr{B}$ in Ω.

Remark 3. We only used the nonlinear interaction of P and SV waves to determine $\mathscr{A}$ and $\mathscr{B}$. It has already been observed to be possible in [1], wherein nonlinear interactions of other types are analyzed as well.

Now assume (Ω, g_P) **satisfies the foliation condition.** As in [7], for any point $q \in \partial\Omega$, there exists a wedge-shaped neighborhood $O_q \subset \Omega$ of q such that any geodesic in (O_q, g_P) has no conjugate points. Then the three null-geodesics $\vartheta^{(1)}, \vartheta^{(2)}, \vartheta^{(0)}$, if $\vartheta^{(1)} \subset ((0,T)\times O_q, -\mathrm{d}t^2 + g_P)$, can intersect only at p (since $c_P > c_S$). We can now recover $\mathscr{A}$ and $\mathscr{B}$ in O_q. Then the foliation condition allows a layer stripping scheme to recover the two parameters in the whole domain Ω. For more details, we refer to [7].

4.2. Determination of $\mathscr{C}$

Finally, we recover the parameter $\mathscr{C}$.

Since $\mathscr{A}$ and $\mathscr{B}$ have already been determined, the displacement-to-traction map now gives

$$
\int_0^T\int_\Omega \mathscr{C}(\nabla\cdot u^{(1)})(\nabla\cdot u^{(2)})(\nabla\cdot v)\mathrm{d}x\mathrm{d}t. \tag{31}
$$

We will construct $u^{(1)}, u^{(2)}$ and v so that they all represent P-waves. Still use Gaussian beam solutions, now concentrating near the same null geodesic ϑ in $(M, -\mathrm{d}t^2 + g_P)$. Let

$$
u_\varrho^{(1)} = u_\varrho^{(2)} = e^{\mathrm{i}\varrho\varphi}\chi(\frac{|z'|}{\delta})(\mathbf{a} + \mathcal{O}(\varrho^{-1})),
$$

$$v_\varrho = e^{-2\mathrm{i}\varrho\overline{\varphi}}\chi(\frac{|z'|}{\delta})(\overline{\mathbf{a}} + \mathcal{O}(\varrho^{-1})).$$

Let $u^{(k)}$, $k = 1, 2$ to be the solution to (4) with

$$f^{(1)} = u_\varrho^{(1)}|_{[0,T]\times\partial\Omega}, \quad f^{(2)} = u_\varrho^{(2)}|_{[0,T]\times\partial\Omega}$$

and v to be the solution to (7) with

$$g = v_\varrho|_{[0,T]\times\partial\Omega}.$$

We extend $\mathscr{C}$ to $\widetilde{\Omega}$ such that $\mathscr{C} = 0$ in $\widetilde{\Omega} \setminus \Omega$. Then the displacement-to-traction map determines

$$\begin{aligned}&\frac{1}{2\mathrm{i}}\varrho^{-3/2}\int\limits_M \mathscr{C}(\nabla\cdot u^{(1)})(\nabla\cdot u^{(2)})(\nabla\cdot v)\mathrm{d}V\\ =&\varrho^{3/2}\int\limits_{\tau_- - \frac{\epsilon}{\sqrt{2}}}^{\tau_+ + \frac{\epsilon}{\sqrt{2}}}\int\limits_{|z'|<\delta}\mathscr{C}e^{-4\varrho\Im\varphi}\chi^3(\frac{|z'|}{\delta})(\nabla\varphi\cdot\mathbf{a})(\nabla\varphi\cdot\mathbf{a})(\overline{\nabla\varphi\cdot\mathbf{a}})c_P^3\mathrm{d}z'\wedge\mathrm{d}\tau + \mathcal{O}(\varrho^{-1})\end{aligned}$$

with δ sufficiently small. Notice

$$(\nabla\varphi\cdot\mathbf{a})(\nabla\varphi\cdot\mathbf{a})(\overline{\nabla\varphi\cdot\mathbf{a}})|_{\vartheta(\tau)} = c|\det Y(\tau)|^{-1}(\det Y(\tau))^{-1/2}c_P^{-15/2}(\tau)\rho^{-3/2}(\tau),$$

where c is some constant. Thus, using method of stationary phase and Lemma 2, we have

$$\begin{aligned}&\lim_{\varrho\to+\infty}\varrho^{3/2}\int\limits_{|z'|<\delta}\mathscr{C}e^{-4\varrho\Im\varphi}\chi^3(\frac{|z'|}{\delta})(\nabla\varphi\cdot\mathbf{a})(\nabla\varphi\cdot\mathbf{a})(\overline{\nabla\varphi\cdot\mathbf{a}})c_P^3\mathrm{d}z'\\ =&c\mathscr{C}(\tau,0)c_P^{-9/2}(\tau,0)\rho(\tau,0)^{-3/2}(\det Y(\tau))^{-1/2}.\end{aligned}$$

Thus we can recover

$$\int\limits_\vartheta \mathscr{C}c_P^{-9/2}(\tau,0)\rho^{-3/2}(\tau,0)(\det Y(\tau))^{-1/2}\mathrm{d}\tau.$$

Remember that Y solves the equation (18). By [27, Corollary 3.5],

$$\frac{\partial^2\overline{g}^{11}}{\partial z^1\partial z^j}\Big|_\vartheta = 0.$$

Thus one can take $Y_{11} = c_0$, $Y_{1j} = Y_{j1} = 0$. Here $c_0 > 0$ is independent of τ. Denote $\widetilde{Y} = (Y_{\alpha\beta})_{\alpha,\beta=2}^3$. Then the 2×2 matrix $\widetilde{Y}$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\widetilde{Y} + \widetilde{D}\widetilde{Y} = 0, \quad \widetilde{Y}(a) = \widetilde{Y}_0, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{Y}(a) = \widetilde{Y}_1,$$

where $\widetilde{D}_{\alpha\beta} = \frac{1}{2}\left(\partial^2_{\alpha\beta}g^{11}|_\vartheta\right)^3_{\alpha,\beta=2}$.

Now let us use some notations and definitions introduced in [40]. We will follow the lines in [28]. Assume $\vartheta(t) = (t, \gamma(t))$ where γ is a geodesic in the Riemannian manifold (Ω, g_P). Denote

$$\dot{\gamma}(t)^{\perp} := \{v \in T_{\gamma(t)}\Omega | g_P(\dot{\gamma}(t), v) = 0\}$$

to be the orthogonal complement at the point $\gamma(t)$ of $\dot{\gamma}$. Define the $(1,1)$-tensor $\Pi_{\gamma(t)}$ to be the projection from $T_{\gamma(t)}M$ onto $\dot{\gamma}(t)^{\perp}$. A $(1,1)$-tensor $L(t)$ is said to be transversal if $\Pi_{\gamma(t)}L(t)\Pi_{\gamma(t)} = L(t)$. Denote $\mathbb{Y}_\gamma$ to be the set of all transversal $(1,1)$-tensors $Y(t)$ that solve the complex Jacobi equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}Y(t) + K(t)Y(t) = 0,$$

subject to the constraint that

$$Y(t_0) \text{ is non-degenerate, } \dot{Y}(t_0)Y(t_0)^{-1} \text{ is symmetric and } \Im(\dot{Y}(t_0)Y(t_0)^{-1}) > 0.$$

Here $K = K^i_j \frac{\partial}{\partial y^i} \otimes \mathrm{d}y^j|_\gamma$ is a tensor on γ, where $K^i_j = R_{jk\ell}{}^i \dot{\gamma}^k \dot{\gamma}^\ell$, and R is the Riemann curvature tensor.

As in [27], we now have the *Jacobi weighted ray transform of the first kind* (cf. [28]) of $\mathscr{C} c_P^{-9/2}\rho^{-3/2} =: f$ along the geodesic γ in (Ω, g_P) passing through $x_0 = \gamma(t_0)$,

$$\mathscr{J}^{(1)}_Y f = \int_{t_-}^{t_+} f(\gamma(t))(\det Y(t))^{-1/2}\mathrm{d}t$$

for any $Y \in \mathbb{Y}_\gamma$.

By [28, Proposition 3], $\mathscr{J}^{(1)}_Y f$ uniquely determines $f(x_0)$ **if** (Ω, g_P) **has no conjugate points**. Therefore, we can recover $\mathscr{C}(x_0)$ since c_P and ρ are already known. Here we use only the ray transform along this single geodesic. It is possible since we have the integral of f along the geodesic with a class of weights.

If (Ω, g_P) **satisfies the foliation condition**, we can adopt a layer stripping method as previously used. One can also directly use the invertibility of weighted geodesic ray transform with a single weight established in [8].

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