

INVERSE PROBLEMS FOR THE FRACTIONAL-LAPLACIAN WITH LOWER ORDER NON-LOCAL PERTURBATIONS

S. BHATTACHARYYA, T. GHOSH, AND G. UHLMANN

ABSTRACT. In this article, we introduce a model featuring a Lévy process in a bounded domain with semi-transparent boundary, by considering the fractional Laplacian operator with lower order non-local perturbations. We study the wellposedness of the model, certain qualitative properties and Runge type approximation. Furthermore, we consider the inverse problem of determining the unknown coefficients in our model from the exterior measurements of the corresponding Cauchy data. We also discuss the recovery of all the unknown coefficients from a single measurement.

1. INTRODUCTION

1.1. Model problem and motivation. In this article we consider direct and inverse problems concerning a non-local operator $\mathcal{L}_{b,q}$, consisting of global and regional non-local operators. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a non-empty bounded, Lipschitz domain. Let us consider the operator $\mathcal{L}_{b,q}$ defined by

$$(1.1) \quad \mathcal{L}_{b,q} := (-\Delta)^t + (-\Delta)_\Omega^{s/2} b (-\Delta)_\Omega^{s/2} + q, \quad 0 < s < t < 1,$$

where the perturbation coefficients b and q are $L^\infty(\Omega)$ functions. For the sake of simplicity, throughout this article we consider all the functions to be real valued. The principal part of the operator $\mathcal{L}_{b,q}$ is given by the fractional Laplacian operator $(-\Delta)^t$ of order $2t$ whose non-locality is over the entire $\mathbb{R}^n$. The lower order term contains another non-local operator, commonly referred to as the regional fractional Laplacian operator $(-\Delta)_\Omega^{s/2}$, of order s , whose non-locality is over Ω . We also add a zero-th order local term defined in Ω . Observe that we allow any $s \in (0, t)$, so that we can have non-local perturbations of order $(2s)$ as close as we wish to the order $(2t)$ of the principal part.

Probabilistically, the fractional Laplacian operator $(-\Delta)^t$ represents the infinitesimal generator of a symmetric $2t$ -stable Lévy process in the entire space [App09]. Here we are interested in the restriction of $(-\Delta)^t$ to a bounded domain Ω . For example, one can think of the homogeneous Dirichlet exterior value problem for the fractional Laplacian operator (*e.g.* $(-\Delta)^t v = g$ in Ω and $v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$) which represents the infinitesimal generator of a symmetric $2t$ -stable Lévy process for

Received by the editors January 1, 2019, and, in revised form, September 6, 2019, November 6, 2019, and February 14, 2020.

2020 *Mathematics Subject Classification.* Primary 35R30, 35R11.

The first and second authors were partly supported by Project no. 16305018 of the Hong Kong Research Grant Council.

The third author was partly supported by NSF, the Si-Yuan Professorship at the IAS-HKUST and the Walker Family Endowed Professorship at the University of Washington.

which particles are killed upon leaving the domain Ω (see [BV16, AVMRTM10]). The fractional Laplacian operator in $\mathbb{R}^n$ is defined by

$$(1.2) \quad (-\Delta)^t u = \mathcal{F}^{-1}\{|\xi|^{2t}\widehat{u}(\xi)\}, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Here $\mathcal{S}$ denotes the Schwartz space in $\mathbb{R}^n$ and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, with the Fourier transform defined by $\widehat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx$. The definition (1.2) is valid for all $t > -\frac{n}{2}$. For $0 < t < 1$, the fractional Laplacian $(-\Delta)^t$ has an equivalent integral representation given by

$$(-\Delta)^t u(x) = C_{n,t} \text{ p.v. } \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2t}} dy, \quad x \in \mathbb{R}^n,$$

where p.v. denotes the principal value and $C_{n,t} > 0$ is a constant depending only on the dimension $n \geq 1$ and $t \in (0, 1)$.

Now we define the regional fractional Laplacian, or censored fractional Laplacian operator, on the domain Ω . For $0 < a < 1$ we define $(-\Delta)_\Omega^a$ on $C^\infty(\overline{\Omega})$ functions as

$$(-\Delta)_\Omega^a u(x) = C_{n,a} \text{ p.v. } \int_\Omega \frac{u(x) - u(y)}{|x - y|^{n+2a}} dy, \quad x \in \Omega.$$

In contrast to the fractional Laplacian, the regional fractional Laplacian $(-\Delta)_\Omega^a$ represents the infinitesimal generator of a censored $2a$ -stable process that is obtained from a symmetric $2a$ -stable Lévy process restricted to the domain Ω . The probabilistic meaning for such a process (and hence the operator) is that it can only jump within the domain. Such process can be also defined using the Feynman-Kac formula; see [CZ95, MK00, GM05, GM06].

We define the operator $\mathcal{L}_{b,q}$ to be a weighted combination of the global and the regional fractional Laplacian operator giving a large class of processes on domains with semi-transparent boundaries. For those processes, after hitting the boundary of Ω , a particle can either go outside the domain Ω or can reflect back into the domain depending on certain parameters. The coefficient $b(x)$ denotes the transparency or permeability. The vanishing of b in Ω means that the domain is transparent/permeable, i.e. if $b \equiv 0$ in Ω , then the non-local part of $\mathcal{L}_{b,q}u(x)$ is $(-\Delta)^t u(x)$, which makes the process a $2t$ -stable Lévy process in the entire space. In other words, $b = 0$ implies that the process can jump anywhere in the space $\mathbb{R}^n$ freely.

Finally we mention that the study of the operator $\mathcal{L}_{b,q}$ is in itself of interest mathematically, since it contains two different types of non-locality and has various interesting properties like unique continuation and Runge type approximations. In the direct problem we prove existence and stability of solutions of $\mathcal{L}_{b,q}u = F$ in Ω having the Dirichlet data $u = f$ defined on $\Omega_e := \mathbb{R}^n \setminus \overline{\Omega}$. Here we take $F \in H^{-t}(\Omega)$ and $f \in \widetilde{H}^t(\Omega_e)$ (cf. Section 2.2). Later in this article we use the direct problem for the operator $(-\Delta)_\Omega^a$, which is defined only on Ω and also has a wellposed Dirichlet exterior value problem on a Lipschitz domain $\mathcal{O} \Subset \Omega$ in the scale of Sobolev spaces. See [BBC03, GM05, GM06] for Dirichlet exterior value problems for the regional fractional Laplacian operator on other function spaces. In Section 2.5 we discuss the direct problem in detail and prove existence, uniqueness and stability results for the operator $\mathcal{L}_{b,q}$ and the regional fractional Laplacian operator $(-\Delta)_\Omega^a$.

1.2. Inverse problems. We consider the recovery of the coefficients b and q in Ω from the non-local exterior data $(f, \mathcal{N}_{b,q}(f))$ measured in some open subset of

$\Omega_e \times \Omega_e$. For $f \in \tilde{H}^t(\Omega_e)$, the non-local Neumann data $\mathcal{N}_{b,q}(f)$ on Ω_e is defined by

$$\mathcal{N}_{b,q}(f) := C_{n,t} \int_{\Omega} \frac{u_f(x) - u_f(y)}{|x - y|^{n+2t}} dy; \quad \forall x \in \Omega_e,$$

where $u_f \in H^t(\mathbb{R}^n)$ is the unique solution of the problem $\mathcal{L}_{b,q}u_f = 0$ in Ω and $u = f$ on Ω_e (see Section 2.6). Let $W, \widetilde{W} \subset \Omega_e$ be any two non-empty open sets. We define the non-local partial Cauchy data corresponding to the operator $\mathcal{L}_{b,q}$ by

$$(1.3) \quad \mathcal{C}_{b,q}(W, \widetilde{W}) := \{(f, \mathcal{N}_{b,q}(f)|_{\widetilde{W}}); f \in \tilde{H}^t(W)\}.$$

We assume that $b \in L^\infty(\Omega)$, $q \in L^\infty(\Omega)$ are compactly supported functions and we also assume throughout this paper that they satisfy the following eigenvalue condition

$$(1.4) \quad \mathcal{L}_{b,q}\varphi = 0 \text{ in } \Omega, \quad \varphi = 0 \text{ in } \Omega_e \quad \text{has only the zero solution.}$$

For non-negative functions b and q the above eigenvalue condition is satisfied. Let $W, \widetilde{W} \subset \Omega_e$ be any two non-empty open subsets of Ω_e . We prove two results (Theorems 1.1 and 1.2) regarding the recovery of the coefficients b, q in Ω from the non-local partial Cauchy data $\mathcal{C}_{b,q}(W, \widetilde{W})$. In the first result (Theorem 1.1) we show that measuring $(f, \mathcal{N}_{b,q}(f)|_{\widetilde{W}})$ for all $f \in \tilde{H}^t(W)$, one can uniquely determine b and q in Ω . We discuss also how much one can recover from a single measurement of the non-local Cauchy data. We prove that given a single measurement of the Cauchy data $\mathcal{C}_{b,q}(W, \widetilde{W})$ corresponding to a single non-zero f , one can determine b and q on certain subsets of Ω . Moreover, given a single measurement $(f, \mathcal{N}f|_{\widetilde{W}})$, we show that the above subsets of Ω (where we can recover b and q) are optimal in the sense that one cannot conclude anything about b and q outside those subsets from that measurement. We state our results for the inverse problems in Theorems 1.1 and 1.2 and prove them in Section 3.2.

These type of inverse problems are often addressed as Calderón type inverse problems. In the standard Calderón problem [Cal80] the objective is to determine the electrical conductivity of a medium from voltage and current measurements made on its boundary. Study of the inverse boundary value problems has a long history, in particular, in the context of electrical impedance tomography; it has applications in seismic and medical imaging, as well as to inverse scattering problems. We refer to [Uhl14] and the references therein for a survey of this topic.

The study of Calderón type inverse problems for non-local operators began with the article [GSU20], where the authors address the inverse problem of determining the potential q in the fractional Schrödinger operator $((-\Delta)^t + q(x))$, $0 < t < 1$, in Ω from the corresponding Dirichlet to Neumann map in the exterior domain Ω_e . In [AS19] the authors study stability estimates for recovering the potential q . Later it has been shown that with a single measurement of $(f, \mathcal{N}_{b,q}(f))$ for non zero f , it is possible to recover and reconstruct the potential q in Ω (see [GRSU20]). The problem of recovering q for the anisotropic fractional elliptic operators has been considered in [GLX17].

1.2.1. All measurements. In this article we are interested in determining two unknowns b and q in the perturbed non-local operator $\mathcal{L}_{b,q}$ where in addition to the zeroth order perturbation we have a $2s$ -order non-local perturbation to the fractional Laplacian operator of order $2t$, $0 < s < t < 1$. Our result for all measurements is:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded Lipschitz domain. Let $b_1, b_2, q_1, q_2 \in L^\infty(\Omega)$ compactly supported in Ω be such that assumption (1.4) is satisfied for $\mathcal{L}_{b_k, q_k}$, $k = 1, 2$. Let $W, \widetilde{W} \subset \Omega_e$ be any two non-empty open subsets.*

If $\mathcal{C}_{b_1, q_1}(W, \widetilde{W}) = \mathcal{C}_{b_2, q_2}(W, \widetilde{W})$, then $q_1 = q_2$ and $b_1 = b_2$ in Ω .

1.2.2. Single measurement. We also consider the problem of recovering b and q on suitable subsets of Ω subject from the non-local Cauchy data $\mathcal{C}_{b_2, q_2}(W, \widetilde{W})$ for only a single non-zero $f \in \widetilde{H}^t(\Omega_e)$. Let us fix a non-zero $f \in \widetilde{H}^t(\Omega_e)$ and let u_f be the unique solution of the problem $\mathcal{L}_{b, q} u_f = 0$ in Ω and $u_f|_{\Omega_e} = f$. If $b \equiv 0$ in Ω , a single non-zero measurement of the Cauchy data is enough to determine q in Ω (see [GRSU20]). If $b \not\equiv 0$ in Ω and $u_f = 0$ in some non-empty open subset $E \subset \Omega$, then we cannot conclude anything about q in E . More precisely, if $b \not\equiv 0$ in Ω and there exists a non-empty open set $E \subset \Omega$ such that $u_f = 0$ in E , then for any $\varphi \in C_c(E)$

$$(1.5) \quad \mathcal{L}_{b, q} u_f = 0 \quad \text{in } \Omega \implies \mathcal{L}_{b, (q+\varphi)} u_f = 0 \quad \text{in } \Omega,$$

with $\mathcal{C}_{b, q}(W, \widetilde{W}) = \mathcal{C}_{b, q+\varphi}(W, \widetilde{W})$. Therefore, it is impossible to recover q on E from the single measurement $(f, \mathcal{N}_{b, q}(f)|_{\widetilde{W}})$. Similarly, if $(-\Delta)_\Omega^{s/2} u_f = 0$ in some non-empty open subset $F \subset \Omega$, then it is impossible to recover b on F from the single measurement $(f, \mathcal{N}_{b, q}(f)|_{\widetilde{W}})$. Therefore, the optimal result would be to recover b and q on the support of $(-\Delta)_\Omega^{s/2} u_f$ and u_f respectively. Our result for a single measurement is:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded Lipschitz domain and $b_1, b_2, q_1, q_2 \in C_c(\Omega)$ continuous functions with compact support inside Ω . We assume $\mathcal{L}_{b_1, q_1}, \mathcal{L}_{b_2, q_2}$ are such that the assumption (1.4) is satisfied. Let $f \in \widetilde{H}^t(\Omega_e)$ be a fixed non-zero function and $(u_f)_j \in H^t(\mathbb{R}^n)$ solves $\mathcal{L}_{b_j, q_j}(u_f)_j = 0$ in Ω with $(u_f)_j = f$ in Ω_e for $j = 1, 2$.*

If $\mathcal{N}_{b, q}(f)_1 = \mathcal{N}_{b, q}(f)_2$ on $\widetilde{W}$, where $\widetilde{W} \subset \Omega_e$ is some non-empty open subset, then $(u_f)_1 \equiv (u_f)_2$ in $\mathbb{R}^n$. Moreover, we have $q_1 = q_2$ on the support of $(u_f)_1$ and $b_1 = b_2$ on the support of $(-\Delta)_\Omega^{s/2}(u_f)_1$ in Ω .

The remainder of this paper is organized as follows: In Section 2 we discuss the direct problem for the regional fractional Laplacian operator $(-\Delta)_\Omega^a$ and the non-local operator $\mathcal{L}_{b, q}$. We first define the fractional order Sobolev spaces on $\mathbb{R}^n$, as well as on a non-empty open domain. Then we study wellposedness of the Dirichlet exterior value problem for the regional fractional Laplacian operator acting on fractional order Sobolev spaces. In the rest of the section we discuss wellposedness of the Dirichlet exterior value problem for the non-local operator $\mathcal{L}_{b, q}$ and finally, we finish our discussion on direct problems by defining the non-local Cauchy data corresponding to the operator $\mathcal{L}_{b, q}$. In Section 3 we prove Theorems 1.1 and 1.2. Having the equality of the non-local Cauchy data corresponding to two non-local operators $\mathcal{L}_{b_k, q_k}$, for $k = 1, 2$, and the unique continuation principle for the fractional Laplacian operator $(-\Delta)^t$ (cf. Proposition 3.2), we derive an integral identity (3.2) relating the perturbation coefficients b_k and q_k , for $k = 1, 2$. Next we discuss the unique recovery of the coefficients from the integral identity (3.2). In order to do that we prove unique continuation (Lemma 3.4) and a Runge type approximation property (Lemma 3.5) for the regional fractional Laplacian operator. Using these we complete the proof of Theorem 1.2. Finally we prove a

Runge type approximation (Lemma 3.6) for the non-local operator $\mathcal{L}_{b,q}$ and using that we complete the proof of Theorem 1.1.

2. DIRECT PROBLEMS

In this section we study the direct problems for the operators $(-\Delta)_{\Omega}^a$ and $\mathcal{L}_{b,q}$. For simplicity we assume that all functions including b and q to be real valued. Let us start with recalling the definition of the fractional order Sobolev spaces $H^r(\mathbb{R}^n)$, for $r \in \mathbb{R}$. We define the space $H^r(\mathbb{R}^n)$ as

$$(2.1) \quad H^r(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n); \langle \xi \rangle^r \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}, \quad \forall r \in \mathbb{R},$$

equipped with the norm $\|u\|_{H^r(\mathbb{R}^n)} := \|\langle \xi \rangle^r \hat{u}(\xi)\|_{L^2(\mathbb{R}^n)}$, where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions, $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ and $\hat{\cdot}$ denotes the Fourier transform. The space $H^{-r}(\mathbb{R}^n)$, for $r > 0$, can also be realized as the dual of the fractional order Sobolev space $H^r(\mathbb{R}^n)$, i.e. $H^{-r}(\mathbb{R}^n) = (H^r(\mathbb{R}^n))^*$.

Here we also provide an equivalent definition of the fractional order Sobolev spaces that does not use the Fourier transform as in (1.2). For $r \in (0, 1)$ one can equivalently define the space $H^r(\mathbb{R}^n)$ as

$$(2.2) \quad H^r(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2} + r}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n)\},$$

with the well-known Aronszajn-Slobodeckij inner product, given for real-valued u, v by [AF03]:

$$\langle u, v \rangle_{H^r(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x) dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2r}} dy dx,$$

for all $u, v \in H^r(\mathbb{R}^n)$. Following that, we can assign the graph norm on $H^r(\mathbb{R}^n)$ by

$$\|u\|_{H^r(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2} + r}} \right\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}^2.$$

2.1. Fractional Laplacian operator. Let $0 < t < 1$, we have defined the fractional Laplacian $(-\Delta)^t$ in $\mathbb{R}^n$ for Schwartz class functions in (1.2). Note that $(-\Delta)^t u$ is not a Schwartz class function due to its lack of decay near infinity; in particular, $(-\Delta)^t u$ decays at infinity as $|x|^{-n-2t}$ (see [Lan72]). The operator $(-\Delta)^t$ satisfies the following integration by parts formula on $\mathbb{R}^n$ in the L^2 sense (i.e. $(-\Delta)^t u \in L^2(\mathbb{R}^n)$, $u \in \mathcal{S}(\mathbb{R}^n)$ for $0 < t < 1$) given by

$$(2.3) \quad \int_{\mathbb{R}^n} ((-\Delta)^t u) v dx = \int_{\mathbb{R}^n} ((-\Delta)^{t/2} u) ((-\Delta)^{t/2} v) dx, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n).$$

Therefore,

$$(2.4) \quad \int_{\mathbb{R}^n} ((-\Delta)^t u) v dx = \int_{\mathbb{R}^n} ((-\Delta)^t v) u dx, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n).$$

There are several equivalent definitions of the fractional Laplacian; see [Kwa17]. For instance, it can be defined using the principal value integral as ($0 < t < 1$)

$$\begin{aligned} (-\Delta)^t u(x) &= C_{n,t} \text{ p.v. } \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2t}} dy \\ &= C_{n,t} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2t}} dy, \end{aligned}$$

where $C_{n,t}$ is a constant given by $\frac{4^t \Gamma(\frac{n}{2}+t)}{\pi^{n/2} \Gamma(-t)}$ (see [DNPV12]), and $B(x, \epsilon)$ is a ball in $\mathbb{R}^n$ centered at x with radius $\epsilon > 0$. The difference $u(x) - u(y)$ in the numerator, which vanishes at the singularity of the kernel, provides a regularization. This together with the averaging of positive and negative parts allows the principal value to exist at least for smooth u with sufficient decay. However, when $t \in (0, \frac{1}{2})$, the integral is not singular near x . Indeed, for $u \in \mathcal{S}(\mathbb{R}^n)$ and $0 < t < \frac{1}{2}$, we have

$$\left| \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2t}} dy \right| \leq 2 \left(\|\nabla u\|_{L^\infty} \int_{\overline{B}(x,1)} \frac{dy}{|x - y|^{n+2t-1}} + \|u\|_{L^\infty} \int_{\mathbb{R}^n \setminus \overline{B}(x,1)} \frac{dy}{|x - y|^{n+2t}} \right),$$

and both of the integrals on the right hand side are finite. Note that here we have used only C^1 regularity and the boundedness of the gradient of u . Moreover, by using the C^2 regularity and boundedness of the second order derivatives as well, in general for $t \in (0, 1)$ we can write the fractional Laplacian as a standard Lebesgue integral given by (see [BV16])

$$(-\Delta)^t u(x) = -\frac{C_{n,t}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2t}} dy, \quad u \in \mathcal{S}(\mathbb{R}^n), x \in \mathbb{R}^n.$$

Next we extend $(-\Delta)^t$ to larger spaces, in particular to Sobolev spaces.

Proposition 2.1. *The fractional Laplacian extends as a bounded map*

$$(2.5) \quad (-\Delta)^t : H^r(\mathbb{R}^n) \rightarrow H^{r-2t}(\mathbb{R}^n)$$

whenever $r \in \mathbb{R}$ and $t \in (0, 1)$.

2.2. Fractional Sobolev spaces on domains. Now we briefly discuss about the fractional order Sobolev spaces defined on any open subset with Lipschitz boundary. Let $\mathcal{O} \subseteq \mathbb{R}^n$ be any non-empty open set (bounded or unbounded) with Lipschitz boundary. We define $H^r(\mathcal{O})$, for $r \in \mathbb{R}$ by

$$H^r(\mathcal{O}) := \{u|_{\mathcal{O}}; u \in H^r(\mathbb{R}^n)\},$$

equipped with the norm

$$\|u\|_{H^r(\mathcal{O})} := \inf_{\substack{v \in H^r(\mathbb{R}^n) \\ v|_{\mathcal{O}} = u}} \|v\|_{H^r(\mathbb{R}^n)}.$$

One can equivalently define the space $H^r(\mathcal{O})$ for $r \in (0, 1)$ by

$$H^r(\mathcal{O}) = \{u \in L^2(\mathcal{O}); \frac{u(x) - u(y)}{|x - y|^{\frac{n}{2}+r}} \in L^2(\mathcal{O} \times \mathcal{O})\},$$

equipped with the inner product $\langle \cdot, \cdot \rangle_r$ for real-valued u, v as

$$\langle u, v \rangle_r = \int_{\mathcal{O}} uv \, dx + \int_{\mathcal{O} \times \mathcal{O}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2r}} dx \, dy.$$

Let $C_c^\infty(\overline{\mathcal{O}})$ denote the restriction of all $C_c^\infty(\mathbb{R}^n)$ (compactly supported smooth functions in $\mathbb{R}^n$) functions to $\overline{\mathcal{O}}$. We note that $C_c^\infty(\overline{\mathcal{O}})$ is dense in $H^r(\mathcal{O})$.

The dual of $H^r(\mathcal{O})$ is

$$(H^r(\mathcal{O}))^* = \{u \in H^{-r}(\mathbb{R}^n); \text{supp } u \subseteq \overline{\mathcal{O}}\}.$$

We define

$$\begin{cases} H_0^r(\mathcal{O}) := \{\text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^r(\mathcal{O})\} \\ \tilde{H}^r(\mathcal{O}) := \{\text{closure of } C_c^\infty(\mathcal{O}) \text{ in } H^r(\mathbb{R}^n)\}. \end{cases}$$

Then we have the following identifications [McL00]:

$$(2.6) \quad \left(\tilde{H}^r(\mathcal{O})\right)^* = H^{-r}(\mathcal{O}) \text{ and } (H^r(\mathcal{O}))^* = \tilde{H}^{-r}(\mathcal{O}), \quad r \in \mathbb{R}.$$

Furthermore, one has [McL00]

$$(2.7) \quad \tilde{H}^r(\mathcal{O}) = H^r(\mathcal{O}) = H_0^r(\mathcal{O}), \quad r < 1/2, \quad \text{and} \quad H^{1/2}(\mathcal{O}) = H_0^{1/2}(\mathcal{O}),$$

$$(2.8) \quad \tilde{H}^r(\mathcal{O}) = H_0^r(\mathcal{O}), \quad r > -\frac{1}{2}, \quad r \neq \left\{\frac{1}{2}, \frac{3}{2}, \dots\right\}.$$

We remark here that the above equivalences require $\mathcal{O} \subset \mathbb{R}^n$ to be a Lipschitz domain in $\mathbb{R}^n$.

Remark 2.2. Here we note that the characteristic function on the set $\mathcal{O}$ namely $\chi_{\mathcal{O}} \in H_0^{1/2}(\mathcal{O})$ but $\chi_{\mathcal{O}} \neq \tilde{H}^{1/2}(\mathcal{O})$.

Next we define the Lions-Magenes space $H_{0,0}^{1/2}(\mathcal{O})$ (see [Tar07, Chapter 33]) as

$$H_{0,0}^{1/2}(\mathcal{O}) := \{u \in H^{1/2}(\mathcal{O}); \frac{u(x)}{d(x, \mathcal{O}^c)^{1/2}} \in L^2(\mathcal{O})\},$$

where $d(x) = d(x, \mathcal{O}^c)$ is a smooth positive extension of the distance to boundary function $\text{dist}(x, \mathcal{O}^c)$ inside $\mathcal{O}$. We have the following equivalence

$$\tilde{H}^{1/2}(\mathcal{O}) = H_{0,0}^{1/2}(\mathcal{O}).$$

We also mention that (see [Tar07, Lemma 37.1])

$$(2.9) \quad u \in H^r(\mathcal{O}) \text{ and } \frac{u}{(d(x, \mathcal{O}^c))^r} \in L^2(\mathcal{O}) \iff u \in \tilde{H}^r(\mathcal{O}), \quad r \in (0, 1).$$

2.3. Regional fractional Laplacian operator. Let us consider a Lipschitz domain $\mathcal{O} \subset \mathbb{R}^n$. Let $0 < a < 1$, recall the definition of the regional fractional Laplacian operator $(-\Delta)_{\mathcal{O}}^a$ on the domain $\mathcal{O}$ over the class of $C_c^\infty(\overline{\mathcal{O}})$ functions by

$$(2.10) \quad (-\Delta)_{\mathcal{O}}^a u(x) = C_{n,a} \lim_{\epsilon \rightarrow 0^+} \int_{\mathcal{O} \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2a}} dy, \quad u \in C_c^\infty(\overline{\mathcal{O}}), x \in \mathcal{O}.$$

Now we state Proposition 2.3 [GM06, Theorem 3.3] for the regional fractional Laplacian operator.

Proposition 2.3 ([GM06][Theorem 3.3]). *For all $u, v \in C_c^\infty(\overline{\mathcal{O}})$ we have*

$$(2.11) \quad \int_{\mathcal{O}} ((-\Delta)_{\mathcal{O}}^a u) v dx = \frac{C_{n,a}}{2} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2a}} dy dx.$$

From the integration by parts formula (2.11) we get Corollary 2.4.

Corollary 2.4.

$$\int_{\mathcal{O}} ((-\Delta)_{\mathcal{O}}^a u) v dx = \int_{\mathcal{O}} ((-\Delta)_{\mathcal{O}}^a v) u dx, \quad \forall u, v \in C_c^\infty(\overline{\mathcal{O}}).$$

Clearly, when $\mathcal{O} = \mathbb{R}^n$ the regional fractional Laplacian coincides with the definition of the usual fractional Laplacian $(-\Delta)^a$, $(0 < a < 1)$. Moreover, for $u \in C_c^\infty(\mathcal{O})$ the regional fractional Laplacian can be identified with the fractional Schrödinger operator $((-\Delta)^a - \varphi_a)$ in $\mathcal{O}$ ($0 < a < 1$)

$$(2.12) \quad \forall x \in \mathcal{O}, \quad (-\Delta)_{\mathcal{O}}^a u(x) = (-\Delta)^a u(x) - \varphi_a(x)u(x), \quad \forall u \in C_c^\infty(\mathcal{O}),$$

where

$$\varphi_a(x) = C_{n,a} \int_{\mathbb{R}^n \setminus \mathcal{O}} \frac{1}{|x-y|^{n+2a}} dy.$$

The potential $\varphi_a \in C_{loc}^{0,1}(\mathcal{O})$ is a locally Lipschitz function and for some constant $C > 1$ (see [Che18, Lemma 2.4])

$$(2.13) \quad \frac{1}{C} (\text{dist } (x, \mathcal{O}^c))^{-2a} \leq \varphi_a(x) \leq C (\text{dist } (x, \mathcal{O}^c))^{-2a}, \quad x \in \mathcal{O}.$$

We extend the definition of $(-\Delta)_{\mathcal{O}}^a$ over the space $H^a(\mathcal{O})$ for $0 < a < 1$. Since $C_c^\infty(\overline{\mathcal{O}})$ is dense in $H^a(\mathcal{O})$, for $u, v \in H^a(\mathcal{O})$ we define $(-\Delta)_{\mathcal{O}}^a u \in H^{-a}(\mathcal{O})$ weakly by

$$(2.14) \quad \langle (-\Delta)_{\mathcal{O}}^a u, v \rangle_{(H^{-a}(\mathcal{O}), H^a(\mathcal{O}))} = \frac{C_{n,a}}{2} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2a}} dy dx.$$

Proposition 2.5. *Let $0 < a < 1$ and $\mathcal{O} \subset \mathbb{R}^n$ be an open subset with Lipschitz boundary, then*

$$(2.15) \quad (-\Delta)_{\mathcal{O}}^a : H^a(\mathcal{O}) \rightarrow H^{-a}(\mathcal{O}) \text{ is continuous.}$$

Proof. From the integration by parts formula (2.11) we get the duality inner-product as (2.14), satisfying

$$\begin{aligned} |\langle (-\Delta)_{\mathcal{O}}^a u, v \rangle_{(H^{-a}(\mathcal{O}), H^a(\mathcal{O}))}| &\leq \left\| \frac{(u(x) - u(y))}{|x-y|^{\frac{n}{2}+a}} \right\|_{L^2(\mathcal{O} \times \mathcal{O})}^2 \left\| \frac{(v(x) - v(y))}{|x-y|^{\frac{n}{2}+a}} \right\|_{L^2(\mathcal{O} \times \mathcal{O})}^2 \\ &\leq \|u\|_{H^a(\mathcal{O})} \|v\|_{H^a(\mathcal{O})}. \end{aligned}$$

□

Moreover, from (2.14) it is also clear that for $0 \leq \delta < a$,

$$(2.16) \quad (-\Delta)_{\mathcal{O}}^a : H^{a+\delta}(\mathcal{O}) \rightarrow H^{-a+\delta}(\mathcal{O}) \text{ is continuous.}$$

Next we examine for $u \in \tilde{H}^a(\mathcal{O})$ ($0 < a < 1$) whether $(-\Delta)_{\mathcal{O}}^{a/2} u \in L^2(\mathcal{O})$.

Lemma 2.6. *For all $\mathcal{O}' \Subset \mathcal{O}$ and for all $u \in \tilde{H}^a(\mathcal{O})$, $0 < a < 1$ we have*

$$(2.17) \quad (-\Delta)_{\mathcal{O}}^{a/2} u \in L^2(\mathcal{O}').$$

Proof. Let us extend the function $u \in \tilde{H}^a(\mathcal{O})$ by 0 in $\mathbb{R}^n$ and denote the extension by u also, $u \in H^a(\mathbb{R}^n)$. Therefore, we have $(-\Delta)^{a/2} u \in L^2(\mathbb{R}^n)$ (see (2.5)), in particular $(-\Delta)^{a/2} u \in L^2(\mathcal{O})$. Now, from (2.12), in $\mathcal{O}$ we have

$$(2.18) \quad (-\Delta)_{\mathcal{O}}^{a/2} u + \varphi_{a/2}(x)u = (-\Delta)^{a/2} u|_{\mathcal{O}} \in L^2(\mathcal{O}).$$

Since $\mathcal{O}' \Subset \mathcal{O}$ implies $\varphi_{a/2}(x)u|_{\mathcal{O}'} \in L^2(\mathcal{O}')$ and hence we get (2.17). □

Lemma 2.7. *Let $\mathcal{O}$ be a bounded Lipschitz domain in $\mathbb{R}^n$. Let $u \in \tilde{H}^a(\mathcal{O})$ for $0 < a < 1$, then $(-\Delta)_{\mathcal{O}}^{a/2} u \in L^2(\mathcal{O})$ and*

$$(2.19) \quad \|u\|_{L^2(\mathcal{O})} + \|(-\Delta)_{\mathcal{O}}^{a/2} u\|_{L^2(\mathcal{O})} \leq \|u\|_{\tilde{H}^a(\mathcal{O})}.$$

Remark 2.8. Note that the above estimate (2.19) holds for $H_{0,0}^{1/2}(\mathcal{O})$ space, whereas, for $u \in H_0^{1/2}(\mathcal{O})$, $(-\Delta)_{\mathcal{O}}^{1/4} u$ might not be in $L^2(\mathcal{O})$. See the counterexample in [Dyd04, Section 2].

Proof of Lemma 2.7. From (2.18) it is enough to show that $\varphi_{a/2}(x)u \in L^2(\mathcal{O})$. Note that from (2.13) we have $\varphi_{a/2}(x) \sim (\text{dist}(x, \mathcal{O}^c))^{-a}$ for $x \in \mathcal{O}$ and $0 < a < 1$. We recall the following fractional Hardy inequalities from [Dyd04] as follows:

For $0 < a < \frac{1}{2}$, $\forall u \in H_0^a(\mathcal{O})$,

$$(2.20) \quad \int_{\mathcal{O}} \frac{|u(x)|^2}{(\text{dist}(x, \mathcal{O}^c))^{2a}} dx \leq C \left(\int_{\mathcal{O}} |u|^2 dx + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx \right),$$

for $a = \frac{1}{2}$, $\forall u \in H_{0,0}^{1/2}(\mathcal{O})$,

$$(2.21) \quad \int_{\mathcal{O}} \frac{|u(x)|^2}{(\text{dist}(x, \mathcal{O}^c))} dx \leq C \left(\int_{\mathcal{O}} |u|^2 dx + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+1}} dy dx \right),$$

for $\frac{1}{2} < a < 1$, $\forall u \in H_0^a(\mathcal{O})$,

$$(2.22) \quad \int_{\mathcal{O}} \frac{|u(x)|^2}{(\text{dist}(x, \mathcal{O}^c))^{2a}} dx \leq C \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx,$$

where $C = C(\mathcal{O}, n, a)$.

Then for $u \in \tilde{H}^a(\mathcal{O})$, in all cases, we have that $(-\Delta)_{\mathcal{O}}^{a/2} u \in L^2(\mathcal{O})$. Moreover, from (2.18) we conclude

$$(2.23) \quad \begin{aligned} \|(-\Delta)_{\mathcal{O}}^{a/2} u\|_{L^2(\mathcal{O})} &\leq \|\varphi_{a/2} u\|_{L^2(\mathcal{O})} + \|(-\Delta)^{a/2} u\|_{L^2(\mathcal{O})} \\ &\leq C \|u\|_{H^a(\mathcal{O})} + \|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

We also have that

$$\begin{aligned} &\|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)}^2 \\ &= C_{n,a} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx \\ &= C_{n,a} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx \quad (\text{since } u = 0 \text{ in } \mathbb{R}^n \setminus \overline{\mathcal{O}}) \\ &\quad + 2C_{n,a} \int_{\mathcal{O}} \left[(u(x))^2 \left(\int_{\mathbb{R}^n \setminus \mathcal{O}} \frac{1}{|x - y|^{n+2a}} dy \right) \right] dx \quad (\text{see (2.13)}) \\ &\leq C_{n,a} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx + C \int_{\mathcal{O}} \frac{(u(x))^2}{\text{dist}(x, \mathcal{O}^c)^{2a}} dx \\ &\leq \|u\|_{H^a(\mathcal{O})}^2. \end{aligned}$$

Therefore, $\|u\|_{L^2(\mathcal{O})} + \|(-\Delta)_{\mathcal{O}}^{a/2} u\|_{L^2(\mathcal{O})} \leq C \|u\|_{H^a(\mathcal{O})}$. □

Lemma 2.9. *Let $u \in \tilde{H}^r(\mathcal{O})$ for $0 < a \leq r$. Then $(-\Delta)_{\mathcal{O}}^{a/2} u \in H^{r-a}(\mathcal{O})$.*

Proof. Let us write

$$(2.24) \quad (-\Delta)_{\mathcal{O}}^{a/2} u = (-\Delta)^{a/2} u - \varphi_{a/2} u$$

for $u \in \tilde{H}^r(\mathcal{O})$. Thanks to (2.5), we already know that $((-\Delta)^{a/2} u)|_{\mathcal{O}} \in H^{r-a}(\mathcal{O})$. Moreover, from [Gru14] we have that $\frac{u}{d(x, \mathcal{O}^c)^a} \in H^{r-a}(\mathcal{O})$ for $u \in \tilde{H}^r(\mathcal{O})$. Since $\varphi_{a/2} \sim d(x, \mathcal{O}^c)^{-a}$, therefore from (2.24) it follows $(-\Delta)_{\mathcal{O}}^{a/2} u \in H^{r-a}(\mathcal{O})$. $\square$

Let $0 < a < \min\{1, \frac{n}{2}\}$, the well-known Hardy-Littlewood-Sobolev inequality follows as (see [Pon16, Proposition 15.5])

$$(2.25) \quad \|u\|_{L^{\frac{2n}{n-2a}}(\mathbb{R}^n)} \leq C \|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)}, \quad \forall u \in C_c^\infty(\mathbb{R}^n),$$

where C depends on n and a . Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $u \in \tilde{H}^a(\mathcal{O})$ with $0 < a < \min\{1, \frac{n}{2}\}$ we have the following inequality

$$(2.26) \quad \|u\|_{L^2(\mathcal{O})} \leq C_{\mathcal{O}} \|u\|_{L^{\frac{2n}{n-2a}}(\mathbb{R}^n)} \leq C \|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)}.$$

Proposition 2.10 ([HSV13, DIV16]). *For $0 < a < 1$ we have the following Poincaré-Wirtinger inequality*

$$(2.27) \quad \left\| u - \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} u \right\|_{L^2(\mathcal{O})} \leq C \left(\int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx \right)^{\frac{1}{2}}, \quad \forall u \in H^a(\mathcal{O}),$$

where $C = C(\mathcal{O}, n, a)$.

Lemma 2.11 (Poincaré inequality). *Let $\mathcal{O} \subset \mathbb{R}^n$, $n \geq 1$ be a bounded open set with Lipschitz boundary and $u \in \tilde{H}^a(\mathcal{O})$; then for some $C = C(\mathcal{O}, n, a)$ we have*

$$(2.28) \quad \|u\|_{L^2(\mathcal{O})} \leq C \left(\int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx \right)^{\frac{1}{2}} \quad \text{whenever } \frac{1}{2} < a < 1.$$

Proof. We have for $u \in \tilde{H}^a(\mathcal{O})$

$$\|(-\Delta)^{a/2} u\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx + 2 \int_{\mathcal{O}} \int_{\mathbb{R}^n \setminus \mathcal{O}} \frac{(u(x) - u(y))^2}{|x - y|^{n+2a}} dy dx.$$

Using (2.13) and (2.22) on the second term of the above identity for $\frac{1}{2} < a < 1$, we get (2.28). $\square$

Remark 2.12. The above inequality is not true for $0 < a \leq \frac{1}{2}$ (see [Dyd04]). For example, the characteristic function $\chi_{\mathcal{O}} \in \tilde{H}^a(\mathcal{O})$ for $a \in (0, \frac{1}{2})$ and (2.28) does not hold in this case. For $a = \frac{1}{2}$ see Remark 2.2.

2.4. Direct problem for the regional fractional Laplacian operator. In this article we discuss two direct problems corresponding to the non-local operators $(-\Delta)_{\Omega}^a$ and $\mathcal{L}_{b,q}$. Let us start with the regional fractional Laplacian operator $(-\Delta)_{\Omega}^a$. Let $0 < a < 1$, $\mathcal{O} \Subset \Omega$, i.e. $\mathcal{O}$ is compactly contained in Ω and $f \in H^{-a}(\mathcal{O})$, $g \in H^a(\Omega \setminus \overline{\mathcal{O}})$; we solve the Dirichlet problem

$$(2.29) \quad (-\Delta)_{\Omega}^a u = f \quad \text{in } \mathcal{O}, \quad \text{and } u = g \quad \text{on } \Omega \setminus \overline{\mathcal{O}}.$$

We define the corresponding bilinear form $\mathcal{B}_\Omega : H^a(\Omega) \times H^a(\Omega) \mapsto \mathbb{R}$ by

$$\mathcal{B}_\Omega(\varphi, \psi) := \int_\Omega \int_\Omega \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{n+2a}} dy dx.$$

We say $u \in H^a(\Omega)$ is a weak solution of (2.29) if $\mathcal{B}_\Omega(u, \psi) = \langle f, \psi \rangle$, for all $\psi \in \tilde{H}^a(\mathcal{O})$ with $u = g$ on $\Omega \setminus \overline{\mathcal{O}}$.

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^n$, bounded Lipschitz domain, and $\mathcal{O} \subset \Omega$ be a non-empty open Lipschitz domain compactly contained in Ω . Let $f \in H^{-a}(\mathcal{O})$ and $G \in H^a(\Omega)$ with $G = g$ in $\Omega \setminus \overline{\mathcal{O}}$, $0 < a < 1$. There exists a unique weak solution $v \in H^a(\Omega)$ solving (2.29). Moreover, it satisfies the following stability estimate*

$$(2.30) \quad \|v\|_{H^a(\Omega)} \leq C (\|f\|_{H^{-a}(\mathcal{O})} + \|G\|_{H^a(\Omega)}).$$

Proof of Theorem 2.13.

Homogeneous case. Let us begin with the homogeneous boundary value problem, i.e. when $g = 0$ in $\Omega \setminus \overline{\mathcal{O}}$. Let $f \in H^{-a}(\mathcal{O})$; we say $v_f \in \tilde{H}^a(\mathcal{O})$, $0 < a < 1$, is the weak solution of

$$(2.31) \quad (-\Delta)_\Omega^a v = f \quad \text{in } \mathcal{O}, \quad v = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}},$$

if for all $w \in C_c^\infty(\mathcal{O})$

$$(2.32) \quad \mathcal{B}_\Omega(v_f, w) = \langle f, w \rangle_{(H^{-a}(\mathcal{O}), \tilde{H}^a(\mathcal{O}))}.$$

A straightforward calculation shows that the bilinear form $\mathcal{B}(\cdot, \cdot)$ is continuous over $\tilde{H}^a(\Omega) \times \tilde{H}^a(\Omega)$. Next we show that $\mathcal{B}_\Omega(\cdot, \cdot)$ is coercive over $\tilde{H}^a(\mathcal{O})$ for $\mathcal{O} \Subset \Omega$ i.e.

$$(2.33) \quad \mathcal{B}_\Omega(\varphi, \varphi) = \int_\Omega \int_\Omega \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2a}} dy dx \geq C \|\varphi\|_{\tilde{H}^a(\mathcal{O})}^2, \quad \forall \varphi \in \tilde{H}^a(\mathcal{O}).$$

Let $\varphi \in C_c^\infty(\mathcal{O})$, using the Poincaré-Wirtinger inequality (2.27) over Ω , we get

$$\begin{aligned} C \left(\int_\Omega \int_\Omega \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2a}} dy dx \right)^{\frac{1}{2}} &\geq \left\| \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \right\|_{L^2(\Omega)} \\ &\geq \|\varphi\|_{L^2(\mathcal{O})} - \frac{1}{|\Omega|^{\frac{1}{2}}} \int_\Omega |\varphi| \\ &\geq \|\varphi\|_{L^2(\mathcal{O})} - \frac{1}{|\Omega|^{\frac{1}{2}}} \int_\Omega \chi_{\mathcal{O}} |\varphi| \\ &\geq \left(1 - \frac{|\mathcal{O}|^{\frac{1}{2}}}{|\Omega|^{\frac{1}{2}}} \right) \|\varphi\|_{L^2(\mathcal{O})}. \end{aligned}$$

Hence, we have (2.33).

Therefore, using the Lax-Milligram theorem, for a given $f \in H^{-a}(\mathcal{O})$, we have a unique weak solution of (2.31) in $\tilde{H}^a(\mathcal{O})$ satisfying the stability estimate

$$\|v_f\|_{\tilde{H}^a(\mathcal{O})} \leq \|f\|_{H^{-a}(\mathcal{O})}.$$

Remark 2.14. Here we remark that, in establishing the coercivity of the bilinear form $\mathcal{B}_\Omega(\cdot, \cdot)$ over the $\tilde{H}^a(\mathcal{O})$ space, the assumption $\mathcal{O} \Subset \Omega$ is crucial. For example, $\mathcal{B}_\Omega(\cdot, \cdot)$ fails to become coercive over the space $\tilde{H}^a(\Omega)$ while $0 < a < \frac{1}{2}$.

Inhomogeneous case. Let $G \in H^a(\Omega)$ then from (2.15) we know $(-\Delta)_\Omega^a G \in H^{-a}(\Omega)$, $0 < a < 1$. Now we are interested in the following inhomogeneous problem

$$(-\Delta)_\Omega^a v = f \quad \text{in } \mathcal{O}, \quad (v - G) \in \tilde{H}^a(\mathcal{O}).$$

Then $w = (v - G) \in \tilde{H}^a(\mathcal{O})$ solves

$$(-\Delta)_\Omega^a w = f - (-\Delta)_\Omega^a G \in H^{-a}(\mathcal{O}),$$

and by the previous discussion we have a unique weak solution in $\tilde{H}^a(\mathcal{O})$, where as the stability estimate (2.30) follows from Proposition 2.5. $\square$

Corollary 2.15. *The operator*

$$((-\Delta)_\Omega^a)^{-1} : H^{-a}(\mathcal{O}) \rightarrow \tilde{H}^a(\mathcal{O})$$

is one-one, onto and bounded.

2.5. Direct problem for $\mathcal{L}_{b,q}$. Here we study the direct problem for the operator $\mathcal{L}_{b,q}$. Let Ω be an open, bounded Lipschitz domain in $\mathbb{R}^n$. We consider the inhomogeneous problem

$$(2.34) \quad \begin{aligned} \mathcal{L}_{b,q} u &:= \left((-\Delta)^t + (-\Delta)_\Omega^{s/2} b (-\Delta)_\Omega^{s/2} + q \right) u = F && \text{in } \Omega, \\ u &= f && \text{in } \Omega_e, \end{aligned}$$

where $F \in H^{-t}(\Omega)$, $f \in \tilde{H}^t(\Omega_e)$ and $b, q \in L^\infty(\Omega)$.

The bilinear form associated to the operator $\mathcal{L}_{b,q}$ is

$$\mathcal{B}_{b,q} : H^t(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \rightarrow \mathbb{R}$$

given by

$$(2.35) \quad \begin{aligned} \mathcal{B}_{b,q}(\varphi, \psi) &:= \int_{\mathbb{R}^n} (-\Delta)^{t/2} \varphi(x) (-\Delta)^{t/2} \psi(x) dx \\ &+ \int_{\Omega} b(x) \left((-\Delta)_\Omega^{s/2} \varphi \right)(x) \left((-\Delta)_\Omega^{s/2} \psi \right)(x) dx + \int_{\Omega} q(x) \varphi(x) \psi(x) dx. \end{aligned}$$

We define $u \in H^t(\mathbb{R}^n)$ to be a weak solution of (2.34) if for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\mathcal{B}_{b,q}(u, \varphi) = \langle F, \varphi \rangle, \quad \text{with } u = f \text{ in } \Omega_e.$$

We state Theorem 2.16 for wellposedness of the Dirichlet problem (2.34)

Theorem 2.16. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a Lipschitz domain and $b, q \in L^\infty(\Omega)$. Let $0 < t < 1$, for any $F \in H^{-t}(\Omega)$ and $f \in \tilde{H}^t(\Omega_e)$ there exist a unique weak solution $u \in H^t(\mathbb{R}^n)$ solving the Dirichlet problem (2.34). It satisfies the following stability estimate*

$$\|u\|_{H^t(\mathbb{R}^n)} \leq C \left(\|F\|_{H^{-t}(\Omega)} + \|f\|_{H^t(\Omega_e)} \right).$$

Proof. By extending $f \in \tilde{H}^t(\Omega_e)$ by 0 in Ω as a $H^t(\mathbb{R}^n)$ function, we observe that the theorem is equivalent to considering the following homogeneous problem for $v = (u - f) \in \tilde{H}^t(\Omega)$ and $\tilde{F} := F - (-\Delta)^t f$,

$$(2.36) \quad \begin{aligned} \mathcal{L}_{b,q} v &= \tilde{F} && \text{in } \Omega, \\ v &= 0 && \text{in } \Omega_e. \end{aligned}$$

Equivalently, in terms of the bilinear form we seek $v \in \tilde{H}^t(\Omega)$ solving

$$(2.37) \quad \mathcal{B}_{b,q}(v, \varphi) = \langle \tilde{F}, \varphi \rangle, \quad \forall \varphi \in \tilde{H}^t(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the usual duality between the spaces $H^{-t}(\Omega)$ and $\tilde{H}^t(\Omega)$. Let us note that,

$$\langle \tilde{F}, \varphi \rangle_{H^{-t}(\Omega), \tilde{H}^t(\Omega)} \leq (\|F\|_{H^{-t}(\Omega)} + \|f\|_{H^t(\Omega_e)}) \|\varphi\|_{\tilde{H}^t(\Omega)}.$$

In order to prove the existence of the solution of (2.34), now, we will show the existence of a solution $v \in \tilde{H}^t(\Omega)$ solving (2.36).

Continuity of the bilinear form $\mathcal{B}_{b,q}(\cdot, \cdot)$. Let $\varphi, \psi \in \tilde{H}^t(\Omega)$, first note that due to (2.17) for $0 < s < t < 1$ with $s \neq \frac{1}{2}$ we have

$$\|(-\Delta)_\Omega^{s/2} \varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{\tilde{H}^s(\Omega)} \leq \|\varphi\|_{\tilde{H}^t(\Omega)}.$$

For $s = \frac{1}{2}$ and $\varphi \in \tilde{H}^t(\Omega)$ with $\frac{1}{2} < t < 1$, we find that $\varphi \in H_{0,0}^{1/2}(\Omega)$ satisfying

$$\|(-\Delta)_\Omega^{1/4} \varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{H_{0,0}^{1/2}(\Omega)} \leq \|\varphi\|_{\tilde{H}^t(\Omega)}.$$

To see this, let $\varphi \in \tilde{H}^r(\Omega)$ for some $\frac{1}{2} < r < 1$, we claim that $\varphi \in H_{0,0}^{1/2}(\Omega)$, satisfying $\|(-\Delta)_\Omega^{1/4} \varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{H_{0,0}^{1/2}(\Omega)} \leq \|\varphi\|_{\tilde{H}^r(\Omega)}$. By using (2.9) we have $\frac{\varphi}{d(x, \Omega^c)^r} \in L^2(\Omega)$, where $d(x, \Omega^c)$ is a smooth positive extension into Ω of $\text{dist}(x, \Omega^c)$ near $\partial\Omega$. Now since $\frac{1}{2} < r < 1$, we have that $\frac{\varphi}{d(x, \Omega^c)^{\frac{1}{2}}}$ $\in L^2(\Omega)$. Hence $\varphi \in H_{0,0}^{1/2}(\Omega)$ and using (2.19) the above estimate follows.

Therefore, we have

$$(2.38) \quad \begin{aligned} |\mathcal{B}_{b,q}(\varphi, \psi)| &\leq \|(-\Delta)^{t/2} \varphi\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{t/2} \psi\|_{L^2(\mathbb{R}^n)} \\ &\quad + \|b\|_{L^\infty(\Omega)} \|(-\Delta)_\Omega^{s/2} \varphi\|_{L^2(\Omega)} \|(-\Delta)_\Omega^{s/2} \psi\|_{L^2(\Omega)} \\ &\quad + \|q\|_{L^\infty(\Omega)} \|\varphi\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}, \\ &\leq C \|\varphi\|_{\tilde{H}^t(\Omega)} \|\psi\|_{\tilde{H}^t(\Omega)}. \end{aligned}$$

Coercivity of the bilinear form $\mathcal{B}_{b,q}$ on $\tilde{H}^t(\Omega)$. Let $0 < s < t < 1$ and $\varphi \in \tilde{H}^t(\Omega)$. Then for $\sigma > \|q\|_{L^\infty(\Omega)}$ we obtain

$$(2.39) \quad \begin{aligned} \mathcal{B}_{b,q}(\varphi, \varphi) + \sigma \langle \varphi, \varphi \rangle_{L^2(\Omega)} &\geq \|(-\Delta)^{t/2} \varphi\|_{L^2(\mathbb{R}^n)}^2 + \sigma \|\varphi\|_{L^2(\Omega)}^2 \\ &\quad - \|b\|_{L^\infty(\Omega)} \|(-\Delta)_\Omega^{s/2} \varphi\|_{L^2(\Omega)}^2 - \|q\|_{L^\infty(\Omega)} \|\varphi\|_{L^2(\Omega)}^2 \\ &\geq \|(-\Delta)^{t/2} \varphi\|_{L^2(\mathbb{R}^n)}^2 + (\sigma - \|q\|_{L^\infty(\Omega)}) \|\varphi\|_{L^2(\Omega)}^2 \\ &\quad - \|b\|_{L^\infty(\Omega)} \|(-\Delta)_\Omega^{s/2} \varphi\|_{L^2(\Omega)}^2 \\ &\geq \tilde{C} \|\varphi\|_{\tilde{H}^t(\Omega)}^2 - \|b\|_{L^\infty(\Omega)} \|\varphi\|_{\tilde{H}^r(\Omega)}^2, \end{aligned}$$

where $r = s$ when $s \neq \frac{1}{2}$, and for $s = \frac{1}{2}$, we take some fixed $r \in (\frac{1}{2}, t)$.

Given the compact inclusions

$$\tilde{H}^t(\Omega) \hookrightarrow \tilde{H}^r(\Omega) \hookrightarrow L^2(\Omega), \quad \text{for } 0 < r < t < 1,$$

we have for $\lambda > 0$ (see [Tem77, Lemma 2.1])

$$\|\varphi\|_{\tilde{H}^r(\Omega)}^2 \leq \frac{1}{\lambda} \|\varphi\|_{\tilde{H}^t(\Omega)}^2 + C_\lambda \|\varphi\|_{L^2(\Omega)}^2, \quad \text{for } \varphi \in \tilde{H}^t(\Omega).$$

Therefore, combining the above estimates we get for $\lambda \geq 2\tilde{C}^{-1}\|b\|_{L^\infty(\Omega)}$

$$(2.40) \quad \mathcal{B}_{b,q}(\varphi, \varphi) + (\sigma + C_\lambda)\|\varphi\|_{L^2(\Omega)}^2 \geq \frac{\tilde{C}}{2}\|\varphi\|_{\tilde{H}^t(\Omega)}^2, \quad \text{for } \varphi \in \tilde{H}^t(\Omega).$$

By the Riesz-representation theorem there exists a unique $w = G_\mu(\tilde{F}) \in \tilde{H}^t(\Omega)$, where $G_\mu : H^{-t}(\Omega) \rightarrow \tilde{H}^t(\Omega)$ is a bounded map, such that

$$\mathcal{B}_{b,q}(w, \varphi) + \mu\langle w, \varphi \rangle = \langle \tilde{F}, \varphi \rangle, \quad \forall \varphi \in \tilde{H}^t(\Omega), \quad \mu \geq (\sigma + C_\lambda).$$

Hence, we have unique $w \in \tilde{H}^t(\Omega)$ satisfying

$$(\mathcal{L}_{b,q} + \mu I)w = \tilde{F}, \quad \text{in } \Omega,$$

where $I : \tilde{H}^t(\Omega) \rightarrow H^{-t}(\Omega)$ is the identity map.

Observe that, $v \in \tilde{H}^t(\Omega)$ satisfies $\mathcal{L}_{b,q}v = \tilde{F}$ if and only if

$$(\mathcal{L}_{b,q} + \mu I)v - \mu Iv = \tilde{F} \iff v - \mu(G_\mu \circ I)v = G_\mu(\tilde{F}).$$

Now, using the Rellich-Kondrachov compact embedding theorem we have

$$I : \tilde{H}^t(\Omega) \rightarrow H^{-t}(\Omega)$$

is compact and consequently $(G_\mu \circ I) : \tilde{H}^t(\Omega) \rightarrow \tilde{H}^t(\Omega)$ is a compact operator. Hence, by the Fredholm alternative theorem, existence of a solution $v \in \tilde{H}^t(\Omega)$ of $v - \mu(G_\mu \circ I)v = G_\mu(\tilde{F})$ in Ω follows from the uniqueness of the trivial solution of $\mu(G_\mu \circ I)v = v$ in Ω in the function space $\tilde{H}^t(\Omega)$ (c.f. assumption (1.4)). Therefore, we get a unique solution $v \in \tilde{H}^t(\Omega)$ of $\mathcal{L}_{b,q}v = \tilde{F}$, in Ω , and consequently a unique solution $u = (v + f) \in H^t(\mathbb{R}^n)$ solving

$$\begin{aligned} \mathcal{L}_{b,q}u &= F \quad \text{in } \Omega, \\ u &= f \quad \text{in } \Omega_e, \end{aligned}$$

for $F = (\tilde{F} + (-\Delta)^t f) \in H^{-t}(\Omega)$ and $f \in \tilde{H}^t(\Omega_e)$.

Stability estimate. Now we will show that if u is the unique solution of (2.34) then the following stability estimate is true:

$$\|u\|_{H^t(\mathbb{R}^n)} \leq C \left(\|F\|_{H^{-t}(\Omega)} + \|f\|_{\tilde{H}^t(\Omega_e)} \right).$$

In order to show that, observe that from (2.40) we get

$$\|v\|_{\tilde{H}^t(\Omega)}^2 \leq C\|v\|_{L^2(\Omega)}^2 + \mathcal{B}_{b,q}(v, v).$$

Now as v solves (2.36) we get $\mathcal{B}_{b,q}(v, v) = \langle \tilde{F}, v \rangle$. Hence,

$$\begin{aligned} \|v\|_{\tilde{H}^t(\Omega)}^2 &\leq C\|v\|_{L^2(\Omega)}^2 + |\langle \tilde{F}, v \rangle| \\ &\leq C \left(\|v\|_{L^2(\Omega)} + \|\tilde{F}\|_{H^{-t}(\Omega)} \right) \|v\|_{\tilde{H}^t(\Omega)} \end{aligned}$$

or,

$$\|v\|_{\tilde{H}^t(\Omega)} \leq C \left(\|v\|_{L^2(\Omega)} + \|\tilde{F}\|_{H^{-t}(\Omega)} \right).$$

Now by putting $u = v + f$ and $\tilde{F} = F - (-\Delta)^t f$ with $\|\tilde{F}\|_{H^{-t}(\Omega)} \leq \|F\|_{H^{-t}(\Omega)} + \|f\|_{\tilde{H}^t(\Omega_e)}$ one gets

$$\|u\|_{H^t(\mathbb{R}^n)} \leq C \left(\|u\|_{L^2(\Omega)} + \|F\|_{H^{-t}(\Omega)} + \|f\|_{\tilde{H}^t(\Omega_e)} \right).$$

Moreover, by using the compactness of the inverse operator $(G_\mu \circ I)$ we obtain

$$\|u\|_{H^t(\mathbb{R}^n)} \leq C \left(\|F\|_{H^{-t}(\Omega)} + \|f\|_{\tilde{H}^t(\Omega_e)} \right).$$

Hence, the stability estimate follows. $\square$

Remark 2.17. In this remark, we will discuss the case where $f \in H^t(\Omega_e)$ instead of $f \in \tilde{H}^t(\Omega_e)$ in (2.34). In this case, we also assume $b \in H^1(\Omega) \cap L^\infty(\Omega)$, and we will see how the higher regularity of b helps to solve the forward problem (2.34) with more general exterior data $f \in H^t(\Omega_e)$. Let us call $\tilde{f} \in H^t(\mathbb{R}^n)$ be a non-zero extension of $f \in H^t(\Omega_e)$, i.e. $\tilde{f}|_{\Omega_e} = f$ satisfying $\|\tilde{f}\|_{H^t(\mathbb{R}^n)} \leq C\|f\|_{H^t(\Omega_e)}$. Now let us consider the modified (w.r.t. (2.37)) homogeneous problem in this case, as $\tilde{v} = u - \tilde{f} \in H^t(\mathbb{R}^n)$ solves

$$(2.41) \quad \mathcal{B}_{b,q}(\tilde{v}, \varphi) = \langle F - (-\Delta)^t \tilde{f}, \varphi \rangle - \langle b(-\Delta)_\Omega^{s/2} \tilde{f}, (-\Delta)_\Omega^{s/2} \varphi \rangle - \langle q\tilde{f}, \varphi \rangle, \quad \forall \varphi \in \tilde{H}^t(\Omega).$$

Let's consider the r.h.s. of (2.41). It's easy to observe that

$$\begin{aligned} \langle q\tilde{f}, \varphi \rangle_{L^2(\Omega)} &\leq \|q\|_{L^\infty(\Omega)} \|\tilde{f}\|_{L^2(\Omega_e)} \|\varphi\|_{L^2(\Omega)} \\ \text{and} \quad \langle F - (-\Delta)^t \tilde{f}, \varphi \rangle_{H^{-t}(\Omega), \tilde{H}^t(\Omega)} &\leq (\|F\|_{H^{-t}(\Omega)} + \|\tilde{f}\|_{H^t(\Omega_e)}) \|\varphi\|_{\tilde{H}^t(\Omega)}. \end{aligned}$$

In the second term, we will use H^r -duality instead of the L^2 -inner product as done earlier. We get that, due to (2.16) $(-\Delta)_\Omega^{s/2}(\tilde{f}|_\Omega) \in H^{-s/2+\delta}(\Omega)$, for any $0 \leq \delta < \frac{s}{2}$ and $\tilde{f}|_\Omega \in H^t(\Omega)$ such that $\frac{s}{2} + \delta \leq t < 1$. On the other hand, $(-\Delta)_\Omega^{s/2} \varphi \in H^{t-s}(\Omega)$ for $\varphi \in \tilde{H}^t(\Omega)$. By choosing $\delta \in [0, \frac{s}{2})$ close to $\frac{s}{2}$, for given $0 < s < t < 1$ it is always possible to have some $\delta = \delta_0$ such that $t - s \geq \frac{s}{2} - \delta_0$. Therefore, for that $\delta_0 \in [0, \frac{s}{2})$ we have $(-\Delta)_\Omega^{s/2} \varphi \in H^{s/2-\delta_0}(\Omega)$. Now by using the regularity of $b \in H^1(\Omega)$, we have $b(-\Delta)_\Omega^{s/2} \varphi \in H^{s/2-\delta_0}(\Omega)$ as well. Thus

$$\begin{aligned} &\langle (-\Delta)_\Omega^{s/2} \tilde{f}, b(-\Delta)_\Omega^{s/2} \varphi \rangle_{H^{-s/2+\delta_0}(\Omega), H^{s/2-\delta_0}(\Omega)} \\ &\leq \|b\|_{H^1(\Omega)} \|(-\Delta)_\Omega^{s/2} \tilde{f}\|_{H^{-s/2+\delta_0}(\Omega)} \|(-\Delta)_\Omega^{s/2} \varphi\|_{H^{s/2-\delta_0}(\Omega)}. \end{aligned}$$

Again by using (2.16), we have $\|(-\Delta)_\Omega^{s/2} \tilde{f}\|_{H^{-s/2+\delta_0}(\Omega)} \leq \|\tilde{f}\|_{H^{s/2+\delta_0}(\Omega)} \leq \|\tilde{f}\|_{H^t(\Omega)}$, since $\frac{s}{2} + \delta_0 < s < t$. By Lemma 2.9, $\|(-\Delta)_\Omega^{s/2} \varphi\|_{H^{s/2-\delta_0}(\Omega)} \leq C\|\varphi\|_{\tilde{H}^{3s/2-\delta_0}(\Omega)} \leq C\|\varphi\|_{\tilde{H}^t(\Omega)}$, since by our choice $t \geq \frac{3s}{2} - \delta_0$. Hence,

$$\langle (-\Delta)_\Omega^{s/2} \tilde{f}, b(-\Delta)_\Omega^{s/2} \varphi \rangle_{H^{-s/2+\delta_0}(\Omega), H^{s/2-\delta_0}(\Omega)} \leq C\|b\|_{H^1(\Omega)} \|\tilde{f}\|_{H^t(\Omega)} \|\varphi\|_{H^t(\Omega)}.$$

So from (2.41) we get $|\mathcal{B}_{b,q}(v, \varphi)| \leq (\|F\|_{H^{-t}(\Omega)} + \|f\|_{H^t(\Omega_e)}) \|\varphi\|_{\tilde{H}^t(\Omega)}$.

Then following the same procedure as we did before, one can solve the homogeneous problem (2.41) in $\tilde{H}^t(\Omega)$, and consequently the inhomogeneous problem (2.34) in $H^t(\mathbb{R}^n)$ with the exterior data in $H^t(\Omega_e)$.

2.6. Non-local Cauchy data. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $n \geq 1$. Let us consider $u_f \in H^t(\mathbb{R}^n)$ be a solution of the Dirichlet exterior value problem (2.34). Let us introduce the operator $\mathcal{N}_{b,q}(f)$, we will call it as Neumann data,

using the “non-local normal derivative” (see [DROV17]) of u_f given

$$(2.42) \quad \mathcal{N}_{b,q}(f)(x) := C_{n,t} \int_{\Omega} \frac{u_f(x) - u_f(y)}{|x-y|^{n+2t}} dy, \quad x \in \Omega_e,$$

where $u_f \in H^t(\mathbb{R}^n)$ is a unique weak solution of (2.34).

Let us now define the non-local Dirichlet to Neumann map for $0 < t < 1$. Under the assumption (1.4) we get a unique solution $u_f \in H^t(\mathbb{R}^n)$ of the Dirichlet exterior value problem (2.34). We define the Dirichlet to Neumann map $\Lambda_{b,q} : \tilde{H}^t(\Omega_e) \mapsto H^{-t}(\Omega_e)$ by

$$\langle \Lambda_{b,q} f, \psi \rangle := \langle \mathcal{L}_{b,q} u_f, \psi \rangle_{\Omega} \quad \forall \psi \in \tilde{H}^t(\Omega_e),$$

where u_f is the unique solution of $\mathcal{L}_{b,q} u_f = 0$ in Ω and $u_f = f$ on Ω_e .

Using the integration by parts formula (2.11) and the fact that $\psi|_{\Omega} = 0$, we get,

$$(2.43) \quad \langle \Lambda_{b,q} f, \psi \rangle = \int_{\mathbb{R}^n} ((-\Delta)^t u_f) \psi + \int_{\Omega} b \left((-\Delta)_{\Omega}^{s/2} u_f \right) \left((-\Delta)_{\Omega}^{s/2} \psi \right) = \int_{\Omega_e} ((-\Delta)^t u_f) \psi.$$

Therefore,

$$(2.44) \quad \Lambda_{b,q} f := (-\Delta)^t u_f, \quad \text{in } \Omega_e.$$

Next we state the relation between $\Lambda_{b,q}(f)$ and the non-local Neumann derivative $\mathcal{N}_{b,q}(f)$ in Proposition 2.18.

Proposition 2.18 ([GSU20]). *We have*

$$\Lambda_{b,q} f = \mathcal{N}_{b,q}(f) - m f + (-\Delta)^t (E_0 f)|_{\Omega_e}, \quad f \in \tilde{H}^t(\Omega_e),$$

where, for $\gamma > -1/2$, $\mathcal{N}$ is the map

$$(2.45) \quad \mathcal{N} : H^{\gamma}(\mathbb{R}^n) \rightarrow H_{\text{loc}}^{\gamma}(\Omega_e), \quad \mathcal{N} u = m u|_{\Omega_e} + (-\Delta)^t (\chi_{\Omega} u)|_{\Omega_e},$$

where $m \in C^{\infty}(\Omega_e)$ is given by $m(x) = c_{n,t} \int_{\Omega} \frac{1}{|x-y|^{n+2t}} dy$, χ_{Ω} is the characteristic function of Ω and E_0 is extension by zero in Ω . If $u \in L^2(\mathbb{R}^n)$, then $\mathcal{N} u \in L_{\text{loc}}^2(\Omega_e)$ is given a.e. by the formula (2.42).

Proof. Since the operators $\Lambda_{b,q}$ (c.f. (2.44)) and $\mathcal{N}$ (c.f. (2.42)) do not involve any regional non-locality from Ω , the proof is the same as in [GSU20, Lemma 3.2]. $\square$

This result shows that knowing $\Lambda_{b,q} f$ is equivalent to knowing $\mathcal{N}_{b,q}(f)$ for $f \in \tilde{H}^t(\Omega_e)$, since they differ by known quantities which are independent of b and q .

Let $W, \widetilde{W} \subset \Omega_e$ be any non-empty open subsets. We define the non-local Cauchy data $\mathcal{C}_{b,q}(W, \widetilde{W})$ (c.f. (1.3)) corresponding to the operator $\mathcal{L}_{b,q}$ by

$$(2.46) \quad \mathcal{C}_{b,q}(W, \widetilde{W}) = \{(f, \mathcal{N}_{b,q}(f)|_{\widetilde{W}}); f \in \tilde{H}^t(W)\}.$$

Hence, $\mathcal{C}_{b,q}(W, \widetilde{W})$ is determined by the non-local DN map $\Lambda_{b,q}|_{\widetilde{W}}$ applied on $\tilde{H}^t(W)$.

The main resemblance of the non-local normal derivative with the local normal derivative can be explained using the following integration by-parts formula [GLX17, DROV17]

$$(2.47) \quad \int_{\Omega} v (-\Delta)^t w dx + \int_{\Omega_e} v \mathcal{N}(w) = \int_{\Omega} w (-\Delta)^t v dx + \int_{\Omega_e} w \mathcal{N}(v)$$

together with the following limiting equivalence as (see [DROV17])

$$\lim_{t \rightarrow 1} \int_{\mathbb{R}^n \setminus \Omega} v \mathcal{N}(w) dx = \int_{\partial \Omega} v \frac{\partial w}{\partial \nu} d\sigma$$

for all $v, w \in \mathcal{S}(\mathbb{R}^n)$; where $\mathcal{N}(w)$ as in (2.45) and ν , $d\sigma$ denote the boundary normal vector and the surface measure respectively.

Here we end our discussion on the direct problems for the regional fractional Laplacian operator $(-\Delta)_{\Omega}^a$ and the non-local operator $\mathcal{L}_{b,q}$. Next we move into studying the inverse problems of recovering the coefficients b, q from the associated non-local Cauchy data $\mathcal{C}_{b,q}(W, \widetilde{W})$.

3. INVERSE PROBLEMS

Here we address the inverse problem:

Does $\mathcal{C}_{b,q}(W, \widetilde{W})$ (c.f. (2.46)) uniquely determine b and q in Ω ?

We will answer this question by proving Theorem 1.1 and Theorem 1.2.

To start with, let us consider the operators

$$\mathcal{L}_{b_k, q_k} := \left((-\Delta)^t + (-\Delta)_{\Omega}^{s/2} b_k (-\Delta)_{\Omega}^{s/2} + q_k \right), \quad k = 1, 2,$$

where b_k, q_k are $L^{\infty}(\Omega)$ functions satisfying (1.4). Let $\mathcal{C}_{b_1, q_1}(W, \widetilde{W}) = \mathcal{C}_{b_2, q_2}(W, \widetilde{W})$. Then for any $f \in \widetilde{H}^t(W)$ we have

$$\mathcal{N}(u_2)|_{\widetilde{W}} = \mathcal{N}(u_1)|_{\widetilde{W}},$$

where

$$(3.1) \quad \mathcal{L}_{b_k, q_k} u_k = 0 \quad \text{in } \Omega, \quad \text{and} \quad u_k|_{\Omega_e} = f \in \widetilde{H}^t(\Omega_e), \quad \text{for } k = 1, 2.$$

Lemma 3.1 (Integral identity). *Let $f \in \widetilde{H}^t(\Omega_e)$ and $u_k \in H^t(\mathbb{R}^n)$ be solutions of (3.1) for $k = 1, 2$. Let $\widetilde{W} \subset \Omega_e$ be a non-empty open set such that*

$$\mathcal{N}(u_1)|_{\widetilde{W}} = \mathcal{N}(u_2)|_{\widetilde{W}}.$$

Then $u_1 = u_2$ in $\mathbb{R}^n$.

Moreover, we have the following integral identity

$$(3.2) \quad \int_{\Omega} (b_1 - b_2) \left((-\Delta)_{\Omega}^{s/2} u_f \right) \left((-\Delta)_{\Omega}^{s/2} \varphi \right) + \int_{\Omega} (q_1 - q_2) \varphi u_f = 0, \quad \forall \varphi \in C_c^{\infty}(\Omega),$$

where $u_f = u_1 = u_2$ in $\mathbb{R}^n$.

Next we prove Lemma 3.1 which follows from the unique continuation property for the fractional Laplacian operator (see [GSU20, Theorem 1.2]).

Proposition 3.2. *Let $u \in H^{-r}(\mathbb{R}^n)$, $r > 0$. If $u = (-\Delta)^t u = 0$ in some non-empty open set $\mathcal{O} \subset \mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n$.*

Proof of Lemma 3.1. By using Proposition 2.18 and (2.44) from $\mathcal{N}_{b,q}(f)|_{\widetilde{W}} = \mathcal{N}_{b,q}(f)|_{\widetilde{W}}$, we obtain $(-\Delta)^t u_1|_{\widetilde{W}} = (-\Delta)^t u_2|_{\widetilde{W}}$. Since $u_1 = u_2 = f$ in Ω_e , so we have

$$(-\Delta)^t (u_1 - u_2)|_{\widetilde{W}} = 0 = (u_1 - u_2)|_{\widetilde{W}}.$$

Therefore, from Proposition 3.2 it follows that $u_1 = u_2$ on $\mathbb{R}^n$.

Let us now denote $u_f = u_1 = u_2$ in $\mathbb{R}^n$ and observe that

$$\begin{aligned} \left((-\Delta)^t + (-\Delta)_\Omega^{s/2} b_1 (-\Delta)_\Omega^{s/2} + q_1 \right) u_f &= 0, \quad \text{in } \Omega, \\ \left((-\Delta)^t + (-\Delta)_\Omega^{s/2} b_2 (-\Delta)_\Omega^{s/2} + q_2 \right) u_f &= 0, \quad \text{in } \Omega, \end{aligned}$$

which implies

$$(3.3) \quad (-\Delta)_\Omega^{s/2} (b_1 - b_2) (-\Delta)_\Omega^{s/2} u_f + (q_1 - q_2) u_f = 0, \quad \text{in } \Omega.$$

One can equivalently write the above equation as

$$\int_\Omega (b_1 - b_2) \left((-\Delta)_\Omega^{s/2} u_f \right) \left((-\Delta)_\Omega^{s/2} \varphi \right) + \int_\Omega (q_1 - q_2) \varphi u_f = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

□

Now our goal is to show $b_1 = b_2$ and $q_1 = q_2$ from the integral identity (3.2). In order to do that we derive a unique continuation property for the regional fractional Laplacian and Runge approximation property for the non-local operators $(-\Delta)_\Omega^a$ and $\mathcal{L}_{b,q}$.

3.1. Unique continuation and Runge approximation. As a consequence of Proposition 3.2 we get the following Runge approximation property (see [GSU20, Theorem 1.3]). Let us consider the set

$$X_{\mathcal{O},W} := \{v|_{\mathcal{O}}; (-\Delta)^t v = 0, \text{ in } \mathcal{O}, v|_{\mathcal{O}_e} = f, \forall f \in C_c^\infty(W)\},$$

where W is some open bounded subset of $\mathcal{O}_e := \mathbb{R}^n \setminus \overline{\mathcal{O}}$.

Proposition 3.3. *The set $X_{\mathcal{O},W}$ is dense in $L^2(\mathcal{O})$.*

For the regional fractional Laplacian $(-\Delta)_\Omega^a$, $0 < a < 1$, first we prove the following unique continuation property.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$, be a bounded Lipschitz domain. Let $v \in H^a(\Omega)$, $0 < a < 1$. If $v = (-\Delta)_\Omega^a v = 0$ on a non-empty open subset $\mathcal{O} \Subset \Omega$, then $v = 0$ in Ω .*

Proof.

Case 1. Let us take $v \in \tilde{H}^a(\Omega)$ and extend it by zero in Ω_e . From (2.12) with using the fact $v = (-\Delta)_\Omega^a v = 0$ in $\mathcal{O}$ we simply obtain

$$v = (-\Delta)^a v = 0 \text{ in } \mathcal{O}.$$

Consequently, from Proposition 3.2 we obtain $v \equiv 0$, or $v = 0$ in Ω .

Case 2. Let $v \in H^a(\Omega)$, and $0 < a < \frac{1}{2}$. Then using the fact that $\tilde{H}^a(\Omega) = H^a(\Omega)$ for $0 < a < \frac{1}{2}$ (see (2.7), (2.8)) we get $v \in \tilde{H}^a(\Omega)$ and using Case 1 we prove the lemma.

Case 3. Now if $v \in H^a(\Omega)$, and $a \geq \frac{1}{2}$, then we define $\tilde{v} = \chi_\Omega v$ in $\mathbb{R}^n$ to have at least $\tilde{v} \in L^2(\mathbb{R}^n)$. Note that, $(-\Delta)_\Omega^a v = (-\Delta)_\Omega^a \tilde{v} = 0$ in Ω . Now, from (2.12) we obtain $(-\Delta)^a \tilde{v} = 0$ in $\mathcal{O}$. Since Proposition 3.2 is also valid for $L^2(\mathbb{R}^n)$ functions, so from $\tilde{v} = (-\Delta)^a \tilde{v} = 0$ in $\mathcal{O} \Subset \Omega$, implies $\tilde{v} \equiv 0$, or $v = 0$ in Ω . □

Using the unique continuation Lemma 3.4, we prove the following Runge approximation result for the regional fractional Laplacian.

Lemma 3.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and $\mathcal{O} \Subset \Omega$ be a non-empty open subset, with Lipschitz boundary, compactly contained in Ω . Then for $0 < a < 1$, we have*

$$X_{\mathcal{O}} := \{v|_{\mathcal{O}}; v \in H^a(\Omega), (-\Delta)_{\Omega}^a v = 0 \text{ in } \mathcal{O}\}$$

is dense in $L^2(\mathcal{O})$.

Proof. It is enough to show that

$$(3.4) \quad \text{if } \langle w, v \rangle_{L^2(\mathcal{O})} = 0 \quad \forall v \in X_{\mathcal{O}}, \quad \text{then } w = 0 \text{ in } \mathcal{O}.$$

Let us assume that, there is a $w \in L^2(\mathcal{O})$ such that $\langle w, v \rangle_{L^2(\mathcal{O})} = 0$ for all $v \in X_{\mathcal{O}}$. Then we consider the function $\varphi \in \tilde{H}^a(\mathcal{O})$ as the unique weak solution of

$$(3.5) \quad \begin{aligned} (-\Delta)_{\Omega}^a \varphi &= w \quad \text{in } \mathcal{O} \\ \varphi &= 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}}. \end{aligned}$$

Now from (3.4) and (3.5) we have

$$\begin{aligned} 0 &= \langle w, v \rangle_{L^2(\mathcal{O})} = \langle (-\Delta)_{\Omega}^a \varphi, v \rangle_{L^2(\mathcal{O})} \\ &= \langle (-\Delta)_{\Omega}^a \varphi, v \rangle_{L^2(\Omega)} - \langle (-\Delta)_{\Omega}^a \varphi, v \rangle_{L^2(\Omega \setminus \overline{\mathcal{O}})}. \end{aligned}$$

Thus for all $v \in X_{\mathcal{O}}$, we conclude

$$(3.6) \quad \langle (-\Delta)_{\Omega}^a \varphi, v \rangle_{L^2(\Omega \setminus \overline{\mathcal{O}})} = \langle (-\Delta)_{\Omega}^a \varphi, v \rangle_{L^2(\Omega)} = \langle \varphi, (-\Delta)_{\Omega}^a v \rangle_{L^2(\mathcal{O})} = 0.$$

Since $v|_{\Omega \setminus \overline{\mathcal{O}}} \in H^a(\Omega \setminus \overline{\mathcal{O}})$ can be chosen arbitrarily, (3.6) implies

$$(3.7) \quad (-\Delta)_{\Omega}^a \varphi = 0 \quad \text{in } \Omega \setminus \overline{\mathcal{O}}.$$

Therefore, we obtain that

$$\varphi = (-\Delta)_{\Omega}^a \varphi = 0 \text{ in } \Omega \setminus \overline{\mathcal{O}},$$

which implies that $\varphi \equiv 0$ in Ω due to the unique continuation (cf. Lemma 3.4) and hence $w = 0$ in $\mathcal{O}$. This proves (3.4) and Lemma 3.5. $\square$

Now we prove a Runge approximation type property for the operator $\mathcal{L}_{b,q}$. Let us recall the operator $\mathcal{L}_{b,q}$ introduced in (1.1). Let $\mathcal{O} \Subset \Omega$ be a non-empty open set, compactly contained in Ω , with Lipschitz boundary and consider the sets

$$X := \{(-\Delta)_{\Omega}^{s/2} v|_{\mathcal{O}}; \mathcal{L}_{b,q} v = 0, \text{ in } \Omega, v|_{\Omega_e} = f, \forall f \in C_c^{\infty}(W)\}$$

and

$$Y := \{v|_{\Omega}; \mathcal{L}_{b,q} v = 0, \text{ in } \Omega, v|_{\Omega_e} = f, \forall f \in C_c^{\infty}(W)\},$$

where W is some non-empty open bounded subset of Ω_e .

Lemma 3.6. *Let $\mathcal{L}_{b,q}, X, Y$ are as above. Then*

- (1) *For any $F \in L^2(\mathcal{O})$ and any $\epsilon > 0$ there exists $u \in X$ such that*

$$\|F - u\|_{L^2(\mathcal{O})} < \epsilon.$$

- (2) *The set Y is dense in $L^2(\Omega)$.*

Proof. (1) We observe that, it is enough to prove the result for $F \in \tilde{H}^s(\mathcal{O})$ for some $0 < s < 1$, since $\tilde{H}^s(\mathcal{O})$ is dense in $L^2(\mathcal{O})$. Let $F \in \tilde{H}^s(\mathcal{O})$ such that $\langle F, \tilde{v} \rangle_{L^2(\mathcal{O})} = 0$, for all $\tilde{v} \in X$, then we show $F = 0$ in $\mathcal{O}$. Since $0 < s < t$, therefore $\tilde{v} = (-\Delta)_\Omega^{s/2} v|_{\mathcal{O}} \in L^2(\mathcal{O})$ for some $v \in H^t(\mathbb{R}^n)$ solving $\mathcal{L}_{b,q}v = 0$ in Ω . Thus we get that

$$\langle F, (-\Delta)_\Omega^{s/2} v \rangle_{L^2(\mathcal{O})} = 0.$$

Now extending F by 0 outside $\mathcal{O}$ we have $F \in \tilde{H}^s(\Omega)$ and consequently $(-\Delta)_\Omega^{s/2} F \in L^2(\Omega)$ thanks to Lemma 2.6. Next we write

$$0 = \langle F, (-\Delta)_\Omega^{s/2} v \rangle_{L^2(\mathcal{O})} = \left\langle \left((-\Delta)_\Omega^{s/2} F \right), v \right\rangle_{L^2(\Omega)}.$$

Since $(-\Delta)_\Omega^{s/2} F \in L^2(\Omega)$, there is $w \in H^t(\mathbb{R}^n)$, $0 < t < 1$ such that

$$\mathcal{L}_{b,q}w = (-\Delta)_\Omega^{s/2} F \quad \text{in } \Omega, \quad \text{with } w = 0 \text{ in } \Omega_e.$$

Therefore, we get

$$0 = \langle \mathcal{L}_{b,q}w, v \rangle_{L^2(\Omega)} = \langle w, \mathcal{L}_{b,q}v \rangle_{L^2(\Omega)} - \langle (-\Delta)^t w, v \rangle_{L^2(\Omega_e)}.$$

Since $\mathcal{L}_{b,q}v = 0$ in Ω , thus

$$\langle (-\Delta)^t w, f \rangle_{\Omega_e} = 0, \quad \forall f \in C_c^\infty(W).$$

Hence, $(-\Delta)^t w = 0 = w$ in $W \subset \Omega_e$. Consequently, by the unique continuation given by Proposition 3.2 we conclude that $w \equiv 0$, that $(-\Delta)_\Omega^{s/2} F = 0$ in Ω . Using the unique continuation for the regional fractional Laplacian operator since $(-\Delta)_\Omega^{s/2} F = 0 = F$ in $\Omega \setminus \overline{\mathcal{O}}$, we get $F = 0$ in Ω (c.f. Lemma 3.4). Hence, part (1) follows.

(2) The proof is similar to part (1). Let $G \in L^2(\Omega)$ and $\langle G, v \rangle_{L^2(\Omega)} = 0$, for all $v \in Y$, then we will show that $G = 0$ in Ω to prove our claim.

Let $w \in H^t(\mathbb{R}^n)$ solves $\mathcal{L}_{b,q}w = G$ in Ω and $w = 0$ in Ω_e . Then we have

$$0 = \langle \mathcal{L}_{b,q}w, v \rangle_{L^2(\Omega)} = \langle w, \mathcal{L}_{b,q}v \rangle_{L^2(\Omega)} - \langle (-\Delta)^t w, v \rangle_{L^2(\Omega_e)}.$$

Since $\mathcal{L}_{b,q}v = 0$ in Ω , we get

$$\langle (-\Delta)^t w, f \rangle_{\Omega_e} = \langle (-\Delta)^t w, v \rangle_{\Omega_e} = 0, \quad \text{for all } f \in C_c^\infty(W).$$

Hence, $(-\Delta)^t w = 0 = w$ in $W \subset \Omega_e$. Consequently, by the unique continuation in Proposition 3.2 we have $w \equiv 0$ and therefore $G = 0$ in Ω . $\square$

3.2. Proof of Theorems 1.1 and 1.2. Let us recall the integral identity from Lemma 3.1, given by

$$(3.8) \quad \int_\Omega (b_1 - b_2) \left((-\Delta)_\Omega^{s/2} u_f \right) \left((-\Delta)_\Omega^{s/2} \varphi \right) + \int_\Omega (q_1 - q_2) \varphi u_f = 0, \quad \forall \varphi \in C_c^\infty(\Omega),$$

where $u_f \in H^t(\mathbb{R}^n)$ is the unique solution of the Dirichlet exterior value problem $\mathcal{L}_{b_k, q_k} u_f = 0$ in Ω and $u_f = f$ in Ω_e , for $k = 1, 2$.

Since b_k, q_k are compactly supported in Ω , let us consider a non-empty open subset $\mathcal{O} \Subset \Omega$, with Lipschitz boundary, containing the compact supports of b_k, q_k in Ω , for $k = 1, 2$. Now, using Lemma 3.5 we know that

$$X_{\mathcal{O}} = \{ \varphi|_{\mathcal{O}} : \varphi \in H^{s/2}(\Omega); (-\Delta)_\Omega^{s/2} \varphi = 0 \text{ in } \mathcal{O} \}$$

is dense in $L^2(\mathcal{O})$. Hence, from the L^2 -density of $C_c^\infty(\Omega)$ in $X_{\mathcal{O}}$ and the integral identity (3.8), we conclude

$$\int_{\mathcal{O}} (q_1 - q_2) u_f \varphi = 0, \quad \forall \varphi \in X_{\mathcal{O}},$$

or equivalently

$$(3.9) \quad (q_1 - q_2) u_f = 0 \quad \text{in } \mathcal{O}.$$

Plugging (3.9) in the integral identity (3.8) we have

$$(3.10) \quad \int_{\Omega} (b_1 - b_2) \left((-\Delta)_{\Omega}^{s/2} u_f \right) \left((-\Delta)_{\Omega}^{s/2} \varphi \right) = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Note that $\mathcal{O} \Subset \Omega$ contains the compact supports of b_1, b_2 in Ω and $(b_1 - b_2)(-\Delta)_{\Omega}^{s/2} u_f \in L^2(\mathcal{O})$. Hence, take $\varphi \in H^{s/2}(\Omega)$ to be the weak solution of (see Theorem 2.13)

$$\begin{aligned} (-\Delta)_{\Omega}^{s/2} \varphi &= (b_1 - b_2)(-\Delta)_{\Omega}^{s/2} u_f \quad \text{in } \mathcal{O} \\ \varphi &= 0 \quad \text{on } \Omega \setminus \overline{\mathcal{O}}. \end{aligned}$$

Then plugging this φ in (3.10), we obtain

$$(3.11) \quad (b_1 - b_2)(-\Delta)_{\Omega}^{s/2} u_f = 0 \quad \text{in } \Omega.$$

Observe the relations obtained in (3.9) and (3.11), that is

$$(3.12) \quad (b_1 - b_2)(-\Delta)_{\Omega}^{s/2} u_f = 0 = (q_1 - q_2) u_f \quad \text{in } \Omega.$$

Note that the above identity (3.12) is true for each $f \in \tilde{H}^t(\Omega_e)$ (and the corresponding solution $u_f \in H^t(\mathbb{R}^n)$) satisfying $\mathcal{N}_{b,q}(f)|_{\widetilde{W}} = \mathcal{N}_{b,q}(f)|_{\widetilde{W}}$.

Proof of Theorem 1.2. Let us assume that the operators $\mathcal{L}_{b_k, q_k}$, for $k = 1, 2$ satisfies the assumption (1.4). Fix a nonzero $f \in \tilde{H}^t(\Omega_e)$ such that

$$(\Lambda_{b_1, q_1} - \Lambda_{b_2, q_2})(f)|_{\widetilde{W}} = 0.$$

Since $(b_1 - b_2), (q_1 - q_2) \in C_c(\Omega)$, then

$$(3.13) \quad B = \{x \in \Omega; (b_1 - b_2)(x) \neq 0\} \quad \text{and} \quad C = \{x \in \Omega; (q_1 - q_2)(x) \neq 0\}$$

are open subsets in Ω . If B, C are non-empty, then from (3.12), we get $(-\Delta)_{\Omega}^{s/2} u_f$ and u_f are zero on the open sets B and C respectively.

From Lemma 3.4, it is clear that B and C are disjoint open sets. If $B \cap C \neq \emptyset$, it will lead to $u_f \equiv 0$ in Ω . We also observe that B, C cannot be complementary in Ω , that if $C = \Omega \setminus \overline{B}$ then the following exterior value problem

$$(-\Delta)_{\Omega}^{s/2} v = 0 \text{ in } B, \quad v = 0 \text{ in } \Omega \setminus \overline{B}$$

has only $v = 0$ solution in Ω (cf. Proposition 2.13). This completes the proof of Theorem 1.2. $\square$

Proof of Theorem 1.1. Let $\mathcal{N}_{b,q}(f)|_{\widetilde{W}} = \mathcal{N}_{b,q}(f)|_{\widetilde{W}}$, for all $f \in \widetilde{H}^t(W)$, where $W, \widetilde{W} \subset \Omega_e$ are any non-empty open sets. Then we have the identity (3.12):

$$(b_1 - b_2)(-\Delta)_\Omega^{s/2} u_f = 0 = (q_1 - q_2)u_f \quad \text{in } \Omega, \quad \forall f \in C_c^\infty(W).$$

Now let us recall the approximation result of Lemma 3.6. Since, $(b_1 - b_2)$ is compactly supported inside Ω , using the density result concerning $(-\Delta)_\Omega^{s/2} u_f$ in the Part (1) of Lemma 3.6 and varying $f \in C_c^\infty(W)$ we obtain $b_1 = b_2$ in Ω . Similarly by using Part (2) of Lemma 3.6 we get $q_1 = q_2$ in Ω . This completes the proof of Theorem 1.1. $\square$

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INSTITUTE FOR ADVANCED STUDY, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG

Email address: arkatifr@gmail.com

INSTITUTE FOR ADVANCED STUDY, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG; AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON

Email address: imaginetuhin@gmail.com

INSTITUTE FOR ADVANCED STUDY, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, HONG KONG; AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON

Email address: gunther@math.washington.edu