THE PERFECT GROUPS OF ORDER UP TO TWO MILLION

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Abstract. We enumerate the 15768 perfect groups of order up to $2 \cdot 10^6$, up to isomorphism, thus also completing the missing cases in [HP89]. The work supplements the by now well-understood computer classifications of solvable groups, illustrating scope and feasibility of the enumeration process for nonsolvable groups.

The algorithmic setup for constructing finite groups of a given order, up to isomorphism, has been well-established, both in theory and in practice, for the construction of groups [BEO02, EHH17]. It proceeds inductively, by constructing extensions of known groups of smaller orders and eliminating isomorphic candidates when they arise. Due to limitations in implementations of underlying routines, this however had been done so far mostly for solvable groups.

The aim of this paper is to show the feasibility of generalizing this approach to the case of nonsolvable groups. Instrumental in this has been the calculation of 2-cohomology through confluent rewriting systems, generalizing the method [HEO05, §8.7.2] for solvable groups that uses a PC presentation.

The construction process is illustrated by revisiting the enumeration of perfect groups that was started in [HP89] and to extend it to order $2 \cdot 10^6$. In total we find 15768 perfect groups, seeded from the 66 nonabelian simple groups of order up to $2 \cdot 10^6$.

Compared with [HP89], this newly provides explicit lists of the groups of orders 61440, 86016, 122880, 172032, 245760, 344064, 368640, 491520, 688128, 737280, 983040 that were omitted in their classification of groups of order up to 10^6. In this range, the calculations also found five groups (in addition to two groups found already in 2005 by Jack Schmidt) that had been overlooked in [HP89]. Besides serving as examples for testing conjectures, such lists of groups are used as seed in algorithms for the calculation of subgroups of a given finite group [Neu60, CCH01, Hul13], or indeed for the construction of all groups of a given order.

All calculations were done using the system `GAP` [GAP20], which also serves as repository of the resulting group data. The program that performed the classification is available at `https://github.com/hulpke/perfect` and should allow for easy generalization or extension.

In addition to the actual classification result, this work also serves as a prototype of enumeration of nonsolvable groups, extending the work of [BEO02] to the nonsolvable case. It illustrates the feasibility range of current implementations of underlying routines for cohomology, extensions, and isomorphism tests, with a number of general-purpose improvements in the system GAP [GAP20] (that will be part of the 4.12 release) by the author having been motivated by this work.

Indeed, the fact, that it took over 30 years since the publication of [HP89] to complete the classification of perfect groups up to order one million, indicates the broad infrastructural requirements of such classifications, with isomorphism tests [CH03] being the most prominent utility (and ultimately the bottleneck of any classification).

1. The construction process

We first briefly summarize the construction process for perfect groups of a given order $n > 1$. This process closely follows the description in [HEO05, §11.3] (and, apart from seeding with nonabelian simple groups, is fundamentally the same strategy as used in [BEO02] for solvable groups).

The construction of perfect groups for a chosen order n consists of two parts, depending on whether the resulting groups have a solvable normal subgroup or not.

1.1. Fitting-free groups.

Groups that have no solvable normal subgroup are called *Fitting-free*. Such a group G embeds into the automorphism group of its socle $S \lhd G$, which in turn is a direct product of simple nonabelian groups. The conjugation action of G on the k direct factors of S induces a permutation representation of G of degree k. For its image to be nontrivial perfect, we would need $k \geq 5$ (and thus $n = |G| \geq 60^6$). For the order range considered, this means that this image is trivial, thus all direct factors of S must be normal in G. But then G/S is isomorphic to a subgroup of the direct product of the automorphism groups of the simple nonabelian socle factors. Such a factor group is solvable (by the Schreier conjecture), showing that we must have $G = S$ as a direct product of simple nonabelian groups.

For $n \leq 2 \cdot 10^6$, the possible direct factors to consider are:

$$A_5, A_6, A_7, \mathrm{PSL}(3,3), \mathrm{PSU}(3,3), M_{11}, A_8, \mathrm{PSL}(3,4), \mathrm{PSp}(4,3), \mathrm{Sz}(8),$$
$$\mathrm{PSU}(3,4), M_{12}, \mathrm{PSU}(3,5), J_1, A_9, \mathrm{PSL}(3,5), M_{22}, J_2, \mathrm{PSp}(4,4), A_{10}, \mathrm{PSL}_3(7).$$

and $\mathrm{PSL}(2,q)$ for prime powers $7 \leq q \leq 157$, $q \neq 128$. (Note that $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$ and $\mathrm{PSL}(2,9) \cong A_6$.)

1.2. Inductive construction.

Groups of order n that possess a solvable normal subgroup can be constructed as extension of groups of smaller order $d \mid n$ by a simple module of order $p^a = n/d$. As factor groups of perfect groups these smaller groups need to be perfect themselves. We thus assume that, by induction, all perfect groups of order dividing n are known. (Of course the existence of perfect groups of order d is only necessary, but not sufficient, for the existence of perfect groups of order $n = p^a \cdot d$.)

We also can assume that $p \mid d$ if $a = 1$, since the action of a perfect group on a 1-dimensional module must be trivial, and any extension for $p = n/d$ and p coprime to d thus would be a direct product and thus not perfect.

This gives the following construction process:

(1) Iterate over all proper divisors $d \mid n$ with $n/d = p^a$, such that $a > 1$ or $p \mid d$. Then iterate over all perfect groups F of order d:
(2) Classify the irreducible a-dimensional F-modules M over $\mathbb{F}_p$. For this, we use the Burnside-Brauer theorem, as described in [HEO05, §7.5.5], to classify all modules, and eliminate those of the wrong dimension. (Clearly it is sufficient to consider modules for the factor group $F/O_p(F)$ by the largest normal p-subgroup. The index of the kernel of the module action is further bounded by $|\mathrm{GL}_a(p)|$, which can eliminate some small dimensions $a > 1$ for groups that have no small proper factors.)

(3) For each such module M calculate the 2-cohomology group $H^2(F, M)$ (Section 1.3).
(4) For each cocycle ζ, representing an element of $H^2(F, M)$, construct the corresponding extension E.
(5) Test each E for being perfect (and discard E, if not).
(6) Eliminate isomorphic groups.

Compared with [HP89], this construction does not aim to construct groups as subdirect products, if possible. Nor does it use further theoretical results to restrict potential orders, primes, or module types. Instead, the generic construction algorithm is used throughout, to have as much of an independent verification of the prior results as possible, and to reduce the potential of missing cases in a lengthy case distinction. (See Section 3 for evidence of this risk.)

The main difficulty, as with any enumerative construction, however remains the elimination of isomorphic groups. (This is stated already in [HP89] as having been the main obstacle.) Since isomorphism tests are expensive, we utilize a number of techniques to reduce the number of tests needed. These techniques include the incorporation of isomorphism elimination already in the construction process (Compatible pairs, Section 1.4), testing for isomorphism invariants that can be computed cheaply and can decide non-isomorphism in many cases (Fingerprints, Section 2.2), and in particular by selecting a "canonical" construction path for groups (Section 2.1).

Since many of these concepts have been described before in other contexts, the following descriptions will be brief and focus on the case of perfect groups and the actual implementation used.

1.3. **Calculating Cohomology.** Assume that a factor group F, and an irreducible $\mathbb{F}_pG$-module M have been chosen. The first step of the calculation is the determination of $H^2(F, M)$, using the confluence condition for rewriting systems. Such an approach has been suggested, e.g., in [Sch10]. A detailed description can be found in [DH21, §7].

First, we determine a confluent rewriting system for F. Such a rewriting system can be composed (with a wreath product ordering) from rewriting systems for the composition factors of F, which in turn can be found using the methods of [Sch10]. We also found it useful, in the case of large primes, to introduce powers of generators as extra generators, e.g. writing x^{50} as y^4x^2 with $y = x^{12}$, as doing so keeps word lengths shorter.

The rewriting system for F can be extended to one for an extension of M by F by adding generators for M, adding relations describing the F-module structure of M (that is $m \cdot f \to f \cdot m^f$), and by modifying each rule $l \to r$ for F by a variable tail $t \in M$ to $l \to r \cdot t$.

The "critical pair" confluence conditions of the Knuth-Bendix algorithm (which hold for the rewriting system in F, since it was assumed to be confluent) then translate into linear conditions on the the tails. If we represent 2-cochains as vectors, composed from sequences of tail values, these conditions form a homogeneous system of linear equations, whose solution space represents the 2-cocycles $Z^2(F, M)$.

We represent the the 2-coboundaries $B^2(F, M)$ by a subspace therein. A generating set for this subspace is obtained by all combinations of changing coset representatives for a generator x of F by basis vectors of M.

Then $H^2(F, M)$ can be represented by a complement subspace to B^2 in Z^2, each of its elements representing a choice of tails to yield a rewriting system (and thus a presentation) for the associated extension of M by F.

1.4. **Compatible pairs.** Isomorphisms that respect the extension structure induce an action on the cohomology group through the group of compatible pairs:

An element $(\kappa, \nu) \in \mathrm{Aut}(F) \times \mathrm{Aut}(M)$ (here $\mathrm{Aut}(M)$ is the group of vector space automorphisms, i.e. $\mathrm{GL}(M)$) is called a *compatible pair* [Rob82], if its action is compatible with the module action of F on M, that is, if for any $f \in F$ and for any $m \in M$ we have that

$$\nu(m^f) = \nu(m)^{\kappa(f)}. \tag{1}$$

The compatible pairs form a group CP. It acts on $H^2(F, M)$, with extension corresponding to cocycles in the same orbit being isomorphic [HEO05, §8.9]. It thus is sufficient to consider only orbit representatives under this action for the construction of groups.

We compute the group of compatible pairs as follows: If $\kappa = 1$, the condition (1) is that $\nu \in \mathrm{Aut}_{\mathbb{F}_pF}(M)$ must be an $\mathbb{F}_pF$-module automorphism. This group $\mathrm{Aut}_{\mathbb{F}_pF}(M)$ of module automorphism can be computed using MeatAxe-style techniques [Smi94].

Furthermore, if two compatible pairs (κ, μ) and (κ, ν) agree in their first component, their quotient $(1, \mu/\nu)$ will be a compatible pair. This means that if will be sufficient to identify the nontrivial κ that can be first component of a compatible pair (κ, ν), and if so to determine a suitable ν. For this, given $\kappa \in \mathrm{Aut}(F)$, we consider the vector space M in a second way as an $\mathbb{F}_pF$-module (which we shall call M_κ), namely with F acting through its image under κ. Then (κ, ν) is a compatible pair, if and only if ν is an $\mathbb{F}_pF$-module isomorphism between these F modules M and M_κ. Again, MeatAxe-style techniques [Smi94] can test for such module isomorphisms constructively, producing a suitable ν.

We then use this test in a backtrack search through $\mathrm{Aut}(F)$, using a faithful permutation representation of $\mathrm{Aut}(F)$ [HEO05, §4.6]. This search determines the image of the projection of the compatible pairs on the first component. For each $\kappa \in \mathrm{Aut}(F)$ tested successfully, we store the compatible pair (κ, ν) for ν produced by the isomorphism test.

These compatible pairs, together with pairs $(1, \nu)$ for ν from a generating set for the group of module automorphisms, generate the group of compatible pairs.

The determination of orbit representatives of CP on $H^2(F, M)$ then is straightforward, as in the cases encountered the dimension of the cohomology group is sufficiently small to list its elements.

1.5. **Permutation Representations.** The cocycles representing H^2 each are collections of values of tails for rewriting rules for the corresponding extension E. We thus get each extension E in the form of a finitely presented group.

We first test each group E obtained this way for it being perfect, and discard those groups that are not. This test can be performed effectively, given only a presentation of E, using a Smith Normal form computation.

Any further checks however will be done far more efficiently, if the group can be described by a faithful permutation representation of moderate degree. (Such a representation also is desirable for a resulting library of perfect groups.) To find such a representation we follow the strategy in [Hul01]:

In a slight abuse of notation we write $M \lhd E$ to denote the normal subgroup associated to the construction. We are searching through subgroups of F of small index [CHSS05], in search of a subgroup $U \leq F$, such that its pre-image $T \leq E$ has an abelianization T/T' that is larger than U/U'. (The abelianization of subgroups of finitely presented groups, indeed an epimorphism $T \to T/T'$, can be found by standard methods [Sim94].) In this

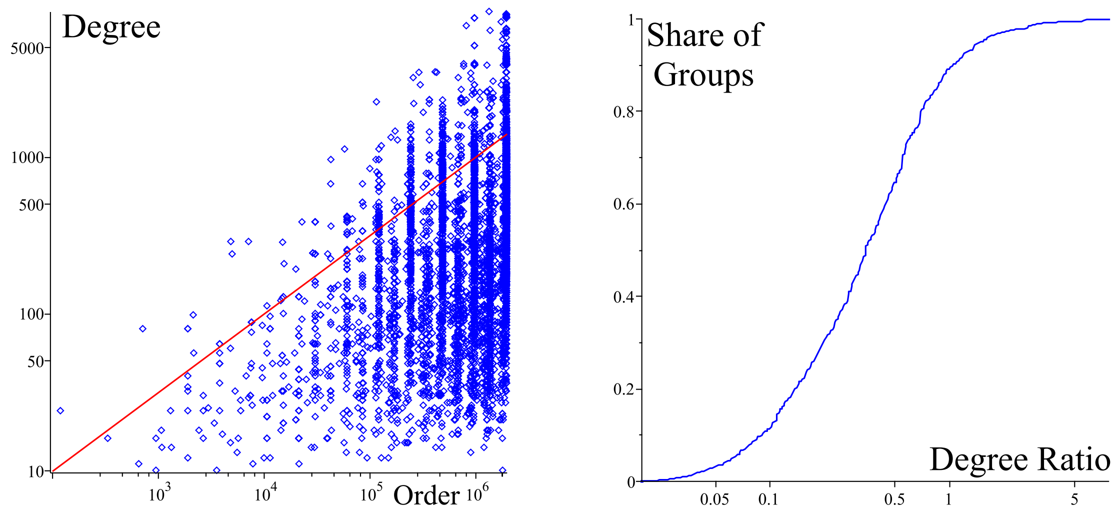

FIGURE 1. Permutation degrees in relation to $\sqrt{|G|}$.

case, there will be a permutation representation λ of T with $T' \leq \ker \lambda$ (that is, we can find λ from the representation $T \to T/T'$) with $M \not\leq \ker \lambda$. The subdirect product of the induced representation $\lambda\uparrow^E$, with a representation $E \to F$, then is a representation of E with a kernel strictly smaller than M. As M is a simple module, the subdirect product of $\lambda\uparrow^E$ with a faithful representation of F will be a faithful representation of E.

We found that this process resulted in reasonable degree permutation representations for the purpose of isomorphism tests. When storing the groups with permutation representations in a library, we however also ran degree reduction heuristics (the GAP command `SmallerDegreePermutationRepresentation`) on the resulting groups to try to get a smaller degree permutation representation.

For about 0.5% of the groups this resulted in permutation representations of unduly larger degree, the worst being degree 36864 for four groups of order 1843200. To make the use of the groups less costly, we re-ran the degree reduction for a longer time for these outlier groups, also testing the action on cosets of a number of randomly generated subgroups. In five remaining stubborn cases, finally, partial subgroup lattice information was used to determine a smaller degree permutation representation by hand. These reductions resulted in a maximal degree of 8448 (occurring for one group of order 1966080). We did not aim to establish minimal faithful permutation degrees for all groups, as this would have come at significant higher cost – essentially the computation of subgroup lattices.

Figure 1 shows the overall statistics of the resulting permutation degrees. The log-log plot on the left hand side displays a blue scatter plot of degrees versus group orders. Its shape, and the least-squares fit $|G|^{0.42}$, suggest to consider the degrees in relation to $\sqrt{|G|}$, which is given by the red line.

The right hand side of Figure 1 shows, for a ratio r given on the x-axis, the share of groups for which $\text{degree}(G)/\sqrt{|G|} \leq r$. It indicates that about 80% of the groups have a permutation degree between $0.1\sqrt{|G|}$ and $\sqrt{|G|}$, while all have a degree bounded by $10\sqrt{|G|}$. This justifies that for the groups constructed a degree of magnitude $\sqrt{|G|}$ can be expected.

2. Isomorphism Rejection

At the heart of any combinatorial classification algorithm is the question of isomorphism rejection. This is needed here, since the same abstract group might be constructed in multiple ways, e.g. through different normal subgroups, and such cases will not be identified by compatible pairs.

Isomorphism tests however are expensive, which makes it imperative to reduce the overall number of tests required. We do so through methods that are common to combinatorial enumerations

2.1. **Canonical construction.** We associate to each group E, constructed as extension of $M \lhd E$ by $E/M \cong F$, a "canonical" construction path. This allows us to discard any group we have constructed, if it turns out that it had been constructed in a non-canonical way.

We define this canonical construction in the following way:

(1) Amongst all minimal normal subgroups $N \lhd E$, M must be of minimal order.
(2) Amongst all minimal normal subgroups $N \lhd E$ with $|N| = |M|$, the isomorphism type of $F = E/M$ is minimal (in a list of isomorphism types of perfect groups of order $d = |F|$).

These conditions imply that a group E, constructed from F and M, gets discarded if there is $N \lhd E$ with $|N| \lneqq |M|$ or with $|N| = |M|$ and a factor group F/N of smaller (in a given list) isomorphism type. Such an isomorphism type does not need to be determined exactly, it is sufficient to know that any possible isomorphism type is smaller (or larger) than the type of F.

Once it has been established that a group has been constructed canonically, the only isomorphisms that we need to test for must be amongst groups constructed for the same F and for modules of the same order. Furthermore, such an isomorphism may not fix M, since it otherwise would result in a compatible pair. This means that the group E must have a second minimal normal subgroup M_2, such that $E/M \cong E/M_2$, and that the E/M-modules M is isomorphic to the E/M_2-module M_2. Explicit isomorphism tests thus only need to be run amongst the extensions that satisfy this property, which means in particular that they must have been constructed from the same factor group $F \cong E/M$.

This restriction significantly reduces the number of isomorphism tests required.

2.2. **Fingerprints.** The second of the "canonicity" conditions requires us to identify the isomorphism type of factor groups. Since we assume that a list of all perfect groups of order $d \mid n$ is known a priori, we can usually identify this type, without (or with minimal) need for isomorphism tests, by calculating a "fingerprint" of the group, composed of values obtained from isomorphism invariants. Groups with different fingerprints cannot be isomorphic, and the hope is that fingerprint information identifies a group uniquely in a given list. (Typically this hope is not fully satisfied, and would be provable only post-factum for an explicitly constructed list of isomorphism types.) In fact, even partial fingerprinting information can be sufficient, as long as it proves that all possible isomorphism types for a given factor group of E are smaller (respectively larger) than the type of F used in the construction.

The same kind of fingerprint information can be used to disprove isomorphism, before attempting an actual search for isomorphisms [CH03] amongst groups constructed for the same F and for modules of the same order.

In the construction of perfect groups of order up to $2 \cdot 10^6$, the following fingerprint properties were used. This list was determined experimentally as providing a decent tradeoff between avoiding isomorphism tests and not being too costly on their own. They however do not guarantee unique identification, and in a handful of cases explicit isomorphism tests have been required to identify factor group types uniquely. (If (sub)groups are given as identification, the actual fingerprint information for each subgroup consists of its order, whether it is perfect, as well as its identification in the small groups library [BEO02] if the order is sufficiently small.) Clearly one can stop computing fingerprint information, as soon a group is uniquely identified in a given list.

- Conjugacy class representatives and centralizers.
- Normal subgroups and their centralizers.
- Maximal, and low index [CHSS05] subgroups.
- Automorphism groups.
- Characteristic subgroups.
- Character table (i.e. classes/characters being permutable to make the resulting tables identical. This is done by a backtrack search for a class permutation whose effect on the character table can be undone by a character permutation and is provided by the GAP command `TransformingPermutationsCharacterTables`). We attempt this test (that requires the calculation of character tables in the first place) for factor groups that have at most 200 classes, respectively full groups (before an isomorphism test) that have at most 500 classes, and abort such a backtrack search for a suitable class permutation if its runtime exceeds 5-10 minutes, as at that point an explicit isomorphism test becomes competitive for the groups considered here. (Again, these parameters were found experimentally as providing a reasonable trade-off between computation cost and potential speedup.)

We note as an aside that character tables will not give a perfect identification for the pool of groups considered. The smallest example for this happening are two groups (Numbers 37 and 38 in the published list) of order 61440 and structure $2^2.2^4.2^4.A_5$, which form a Brauer pair. (These two groups differ in the number of classes of subgroups – 15731 versus 15715 – but this invariant count turns out to be more expensive to compute than an isomorphism test, which is the reason that we do not use it in the classification.)

3. Comparisons with [HP89]

The implementation of the algorithm was tested by revisiting the constructions for the orders considered in [HP89].

In checking results, we found five new groups, of order 243000 (two groups), 729000, 871200, and 878460 that had been overlooked in the earlier classification. (All of them are "obvious" constructions without any need of isomorphism testing, and their omission is unlikely to be the result of a calculation error, but seems to stem from sub-cases of a complicated construction having been overlooked accidentally. That is, they do not indicate conceptual errors in the earlier classification, but are simply clerical errors.) We also confirmed the two missing groups (or order 450000 and 962280) found in 2005 by Jack Schmidt.

All other results of [HP89] were confirmed. Given the substantial progress in both hardware and software since then, this makes the results of [HP89] even more impressive and

Order	61440	86016	122880	172042	245760	344064
Count	98	52	258	154	582	291
Order	368640	491520	688128	737280	983040	
Count	46	1004	508	54	1880	

TABLE 1. Newly found perfect groups of order $\leq 10^6$, counts

should be considered as a resounding validation of that work. The discovery of a small number of overlooked groups however also serves as corroboration of the more basic construction approach used here, in which we do not aim to use special methods for particular cases – such reductions do not apply to the hardest cases, and thus have little impact on the overall time required, but they increase the risk of accidental omissions.

As for the orders $< 10^6$, for which [HP89] omitted lists, we found counts of groups as given in table 1 (the underlined numbers already were determined in [HP89], though not the actual groups). The calculations for these 11 orders took about 2 (single-core) weeks on a Dell R740, Xeon Gold 6132, 2.60 GHz Processor (Geekbench 5 Single-Core 910) and used about 20GB of memory.

Lists of the groups will be made available as part of the system GAP [GAP20], release 4.12.

4. Results

We furthermore enumerated the perfect groups of orders between 10^6 and $2 \cdot 10^6$. Their counts are given [1] in table 2, giving in total 15768 perfect groups of order up to $2 \cdot 10^6$.

The total calculation time for these groups has been about 11 (single-core, on the same machine as above) weeks, with over 90% of the time taken by by order 1966080. Most of this time was (unsurprisingly) taken by work for isomorphism rejection, namely in identifying the type of factor groups in the test for the canonical construction described in Section 2.1, while only comparatively few explicit isomorphism tests have been necessary.

The largest cohomology groups encountered were of dimension 12, with up to 256 orbits remaining under the action on compatible pairs. (This occurs for example for the trivial module of group 957 of order 983040). The increase in the number of orbits on the cohomology groups, and the time required indicates that any significant extension of the lists would require a magnitude more of resources, which ultimately is the reason for stopping the classification at order $2 \cdot 10^6$.

With most of the groups being distributed amongst a small number of orders – primarily products of 60 of 168 with powers of 2, and with most groups being extensions of groups F of order $n/2$ by trivial modules M of order 2 – it of course would be easy for an interested researcher to calculate the groups of other, larger, orders with the provided program, as long as their total number was small.

We also tried to compare the experimental counts of groups with theoretical predictions in [Hol89]. For this, denote by $\mathrm{Perf}(\leq n)$ the number of perfect groups of order $\leq n$. The

[1]Due to an spreadsheet editing error, an earlier version of this note gave some wrong values in the table and subsequently a wrong total.

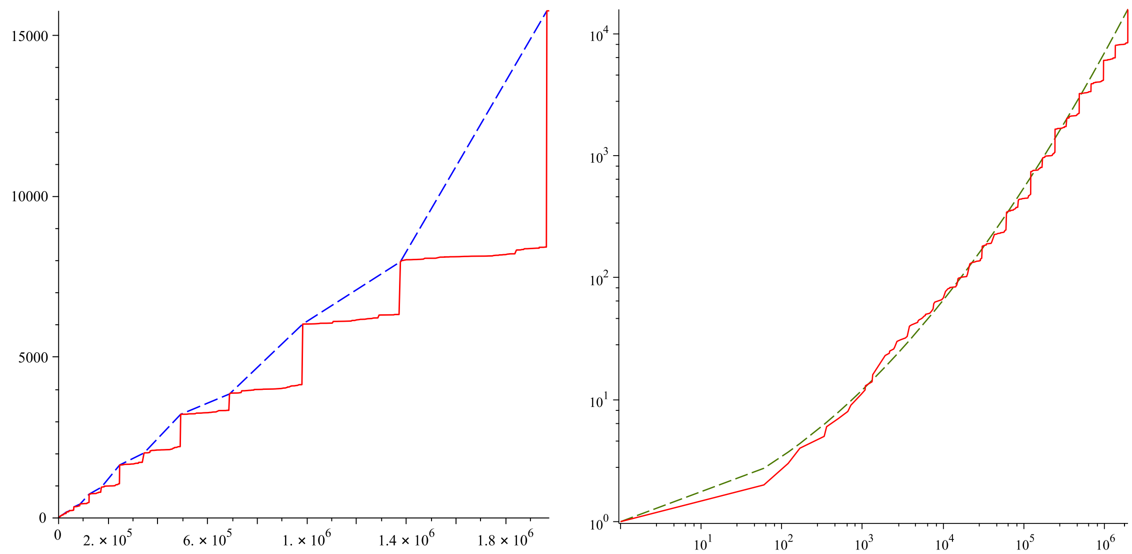

FIGURE 2. Counts of perfect groups, Perf($\leq n$)

estimates in [Hol89, Theorem A],

$$n^{\log_2(n)^2/108-c\cdot\log_2(n)} \leq \mathrm{Perf}(\leq n) \leq n^{\log_2(n)^2/48+\log_2(n)}$$

give for $n = 2\cdot 10^6$ only that

$$1.8\cdot 10^{-15} \leq \mathrm{Perf}(\leq 2\cdot 10^6) \leq 2.6\cdot 10^{189},$$

and thus agree with the result, but do not provide a meaningful validation. The exceedingly tiny lower bound here stems from choosing the parameter $c = 11/36$, which is indicated in [Hol89] as a correct, but unlikely to be optimal, estimate.

Figure 2, left, shows a plot of Perf($\leq n$) (red), as well as an "envelope" (blue, dashed) that accounts for the main growth contributors from a few orders. The changing growth of this envelope is due to different series of extensions with factor groups A_5, respectively $L_3(2)$, and the fact that the simple groups do not have irreducible modules in every dimension. (E.g. perfect groups of order n with radical factor A_5 give rise to extensions of order $2n$ and $16n$, but not $4n$ or $8n$.)

Theorem A in [Hol89] indicates that Perf($\leq n$) behaves asymptotically similar to a function $n^{p(\log_2(n))}$ with p a polynomial of degree 2, thus $\log_2(\mathrm{Perf}(\leq n))$ should be a polynomial of degree 3 in $\log_2(n)$. A least squares fit of the values for the envelope (to account for the growth mainly occurring in a few orders – we also attempted a fit for all data and got similar results with a worse match) of Perf($\leq n$) finds a degree 3 best fit polynomial

$$-0.00060x^3 + 0.04547x^2 - 0.03027x - 0.07598$$

Order	Count	Order	Count	Order	Count	Order	Count
1008000	1	1233792	1	1467648	2	1774080	9
1008420	1	1244160	15	1468800	1	1785600	3
1016064	1	1253376	4	1474560	26	1787460	1
1020096	1	1260000	2	1512000	1	1788864	4
1024128	1	1267200	15	1518480	1	1800000	3
1030200	1	1270080	2	1536000	33	1806336	13
1036800	3	1277760	2	1548288	1	1814400	6
1044480	4	1285608	1	1555200	3	1815000	3
1048320	2	1290240	88	1572480	1	1822500	2
1053696	9	1294920	1	1574640	4	1837080	1
1080000	1	1296000	1	1592136	1	1843200	113
1083000	1	1310400	1	1614720	3	1843968	3
1088640	1	1330560	2	1615680	1	1845120	3
1092624	1	1342740	1	1632960	7	1858560	1
1100736	1	1350000	1	1645056	1	1866240	13
1102248	3	1351680	8	1651104	1	1872000	1
1105920	49	1354752	3	1653900	1	1875000	22
1123980	1	1370880	1	1658880	2	1876896	1
1125000	1	1376256	1639	1663200	1	1886976	1
1149120	3	1382400	38	1693440	2	1920000	15
1166400	4	1386240	3	1713660	1	1924560	2
1176120	3	1399680	21	1721400	1	1934868	1
1179360	4	1414944	1	1723680	1	1935360	26
1180980	14	1425600	3	1728000	1	1953000	1
1192464	1	1425720	1	1728720	1	1959552	6
1200000	17	1441440	3	1742400	2	1964160	1
1209600	8	1442784	1	1747200	1	1966080	7344
1215000	9	1451520	3	1749600	8	1975680	1
1224120	1	1457280	1	1756920	8	1980000	1
1224936	1	1461600	3	1762560	3		
1231200	1	1463340	1	1771440	1		

TABLE 2. Counts of perfect groups beyond 10^6

whose coefficients (in magnitude, as well as the negative coefficient of the leading term) seem to indicate an ill fit. Tries of several polynomials make a quadratic fit $0.02807x^2 + 0.08021x$ seem the best match. Figure 2, right, gives a log-log plot of $\mathrm{Perf}(\leq n)$ (red) and the (green, dashed) fit curve

$$n^{0.02807 \log_2(n)+0.08021}.$$

This discrepancy to the asymptotic results [Hol89] in the degree in the exponent is likely due to the orders considered being still too small to be representative of the asymptotic behavior, and the logarithmic curve fitting giving more weight to the group of small order.

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