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Law of two-sided exit by a spectrally positive strictly stable process

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Abstract

For a spectrally positive strictly stable process with index in (1, 2), we obtain (i) the sub-probability density of its first exit time from an interval by hitting the interval's lower end before jumping over its upper end, and (ii) the joint distribution of the time, undershoot, and jump of the process when it makes the first exit the other way around. The density of the exit time is expressed in terms of the roots of a Mittag-Leffler function. Some theoretical applications of the results are given.

MSC: primary 60G51; secondary 60E07

Keywords: Two-sided exit problem; Lévy process; stable; spectrally positive; Mittag-Leffler

1. Introduction

The so-called exit problems, which concern the event that a stochastic process gets out of a set for the first time, occupy a prominent place in the study of Lévy processes. For spectrally one-sided Lévy processes, years of intensive research have revealed many remarkable facts about the first exit from a bounded interval [2,10,14]. An essential tool for the investigation is the scale function. Since the function can be analytically extended to the entire $\mathbb{C}$ ([14], Lemma 8.3), it is amenable to treatments by complex analysis. By combining the scale function and residual calculus, this paper obtains series expressions of the distribution of the first exit of a spectrally positive strictly stable process with index in $(1, 2)$.

Let X be a real-valued Lévy process. It is called stable if for each $t > 0$, there is a constant $C(t)$ such that $X_t \sim t^{1/\alpha} X_1 + C(t)$, and strictly stable if $C(t) \equiv 0$ ([22], p. 69). On the other hand, X is called spectrally one-sided if all its jumps have the same sign, and depending on the

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sign, it is further called spectrally positive or negative ([14], p. 58). It is well-known that X is spectrally positive and strictly stable with index $\alpha \in (1, 2)$ if and only if $\ln \mathbb{E}(e^{-qX_1}) = Cq^\alpha$ for all $q \geq 0$, where $C > 0$ is a constant (cf. [14], p. 89). Henceforth, without loss of generality, assume

$$\mathbb{E}(e^{-qX_t}) = e^{q^\alpha t}, \quad q \geq 0, \ \alpha \in (1, 2). \tag{1}$$

Given $b, c > 0$, there are only two ways for X to exit from $[-b, c]$, either by making a continuous downward passage of $-b$ or by making an upward jump across c ([14], p. 232). These two possibilities will be referred to as the first exit at the lower end and the first exit at the upper end, respectively. Their probabilities and the related scale functions have become well-known [2,14]. However, not much is known about the probability density functions (p.d.f.) involved.

In [8], the distribution of X's first upward passage of a fixed level is obtained. It turns out that the method used there can be extended to the first exit from $[-b, c]$. Section 3 considers the first exit at the lower end. It will be shown that the (sub-)p.d.f. of the exit time has an expression in terms of the residuals of a meromorphic function at the roots of a Mittag-Leffler function, and as a result, is of the form $\sum_\varsigma p_\varsigma(t)e^{\varsigma t}$, where the sum runs over the roots and for each root ς, $p_\varsigma(t)$ is a polynomial in t whose coefficients are determined by ς and several Mittag-Leffler functions. For all but a finite number of ς, $p_\varsigma(t)$ is a constant. The result is an extension of a known result on the first exit of a standard Brownian motion. It also highlights the importance of precise knowledge on the roots of Mittag-Leffler functions [20], which unfortunately is still in short supply. The section also obtains the asymptotic of the p.d.f. near time 0 based on an alternative expression. Section 4 derives the joint (sub-)p.d.f. of the time of the first exit by X at the upper end and its undershoot and jump at that moment. It will be shown that conditional on the undershoot, the exit time and jump are independent. This allows the joint distribution to be factorized into the marginal p.d.f. of the undershoot, and the marginal conditional p.d.f.'s of the exit time and jump given the undershoot. The marginal conditional p.d.f. of the exit time again can be written in the form $\sum p_\varsigma(t)e^{\varsigma t}$. This is in contrast to the power series expression of the time of the first upward passage of c [1,8,19,23], even though the latter can be regarded as the limit of the first exit from $[-b, c]$ as $b \to \infty$. This section also obtains an asymptotic of the p.d.f. near zero.

Basically, the reason that X's first exit time has a residual-based expression for its p.d.f. is two-fold. First, the Laplace transform of the p.d.f. can be analytically extended to a meromorphic function, specifically, a rational function of scale functions. Second, by contour integral, the inversion of the Fourier transform of the p.d.f. is reduced to the sum of the residuals of the meromorphic function at its poles, which happen to be the roots of a Mittag-Leffler function. This rather generic explanation suggests that the same analysis may be carried out for other fluctuation identities, such as those related to a reflected process, provided the corresponding scale functions are available (cf. [14], Chapter 8). On the other hand, as precise knowledge on the roots of Mittag-Leffler functions is scarce, the residual-based expressions may be harder to use in practice than direct numerical methods such as Fourier transform. Nevertheless, as will be seen in Section 5, they can be applied to provide some rather detailed information about X.

2. Notation and preliminaries

Let $u(x)$ and $v(x)$ be functions of x, both of which as well as x may be complex-valued. As $x \to x_0$, by $u(x) \asymp v(x)$ it means $u(x) = O(v(x))$ and $v(x) = O(u(x))$, i.e., for some constant

$C > 0$ and all x in a neighborhood of x_0, $|u(x)| \le C|v(x)|$ and $|v(x)| \le C|u(x)|$. On the other hand, by $u(x) \sim v(x)$ it means $u(x)/v(x) \to 1$. For elements of complex analysis, see [21].

2.1. Integral transforms

For $f \in L^1(\mathbb{R})$, denote its Laplace transform and Fourier transform, respectively, by

$$\widetilde{f}(z) = \int e^{-zt} f(t)\,\mathrm{d}t, \qquad \widehat{f}(y) = \widetilde{f}(-\mathrm{i}y), \quad y \in \mathbb{R}.$$

The domain of $\widetilde{f}$ is $\{z \in \mathbb{C} : \int |e^{-zt} f(t)|\,\mathrm{d}t < \infty\}$. Similarly, for a finite measure μ on $\mathbb{R}$, denote its Laplace transform and Fourier transform, respectively, by

$$\widetilde{\mu}(z) = \int e^{-zt} \mu(\mathrm{d}t) \quad \text{and} \quad \widehat{\mu}(y) = \widetilde{\mu}(-\mathrm{i}y), \quad y \in \mathbb{R}.$$

The domain of $\widetilde{\mu}$ is $\{z \in \mathbb{C} : \int |e^{-zt}|\,\mu(\mathrm{d}t) < \infty\}$.

Let $D \subset \mathbb{C}$ be an open set and $z_0 \in D$. If g is an analytic function in $D \setminus \{z_0\}$ and has z_0 as a pole, possibly removable, then the residual of g at z_0 is

$$\mathrm{Res}(g(z), z_0) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma g(z)\,\mathrm{d}z,$$

where γ is any counterclockwise simple closed contour in $D \setminus \{z_0\}$ ([21], p. 224).

2.2. Some properties of the Mittag–Leffler function

The Mittag-Leffler function with parameters $a > 0$ and $b \in \mathbb{C}$ is the entire function

$$E_{a,b}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an+b)}, \quad z \in \mathbb{C}.$$

A classical review of the function is in [11]; also see Section 10.46 of [18] and references therein. There are some papers on the computation of the Mittag-Leffler function [13,24], however, little seems to have been done on the computation of its roots. As we will heavily rely on [20], for convenience, some of its key results are collected below. By Theorem 1.2.1 in [20], for $a \in (0, 2)$, $b \in \mathbb{C}$, and $m \in \mathbb{N}$, as $|z| \to \infty$,

$$E_{a,b}(z) = a^{-1} z^{(1-b)/a} \exp(z^{1/a}) - \sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(b-an)} + O(|z|^{-m-1}), \tag{2}$$

where the principle value $\arg z$ is assumed, i.e., if $z = |z|e^{\mathrm{i}\theta} \neq 0$, then θ is assumed to be in $(-\pi, \pi]$. The implicit coefficient in $O(\cdot)$ is uniform for $\arg z$. Indeed, for $a \in (0, 4/3]$, this is immediate from Theorems 1.4.1–1.4.2 in [20]. For $a \in (4/3, 2)$, by Theorem 1.5.1 in [20],

$$E_{a,b}(z) = a^{-1} \sum_{n:|\arg z + 2\pi n| \le 3\pi a/4} (z^{1/a} e^{2\pi \mathrm{i} n/a})^{1-b} \exp(z^{1/a} e^{2\pi \mathrm{i} n/a}) - \sum_{n=1}^{m} \frac{z^{-n}}{\Gamma(b-an)} + R_m(z)$$

with $\sup_{|z|\ge 2} |z^{m+1} R_m(z)| < \infty$, again yielding the uniformity. Since $\mathrm{Re}(z^{1/a}) = |z|^\alpha \cos(\arg z/a)$, by (2) and the continuity of $E_{a,b}$, there is a constant $C > 0$ such that

$$|E_{a,b}(z)| \le \min(C, |z|^{(1-b)/a}) \quad \text{if } |\arg z| \in [a\pi/2, \min(a, 1)\pi]. \tag{3}$$

From [20], p. 333,

$$azE'_{a,b}(z) + (b-1)E_{a,b}(z) = E_{a,b-1}(z). \tag{4}$$

The paper will mostly focus on $a = b = \alpha \in (1, 2)$. By (2), as $|z| \to \infty$,

$$E_{\alpha,\alpha}(z) = \alpha^{-1} z^{1/\alpha - 1} \exp(z^{1/\alpha}) - \frac{(\alpha-1)\alpha z^{-2}}{\Gamma(2-\alpha)} + O(|z|^{-3}). \tag{5}$$

Denote $\mathscr{Z}_{a,b} = \{z \in \mathbb{C} : E_{a,b}(z) = 0\}$. From Theorems 2.1.1 and 4.2.1, and Chapter 6 in [20],

$$\mathscr{Z}_{\alpha,\alpha} \subset \{z : |\arg z| > \alpha\pi/2\}, \tag{6}$$

$E_{\alpha,\alpha}(z)$ has at least one real root, all real roots are negative, and all roots with large enough modulus are simple and can be enumerated as $\varsigma_{\pm n}$ such that

$$\varsigma_{\pm n}^{1/\alpha} = \pm 2\pi i n - (\alpha+1)\ln(\pm 2\pi i n) + \ln \frac{\alpha^2(\alpha-1)}{\Gamma(2-\alpha)} + O(\ln n/n) \quad \text{as } n \to \infty, \tag{7}$$

where the principal branch of the logarithmic function is used, i.e., $\ln z = \ln|z| + \mathrm{i} \arg z$.

2.3. First exit, passage, or hitting time

Recall that X is assumed to be a Lévy process satisfying (1). The first exit time from a Borel set A by X is defined to be $\inf\{t > 0 : X_t \notin A\}$. For $c > 0$ and $x \in \mathbb{R}$, denote

$$T_c = \inf\{t > 0 : X_t > c\}, \quad \tau_x = \inf\{t > 0 : \ X_t = x\}.$$

The definitions follow [14] and are somewhat different from those in [2], which take the infima over $t \geq 0$. However, for X considered here, they specify the same random times. Both T_c and τ_x have p.d.f.'s [1,2,19,23]. The distribution of τ_x is classical for $x < 0$ ([2], Theorem VII.1) and is known for $x > 0$ [23]. Almost surely, $T_c < \tau_c < \infty$ and $X_{T_c} > c > X_{T_c -}$ [23], and given $b > 0$, $\tau_{-b} = \inf\{t > 0 : X_t < -b\}$, so the first exit time from $[-b, c]$ is $\min(\tau_{-b}, T_c)$ ([10], Theorem 5.17). When $\tau_{-b} < T_c$ (resp. $\tau_{-b} > T_c$), X is said to exit at the lower (resp. upper) end.

We will heavily rely on the scale function $W^{(q)}$ of $-X$ given by

$$W^{(q)}(x) = x_+^{\alpha-1} E_{\alpha,\alpha}(q x_+^\alpha), \quad q \geq 0, \tag{8}$$

where $x_+ = \max(x, 0)$ ([14], p. 250). For the scale function in general, see [2,10,14]. One important fact is that given $x > 0$, $W^{(q)}(x)$ as a function of $q > 0$ can be analytically extended to the entire $\mathbb{C}$ to become $W^{(z)}(x) = \sum_{k=0}^\infty z^k W^{*(k+1)}(x)$, where W^{*n} is the n-fold convolution of the scale function $W = W^{(0)}$. For $-X$, $W^{*n}(x) = x^{n\alpha-1}$.

3. Distribution of first exit time at lower end

Given $c > 0$ and $x < c$, denote

$$k_{x,c}(t) = \mathbb{P}\{\tau_x \in \mathrm{d}t,\ T_c > \tau_x\}/\mathrm{d}t.$$

Since τ_x has a p.d.f., $k_{x,c}(t)$ is well-defined, and since its integral over t is $\mathbb{P}\{T_c > \tau_x\} < 1$, it is actually a sub-p.d.f. Given $b > 0$, let $d = b + c$. It is well-known that

$$\widetilde{k}_{-b,c}(q) = \frac{W^{(q)}(c)}{W^{(q)}(d)} = \frac{c^{\alpha-1}}{d^{\alpha-1}} \frac{E_{\alpha,\alpha}(c^\alpha q)}{E_{\alpha,\alpha}(d^\alpha q)} \tag{9}$$

([14], Theorem 8.1). The main result of this section is the following.

Proposition 1. *Given b, $c > 0$, let $d = b + c$. Then for $t > 0$,*

$$k_{-b,c}(t) = \frac{c^{\alpha-1}}{d^{\alpha-1}} \sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}} \operatorname{Res}\left(\frac{E_{\alpha,\alpha}(c^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)} e^{zt}, \frac{\varsigma}{d^\alpha}\right) = \frac{c^{\alpha-1}}{d^{2\alpha-1}} \psi_{c^\alpha d^{-\alpha}}\left(\frac{t}{d^\alpha}\right), \tag{10}$$

where for $s \in [0, 1)$,

$$\psi_s(t) = \sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}} \operatorname{Res}(H_s(z)e^{zt}, \varsigma), \quad t > 0, \tag{11}$$

is a p.d.f. concentrated on $[0, \infty)$ and

$$H_s(z) = \frac{E_{\alpha,\alpha}(sz)}{E_{\alpha,\alpha}(z)}, \quad z \in \mathbb{C}. \tag{12}$$

Furthermore, $\psi_s \in C^\infty(\mathbb{R})$ such that for all $n \ge 1$, $\psi_s^{(n)}(t) \to 0$ as $t \downarrow 0$ or $t \to \infty$.

Remark. Since $\mathbb{P}\{\tau_{-b} < T_c\} = (c/d)^{\alpha-1}$ ([2], Theorem VII.8), by (10), conditional on X exiting from $[-b, c]$ at $-b$, the scaled exit time $d^\alpha \tau_{-b}$ has p.d.f. ψ_s with $s = (c/d)^\alpha$.

The main feature of Proposition 1 is that it expresses the p.d.f. of the first exit time in terms of the roots of the Mittag-Leffler function $E_{\alpha,\alpha}(z)$. As noted earlier, this results from residual calculus for (9). However, since currently there is little precise knowledge on the roots of $E_{\alpha,\alpha}(z)$, the contour involved in the calculation has to be chosen carefully. For each term in the sum (11), if $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ has multiplicity n, then in a neighborhood of ς, $H_s(z) = g(z)(z-\varsigma)^{-n}$, where $g(z)$ is analytic. As a result, $\operatorname{Res}(H_s(z)e^{zt}, \varsigma) = (g(z)e^{zt})^{(n)}|_{z=\varsigma}/(n-1)! = \sum_{k=0}^{n-1} c_k(\varsigma)t^{n-1-k}e^{\varsigma t}$. Moreover, from Section 2.2, if $|\varsigma|$ is large enough, then ς is a simple root, giving

$$\operatorname{Res}(H_s(z)e^{zt}, \varsigma) = \frac{E_{\alpha,\alpha}(s\varsigma)e^{\varsigma t}}{E'_{\alpha,\alpha}(\varsigma)}.$$

Proposition 1 extends a result for Brownian motions. If $\alpha = 2$, then by $\mathbb{E}[e^{-qX_t}] = e^{q^2 t}$, $q > 0$, $X_t = B_{2t}$ with B_t a standard Brownian motion. By $E_{2,2}(z) = \sinh(\sqrt{z})/\sqrt{z}$, $\mathscr{Z}_{2,2} = \{-k^2\pi^2, k \in \mathbb{N}\}$ and $E'_{2,2}(z) = [\cosh(\sqrt{z}) - E_{2,2}(z)]/(2z)$. Since $E'_{2,2}(-k^2\pi^2) = (-1)^{k-1}/(2k^2\pi^2)$, each root of $E_{2,2}(z)$ is simple. Note that Proposition 1 does not cover $\alpha = 2$. Nevertheless, by $T_c = \tau_c$ and the above display with $s = (c/d)^2$, a formal application yields

$$\frac{\mathbb{P}\{\tau_{-b} \in \mathrm{d}t,\ \tau_c > \tau_{-b}\}}{\mathrm{d}t} = \frac{2\pi}{d^2} \sum_{k=1}^\infty (-1)^{k-1} k \sin\left(\frac{k\pi c}{d}\right) \exp\left\{-\frac{k^2\pi^2 t}{d^2}\right\}.$$

The series is different from the one in the classical book [5] (p. 212, 3.0.6). However, it can be proved rigorously using a heat equation method ([17], section 7.4); see for example [7].

3.1. Basic properties of scaled first exit time at lower end

This subsection proves the smoothness of ψ_s asserted at the end of Proposition 1. As a by-product, some properties of $H_s(z)$ in (12) are obtained.

Given $s \in (0, 1)$, by (10) and $s^{1-1/\alpha} = \mathbb{P}\{\tau_{s^{1/\alpha}-1} < T_{s^{1/\alpha}}\}$, $\psi_s(t)$ is the density of the (proper) probability measure

$$\mu_s(\mathrm{d}t) = \mathbb{P}\{\tau_{s^{1/\alpha}-1} \in \mathrm{d}t \mid \tau_{s^{1/\alpha}-1} < T_{s^{1/\alpha}}\}, \quad t > 0,$$

and by (9) $H_s(q)$ is the Laplace transform of μ_s, i.e., for $q \ge 0$,

$$\widetilde{\mu}_s(q) = H_s(q) = \frac{E_{\alpha,\alpha}(sq)}{E_{\alpha,\alpha}(q)}. \tag{13}$$

Eq. (13) still holds if q is changed to z with $\mathrm{Re}(z) \ge 0$. By $|\widetilde{\mu}_s(z)| \le \widetilde{\mu}_s(\mathrm{Re}(z)) \le 1$, $|E_{\alpha,\alpha}(sz)| \le |E_{\alpha,\alpha}(z)|$. Then given $|\theta| \le \pi/2$, $|E_{\alpha,\alpha}(re^{\mathrm{i}\theta})|$ is increasing in $r \ge 0$, in particular, if $\mathrm{Re}(z) \ge 0$, then $|E_{\alpha,\alpha}(z)| \ge E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$.

Fix $s \in (0,1)$. For $y \in \mathbb{R}$,

$$\widehat{\mu}_s(y) = H_s(-\mathrm{i}y) = \frac{E_{\alpha,\alpha}(-\mathrm{i}sy)}{E_{\alpha,\alpha}(-\mathrm{i}y)}. \tag{14}$$

Since $|e^{(-\mathrm{i}y)^{1/\alpha}}| = e^{\kappa|y|^{1/\alpha}}$ with $\kappa = \cos(\alpha^{-1}\pi/2) > 0$, by (5), as $|y| \to \infty$, $|E_{\alpha,\alpha}(-\mathrm{i}y)| \sim \alpha^{-1}|y|^{1/\alpha-1}e^{\kappa|y|^{1/\alpha}}$ and so $|\widehat{\mu}_s(y)| \sim s^{1/\alpha-1}e^{\kappa(s^{1/\alpha}-1)|y|^{1/\alpha}}$. As a result, $\int |\widehat{\mu}_s(y)||y|^n\,\mathrm{d}y < \infty$ for all $n \ge 0$, so μ_s has a p.d.f. in $C^\infty(\mathbb{R})$ with vanishing derivative of any order at $\pm\infty$ ([22], Proposition 28.1). By (9), the p.d.f. is exactly ψ_s in Proposition 1. Since ψ_s is supported on $[0,\infty)$, $\psi_s^{(n)}(x) \to 0$ as $x \to 0+$. ψ_s cannot be analytically extended to a neighborhood of 0, for otherwise it would be constant 0. On the other hand, from (13), μ_s has finite moment of any order with its nth moment equal to $(-1)^n H_s^{(n)}(0)$ and by Fourier inversion ([22], Proposition 2.5(xii))

$$\psi_s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\mu}_s(y)e^{-\mathrm{i}yt}\,\mathrm{d}y = \frac{1}{2\pi}\lim_{M\to\infty}\int_{-M}^{M} \widehat{\mu}_s(y)e^{-\mathrm{i}yt}\,\mathrm{d}y. \tag{15}$$

From (14) and the Continuity Theorem of characteristic functions (cf. [6], Theorem 8.28), as $s \to 0+$, μ_s weakly converges to a probability distribution μ_0 with

$$\widehat{\mu}_0(y) = H_0(-\mathrm{i}y) = \frac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(-\mathrm{i}y)}, \quad y \in \mathbb{R}.$$

Similar to μ_s with $s \in (0,1)$, μ_0 has a p.d.f. $\psi_0 \in C^\infty(-\infty,\infty)$ with support on $[0,\infty)$ such that all its derivatives $\psi_0^{(n)}(x)$ vanish as $x \to 0+$ or $x \to \infty$.

3.2. *Contour integration*

In view of last subsection, to prove Proposition 1, it only remains to show (11).

Proof of Eq. (11). Fix $s \in [0,1)$. Define function

$$\sigma(\theta) = \frac{1}{|\sin(\theta/\alpha)|}.$$

Since $\alpha > 1$, $\sigma(\theta)$ is bounded on $[-\pi,-\pi/2] \cup [\pi/2,\pi]$. Put $\sigma_0 = \sigma(\pi/2)$. For $R > 0$, let C_R be the contour that travels along the curve

$$\{[R\sigma(\theta)]^\alpha e^{\mathrm{i}\theta} : \pi/2 \le |\theta| \le \pi\} \tag{16}$$

starting from its top point $\mathrm{i}(R\sigma_0)^\alpha$ and ending at its bottom point $-\mathrm{i}(R\sigma_0)^\alpha$; C_R is smooth except at its intersection with $(-\infty,0)$, and its length is proportional to R^α. Fix $\beta \in (\pi/2, \alpha\pi/2)$. Let $C_{R,1} = C_R \cap \{z \in \mathbb{C} : \pi/2 \le |\arg z| \le \beta\}$ and $C_{R,2} = C_R \cap \{z \in \mathbb{C} : \beta \le |\arg z| \le \pi\}$. Fig. 1 shows the shapes of C_R, $C_{R,1}$, and $C_{R,2}$ as well as the relative scale of C_R with different $\alpha \in (1,2)$.

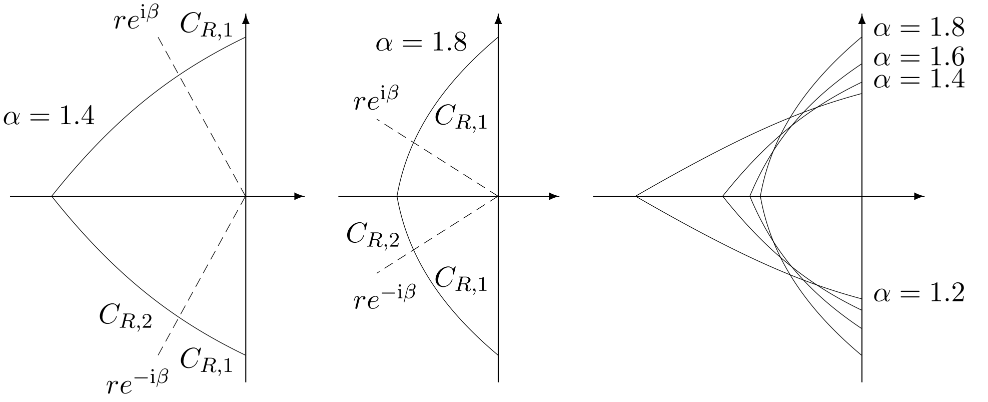

Fig. 1. C_R, $C_{R,1}$, and $C_{R,2}$ according to different $\alpha \in (1, 2)$. Left: $\alpha = 1.4$. β is any fixed value in $(\pi/2, \alpha\pi/2)$. Middle: $\alpha = 1.8$. Right: C_R defined by (16) with the same R but different α.

For $z = re^{i\theta}$, where $\theta = \arg z$, $|\exp(z^{1/\alpha})| = \exp\{r^{1/\alpha}\cos(\theta/\alpha)\}$. If $z \in C_{R,1}$, then $|\theta/\alpha| \le \beta/\alpha < \pi/2$, and so $\cos(\theta/\alpha) \ge \lambda := \cos(\beta/\alpha) > 0$. As a result, for $z \in C_{R,1}$,

$$|\exp(z^{1/\alpha})| \ge \exp(\lambda|z|^{1/\alpha}). \tag{17}$$

Then by (5), as $R \to \infty$, if $s \in (0, 1)$,

$$\begin{aligned} H_s(z) = \frac{E_{\alpha,\alpha}(sz)}{E_{\alpha,\alpha}(z)} &= (1 + o(1))\frac{(sz)^{1/\alpha-1}\exp((sz)^{1/\alpha})}{z^{1/\alpha-1}\exp(z^{1/\alpha})} \\ &= (1 + o(1))s^{1/\alpha-1}\exp\{(s^{1/\alpha} - 1)z^{1/\alpha}\}, \quad z \in C_{R,1}, \end{aligned}$$

where the $o(1)$ term converges to 0 uniformly for $z \in C_{R,1}$, and if $s = 0$,

$$H_s(z) = \frac{1}{\Gamma(\alpha)E_{\alpha,\alpha}(z)} = O(1)z^{1-1/\alpha}\exp\{-z^{1/\alpha}\}, \quad z \in C_{R,1},$$

where the implicit coefficient in $O(1)$ is uniform for $z \in C_{R,1}$. Since $|z| \ge R^\alpha$, from (17),

$$\sup_{z\in C_{R,1}} |H_s(z)| = O(\exp\{-\lambda(1 - s^{1/\alpha})R/2\}). \tag{18}$$

We also need a bound for $H_s(z) = E_{\alpha,\alpha}(sz)/E_{\alpha,\alpha}(z)$ on $C_{R,2}$. However, since $E_{\alpha,\alpha}(z)$ has infinitely many roots in $\{z \in \mathbb{C} : \beta \le |\arg z| \le \pi\}$, R cannot be an arbitrary large number. To select R appropriately, we need the following.

Lemma 2. *Let $R_n = 2\pi n$, $n = 1, 2, \ldots$. Then given any $A \in \mathbb{R} \setminus \{0\}$,*

$$\varliminf_{n\to\infty} \inf_{z\in C_{R_n,2}} |z^{1/\alpha+1}\exp(z^{1/\alpha}) - A| > 0. \tag{19}$$

Proof. For $z = [R\sigma(\theta)]^\alpha e^{i\theta} \in C_{R,2}$ with $\theta = \arg z$,

$$\begin{aligned} z^{1/\alpha+1}\exp(z^{1/\alpha}) &= [R\sigma(\theta)]^{1+\alpha}e^{i(1/\alpha+1)\theta}\exp\{R\sigma(\theta)e^{i\theta/\alpha}\} \\ &= [R\sigma(\theta)]^{1+\alpha}e^{R\sigma(\theta)\cos(\theta/\alpha)}e^{i[(1/\alpha+1)\theta+R\sigma(\theta)\sin(\theta/\alpha)]} \\ &= [R\sigma(\theta)]^{1+\alpha}e^{R\sigma(\theta)\cos(\theta/\alpha)}e^{i[(1/\alpha+1)\theta+R\operatorname{sign}(\theta)]}. \end{aligned}$$

Put $a(\theta, R) = [R\sigma(\theta)]^{1+\alpha} e^{R\sigma(\theta)\cos(\theta/\alpha)}$. Then for $z \in C_{R_n,2}$, by $R_n = 2\pi n$, $z^{1/\alpha+1}\exp(z^{1/\alpha}) = a(\theta, R_n)e^{\mathrm{i}(1/\alpha+1)\theta}$. If there were $z_n = [R\sigma(\theta_n)]^\alpha e^{\mathrm{i}\theta_n} \in C_{R_n,2}$ such that $z_n^{1/\alpha+1}\exp(z_n^{1/\alpha}) \to A$, then taking modulus, $a(\theta_n, R_n) = [R_n\sigma(\theta_n)]^{1+\alpha} e^{R_n\sigma(\theta_n)\cos(\theta_n/\alpha)} \to |A| > 0$. By $|R_n\sigma(\theta_n)| \to \infty$, it follows that $\cos(\theta_n/\alpha) \to 0$, as any sequence n with $R_n\sigma(\theta_n)\cos(\theta_n/\alpha) \to \infty$ (resp. $-\infty$) has $a(\theta_n, R_n) \to \infty$ (resp. 0). Because $|\theta_n|/\alpha \in (\pi/(2\alpha), \pi/\alpha]$, this implies $\theta_n/\alpha = k_n\pi/2 + \epsilon_n$ with $k_n = \pm 1$ and $\epsilon_n \to 0$. But then

$$z_n^{1/\alpha+1}\exp(z_n^{1/\alpha}) = a(\theta_n, R_n)e^{\mathrm{i}(1/\alpha+1)\theta_n} = |A|e^{\mathrm{i}(1+\alpha)k_n\pi/2} + o(1) \not\to A,$$

a contradiction. □

Continuing the proof of Eq. (11), let $A = \alpha^2(\alpha-1)/\Gamma(2-\alpha)$. By (5),

$$E_{\alpha,\alpha}(z) = \alpha^{-1}z^{-2}[z^{1/\alpha+1}\exp(z^{1/\alpha}) - A] + O(|z|^{-3}).$$

Then by Lemma 2, there is $\epsilon > 0$, such that for all large n and $z \in C_{R_n,2}$, $|E_{\alpha,\alpha}(z)| \ge \epsilon|z|^{-2}$. Let $m_0 = \sup_{\pi/2\le|\theta|\le\pi}\sigma(\theta)$. Then by $|z| \le (m_0R_n)^\alpha$,

$$|E_{\alpha,\alpha}(z)| \ge \epsilon m_0^{-2\alpha}R_n^{-2\alpha}. \tag{20}$$

On the other hand, since all the coefficients in the power series expansion of $E_{\alpha,\alpha}(z)$ are positive and $s \in [0, 1)$, $|E_{\alpha,\alpha}(sz)| \le E_{\alpha,\alpha}(|z|) \le E_{\alpha,\alpha}(m_0^\alpha R_n^\alpha)$. Then again by (5),

$$|E_{\alpha,\alpha}(sz)| = O(R_n^{1-\alpha}\exp(m_0R_n)).$$

Combining with the lower bound, this implies

$$\sup_{z\in C_{R_n,2}} |H_s(z)| = O(R_n^{1+\alpha}e^{m_0R_n}), \quad n \to \infty. \tag{21}$$

Let D_R be the domain bounded by C_R and $\{\mathrm{i}y : |y| \le (R\sigma_0)^\alpha\}$. Let $t > 0$. If $C_R \cap \mathscr{Z}_{\alpha,\alpha} = \emptyset$, then by (14) and residual theorem,

$$\begin{aligned}\frac{1}{2\pi}\int_{-(R\sigma_0)^\alpha}^{(R\sigma_0)^\alpha} \widehat{\mu}_s(y)e^{-\mathrm{i}yt}\,\mathrm{d}y &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}(R\sigma_0)^\alpha}^{\mathrm{i}(R\sigma_0)^\alpha} H_s(z)e^{zt}\,\mathrm{d}z \\ &= \sum_{\varsigma\in D_R\cap\mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}(H_s(z)e^{zt}, \varsigma) - \frac{1}{2\pi\mathrm{i}}\int_{C_R} H_s(z)e^{zt}\,\mathrm{d}z. \end{aligned} \tag{22}$$

Consider the contour integral along C_R. For $z = re^{\mathrm{i}\theta} \in C_R$ with $\theta = \arg z$, by $\pi/2 \le |\theta| \le \pi$, $|e^{zt}| = e^{rt\cos\theta} \le 1$. Then by (18),

$$\begin{aligned}\left|\int_{C_{R,1}} H_s(z)e^{zt}\,\mathrm{d}z\right| &\le \mathrm{Length}(C_{R,1}) \times O(e^{-\lambda(1-s^{1/\alpha})R}) \\ &= O(1)R^\alpha e^{-\lambda(1-s^{1/\alpha})R/2}.\end{aligned}$$

On the other hand, if $z \in C_{R,2}$, then by $\beta \le |\theta| \le \pi$ and $r \ge R^\alpha$, $|e^{zt}| \le e^{-b_0R^\alpha t}$, where $sb_0 = -\cos\beta > 0$. Then by (21),

$$\begin{aligned}\left|\int_{C_{R_n,2}} H_s(z)e^{zt}\,\mathrm{d}z\right| &\le \mathrm{Length}(C_{R_n,2}) \times O(R_n^{1+\alpha}e^{m_0R_n - b_0R_n^\alpha t}) \\ &= O(1)R_n^{1+2\alpha}e^{m_0R_n - b_0R_n^\alpha t}.\end{aligned} \tag{23}$$

By $\alpha > 1$, combining the above two bounds yields

$$\int_{C_{R_n}} H_s(z)e^{zt}\,\mathrm{d}z \to 0.$$

Then by the Fourier inversion (15) and (22),

$$\psi_s(t) = \lim_{n\to\infty} \sum_{\varsigma\in D_{R_n}\cap \mathscr{Z}_{\alpha,\alpha}} \operatorname{Res}(H_s(z)e^{zt}, \varsigma), \quad t > 0.$$

To complete the proof of (11), it only remains to show that the series on the r.h.s. of (11) converges absolutely. It suffices to show that for a large enough $M > 0$,

$$\sum_{|\varsigma|>M} |\operatorname{Res}(H_s(z)e^{zt}, \varsigma)| < \infty,$$

as the number of roots ς with $|\varsigma| \le M$ is finite. By Section 2.2, fix $M > 0$ so that all $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ with $|\varsigma| > M$ are simple and can be enumerated as $\varsigma_{\pm n} \asymp n^\alpha$, $n \in \mathbb{N}$. For each such ς,

$$\operatorname{Res}(H_s(z)e^{zt}, \varsigma) = \operatorname{Res}\left(\frac{E_{\alpha,\alpha}(sz)e^{zt}}{E_{\alpha,\alpha}(z)}, \varsigma\right) = \frac{E_{\alpha,\alpha}(s\varsigma)e^{\varsigma t}}{E'_{\alpha,\alpha}(\varsigma)}.$$

Put $r = |\varsigma|$ and $\theta = \arg\varsigma$. To bound the r.h.s., by $E_{\alpha,\alpha}(\varsigma) = 0$ and (5),

$$\alpha^{-1}\varsigma^{1/\alpha-1}\exp(\varsigma^{1/\alpha}) = \frac{\alpha(\alpha-1)\varsigma^{-2}}{\Gamma(2-\alpha)} + O(r^{-3}).$$

On the other hand, by (2),

$$E_{\alpha,\alpha-1}(\varsigma) = \alpha^{-1}\varsigma^{2/\alpha-1}\exp(\varsigma^{1/\alpha}) + \frac{\alpha(\alpha^2-1)\varsigma^{-2}}{\Gamma(2-\alpha)} + O(r^{-3}).$$

As a result, there is $c > 0$, such that $|E_{\alpha,\alpha-1}(\varsigma)| \ge cr^{1/\alpha-2}$, so by (4),

$$|E'_{\alpha,\alpha}(\varsigma)| = |E_{\alpha,\alpha-1}(\varsigma)/(\alpha\varsigma)| \ge (c/\alpha)r^{1/\alpha-3}. \tag{24}$$

Next, by (3) and (6), $\sup_{s\ge 0,\varsigma}|E_{\alpha,\alpha}(s\varsigma)| < \infty$ and $|e^{\varsigma t}| = e^{rt\cos\theta} \le e^{-\lambda rt}$, where $\lambda = -\cos(\alpha\pi/2) > 0$. Putting all the bounds together, there is a constant $C > 0$, such that

$$\left|E_{\alpha,\alpha}(s\varsigma)e^{\varsigma t}/E'_{\alpha,\alpha}(\varsigma)\right| \le Cr^{3-1/\alpha}e^{-\lambda rt}. \tag{25}$$

Taking the sum of (25) over $\varsigma_{\pm n}$ then yields the desired absolute convergence. □

3.3. Alternative expression, asymptotic at time zero, and approximation

Following a general heuristic applicable to Lévy processes (cf. [17], p. 217), one can get an expression of $k_{-b,c}$ analogous to one for a standard Brownian motion ([5], p. 212, 3.0.6). Denote by f_x the p.d.f. of τ_x and let $d = b + c$. Then

$$\begin{aligned} k_{-b,c} &= f_{-b} - f_c * f_{-d} + f_{-b} * f_d * f_{-d} - f_c * f_{-d} * f_d * f_{-d} + \cdots \\ &= \sum_{n=0}^{\infty} f_{-b} * (\delta - f_c * f_{-c}) * (f_d * f_{-d})^{*n}, \end{aligned} \tag{26}$$

where δ is the Dirac measure at 0 and $p^{*0} := \delta$ for any p.d.f. p. Indeed, by $k_{-b,c}(t) = f_{-b}(t) - \mathbb{P}\{\tau_{-b} \in \mathrm{d}t, T_c < \tau_{-b}\}/\mathrm{d}t = f_{-b}(t) - \mathbb{P}\{\tau_{-b} \in \mathrm{d}t, \tau_c < \tau_{-b}\}/\mathrm{d}t$ and strong Markov property,

$$k_{-b,c}(t) = f_{-b}(t) - (k_{c,-b} * f_{-d})(t), \tag{27}$$

where we have defined $k_{x,c}(t) = \mathbb{P}\{\tau_x \in \mathrm{d}t, \tau_c > \tau_x\}/\mathrm{d}t$ for all $x, c \in \mathbb{R}$. Likewise, $k_{c,-b}(t) = f_c(t) - (k_{-b,c} * f_d)(t)$. Plug the identity into (27) to get $k_{-b,c} = \varphi_0 + \varphi_1 * k_{-b,c}$, where $\varphi_0 = f_{-b} - f_c * f_{-d} = f_{-b} * (\delta - f_c * f_{-c})$ and $\varphi_1 = f_d * f_{-d}$. Then $k_{-b,c} = \sum_{n=0}^{N-1} \varphi_0 * \varphi_1^{*n} + r_N$ for each $N \in \mathbb{N}$, where $r_N = \varphi_1^{*N} * k_{-b,c}$, and (26) follows if $r_N(t) \to 0$ as $N \to \infty$. Indeed, given $q > 0$, by $\widetilde{f}_d(q)\widetilde{f}_{-d}(q) < 1$, $\widetilde{r}_N(q) = [\widetilde{f}_d(q)\widetilde{f}_{-d}(q)]^N \widetilde{k}_{-b,c}(q) \to 0$, so $\int_0^t r_N(s)\,\mathrm{d}s \le e^{qt}\widetilde{r}_N(q) \to 0$. Since φ_1 is bounded, then $r_N(t) = \varphi_1 * r_{N-1}(t) \le \sup \varphi_1 \times \int_0^t r_{N-1}(s)\,\mathrm{d}s \to 0$.

Based on (26), it is quite easy to get that as $t \downarrow 0$,

$$k_{-b,c}(t) \sim f_{-b}(t), \tag{28}$$

in particular, by Eq. (14.35) in [22], $\ln k_{-b,c}(t) \sim -Cb^{\alpha/(\alpha-1)} t^{-1/(\alpha-1)}$, where $C > 0$ is constant. First, by (27), $0 < f_{-b}(t) - k_{-b,c}(t) = (k_{c,-b} * f_{-b-c})(t) < (f_c * f_{-d})(t) = (u * f_{-b})(t)$, where $u = f_c * f_{-c}$ is a p.d.f. and we have used $k_{c,-b} < f_c$ and $f_{-d} = f_{-b} * f_{-c}$. Next,

$$(u * f_{-b})(t) = \int_0^t u(s) f_{-b}(t-s)\,\mathrm{d}s \le \sup_{s \le t} f_{-b}(s) \times \int_0^t u(s)\,\mathrm{d}s = o(1) \sup_{s\le t} f_{-b}(s), \quad t \downarrow 0.$$

Since f_{-b} is unimodal ([22], p. 416), for $0 < t \ll 1$, $\sup_{s \le t} f_{-b}(s) = f_{-b}(t)$, implying (28).

An issue that may arise concerns approximation. In practice, the series expression (10) of $k_{-b,c}(t)$ has to be approximated by a sum over $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ with $|\varsigma|$ less than a certain cut-off, and likewise, if Fourier inversion is used to evaluate $k_{-b,c}(t)$, it has to be approximated by an integral of $\widetilde{k}_{-b,c}(\mathrm{i}y)e^{\mathrm{i}yt}$ over y with $|y|$ less than a certain cut-off. How do the errors of these two approximations compare? For brevity, consider the ones for $\psi_s(t)$ with fixed $s \in [0, 1)$ and $t > 0$. From (22), it is seen that to make a fair comparison, the cut-offs in the approximations should be of the same order. Then as a first step, it is reasonable to compare

$$r_M = \sum_{|\varsigma| > M} |\mathrm{Res}(H_s(z)e^{zt}, \varsigma)| \quad \text{and} \quad \widehat{r}_M = \frac{1}{2\pi} \int_{|y|>M} |H_s(\mathrm{i}y)|\,\mathrm{d}y.$$

Corollary 3. *Fix $s \in [0, 1)$ and $t > 0$. Given any $\theta \in (1/2, 1)$, as $M \to \infty$,*

$$r_M = O(M^2 t^{-1} e^{-\theta\lambda M t}), \qquad \widehat{r}_M = O((1-s)^{-1} e^{-\theta(1-s)\lambda_0 M^{1/\alpha}}),$$

where $\lambda = -\cos(\alpha\pi/2)$, $\lambda_0 = \cos(\pi/(2\alpha))$, and the implicit coefficients in the $O(\cdot)$ terms only depend on α.

Remark. The main difference between the bounds is in the power of M in the exponents in the $O(\cdot)$ terms. Since $\alpha > 1$, as $M \to \infty$, the bound on r_M vanishes much faster. However, while it is free of s, for t close to 0, it is small only when M is large. In contrast, the bound on $\widehat{r}_M$ is free of t, however, for s close to 1, it is small only when M is large. Meanwhile, the approximation based on residuals requires the calculation of many ς, so it may actually have much higher computational complexity than the Fourier inversion.

Proof of Corollary 3. The bound on r_M results from a minor refinement of the last part of the proof of Proposition 1. For $M \gg 1$, all the roots $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ with $|\varsigma| > M$ are simple and can be enumerated as $\varsigma_{\pm n} = (2\pi \mathrm{i} n)^\alpha [1 + o(1)]$. Then by (25), letting $\theta' = \theta^{1/2}$,

$$\begin{aligned} r_M &= O(1) \sum_{(2\pi n)^\alpha > \theta' M} n^{3\alpha-1} e^{-\theta'\lambda(2\pi n)^\alpha t} = O(1) \int_{(\theta' M)^{1/\alpha}}^\infty u^{3\alpha-1} e^{-\theta'\lambda u^\alpha t}\,\mathrm{d}u \\ &= O(1) \int_{\theta' M}^\infty y^2 e^{-\theta'\lambda y t}\,\mathrm{d}y = O(M^2 t^{-1} e^{-\theta\lambda M t}). \end{aligned}$$

On the other hand, since $\mathrm{Re}((\mathrm{i}y)^{1/\alpha}) = \lambda_0 y^{1/\alpha}$, by (5), $H_s(\mathrm{i}y) = O(e^{-(1-s)(\mathrm{i}y)^{1/\alpha}[1+o(1)]})$, so

$$\begin{aligned}\widehat{r}_M &= O(1)\int_M^\infty e^{-\theta(1-s)\lambda_0 y^{1/\alpha}}\,\mathrm{d}y\\ &= O(1)\int_{M^{1/\alpha}}^\infty u^{\alpha-1}e^{-\theta(1-s)\lambda_0 u}\,\mathrm{d}u = O((1-s)^{-1}e^{-\theta(1-s)\lambda_0 M^{1/\alpha}}).\end{aligned}$$

Throughout, the implicit coefficients in the $O(\cdot)$ terms only depend on α. □

4. Distribution of first exit at upper end

The main result of this section is Theorem 4. It provides a factorization of the joint sub-p.d.f. of the time T_c, undershoot X_{T_c-}, and jump $\Delta_{T_c} = X_{T_c} - X_{T_c-}$ when X makes its first exit from $[-b, c]$ by jumping across c. For $x \in [-b, c]$ and $t > 0$, define

$$\begin{aligned}&l_{x,-b,c}(t) = \mathbb{P}\{X_t \in \mathrm{d}x,\ X_s \in [-b, c]\,\forall s \le t\}/\mathrm{d}x,\\ &L_{-b,c}(t) = \mathbb{P}\{X_s \in [-b, c]\,\forall s \le t\}.\end{aligned}$$

While the functions can be defined for any process that has a p.d.f. at any time point, in the case of a spectrally one-sided strictly stable process, they have explicit representations.

To start with, letting $d = b + c$, it is known that for $q \ge 0$ ([14], Theorem 8.7),

$$\begin{aligned}\widetilde{l}_{x,-b,c}(q) &= \frac{W^{(q)}(c)W^{(q)}(b+x)}{W^{(q)}(d)} - W^{(q)}(x_+)\\ &= \frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}}\frac{E_{\alpha,\alpha}(c^\alpha q)E_{\alpha,\alpha}((b+x)^\alpha q)}{E_{\alpha,\alpha}(d^\alpha q)} - x_+^{\alpha-1}E_{\alpha,\alpha}(x_+^\alpha q).\end{aligned} \tag{29}$$

Theorem 4. *Fix $b > 0$ and $c > 0$. Let $d = b + c$. Then for $x \in \mathbb{R}$,*

$$\mathbb{P}\{T_c < \tau_{-b},\ X_{T_c-} \in \mathrm{d}x\} = \frac{\mathbf{1}\{x \in (-b, c)\}}{(c-x)^\alpha}\cdot\frac{|\sin(\alpha\pi)|}{\pi}\left[\frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}} - x_+^{\alpha-1}\right]\mathrm{d}x, \tag{30}$$

and for $x \in (-b, c)$, conditional on $\{T_c < \tau_{-b}\} \cap \{X_{T_c-} = x\}$, Δ_{T_c} and T_c are independent, such that Δ_{T_c} has the Pareto p.d.f. $\pi(u) = \alpha(c-x)^\alpha u^{-\alpha-1}\mathbf{1}\{u > c - x\}$ and T_c has p.d.f.

$$p(t) = \Gamma(\alpha)\left[\frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}} - x_+^{\alpha-1}\right]^{-1} l_{x,-b,c}(t) \tag{31}$$

with

$$l_{x,-b,c}(t) = \frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}}\sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}}\mathrm{Res}\left(\frac{E_{\alpha,\alpha}(c^\alpha z)E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)}e^{zt}, \frac{\varsigma}{d^\alpha}\right). \tag{32}$$

Finally, given $t > 0$, the mapping $x \mapsto l_{x,-b,c}(t)$ can be analytically extended to $\mathbb{C}\setminus(-\infty, -b]$.

Corollary 5. *Under the same setting as above,*

$$L_{-b,c}(t) = (c/d)^{\alpha-1}\sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}}\mathrm{Res}\left(\frac{E_{\alpha,\alpha}((c/d)^\alpha z)E_{\alpha,\alpha+1}(z)}{E_{\alpha,\alpha}(z)}e^{zt/d^\alpha}, \varsigma\right), \tag{33}$$

while for z with $\mathrm{Re}(z) \ge 0$

$$\widetilde{L}_{-b,c}(z) = \frac{dc^{\alpha-1}E_{\alpha,\alpha}(c^\alpha z)E_{\alpha,\alpha+1}(d^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)} - c^\alpha E_{\alpha,\alpha+1}(c^\alpha z). \tag{34}$$

In practice $l_{x,-b,c}(t)$ and $L_{-b,c}(t)$ can be evaluated by Fourier inversion, and similar to $k_{-b,c}(t)$, the issue of approximation error due to finite cut-off may arise either for the Fourier inversion or for the residual-based expressions (32)–(33). Bounds on the approximation errors can be obtained similarly as in Corollary 3. For brevity, we omit a detailed discussion on this.

4.1. Factorization and conditional independence

The factorization in Theorem 4 follows from the next result.

Lemma 6. *Denote by $\Pi(\mathrm{d}u)$ the Lévy measure of X. Then given $b, c > 0$, for $t > 0$ and $x \in \mathbb{R}$,*

$$\begin{aligned}&\mathbb{P}\{T_c < \tau_{-b},\ T_c \in \mathrm{d}t,\ X_{T_c-} \in \mathrm{d}x,\ \Delta_{T_c} \in \mathrm{d}u\}\\ &\quad = \mathbf{1}\{x > -b,\ u > c - x > 0\}\,\mathrm{d}t\,\mathbb{P}\{X_t \in \mathrm{d}x,\ X_s \in [-b, c]\,\forall s \le t\}\,\Pi(\mathrm{d}u).\end{aligned}\tag{35}$$

Proof. The proof follows the one on p. 76 of [2]. As noted earlier, $\mathbb{P}\{X_{T_c} > c\} = 1$. If $T_c < \tau_{-b}$, then for all $s < X_{T_c}$, $X_s \in [-b, c]$ and so $X_{T_c-} \ge -b$. Therefore, almost surely, for any bounded function $r(t, x, u) \ge 0$,

$$\begin{aligned}&r(T_c,\ X_{T_c-},\ \Delta_{T_c})\mathbf{1}\{T_c < \tau_{-b}\}\\ &\quad = \sum_t r(t,\ X_{t-},\ \Delta_t)\mathbf{1}\{\Delta_t > c - X_{t-} > 0,\ X_{t-} \ge -b,\ X_s \in [-b, c]\,\forall s < t\}.\end{aligned}$$

The sum is well-defined as it runs over the set of t's where X has a jump, which is countable. The rest of the proof then applies the compensation formula to get the expectation of the sum as an integral of $r(t, x, u)$ with respect to the measure on the r.h.s. of (35). Since the argument has become standard, it is omitted for brevity. □

Proof of Theorem 4, part one. The Lévy measure of X is $\mathbf{1}\{x > 0\}\,x^{-\alpha-1}\mathrm{d}x/\Gamma(-\alpha)$. By Lemma 6,

$$\begin{aligned}&\mathbb{P}\{T_c < \tau_{-b},\ T_c \in \mathrm{d}t,\ X_{T_c-} \in \mathrm{d}x,\ \Delta_{T_c} \in \mathrm{d}u\}\\ &\quad = \mathbf{1}\{c > x > -b\}\,l_{-x,b,c}(t)\,\mathrm{d}t\,\mathrm{d}x\,\frac{\mathbf{1}\{u > c - x\}\,\alpha(\alpha-1)\mathrm{d}u}{\Gamma(2-\alpha)u^{\alpha+1}}.\end{aligned}\tag{36}$$

Letting $q = 0$ in (29) gives

$$\int_0^\infty l_{x,-b,c}(t)\,\mathrm{d}t = \widetilde{l}_{x,-b,c}(0) = \frac{1}{\Gamma(\alpha)}\left[\frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}} - x_+^{\alpha-1}\right].\tag{37}$$

Then by (36), for $x \in (-b, c)$,

$$\begin{aligned}\mathbb{P}\{T_c < \tau_{-b},\ X_{T_c-} \in \mathrm{d}x\} &= \mathrm{d}x\int_0^\infty l_{x,-b,c}(t)\,\mathrm{d}t\int_{c-x}^\infty \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\frac{\mathrm{d}u}{u^{\alpha+1}}\\ &= \left[\frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}} - x_+^{\alpha-1}\right]\frac{(\alpha-1)}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{\mathrm{d}x}{(c-x)^\alpha}.\end{aligned}$$

By Lemma 6, $\mathbb{P}\{X_{T_c-} \in (-b, c)\} = 1$. Then (30) follows. Next, given $x \in (-b, c)$, by (36),

$$\mathbb{P}\{T_c \in \mathrm{d}t,\ \Delta_{T_c} \in \mathrm{d}u \mid T_c < \tau_{-b},\ X_{T_c-} = x\} = Cl_{-x,b,c}(t)\,\mathrm{d}t \times \frac{\mathbf{1}\{u > c - x\}\,\mathrm{d}u}{u^{\alpha+1}},$$

for some constant $C = C(x)$. It follows that conditional on $T_c < \tau_{-b}$ and $X_{T_c -} = x$, T_c and Δ_{T_c} are independent, with Δ_{T_c} following a Pareto distribution and T_c having a p.d.f. in proportion to $l_{-x,b,c}(t)$. By normalizing $l_{-x,b,c}(t)$ by (37), (32) follows. The main step of the proof is to derive the expression of $l_{x,-b,c}(t)$ for given $x \in (-b, c)$, which will be dealt with in the next subsection. □

4.2. Contour integration

Define

$$h_{x,c}(t) = \mathbb{P}\{X_t \in \mathrm{d}x,\ X_s \le c\, \forall s \le t\}/\mathrm{d}x. \tag{38}$$

In [8], $h_{x,c}(t)$ plays a critical role in deriving the distribution of the triple $(T_c, X_{T_c -}, X_{T_c})$, known as Gerber–Shiu distribution ([14], Chapter 10).

Lemma 7. *Fix $b > 0$ and $c > 0$.*

(a) Given $x \in (-b, c)$, $l_{x,-b,c}(t) = h_{x,c}(t) - (k_{-b,c} * h_{b+x,b+c})(t)$, where all the functions involved are treated as functions of t.
(b) Given t, the mapping $x \mapsto l_{x,-b,c}(t)$ is continuous on $(-b, c)$.

Proof. (a) Let $r(x) \ge 0$ be a function with support in $[-b, c]$. Then for any $t > 0$,

$$\mathbb{E}[r(X_t)\mathbf{1}\{X_s \in [-b, c]\, \forall s \le t\}] = \int_{-b}^{c} r(x) l_{x,-b,c}(t)\, \mathrm{d}x.$$

On the other hand, the l.h.s. can be decomposed as the difference of two expectations

$$\mathbb{E}[r(X_t)\mathbf{1}\{X_s \le c\, \forall s \le t\}] - \mathbb{E}[r(X_t)\mathbf{1}\{\tau_{-b} \le t,\ X_s \le c\, \forall s \le t\}].$$

The first expectation is equal to $\int_{-b}^{c} r(x) h_{x,c}(t)\, \mathrm{d}x$. By the strong Markov property of X, the second expectation is equal to

$$\begin{aligned}
&\int_{x=-b}^{c} \int_{u=0}^{t} r(x)\mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_{-b} \in \mathrm{d}u,\ X_s \le c\, \forall s \le t\} \\
&= \int_{x=-b}^{c} \int_{u=0}^{t} r(x)\mathbb{P}\{X_t \in \mathrm{d}x + b,\ X_s \le b + c\, \forall s \le t - u\}\mathbb{P}\{\tau_{-b} \in \mathrm{d}u,\ X_s \le c\, \forall s \le u\} \\
&= \int_{-b}^{c} r(x) \left[\int_0^t h_{x+b,b+c}(t-u) k_{-b,c}(u)\, \mathrm{d}u \right] \mathrm{d}x.
\end{aligned}$$

Comparing the integrals and by $r(x) \ge 0$ being arbitrary, the claimed identity follows.

(b) Given $b, c > 0$, from [8], the mapping $(x, t) \mapsto h_{x,c}(t)$ is continuous on $(-\infty, c) \times [0, \infty)$, while from Section 3, the mapping $t \mapsto k_{-b,c}(t)$ is continuous on $[0, \infty)$. Then given $t > 0$, by dominated convergence, the mapping $x \mapsto (k_{-b,c} * h_{b+x,b+c})(t) = \int_0^t k_{-b,c}(s) h_{b+x,b+c}(t-s)\, \mathrm{d}s$ is continuous on $(-\infty, c)$. By (a), the proof is complete. Note that $l_{x,-b,c}(t)$ is only defined for $x \in (-b, c)$, but the proof implies that it can be continuously extended to $x \le -b$. □

Proof of Theorem 4, part two. Fix b, $c > 0$. For $x \in (-b, c)$, put

$$F_x(z) = \frac{c^{\alpha-1}(b+x)^{\alpha-1}}{d^{\alpha-1}} \frac{E_{\alpha,\alpha}(c^\alpha z) E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)}.$$

First, suppose $x \neq 0$. From (37), $l_{x,-b,c}(t)$ is an integrable function of t. Therefore, to show (32), it suffices to show that $\widehat{l}_{x,-b,c}(y) = \widetilde{l}_{x,-b,c}(-\mathrm{i}y)$ is an integrable function of y and that

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} [F_x(z) - x_+^{\alpha-1} E_{\alpha,\alpha}(x_+^\alpha z)] e^{zt} \,\mathrm{d}z = \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}(F_x(z) e^{zt}, \varsigma/d^\alpha).$$

Let $s = c/d$, $v = (b+x)/d$, and $w = x/d = s+v-1$. By the same contour integral argument in the proof of Proposition 1, if we make change of variables $z' = d^\alpha z$ and $t' = t/d^\alpha$ and let

$$G(z) = G_0(z) - w_+^{\alpha-1} E_{\alpha,\alpha}(w_+^\alpha z),$$

where $G_0(z) = (sv)^{\alpha-1} E_{\alpha,\alpha}(s^\alpha z) E_{\alpha,\alpha}(v^\alpha z)/E_{\alpha,\alpha}(z)$, then it boils down to showing that

$$\int_{-\infty}^{\infty} |G(\mathrm{i}y)| \,\mathrm{d}y < \infty \tag{39}$$

and given any $t > 0$,

$$\int_{C_{R_n}} G(z) e^{zt} \,\mathrm{d}z \to 0, \quad n \to \infty, \tag{40}$$

where the contour C_R and the numbers R_n are defined in the proof of Proposition 1.

Fix $\beta \in (\pi/2, \alpha\pi/2)$. Then $\lambda := \cos(\beta/\alpha) > 0$. Put $\Omega = \{z : |\arg z| \in [\pi/2, \beta]\}$. For $z \in \Omega$, $\mathrm{Re}(z^{1/\alpha}) = |z|^{1/\alpha} \cos(\arg z/\alpha) \geq |z|^{1/\alpha}\lambda$. Then by (5), given $c > 0$, for $z \in \Omega$, as $|z| \to \infty$,

$$c^{\alpha-1} E_{\alpha,\alpha}(c^\alpha z) = \alpha^{-1} z^{1/\alpha-1} \exp(c z^{1/\alpha})[1 + r_c(z)], \tag{41}$$

where $r_c(z) = O(z^{-1-1/\alpha} e^{-cz^{1/\alpha}}) = o(1)$, with the implicit coefficient in $O(\cdot)$ being uniform for $\arg z$. By (41) and $s + v - 1 = w$,

$$\begin{aligned} G_0(z) &= \alpha^{-1} z^{1/\alpha-1} \exp(w z^{1/\alpha})\{1 + [r_s(z) + r_v(z) - r_1(z)][1 + o(1)]\} \\ &= \alpha^{-1} z^{1/\alpha-1} \exp(w z^{1/\alpha})[1 + O(z^{-1-1/\alpha} e^{-\min(s,v) z^{1/\alpha}})]. \end{aligned} \tag{42}$$

If $w < 0$, then $G(z) = G_0(z)$. Letting $z = \mathrm{i}y$ with $y \in \mathbb{R}$ in (42), (39) follows. If $w > 0$, then applying (41) to $c = w_+ = w$ combined with (42) and $0 < w < \min(s, v)$ yields

$$G(z) = O(z^{-2}). \tag{43}$$

Letting $z = \mathrm{i}y$ with $y \in \mathbb{R}$ in (43), (39) again follows. To show (40) for $w \neq 0$, as in the proof of Proposition 1, let $C_{R,1} = C_R \cap \Omega$ and $C_{R,2} = C_R \setminus C_{R,1}$. By (42) and (43),

$$\sup_{z \in C_{R,1}} |G(z)| = \begin{cases} O(R^{1-\alpha} e^{w\lambda R}) & \text{if } w < 0, \\ O(R^{-2\alpha}) & \text{if } w > 0. \end{cases}$$

Meanwhile, $|e^{zt}| \leq 1$ for $z \in C_{R,1}$ and $\mathrm{Length}(C_{R,1}) = O(R^\alpha)$. Then as $R \to \infty$,

$$\int_{C_{R,1}} G(z) e^{zt} \,\mathrm{d}z = \begin{cases} O(R e^{w\lambda R}) & \text{if } w < 0, \\ O(R^{-\alpha}) & \text{if } w > 0. \end{cases}$$

Therefore, if $w \neq 0$, then $\int_{C_{R,1}} G(z) e^{zt} \,\mathrm{d}z \to 0$ as $n \to \infty$. On the other hand, following the derivation of (21), for some $m_0 > 0$,

$$\begin{aligned} \sup_{z \in C_{R_n,2}} \left| \frac{E_{\alpha,\alpha}(s^\alpha z) E_{\alpha,\alpha}(v^\alpha z)}{E_{\alpha,\alpha}(z)} \right| &\leq \sup_{z \in C_{R_n,2}} |H_{s^\alpha}(z)| \sup_{z \in C_{R_n,2}} |E_{\alpha,\alpha}(v^\alpha z)| \\ &= O(R_n^{1+\alpha} e^{m_0 R_n}) \cdot O(R_n^{1-\alpha} e^{m_0 R_n}) = O(R_n^2 e^{2m_0 R_n}), \end{aligned}$$

and, since $w_+ < 1$, $|w_+^{\alpha-1} E_{\alpha,\alpha}(w_+^\alpha z)| \le E_{\alpha,\alpha}(|z|) = O(R_n^{1-\alpha} e^{m_0 R_n})$. Meanwhile, from the derivation of (23), for some $b_0 > 0$, $|e^{zt}| \le e^{-b_0 R^\alpha t}$ for $z \in C_{R,2}$. Then by $\alpha > 1$ and $t > 0$,

$$\int_{C_{R_n,2}} G(z) e^{zt}\,\mathrm{d}z = O(R_n^{2+\alpha} e^{2m_0 R_n - b_0 R_n^\alpha t}) \to 0, \quad n \to \infty.$$

The desired convergence in (40) then follows and hence (32) is proved in the case $x \ne 0$.

It only remains to show that for given $t > 0$, $R(x) := \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} \operatorname{Res}(F_x(z)e^{zt}, \varsigma/d^\alpha)$ has an analytic extension to $D := \mathbb{C}\setminus(-\infty, -b]$. Once this is done, since by Lemma 7, $x \mapsto l_{x,-b,c}(t)$ is continuous at 0 and since it was just shown that the two functions are equal on $(-b, c)\setminus\{0\}$, they must be equal at 0 and can be analytically extended to D, finishing the proof.

For each $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$, by change of variable in the contour integral representation of residual and dominated convergence,

$$\begin{aligned} w_\varsigma(x) &:= \operatorname{Res}\left(\frac{E_{\alpha,\alpha}(c^\alpha z) E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)} e^{zt}, \frac{\varsigma}{d^\alpha}\right) \\ &= \frac{d^{-\alpha}}{2\pi\mathrm{i}} \sum_{n=0}^\infty \frac{((b+x)/d)^{n\alpha}}{\Gamma(n\alpha+\alpha)} \oint_{\gamma_\varsigma} \frac{E_{\alpha,\alpha}(s^\alpha z) z^n}{E_{\alpha,\alpha}(z)} e^{zt'}\,\mathrm{d}z, \end{aligned} \tag{44}$$

where $s = c/d$, $t' = t/d^\alpha$, and $\gamma_\varsigma \subset \mathbb{C} \setminus \mathscr{Z}_{\alpha,\alpha}$ is a counterclockwise circle that encloses ς but no other root in $\mathscr{Z}_{\alpha,\alpha}$. It is then not hard to see that w_ς has an analytic extension from $(-b, c)$ to D. All $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ with large enough modulus are simple roots of $E_{\alpha,\alpha}(z)$ and have $|\arg\varsigma|$ arbitrarily close but strictly greater than $\alpha\pi/2$. For each such ς and each $z \in D$,

$$w_\varsigma(z) = \frac{E_{\alpha,\alpha}(s^\alpha \varsigma) e^{\varsigma t'}}{d^\alpha E'_{\alpha,\alpha}(\varsigma)} \times E_{\alpha,\alpha}(v^\alpha \varsigma),$$

where $v = (b+z)/d$. By (25), for a constant $C > 0$, $\left|E_{\alpha,\alpha}(s^\alpha\varsigma) e^{\varsigma t'}/d^\alpha E'_{\alpha,\alpha}(\varsigma)\right| = O(1) e^{-C|\varsigma|}$. On the other hand, by (5), there is a constant $C' > 0$ such that

$$|E_{\alpha,\alpha}(v^\alpha\varsigma)| \le E_{\alpha,\alpha}((|v|+1)^\alpha|\varsigma|) = O(1)\exp\{C'(|v|+1)|\varsigma|^{1/\alpha}\}. \tag{45}$$

Together, these two bounds imply that for *all* $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$,

$$|w_\varsigma(z)| = O(1) e^{-C|\varsigma| + C'(|b+z|/d+1)|\varsigma|^{1/\alpha}} \tag{46}$$

and hence $\sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} |w_\varsigma(z)|$ converges uniformly in any compact subset of D. As a result, $R(x) = [c^{\alpha-1}(b+x)^{\alpha-1}/d^{\alpha-1}] \sum_{\varsigma \in E_{\alpha,\alpha}} w_\varsigma(x)$ can be extended to a continuous function $R(z)$ on D, and by dominated convergence, the integral of $R(z)$ along any simple closed contour in D is 0. Then by Morera's theorem ([21], p. 208), $R(z)$ is analytic in D. □

Proof of Corollary 5. Define $w_\varsigma(x)$ by (44). From (46), for all $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ and $x \in [-b, c]$, $|w_\varsigma(x)| = O(1) e^{-c'|\varsigma| + 2C|\varsigma|^{1/\alpha}}$. Then by dominated convergence and Fubini's theorem,

$$\begin{aligned} L_{-b,c}(t) &= \int_{-b}^c l_{x,-b,c}(t)\,\mathrm{d}x = \sum_{\varsigma \in E_{\alpha,\alpha}} \int_{-b}^c [c^{\alpha-1}(b+x)^{\alpha-1}/d^{\alpha-1}] w_\varsigma(x)\,\mathrm{d}x \\ &= \frac{c^{\alpha-1}}{d^{\alpha-1}} \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} \int_{-b}^c (b+x)^{\alpha-1} \left(\oint_{d^{-\alpha}\gamma_\varsigma} \frac{E_{\alpha,\alpha}(c^\alpha z) E_{\alpha,\alpha}((b+x)^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)} e^{zt}\,\mathrm{d}z\right) \mathrm{d}x \\ &= \frac{c^{\alpha-1}}{d^{\alpha-1}} \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} \oint_{d^{-\alpha}\gamma_\varsigma} \left[\int_0^d y^{\alpha-1} E_{\alpha,\alpha}(y^\alpha z)\,\mathrm{d}x\right] \frac{E_{\alpha,\alpha}(c^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)} e^{zt}\,\mathrm{d}z. \end{aligned}$$

By using the series expression of $E_{\alpha,\alpha}$ and integration term-by-term,

$$\int_0^d y^{\alpha-1} E_{\alpha,\alpha}(y^\alpha z)\,\mathrm{d}y = d^\alpha E_{\alpha,\alpha+1}(d^\alpha z). \tag{47}$$

Then

$$L_{-b,c}(t) = \frac{c^{\alpha-1}}{d^{\alpha-1}} \sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}} \oint_{d^{-\alpha}\gamma_\varsigma} d^\alpha E_{\alpha,\alpha+1}(d^\alpha z)\frac{E_{\alpha,\alpha}(c^\alpha z)}{E_{\alpha,\alpha}(d^\alpha z)} e^{zt}\,\mathrm{d}z,$$

yielding (33) by a simple change of variable. Finally, by Fubini's theorem, for z with $\mathrm{Re}(z) \ge 0$, $\widetilde{L}_{-b,c}(z) = \int_{-b}^c \widetilde{l}_{x,-b,c}(z)\,\mathrm{d}x$. Then by plugging in (29) and applying (47), (34) follows. □

4.3. Asymptotic near time zero

Denote by g_t the p.d.f. of X_t and by f_x the p.d.f. of τ_x.

Proposition 8. *Given $b > 0$, $c > 0$, and $x \in (-b, c)$, as $t \downarrow 0$, $l_{x,-b,c}(t) \sim g_t(x)$.*

Proof. It is clear that $l_{x,-b,c}(t) < g_t(x)$. On the other hand,

$$g_t(x) - l_{x,-b,c}(t) \le \mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_{-b} < t\}/\mathrm{d}x + \mathbb{P}\{X_t \in \mathrm{d}x,\ T_c < t\}/\mathrm{d}x.$$

By the continuity of X's downward movement and time reversal, $\mathbb{P}\{X_t \in \mathrm{d}x,\ T_c < t\} = \mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_c < t\} = \mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_{-(c-x)} < t\}$. Then

$$g_t(x) - l_{x,-b,c}(t) \le \mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_{-b} < t\}/\mathrm{d}x + \mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_{-(c-x)} < t\}/\mathrm{d}x.$$

Since both b and $c - x$ are greater than $(-x)_+$, it suffices to show that for any $\theta > (-x)_+$, $j(t) := \mathbb{P}\{X_t \in \mathrm{d}x,\ \tau_{-\theta} < t\}/\mathrm{d}x = o(g_t(x))$ as $t \downarrow 0$. Given y, as $t \downarrow 0$,

$$g_t(y) = \begin{cases} t^{-1/\alpha} g_1(t^{-1/\alpha} y) \asymp t^{-1/\alpha}(t^{-1/\alpha} y)^{-\alpha-1} \asymp t & \text{if } y > 0, \\ t^{-1/\alpha} g_1(0) \asymp t^{-1/\alpha} & \text{if } y = 0, \\ t f_y(t)/|y| & \text{if } y < 0, \end{cases}$$

(cf. [2], Corollary VII.3). Since $\theta + x > 0$, by strong Markov property and $g_t(\theta + x) = O(t)$, $j(t) = \int_0^t f_{-\theta}(t-s) g_s(\theta + x)\,\mathrm{d}s = O(t^2)\sup_{s\le t} f_{-\theta}(s)$. Since $f_{-\theta}$ is unimodal ([22], p. 416), $j(t) = O(t^2) f_{-\theta}(t)$. By Eq. (14.35) in [22], $\ln f_{-\theta}(t) \sim -C\theta^{\alpha/(\alpha-1)} t^{-1/(\alpha-1)}$, where $C > 0$ is a constant. If $x \ge 0$, then by $t = O(g_t(x))$, $f_{-\theta}(t) = o(g_t(x))$. If $x < 0$, as $|x| < \theta$, $f_{-\theta}(t) = o(f_{-|x|}(t)) = o(f_x(t))$. In either case, $j(t) = o(g_t(x))$. □

5. Applications

From Section 2.2, if $-\varrho$ is the largest real root of $E_{\alpha,\alpha}(z)$, then $\varrho > 0$. It is known that $-\varrho$ is a simple root of $E_{\alpha,\alpha}(z)$ and as $t \to \infty$, $\mathbb{P}\{X_s \in [-b, c]\ \forall s \le t\} \sim C e^{-\varrho t/d^\alpha}$ for some $C > 0$, where $d = b + c$. This directly follows from Theorem 2 of [4], which considers the exponential decay of a general spectrally negative Lévy process killed at the exit from a bounded interval. As applications of the results in last sections, several refined results on ϱ and the spectrum of the seminar group of X killed at the exit from $[-b, c]$ will be obtained.

From [4], any $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ has $\mathrm{Re}(\varsigma) \le -\varrho$. By combining Theorem 2 of [4] and Theorem 4 of the paper, this can be strengthened as follows.

Proposition 9. *For any $\varsigma \in E_{\alpha,\alpha} \setminus \{-\varrho\}$, $\mathrm{Re}(\varsigma) < -\varrho$.*

The following corollary refines the results in [3,4] on the tail behavior of the first exit time.

Corollary 10. *Fix $b, c > 0$ and let $d = b + c$. Then as $t \to \infty$,*

$$\mathbb{P}\{\tau_{-b} \in \mathrm{d}t,\ \tau_{-b} < T_c\}/\mathrm{d}t \sim \kappa e^{-\varrho t/d^\alpha},$$

$$\mathbb{P}\{T_c \in \mathrm{d}t,\ \tau_{-b} > T_c\}/\mathrm{d}t \sim [\varrho E_{\alpha,\alpha+1}(-\varrho) - 1]\kappa e^{-\varrho t/d^\alpha},$$

where $\kappa = (c^{\alpha-1}/d^{2\alpha-1})E_{\alpha,\alpha}(-c^\alpha \varrho/d^\alpha)/E'_{\alpha,\alpha}(-\varrho)$.

From the second asymptotic in Corollary 10, $\varrho E_{\alpha,\alpha+1}(-\varrho) \ge 1$. The strict inequality is likely to be true, however, it is unclear how to prove it by the approach of the paper.

According to (13), $H_s(z) = E_{\alpha,\alpha}(sz)/E_{\alpha,\alpha}(z)$ is the Laplace transform of a probability distribution. Since $E_{\alpha,\alpha}(z) \ne 0$ if $\mathrm{Re}(z) > -\varrho$, ϱ is the radius of convergence of the power series expansion of $H_s(z)$ around 0. A refined characterization of ϱ along this line is as follows.

Corollary 11. *As $k \to \infty$, $H_0^{(k)}(0) \sim (-1)^k \varrho^{-k-1} k!/[\Gamma(\alpha)E'_{\alpha,\alpha}(-\varrho)]$. In particular, ϱ is the limit of both $-kH_0^{(k-1)}(0)/H_0^{(k)}(0)$ and $(k/e)|H_0^{(k)}(0)|^{-1/k}$.*

Consider the distribution of X given that it has stayed in an interval for a long time. In this context, it is convenient and without loss of generality to let the interval be (0, 1). Denote

$$\mathcal{A}_t = \{X_s \in [0, 1]\, \forall s \le t\}.$$

Since much of the discussion can be done for a general spectrally positive Lévy process, we will often denote the scale function by $W^{(z)}(x)$ instead of $x^{\alpha-1}E_{\alpha,\alpha}(x^\alpha z)$, and use $[W^{(z)}(x)]'$, $[W^{(z)}(x)]''$, …, $[W^{(z)}(x)]^{(k)}$ to denote derivatives in z with x being fixed.

Corollary 12. *Fix any $x_0 \in (0, 1)$. Let Y^B be the process of X conditional on $\mathcal{A}_B$ and $X_0 = x_0$. Then as $B \to \infty$, Y^B converges in finite dimensional distribution to a Markov process Y with*

$$\frac{\mathbb{P}\{Y_{s+t} \in \mathrm{d}y \mid Y_s = x\}}{\mathrm{d}y} = \frac{W^{(-\varrho)}(1-y)}{W^{(-\varrho)}(1-x)} \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}\left(\frac{W^{(z)}(1-x)W^{(z)}(y)}{W^{(z)}(1)} e^{(z+\varrho)t}, \varsigma\right).$$

Theorem 3.1 of [15] furnishes the resolvent density of the transition kernel for a general spectrally one-sided Lévy process. It also obtains the p.d.f. of the corresponding stationary distribution as $W^{(-\varrho)}(x)W^{(-\varrho)}(1-x)/[W^{(-\varrho)}(1)]'$, which also easily follows from Corollary 12 in the case of X.

Finally, consider the semigroup of X killed at the exit from (0, 1). Denote by $\mathbb{P}^x$ the law of X when $X_0 = x$ and $\mathbb{E}^x$ the expectation under $\mathbb{P}^x$. By (32) and (33),

$$\frac{\mathbb{P}^x\{X_t \in \mathrm{d}y,\ \mathcal{A}_t\}}{\mathrm{d}y} = r_{t,\varsigma}(x, y) := \sum_{\varsigma \in \mathscr{Z}_{\alpha,\alpha}} a_{t,\varsigma}(x, y),$$

where given $t > 0$, x and y, the series is absolutely convergent and for $t \ge 0$,

$$a_{t,\varsigma}(x, y) = \mathrm{Res}\left(\frac{W^{(z)}(1-x)W^{(z)}(y)e^{zt}}{W^{(z)}(1)}, \varsigma\right).$$

The semigroup associated with X killed at the exit from [0, 1] is $(Z_t)_{t\ge0}$, where

$$Z_t f(x) = \mathbb{E}^x[f(X_t)\mathbf{1}\{\mathcal{A}_t\}]$$

for $t > 0$ and $Z_0 f(x) = f(x)$. Furthermore, for $\varsigma \in E_{\alpha,\alpha}$ and $t \ge 0$, define operator

$$Z_{t,\varsigma} f(x) = \int_0^1 a_{t,\varsigma}(x, y) f(y)\, dy.$$

Denote by C_0 the Banach space $\{f \in C([0,1]) : f(0) = f(1) = 0\}$ equipped with the sup-norm. From [4,9], when acting on C_0, Z_t has the Feller property, i.e., for any $f \in C_0$, $Z_t f \in C_0$ and $Z_t f \to f$ as $t \to 0$. Meanwhile, Z_t has the strong Feller property, i.e., for any Borel bounded function f on $[0,1]$ and $t > 0$, the restriction of $Z_t f$ on $(0,1)$ is continuous. Denote by $d(\varsigma)$ the multiplicity of ς as a root of $E_{\alpha,\alpha}$. The structure of Z_t and its spectrum are as follows. For symmetric Lévy processes, similar spectral results have long been known [12].

Proposition 13. *For each $t > 0$, the following are true.*

(a) Z_t, $Z_{t,\varsigma}$, $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ are all compact maps of $L^1([0,1]) \to C_0$ with $\sum_{|\varsigma|\le M} Z_{t,\varsigma} \to Z_t$ under the operator norm as $M \to \infty$. Moreover, for $s \ge 0$ and $\eta \in \mathscr{Z}_{\alpha,\alpha}$, $Z_{s,\varsigma} Z_{t,\eta} = \mathbf{1}\{\eta = \varsigma\}\, Z_{s+t,\varsigma}$.

(b) Given $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$, define functions

$$g_{j,\varsigma}(x) = [W^{(\varsigma)}(1-x)]^{(j-1)} = (1-x)^{j\alpha-1} E^{(j-1)}_{\alpha,\alpha}((1-x)^\alpha \varsigma) \tag{48}$$

for $j = 1, \dots, d(\varsigma)$. Then $Z_{t,\varsigma}$ maps $L^2([0,1])$ into $V_\varsigma = \mathrm{span}(g_{1,\varsigma}, \dots, g_{d(\varsigma),\varsigma})$ and $\sum_\varsigma V_\varsigma$ is dense in $L^2([0,1])$.

(c) Let $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ and $j = 1, \dots, d(\varsigma)$. Then $(Z_t - e^{\varsigma t})^k g_{j,\varsigma} = 0$ if and only if $k \ge j$.

(d) For $a \in \{e^{t\varsigma}, \varsigma \in \mathscr{Z}_{\alpha,\alpha}\}$ and $j \ge 1$, the null space of $(Z_t - a)^j$ on $L^2([0,1])$ is spanned by $\{g_{i,\varsigma} : e^{\varsigma t} = a, 1 \le i \le \min(j, d(\varsigma))\}$.

(e) The spectrum of Z_t acting on $L^2([0,1])$ is $\{0\} \cup \{e^{\varsigma t}, \varsigma \in \mathscr{Z}_{\alpha,\alpha}\}$, with 0 the only element that is not an eigenvalue.

5.1. Proof of Proposition 9 and its corollaries

To start with, if $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ with $d(\varsigma) = k$, then in its neighborhood, $W^{(z)}(1) = E_{\alpha,\alpha}(z) = (z-\varsigma)^k g(z)$, where g is analytic with $g(\varsigma) \ne 0$. As a result,

$$\begin{aligned} a_{t,\varsigma}(x,y) = \mathrm{Res}\left(\frac{W^{(z)}(1-x) W^{(z)}(y) e^{zt}}{(z-\varsigma)^k g(z)}, \varsigma\right) &= \frac{1}{(k-1)!}\left[\frac{W^{(\varsigma)}(1-x) W^{(\varsigma)}(y) e^{\varsigma t}}{g(\varsigma)}\right]^{(k-1)} \\ &= \sum_{j=0}^{k-1} c_j [W^{(\varsigma)}(1-x) W^{(\varsigma)}(y) e^{\varsigma t}]^{(j)}, \end{aligned} \tag{49}$$

where $c_j = c_j(\varsigma)$ are constants. Then given $t > 0$, $a_{t,\varsigma}(x,y)$ is a linear combination of functions $[W^{(\varsigma)}(1-x)]^{(j)}[W^{(\varsigma)}(y)]^{(l)}$, $0 \le j, l < d(\varsigma)$, so $a_{t,\varsigma} \in C([0,1] \times [0,1])$. By $[W^{(\varsigma)}(1)]^{(j)} = 0$ for $j < d(\varsigma)$, $a_{t,\varsigma}(0,y) = 0$. Since $[W^{(\varsigma)}(1-x)]^{(j)}$ is a weighted sum of $W^{*n}(1-x)$ over $n \ge 1$ and $W^{*n}(0) = 0$, $a_{t,\varsigma}(1,y) = 0$ as well.

Lemma 14. *For each $t > 0$, $\sup_{x,y\in[0,1]} |r_t(x,y) - \sum_{|\varsigma|\le M} a_{t,\varsigma}(x,y)| \to 0$ as $M \to \infty$. As a result, $r_t \in C([0,1] \times [0,1])$ with $r_t(0,y) = r_t(1,y) = 0$.*

Proof. Denote $\|f\| = \sup_{x,y\in[0,1]} |f(x,y)|$. Then $\|r_t - \sum_{|\varsigma|\le M} a_{t,\varsigma}\| \le \sum_{|\varsigma|>M} \|a_{t,\varsigma}\|$. If $|\varsigma| \gg 1$, then $d(\varsigma) = 1$, so from (49), $a_{t,\varsigma}(x,y) = W^{(\varsigma)}(1-x) W^{(\varsigma)}(y) e^{\varsigma t}/[W^{(\varsigma)}(1)]'$. Then from the derivation of (25), $\|a_{t,\varsigma}\| = O(|\varsigma|^{3-1/\alpha} e^{\mathrm{Re}(\varsigma)t})$. By (7), the sum of the bounds is finite. Then the uniform convergence follows. □

We also need an elementary result. Let S be a finite set of real numbers and for each $s \in S$, let $c_s \in \mathbb{C}$ be a constant. Suppose $F(t) := \sum_{s\in S} c_s e^{\mathrm{i}st} \to 0$ as $t \to \infty$. Then $t^{-1}\int_0^t |F|^2 \to 0$. However, it is always true that $t^{-1}\int_0^t |F|^2 \to \sum_s |c_s|^2$. Thus $c_s = 0$.

Proof of Proposition 9. The goal is to show $S = \{\varsigma \in \mathscr{Z}_{\alpha,\alpha} : \mathrm{Re}(\varsigma) = -\varrho\}$ only contains $-\varrho$. By (6), $S \subset \{z : |\arg z| > \alpha\pi/2\}$. Then $|S| < \infty$. Given a Borel set $B \subset [0, 1]$, as $t \to \infty$

$$\mathbb{P}^x\{X_t \in B, \mathcal{A}_t\} \sim Ce^{-\varrho t}, \tag{50}$$

where $C > 0$ is a constant ([4], Theorem 4). On the other hand, by Lemma 14, the l.h.s. equals

$$\sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}} \int_B a_{t,\varsigma}(x, y)\,\mathrm{d}y \sim \sum_{\varsigma\in S} \int_B a_{t,\varsigma}(x, y)\,\mathrm{d}y.$$

Assume $\max_{\varsigma\in S} d(\varsigma) = k > 1$ and let $S_0 = \{\varsigma \in S : d(\varsigma) = k\}$. Then from (49), for $\varsigma \in S_0$,

$$\int_B a_{t,\varsigma}(x, y)\,\mathrm{d}y = e^{\varsigma t}\left[t^{k-1}\frac{W^{(\varsigma)}(1-x)}{E^{(k)}_{\alpha,\alpha}(\varsigma)}\int_B W^{(\varsigma)}(y)\,\mathrm{d}y + p_\varsigma(t)\right],$$

while for $\varsigma \in S \setminus S_0$, $\int_B a_{t,\varsigma}\,\mathrm{d}y = e^{\varsigma t} p_\varsigma(t)$, where each $p_\varsigma(t)$ is a polynomial of t of degree at most $k-2$. Since each $\varsigma \in S$ has $\mathrm{Re}(\varsigma) = -\varrho$, by comparing with (50), it follows that

$$\sum_{\varsigma\in S_0} e^{(\varsigma+\varrho)t}\frac{W^{(\varsigma)}(1-x)}{E^{(k)}_{\alpha,\alpha}(\varsigma)}\int_B W^{(\varsigma)}(y)\,\mathrm{d}y \to 0.$$

Since $\varsigma+\varrho$ is a pure imaginary number, from the elementary fact mentioned prior to the proof,

$$W^{(\varsigma)}(1-x)\int_B W^{(\varsigma)}(y)\,\mathrm{d}y = 0$$

for each $\varsigma \in S_0$. Since B is arbitrary and $W^{(\varsigma)}(y)$ is continuous in y, then $W^{(\varsigma)}(1-x)W^{(\varsigma)}(y) \equiv 0$, and so $E_{\alpha,\alpha}((1-x)^\alpha\varsigma)E_{\alpha,\alpha}(y^\alpha\varsigma) = 0$ for all $x, y \in (0, 1)$. Since $E_{\alpha,\alpha}(z)$ is analytic, this implies $E_{\alpha,\alpha}(z) \equiv 0$, which is impossible. The contradiction implies that all $\varsigma \in S$ are simple. Then with exactly the same argument,

$$\sum_{\varsigma\in S} e^{(\varsigma+\varrho)t}\frac{W^{(\varsigma)}(1-x)}{E'_{\alpha,\alpha}(\varsigma)}\int_B W^{(\varsigma)}(y)\,\mathrm{d}y - C \to 0,$$

which implies that $S = \{-\varrho\}$. □

Proof of Corollary 10. The first asymptotic result is a direct consequence of Propositions 1 and 9, the second one follows by combining the first one with $\mathbb{P}\{T_c \in \mathrm{d}t,\ \tau_{-b} > T_c\}/\mathrm{d}t = -L'_{-b,c}(t) - \mathbb{P}\{\tau_{-b} \in \mathrm{d}t, \tau_{-b} < T_c\}/\mathrm{d}t$ and Corollary 5. □

Proof of Corollary 11. By Proposition 1, $H_0(q) = [\Gamma(\alpha)E_{\alpha,\alpha}(q)]^{-1}$ is the Laplace transform of $\psi_0(t) = \sum_{\varsigma\in\mathscr{Z}_{\alpha,\alpha}} \mathrm{Res}(H_0(z)e^{zt}, \varsigma)$. Thus $H^{(k)}(0) = \int_0^\infty (-t)^k \psi_0(t)\,\mathrm{d}t$. For each $\varsigma \in \mathscr{Z}_{\alpha,\alpha} \setminus \{-\varrho\}$, by $\mathrm{Re}(\varsigma) < -\varrho$, $\int_0^\infty t^k \mathrm{Res}(H_0(z)e^{zt}, \varsigma)\,\mathrm{d}t = o(k!\varrho^{-k-1})$ as $k \to \infty$. Fix $M > 0$, such that all ς with $|\varsigma| > M$ can be enumerated as $\varsigma_{\pm n} \asymp n^\alpha$ and by (24), have $\int_0^\infty t^k \mathrm{Res}(H_0(z)e^{zt}, \varsigma)\,\mathrm{d}t = O(1/E'_{\alpha,\alpha}(\varsigma))\int_0^\infty t^k e^{\varsigma t}\,\mathrm{d}t = O(k!|\varsigma|^{2-k-1/\alpha})$. Then by dominated convergence, for $k > 2$, $\int_0^\infty (-t)^k \psi_0(t)\,\mathrm{d}t = \Sigma_1 + \Sigma_2 + I$, where Σ_1 is the sum of $\int_0^\infty (-t)^k \mathrm{Res}(H_0(z)e^{zt}, \varsigma)\,\mathrm{d}t$ over $\varsigma \neq -\varrho$ with $|\varsigma| \le M$, Σ_2 is the one over ς with $|\varsigma| > M$, and $I = \int_0^\infty (-t)^k \mathrm{Res}(H_0(z)e^{zt}, -\varrho)\,\mathrm{d}t$. Since there are only a finite number of

ς with $|\varsigma| \le M$, from the above discussion, $\Sigma_1 = o(k!\varrho^{-k-1})$. On the other hand, $\Sigma_2 = O(k!)\sum_{M=O(n^\alpha)} n^{-(k-2)\alpha-1} = o(k!\varrho^{-k-1})$. Finally, $I = \int_0^\infty (-t)^k [\Gamma(\alpha) E'_{\alpha,\alpha}(-\varrho)]^{-1} e^{-\varrho t}\,\mathrm{d}t$, yielding the claim. □

Proof of Corollary 12. Given $0 = t_0 < t_1 < \cdots < t_n < \infty$. If $B > t_n$, then by the Markov property of X, for $x_0, x_1, \ldots, x_n \in (0, 1)$,

$$\begin{aligned}\mathbb{P}^{x_0}\{Y^B_{t_1} \in \mathrm{d}x_1, \ldots, Y^B_{t_n} \in \mathrm{d}x_n\} &= \mathbb{P}^{x_0}\{X_{t_1} \in \mathrm{d}x_1, \ldots, X_{t_n} \in \mathrm{d}x_n \mid \mathcal{A}_B\} \\ &= \prod_{i=1}^n \mathbb{P}^{x_{i-1}}\{X_{t_i} \in \mathrm{d}x_i - x_{i-1}, \mathcal{A}_{t_i - t_{i-1}}\} \times \frac{\mathbb{P}^{x_n}(\mathcal{A}_{B-t_n})}{\mathbb{P}^{x_0}(\mathcal{A}_B)} \\ &= \prod_{i=1}^n l_{x_i - x_{i-1}, -x_{i-1}, 1-x_{i-1}}(t_i - t_{i-1}) \times \frac{L_{-x_n,1-x_n}(\mathcal{A}_{B-t_n})}{L_{-x_0,1-x_0}(\mathcal{A}_B)}.\end{aligned}$$

Combining Corollary 5 and Proposition 9, as $B \to \infty$

$$\mathbb{P}^{x_0}\{Y^B_{t_1} \in \mathrm{d}x_1, \ldots, Y^B_{t_n} \in \mathrm{d}x_n\} \to \prod_{i=1}^n l_{x_i - x_{i-1}, -x_{i-1}, 1-x_{i-1}}(t_i - t_{i-1}) \times \frac{W^{(-\varrho)}(1 - x_n) e^{\varrho t_n}}{W^{(-\varrho)}(1 - x_0)},$$

yielding that Y^B converges to a Markov process Y with transition kernel

$$\frac{\mathbb{P}\{Y_{s+t} \in \mathrm{d}y \mid Y_s = x\}}{\mathrm{d}y} = l_{y-x,-x,1-x}(t) \times \frac{W^{(-\varrho)}(1 - y) e^{\varrho t}}{W^{(-\varrho)}(1 - x)}$$

in the sense of finite dimensional distribution. By Theorem 4, the proof is complete. □

5.2. *Proof of Proposition 13*

Lemma 15. *Fix $z_0 \in \mathbb{C}$. Then for $z \in \mathbb{C} \setminus \{z_0\}$, $j \in \mathbb{N}$, $x > 0$, and integer $j \ge 0$,*

$$\int_0^1 W^{(z_0)}(x - y)[W^{(z)}(y)]^{(j)}\,\mathrm{d}y = \frac{\mathrm{d}^j}{\mathrm{d}z^j}\left[\frac{W^{(z)}(x) - W^{(z_0)}(x)}{z - z_0}\right].$$

Proof. Since $W^{(z)}(x) = \sum_{k=0}^\infty z^k W^{*(k+1)}(x)$, by dominated covergence,

$$W^{(z_0)} * W^{(z)}(x) = \sum_{k=0}^\infty \sum_{n=0}^k z_0^n z^{k-n} W^{*(n+1)} * W^{*(k-n+1)}(x) = \sum_{k=0}^\infty \frac{z_0^{k+1} - z^{k+1}}{z_0 - z} W^{*(k+2)}(x).$$

The r.h.s. is exactly $(W^{(z)}(x) - W^{(z_0)}(x))/(z - z_0)$. Differentiating the equality j times in z then yields the proof. □

Lemma 16. *Fix $x \ge 0$ and $t \ge 0$. Let $\eta, \varsigma \in \mathscr{Z}_{\alpha,\alpha}$. If $d(\varsigma) = k$, then for $0 \le j < k$,*

$$\int_0^1 a_{t,\eta}(x, y)[W^{(\varsigma)}(1 - y)]^{(j)}\,\mathrm{d}y = \mathbf{1}\{\eta = \varsigma\} \sum_{s=0}^j \binom{j}{s} t^s [W^{(\varsigma)}(1 - x)]^{(j-s)} e^{\varsigma t}.$$

Proof. From the definition of $a_{t,\varsigma}$ and Fubini's theorem, the integral is equal to

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{W^{(z)}(1 - x) e^{zt}}{W^{(z)}(1)} \left(\int_0^1 W^{(z)}(y)[W^{(\varsigma)}(1 - y)]^{(j)}\,\mathrm{d}y\right) \mathrm{d}z,$$

where $\gamma \in \mathbb{C} \setminus \mathscr{Z}_{\alpha,\alpha}$ is a counterclockwise simple contour enclosing η but no other roots in $\mathscr{Z}_{\alpha,\alpha}$. Then by Lemma 15, the integral is equal to

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{W^{(z)}(1-x)e^{zt}}{W^{(z)}(1)} \frac{\mathrm{d}^j}{\mathrm{d}w^j} \left[\frac{W^{(z)}(1) - W^{(w)}(1)}{z-w}\right]_{w=\varsigma} \mathrm{d}z.$$

Since

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}w^j}\left[\frac{W^{(z)}(1) - W^{(w)}(1)}{z-w}\right] &= \sum_{s=0}^{j} \binom{j}{s} \frac{\mathrm{d}^s}{\mathrm{d}w^s}[W^{(z)}(1) - W^{(w)}(1)] \frac{\mathrm{d}^{j-s}}{\mathrm{d}w^{j-s}}\left(\frac{1}{z-w}\right)\\ &= \frac{j!W^{(z)}(1)}{(z-w)^{j+1}} - \sum_{s=0}^{j} \frac{j![W^{(w)}(1)]^{(s)}}{s!(z-w)^{j+1-s}},\end{aligned}$$

and $j < d(\varsigma) = k$, when evaluated at $w = \varsigma$, with $W^{(\varsigma)}(1) = [W^{(\varsigma)}(1)]^{(s)} = 0$ for $s = 0, \ldots, j$, the derivative on the l.h.s. is equal to $j!W^{(z)}(1)/(z-\varsigma)^{j+1}$. Therefore, the integral is equal to

$$\frac{j!}{2\pi \mathrm{i}} \oint_\gamma \frac{W^{(z)}(1-x)e^{zt}}{(z-\varsigma)^{j+1}} \mathrm{d}z = \mathbf{1}\{\eta = \varsigma\} [W^{(\varsigma)}(1-x)e^{\varsigma t}]^{(j)},$$

hence the claim. □

Proof of Proposition 9. (a) For $t > 0$, the kernel of Z_t is $r_t(x, y)$. Then the first half of part (a) is a direct consequence of Lemma 14 and [16], Theorem 22.3. Given $s \geq 0$ and $\eta \in \mathscr{Z}_{\alpha,\alpha}$, the kernel of $Z_{s,\varsigma} Z_{t,\eta}$ is $\int_0^1 a_{s,\varsigma}(x, u) a_{t,\eta}(u, y)\, \mathrm{d}u$. If $\eta \neq \varsigma$, then by expressing $a_{t,\eta}(u, y)$ by (49) and applying Lemma 16, it is seen $Z_{s,\varsigma} Z_{t,\eta} = 0$. To show $Z_{s,\varsigma} Z_{t,\varsigma} = Z_{s+t,\varsigma}$, let $\gamma, \gamma' \in \mathbb{C} \setminus \mathscr{Z}_{\alpha,\alpha}$ be two counterclockwise simple contours that enclose ς but no other roots in $\mathscr{Z}_{\alpha,\alpha}$, with γ disjoint from and enclosed by γ'. Then by Fubini's theorem and Lemma 15,

$$\begin{aligned}&\int_0^1 a_{s,\varsigma}(x, u) a_{t,\varsigma}(u, y)\, \mathrm{d}u\\ &= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma'} \left\{\oint_\gamma \frac{W^{(z)}(1-x)W^{(z')}(y)}{W^{(z)}(1)W^{(z')}(1)} \left[\int_0^1 W^{(z)}(u)W^{(z')}(1-u)\, \mathrm{d}u\right] e^{zs} \mathrm{d}z\right\} e^{z't}\, \mathrm{d}z'\\ &= \frac{1}{(2\pi \mathrm{i})^2} \oint_{\gamma'} \left\{\oint_\gamma \frac{W^{(z)}(1-x)W^{(z')}(y)}{z-z'} \left[\frac{1}{W^{(z')}(1)} - \frac{1}{W^{(z)}(1)}\right] e^{sz} \mathrm{d}z\right\} e^{tz'} \mathrm{d}z'.\end{aligned}$$

Given $z' \in \gamma'$, since it is outside of the region enclosed by γ, $W^{(z)}(1-x)e^{sz}/(z-z')$ is analytic in the region. This combined with Fubini's theorem yields that the integral is equal to

$$\frac{1}{(2\pi \mathrm{i})^2} \oint_\gamma \left[\oint_{\gamma'} \frac{W^{(z)}(1-x)W^{(z')}(y)}{(z'-z)W^{(z)}(1)} e^{z't} \mathrm{d}z'\right] e^{zs} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{W^{(z)}(1-x)W^{(z)}(y)}{W^{(z)}(1)} e^{(s+t)z} \mathrm{d}z,$$

which is $a_{s+t}(x, y)$, as claimed.

(b) Denote $Rh(x) = h(1-x)$ and $\langle f, h\rangle = \int_0^1 f\bar{h}$ for $f, h \in L^2([0, 1])$. From (49), $a_{t,\varsigma}(x, y) = \sum_{j,k \leq d(\varsigma)} c_{jk} g_{j,\varsigma}(x) R g_{k,\varsigma}(y)$ with $c_{jk} = c_{jk}(\varsigma, t)$. Then $Z_{t,\varsigma} h = \sum_{j,k\leq d(\varsigma)} c_{jk} \langle g_{k,\varsigma}, R\bar{h}\rangle g_{j,\varsigma} \in V_\varsigma$. If $\langle g_{j,\varsigma}, h\rangle = 0$ for all ς and $j \leq d(\varsigma)$, then for all $t > 0$, $Z_{t,\varsigma} R\bar{h} = 0$, so for all $f \in C_0$, $\langle f, Z_{t,\varsigma} R\bar{h}\rangle = 0$. Since $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$ if and only if $\bar{\varsigma} \in \mathscr{Z}_{\alpha,\alpha}$ and $a_{t,\varsigma}(x, y) = a_{t,\varsigma}(1-y, 1-x) = \overline{a_{t,\bar{\varsigma}}(x, y)}$,

$$\begin{aligned}\langle f, Z_{t,\varsigma} R\bar{h}\rangle &= \int_0^1 \left[\int_0^1 f(1-x)\overline{a_{t,\varsigma}(x,y)}\,\mathrm{d}x\right] h(1-y)\,\mathrm{d}y \\ &= \int_0^1 \left[\int_0^1 f(1-x) a_{t,\bar{\varsigma}}(1-y,1-x)\,\mathrm{d}x\right] h(1-y)\,\mathrm{d}y = \langle Z_{t,\bar{\varsigma}} f, \bar{h}\rangle.\end{aligned}$$

As a result, $\langle Z_{t,\varsigma} f, \bar{h}\rangle = 0$ for all ς and $t > 0$. Then by (a), $\langle Z_t f, \bar{h}\rangle = 0$. Let $t \to 0$. By the Feller property of Z_t and dominated convergence, $\langle f, \bar{h}\rangle = 0$. Since C_0 is dense in $L^2([0,1])$, then $h = 0$.

(c) First, consider $(Z_{t,\varsigma} - e^{\varsigma t})^k g_{j,\varsigma}$ instead of $(Z_t - e^{\varsigma t})^k g_{j,\varsigma}$. By Lemma 16, for $j = 1, \ldots, d(\varsigma)$,

$$\begin{aligned}Z_{t,\varsigma} g_{j,\varsigma}(x) &= \int_0^1 a_{t,\varsigma}(x,y)[W^{(\varsigma)}(1-y)]^{(j-1)} \\ &= \sum_{s=1}^{j} \binom{j-1}{j-s} t^{j-s} [W^{(\varsigma)}(1-x)]^{(s-1)} e^{\varsigma t} = \sum_{s=1}^{j} \binom{j-1}{j-s} t^{j-s} e^{\varsigma t} g_{s,\varsigma}(x).\end{aligned}$$

If $j = 1$, then $Z_{t,\varsigma} g_{1,\varsigma} = e^{\varsigma t} g_{1,\varsigma}$, so $(Z_{t,\varsigma} - e^{\varsigma t}) g_{1,\varsigma} = 0$. Clearly, $g_{1,\varsigma} \neq 0$. If $j > 1$, then

$$(Z_{t,\varsigma} - e^{\varsigma t}) g_{j,\varsigma} = (j-1) t e^{\varsigma t} g_{j-1,\varsigma} + \sum_{s=1}^{j-2} \binom{j-1}{j-s} t^{j-s} e^{\varsigma t} g_{s,\varsigma}.$$

By induction, $(Z_{t,\varsigma} - e^{\varsigma t})^{j-1} g_{j,\varsigma} = (j-1)!(t e^{\varsigma t})^{j-1} g_{1,\varsigma}$, giving $(Z_{t,\varsigma} - e^{\varsigma t})^k g_{j,\varsigma} = 0$ if and only if $k \geq j$. Now, from (b), V_ς is invariant under $Z_{t,\varsigma} - e^{\varsigma t}$. Meanwhile, by Lemma 16 and (a), $\sum_{\eta \neq \varsigma} Z_{t,\eta} V_\varsigma = \{0\}$. Then $(Z_t - e^{\varsigma t})^k g_{j,\varsigma} = (Z_{t,\varsigma} - e^{\varsigma t} + \sum_{\eta \neq \varsigma} Z_{t,\eta})^k g_{j,\varsigma} = (Z_{t,\varsigma} - e^{\varsigma t})^k g_{j,\varsigma}$. This then leads to the proof.

(d) Suppose $0 \neq f \in L^2([0,1])$ is in the null space of $(Z_t - a)^j$. For each $\varsigma \in \mathscr{Z}_{\alpha,\alpha}$, by (a) and induction on j, $(Z_{t,\varsigma} - a)^j Z_{0,\varsigma} f = Z_{0,\varsigma}(Z_t - a)^j f = 0$, so $Z_{0,\varsigma} f$ is in the null space of $(Z_{t,\varsigma} - a)^j$. By (b), $Z_{0,\varsigma} f \in V_\varsigma$. Since V_ς is finite dimensional, $Z_{t,\varsigma}$ restricted on V_ς can be regarded as a matrix with $e^{\varsigma t}$ being its only eigenvalue. As a result, if $e^{\varsigma t} \neq a$, then $Z_{0,\varsigma} f = 0$, so $Z_{t,\varsigma} f = Z_{t,\varsigma} Z_{0,\varsigma} f = 0$. Put $W = \sum_{e^{\varsigma t} = a} Z_{t,\varsigma}$. Then $(W - a)^j f = (Z_t - a)^j f = 0$. By expanding $(W - a)^j f$ and applying (a), it is seen that f is a linear combination of $Z_{t,\varsigma}^k f \in V_\varsigma$, $1 \leq k \leq j$, $e^{\varsigma t} = a$. As a result, $f \in \sum_{e^{\varsigma t} = a} V_\varsigma$. Then by considering the restriction of Z_t on the finite dimensional space $\sum_{e^{\varsigma t} = a} V_\varsigma$, the proof follows from standard matrix algebra.

(e) By (a), Z_t is compact mapping $L^1([0,1]) \to C_0$. Since the identity maps of $C_0 \to L^2([0,1])$ and $L^2([0,1]) \to L^1([0,1])$ are continuous, Z_t is compact mapping $L^2([0,1]) \to L^2([0,1])$. Then by Riesz's spectral theorem ([16], p. 238), 0 is in the spectrum of Z_t and every nonzero element in the spectrum is an eigenvalue of Z_t. From (c), it therefore suffices to show that if $h \in L^2([0,1])$ and $Z_t h = a h$ for some $a \notin \{e^{\varsigma t}, \varsigma \in \mathscr{Z}_{\alpha,\alpha}\}$, then $h = 0$. Indeed, for any ς, $Z_{t,\varsigma}(Z_{0,\varsigma} h) = Z_t Z_{0,\varsigma} h = Z_{0,\varsigma} Z_t h = a Z_{0,\varsigma} h$. Since $Z_{0,\varsigma} h \in V_\varsigma$ and from (d), $e^{\varsigma t}$ is the only eigenvalue of $Z_{t,\varsigma}$ when acting on V_ς, then $Z_{0,\varsigma} h = 0$. Then for any $t > 0$, $Z_{t,\varsigma} h = Z_{t,\varsigma} Z_{0,\varsigma} h = 0$. The proof of (b) already shows that in this case $h = 0$. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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