SUB-NYQUIST MULTICHANNEL BLIND DECONVOLUTION

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ABSTRACT

We consider a continuous-time sparse multichannel blind deconvolution problem. The signal at each channel is expressed as the convolution of a common source signal and its impulse response given as a sparse filter. The objective is to identify these sparse filters from sub-Nyquist samples of channel outputs by leveraging the correlation across channels. We present necessary and sufficient conditions for the unique identification. In particular, the sparse filters should not share a common sparse convolution factor and it is necessary to have $2L$ or more samples per channel from at least two distinct channels. We also show that L-sparse filters are uniquely identifiable from two channels provided that there are $2L^2$ Fourier measurements per channel, which can be computed from sub-Nyquist samples. Additionally, in the asymptotic of the number of channels, $2L$ Fourier measurements per channel are sufficient. The results are applicable to the design of multi-receiver, low-rate, sensors in applications such as radar, sonar, ultrasound, and seismic exploration.

***Index Terms*—** Sub-Nyquist sampling, correlated signals, sparse signals, continuous-time blind deconvolution, multichannel signals

1. INTRODUCTION

In many applications, a common signal is measured from multiple receivers or channels. The diversity of the measurements across sensors aids in improved signal recovery. For example, a multiple-input multiple-output (MIMO) radar system improves spatial and Doppler resolution [1]. In such applications, the implementation cost critically depends on the number of receivers and the sampling rates at each receiver. Therefore it is highly desirable to lower these factors.

The sampling rate can be reduced by utilizing correlation among the measurements, due to the common source signal. The multichannel sampling framework by Papoulis [2] is one of early results in this context. He showed that if a bandlimited signal is measured via $N(\geq 2)$ *known* filters, then the signal can be exactly reconstructed from the uniform samples of the filtered signals provided that the filters satisfy certain invertibility conditions. The sampling rate of each filtered signal is greater than or equal to $\frac{1}{N}$ of the corresponding Nyquist rate. The overall sampling rate is equal to the Nyquist rate. A generalization to this model is to assume that the filters are *unknown* and sparse. This model is ubiquitous in many applications such as radar imaging [3, 4], seismic signal processing [5], room impulse response modeling [6], sonar imaging [7], and ultrasound imaging [8, 9]. In these applications, a source signal is reflected from sparsely located targets and the reflected signal is observed from multiple receivers. Even though in many applications the source signal is assumed to be known, typically, the signal is distorted during propagation and hence unknown at the receiver end. The problem of identifying the unknown source and the sparse filters is known as the sparse multichannel blind deconvolution (S-MBD) problem.

Existing S-MBD results consider a finite-dimensional problem. In [10–12], the identifiability conditions were established by considering all the discrete measurements available in each channel and the desired number of channels is proportional to the number of measurements. Recently, we proposed a compressive S-MBD framework where we show that L-sparse filters could be identified by using only $2L^2$ measurements per channel, and two-channels are sufficient for filter identifiability [13]. To convert a continuous-time (CT) S-MBD problem to a discrete problem and apply the existing results, the sampling rate at each receiver should be inversely proportional to the resolution of the sparse filters. The resolution of the filters denotes the minimum distance between any two non-zero values. Hence, high-resolution filter estimation requires a higher sampling rate.

To date, only a few theoretical results for continuous-time S-MBD are available in the literature. Xia and Li [14] proposed an algorithm for CT-S-MBD in the context of MIMO channel estimation and assumed that the sparse filters have common support. To estimate the L-sparse filters, their algorithm requires $N \geq L$ channels and at least $L + 3$ Fourier measurements from each channel. A common support of the filters is used to build a joint-annihilation framework to estimate the filters. However, it is often the case that the common support assumption is not satisfied in practical applications. For example, in a multi-receiver radar system, configurations of receivers with respect to the target may result in sparse filters with non-identical supports. Da Costa and Chi [15] proposed an atomic-norm-based optimization to solve CT-S-MBD problem. They showed that the existence of a dual certificate in a specific form guarantees the exact recovery of signals. However, conditions on the number of measurements and the sparse filter to guarantees the existence of such a dual certificate were not derived.

In this paper, we consider the identifiability of the CT-S-MBD problem from sub-Nyquist samples. Specifically, we show that by using the correlation of the signals across channels, L-sparse filters are uniquely identified from $2L^2$ Fourier measurements of the output signals. These Fourier measurements can be computed from the same number of time samples of the signals where the output signals are time-limited. Note that the sampling rate is independent of the (approximate) bandwidth of the signal and therefore sub-Nyquist. We also establish necessary conditions for the identifiability of CT-

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S-MBD, that is, $2L$ Fourier measurements per channel are necessary for the unique identification of the filter. Although there is a gap between the sufficient and the necessary conditions, in the asymptotic of the number of filters, the sufficient number of samples per channel converges to $2L$.

Our main results can be compared to relevant previous work as follows: CT-S-MBD generalizes the recovery of signals at finite rate of innovations [16]. We consider a more challenging scenario where the measurements are taken from spike-model signals after convolution with a common unknown source signal. However, unlike [17], we did not impose a stringent condition on the supports of the multiple sparse filters. Recent results (e.g. [18]) analyzed the sensitivity of recovering spike-model signals, which shows the dependence on the minimum separation. Similar to Vetterli et al. [16], our results show that the number of samples sufficient for unique identification grows in the number of spikes L but is independent of the minimum separation requirement. However, we expect that sensitivity analysis of CT-S-MBD in the presence of noise will involve the minimum separation too.

The remainder of the paper is organized as follows. In Section 2, we introduce the mathematical formulation of CT-S-MBD and define identifiability up to fundamental ambiguity. In Section 3, we present our main results that include necessary conditions for identifying a CT-S-MBD without sampling, and sufficient and necessary conditions for identifiability from sub-Nyquist sampling. We conclude the paper with final remarks in Section 4.

Notations: Signals in the time domain are labelled by lowercase letters and their Fourier transforms are labelled by the corresponding uppercase letters. For any positive integer N, the set $\{1, 2, \cdots, N\}$ is denoted as $[N]$.

2. PROBLEM FORMULATION

Consider a set of N continuous-time correlated signals $\{y_n(t)\}_{n=1}^N$ given as

$$y_n(t) = (f * h_n)(t), \tag{1}$$

where $f(t)$ and $h_n(t)$ denote a common source signal and the filter corresponding to the n-th channel, respectively. MBD refers to the identification of the filters and/or source from measurements of $\{y_n(t)\}_{n=1}^N$.

Any MBD problem suffers from the fundamental ambiguities of scaling and shift. Let $\left(f(t), \{h_n(t)\}_{n=1}^N\right)$ be the source-filters pair corresponding to the ground-truth signals. Then the ambiguity class is defined as the set of source-filters pairs given by

$$\mathcal{C}\left(f(t), \{h_n(t)\}_{n=1}^N\right) = \left\{\left(\alpha^{-1} f(t-\tau), \{\alpha h_n(t+\tau)\}_{n=1}^N\right) | \alpha \neq 0, \tau \in \mathbb{R}\right\}. \tag{2}$$

Any source and filter pair in $\mathcal{C}(f(t), \{h_n(t)\}_{n=1}^N)$ generates the same set of output signals, $\{y_n(t)\}_{n=1}^N$, as the ground-truth. Hence, unique identifiability of the MBD problem implies that every source-filters pair $\left(\hat{f}(t), \{\hat{h}_n(t)\}_{n=1}^N\right)$ satisfying $\{y_n(t) = (\hat{f} * \hat{h}_n)(t)\}_{n=1}^N$ belongs to $\mathcal{C}\left(f(t), \{h_n(t)\}_{n=1}^N\right)$.

We aim to identify the sparse filters from sub-Nyquist samples of $\{y_n(t)\}_{n=1}^N$. Let $\mathcal{S}_L$ denote the set of all CT L-sparse signals defined by

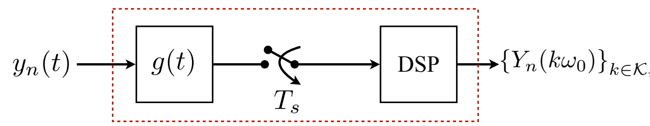

Fig. 1. Sub-Nyquist sampling framework for computing Fourier samples of an CT signal. The minimum sampling rate is $|\mathcal{K}_n|\omega_0$ rad./sec.

$$\mathcal{S}_L := \left\{\sum_{\ell=1}^{L} a_\ell \delta(t-\tau_\ell) \ : \ a_\ell \in \mathbb{R}, \ \tau_\ell \in [0, \tau_{\max})\right\},$$

where $\delta(t)$ denotes Dirac's impulse and $\tau_{\max}$ is the maximum delay known a priori. We assume that $h_n(t) \in \mathcal{S}_L$ for all $n \in [N]$. In other words, we would like to identify a set of parameters $\{a_{n,\ell}\}_{\ell=1}^L \in \mathbb{C}^L$ and $\{\tau_{n,\ell}\}_{\ell=1}^L \in \mathbb{R}^L$ such that

$$h_n(t) = \sum_{\ell=1}^{L} a_{n,\ell}\delta(t - \tau_{n,\ell}). \tag{3}$$

We assume that the source signal $f(t)$ has compact support in time. This results in non-bandlimited measurements.

As in finite-rate-of-innovation (FRI) reconstruction [19], we consider a frequency-domain approach that has the following benefits: (i) the Fourier-domain samples of the output signal, that are required to estimate the unknown parameters, can be effectively computed from sub-Nyquist samples of the signals by applying appropriate sampling kernels [8, 20] and (ii) since the filters are sparse in the time domain they are well spread in the frequency domain which helps in determining the filters from fewer Fourier measurements.

To avail of the advantages, we consider a Fourier-domain filter identification problem. Consider the following Fourier measurements of the output signals $\{y_n(t)\}_{n=1}^N$:

$$\{Y_n(k\omega_0) = F(k\omega_0)H_n(k\omega_0) : k \in \mathcal{K}_n, n \in [N]\}, \tag{4}$$

where $\mathcal{K}_n$ is a set of integers and ω_0 denotes the sampling interval in the Fourier domain. The Fourier measurements are determined from sub-Nyquist samples. Specifically, we apply kernel-based sub-Nyquist sampling mechanism (cf. Fig. 1) proposed in [8, 20]. In this framework, the design of the kernel, the sampling rate, and the digital signal processing (DSP) unit are a function of the Fourier-domain sampling locations $\{k\omega_0 : k \in \mathcal{K}_n\}$. The minimum sampling rate is independent of the bandwidth of $y_n(t)$ and typically much lower than the Nyquist rate. The minimum sampling rate is $|\mathcal{K}_n|\omega_0$. In addition, Tur et al. [8] showed that $|\mathcal{K}_n|$ Fourier coefficient samples are computed from $|\mathcal{K}_n|$ time-samples of the filtered signal provided that the filter satisfies certain alias cancellation conditions. Since a lower sampling rate is desirable, hence it is crucial to minimize $|\mathcal{K}_n|$ for $n \in [N]$.

Given that the Fourier measurements in (4) are computed from sub-Nyquist samples, the objective is to derive a set of conditions for the unique identification of the filters up to a scaling and shift ambiguity. More precisely, filter identifiability is defined as follows.

Definition 1 (Filter Identifiability)**.** *Suppose there exists a set of L-sparse filters $\{\hat{h}_n(t)\}_{n\in[N]}$ and a set of Fourier samples $\{\hat{F}(k\omega_0) :$*

$k \in \bigcup_{n\in[N]} \mathcal{K}_n\}$ *such that*

$$Y_n(k\omega_0) = \hat{F}(k\omega_0)\hat{H}_n(k\omega_0) : k \in \mathcal{K}_n, n \in [N]. \tag{5}$$

Then there exists $\tilde{f}(t)$ *such that*

1. $\left(\tilde{f}(t), \{\hat{h}_n(t)\}_{n=1}^N\right) \in \mathcal{C}\left(f(t), \{h_n(t)\}_{n=1}^N\right)$.
2. $\tilde{F}(k\omega_0) = \hat{F}(k\omega_0) : k \in \bigcup_{n\in[N]} \mathcal{K}_n$.

This definition implies that the Fourier measurements of any feasible solution lie within the fundamental ambiguity class of the ground truth.

Since the measurements, as well as the overall sampling rate, are functions of N, ω_0 and $\{\mathcal{K}_n\}_{n=1}^N$ in addition to the source and the filters, we seek answers to the following questions in the context of unique identifiability of the filters:

Q1 What are the conditions on the filters in addition to sparsity?

Q2 What are the conditions on the source?

Q3 How to choose the sampling interval ω_0?

Q4 How to design the sampling sets $\{\mathcal{K}_n\}_{n=1}^N$?

Q5 What is the minimum number of channels N?

3. IDENTIFIABILITY RESULTS

3.1. Unique Identifiability Conditions for CT-S-MBD

We first present necessary conditions, which need to be satisfied by the filters regardless of the sampling mechanism. To this end, we introduce the coprime structure in a set of filters as follows.

Definition 2 ($\mathcal{S}_L$-Coprime Filters)**.** *Consider a set of N filters $h_n(t) \in \mathcal{S}_L$, $n \in [N]$. The filters are coprime if they can not be decomposed as*

$$h_n(t) = (\hat{h}_n * h_0)(t), \quad n \in [N], \tag{6}$$

such that $\hat{h}_n(t) \in \mathcal{S}_L$ and $\hat{h}_0(t) \notin \mathcal{S}_1$.

In other words, the coprime filters can not have a common convolutional factor $h_0(t)$ which can not be represented as scaling and shift, and L-sparse distinct factors $\hat{h}_n(t)$. With this definition, we state the necessary condition to identify the filters in a CT-MBD problem in the following theorem.

Theorem 1 (CT-MBD Necessary Condition)**.** *Consider a set of N signals $\{y_n(t) = (f * h_n)(t)\}_{n=1}^N$, where $\{h_n(t) \in \mathcal{S}_L, n \in [N]\}$. If the filters are uniquely identifiable, then they need to be $\mathcal{S}_L$-coprime.*

Proof. We prove the theorem by contradiction. Assume that the filters are not $\mathcal{S}_L$-coprime and have a decomposition as in (6). Then the output signals are decomposed as

$$y_n(t) = (f * h_n)(t) = (f * h_0 * \hat{h}_n)(t) = (\hat{f} * \hat{h}_n)(t), \tag{7}$$

where $\hat{f}(t) = (f * h_0)(t)$. Since $h_0(t) \notin \mathcal{S}_1$, it follows that $\hat{f}(t)$ and $\hat{h}_n(t)$ cannot be written as scaled and shifted versions of $f(t)$ and $h_n(t)$. In other words, $\left(\hat{f}(t), \{\hat{h}_n(t)\}_{n=1}^N\right) \notin \mathcal{C}\left(f(t), \{h_n(t)\}_{n=1}^N\right)$. Since the source signal is unconstrained, the alternative source, $\hat{f}(t)$, is also feasible. In addition, since $\hat{h}_n(t) \in \mathcal{S}_L$, the alternative solution $\left(\hat{f}(t), \{\hat{h}_n(t)\}_{n=1}^N\right)$ is a feasible source-filter pair that satisfies the measurements and not within the ambiguity class. Hence, the problem is not uniquely identifiable. □

In essence, without any restriction on the source, especially on the support of the source, the L-sparse filters should be coprime to be uniquely identifiable. Hence, we make the following assumption that also answers the first question that we put forward in the previous section.

A1 The filters, $\{h_n(t)\}_{n=1}^N$, are $\mathcal{S}_L$-coprime.

3.2. Identifiability of CT-S-MBD from Fourier Measurements

From (4), we observe that if $F(k\omega_0) = 0$ for any $k \in \bigcup_{n\in[N]} \mathcal{K}_n$ then the corresponding measurement $Y_n(k\omega_0)$ is zero across all the channels and is not useful. Hence, to avoid zero measurements across the channels, we impose the following Fourier-domain constraint on the source:

A2 Source constraint: $F(k\omega_0) \neq 0, \quad \forall k \in \bigcup_{n\in[N]} \mathcal{K}_n$.

In this work, this is the only restriction imposed on the source and this answers our second question. The assumption **A2** is readily satisfied by both time-limited and band-limited sources. For a time-limited source $f(t)$, its spectrum $F(\omega)$ has infinite support. Hence, one can always select the frequencies $\left\{k\omega_0, k \in \bigcup_{n\in[N]} \mathcal{K}_n\right\}$ such that **A2** is satisfied. Similarly, for a band-limited source, **A2** is satisfied if the frequencies $\left\{k\omega_0, k \in \bigcup_{n\in[N]} \mathcal{K}_n\right\}$ are restricted to support of $F(\omega)$.

To design the sets $\{\mathcal{K}_n\}_{n=1}^N$ and frequency interval ω_0, consider the Fourier samples of the filters

$$H_n(k\omega_0) = \sum_{\ell=1}^{L} a_{n,\ell}\, e^{\mathrm{j}k\omega_0\tau_{n,\ell}}, \quad k \in \mathcal{K}_n,\, n \in [N]. \tag{8}$$

Let us consider a problem of estimating $\{a_{n,\ell}, \tau_{n,\ell}\}_{\ell=1}^L$ from the measurements in (8). This is indeed the problem considered in the FRI framework where the source is assumed to be known. To uniquely identify the time-delays, ω_0 is chosen such that the elements in the set $\{\omega_0\tau_{n,\ell}\}_{\ell=1}^L$ are distinct. One possible choice of ω_0 is $\frac{2\pi}{\tau_{\max}}$. We stick to this choice in the rest of the paper and make the following assumption:

A3 Choice of frequency-interval: $\omega_0 = \frac{2\pi}{\tau_{\max}}$.

Furthermore, the parameters $\{a_{n,\ell}, \tau_{n,\ell}\}_{\ell=1}^L$ can be estimated from $\{H_n(k\omega_0)\}_{k\in\mathcal{K}_n}$ by using Prony's method [21] or annihilating filter if $|\mathcal{K}_n| \geq 2L$ (cf. [19, Ch. 15]). It can be shown that $|\mathcal{K}_n| \geq 2L$ is necessary and sufficient to uniquely determine $\{a_{n,\ell}, \tau_{n,\ell}\}_{\ell=1}^L$ from the measurements in (8) irrespective of the recovery algorithm. The results are stated in the following theorem.

Theorem 2 (Identifying Sum of Exponentials)**.** *Consider the sum of exponential sequence in* (8) *for any $n \in [N]$. Let the assumption **A3** holds. The parameters $\{a_{n,\ell}, \tau_{n,\ell}\}_{\ell=1}^L$ are uniquely determined from the measurements iff $|\mathcal{K}_n| \geq 2L$.*

The necessary and sufficient conditions that $|\mathcal{K}_n| \geq 2L$ is for a known source case. With an unknown source, the necessary conditions still remain the same as one can not do better in terms of lower bound on cardinality $|\mathcal{K}_n|$. With this argument, we state the necessary condition for identifying the filters from the Fourier measurements in the following theorem.

Theorem 3 (Fourier-Domain Necessary Conditions). *For $N \geq 2$, consider the sparse MBD signals $\{y_n(t) = (f * h_n)(t)\}$ such that $h_n(t) \in \mathcal{S}_L$. Consider the Fourier measurements as in* (4)*. Suppose that A1, A2, and A3 hold. Then the filters are not identifiable if $|\mathcal{K}_n| < 2L$ for all $n \in [N]$.*

Theorem 3 provides a partial answer to **Q4**, that is, we know what is the minimum cardinally of the frequency sets but not yet sure about how to design such sets. To obtain a complete answer to this question as well as **Q5**, we present the sufficient conditions for filter identifiability from the Fourier measurements below.

Theorem 4 (Fourier-Domain Sufficient Conditions). *For $N \geq 2$ consider the Fourier measurements as in* (4) *where the filters are L-sparse. Suppose that (A1)–(A3) hold. Then the filters are identifiable provided that there exist $n_1 \neq n_2 \in [N]$ for which the following conditions hold:*

1. *$\mathcal{K}_{n_1} = \mathcal{K}_{n_2}$ and $|\mathcal{K}_{n_1}| \geq 2L^2$.*
2. *$\mathcal{K}_n \subseteq \mathcal{K}_{n_1}$ and $|\mathcal{K}_n| \geq 2L$ for all $n \in [N] \backslash \{n_1, n_2\}$.*

Proof. Let $\hat{f}(t)$ and $\{\hat{h}_n(t) \in \mathcal{S}_L\}_{n=1}^N$ satisfy $y_n(t) = (\hat{f} * \hat{h}_n)(t)$ for all $n \in [N]$. Then we have

$$Y_n(k\omega_0) = \hat{F}(k\omega_0)\hat{H}_n(k\omega_0), \quad n = n_1, n_2, \quad k \in \mathcal{K}_{n_1}. \tag{9}$$

Moreover, by the measurement model in (4), we also have

$$Y_{n_1}(k\omega_0)H_{n_2}(k\omega_0) - Y_{n_2}(k\omega_0)H_{n_1}(k\omega_0) = 0, \quad k \in \mathcal{K}_{n_1}. \tag{10}$$

By plugging in (9) into (10), we obtain

$$F(k\omega_0)\left(H_{n_1}(k\omega_0)\hat{H}_{n_2}(k\omega_0) - \hat{H}_{n_1}(k\omega_0)H_{n_2}(k\omega_0)\right) = 0. \tag{11}$$

Since $F(k\omega_0) \neq 0$ for all $k \in \mathcal{K}_{n_1}$, it follows that (11) implies

$$H_{n_1}(k\omega_0)\hat{H}_{n_2}(k\omega_0) = \hat{H}_{n_1}(k\omega_0)H_{n_2}(k\omega_0), \quad k \in \mathcal{K}_{n_1}. \tag{12}$$

Since each of the sequences $H_{n_1}(k\omega_0)$, $\hat{H}_{n_2}(k\omega_0)$, $\hat{H}_{n_1}(k\omega_0)$, and $H_{n_2}(k\omega_0)$ consists of sum of L complex exponentials as in (8), the products $H_{n_1}(k\omega_0)\hat{H}_{n_2}(k\omega_0)$ and $\hat{H}_{n_1}(k\omega_0)H_{n_2}(k\omega_0)$ consist of maximum of L^2 exponentials. Following Theorem 2, if $|\mathcal{K}_{n_1}| \geq 2L^2$, then both $H_{n_1}(\omega)\hat{H}_{n_2}(\omega)$ and $\hat{H}_{n_1}(\omega)H_{n_2}(\omega)$ are uniquely determined. Therefore , we have

$$H_{n_1}(\omega)\hat{H}_{n_2}(\omega) = \hat{H}_{n_1}(\omega)H_{n_2}(\omega). \tag{13}$$

This implies that there exists a function $H_0(\omega)$ such that

$$\hat{H}_n(\omega) = H_0(\omega)H_n(\omega), \quad n = n_1, n_2. \tag{14}$$

However, since $\hat{H}_{n_1}(\omega)$ and $\hat{H}_{n_2}(\omega)$ are Fourier transforms of $\mathcal{S}_L$-coprime filters, $H_0(\omega)$ is Fourier transform of $h_0(t)$ that is in $\mathcal{S}_1$. Hence, the filters $h_{n_1}(t)$ and $h_{n_2}(t)$ are identified up to the fundamental ambiguities, i.e. there exists $\tilde{f}(t)$ satisfying

$$\left(\tilde{f}(t), \{\hat{h}_{n_1}(t), \hat{h}_{n_2}(t)\}\right) \in \mathcal{C}\left(f(t), \{h_{n_1}(t), h_{n_2}(t)\}\right)$$

and

$$\tilde{F}(k\omega_0) = \hat{F}(k\omega_0), \quad k \in \mathcal{K}_{n_1}. \tag{15}$$

The identity in (15) implies that we have already computed the Fourier measurements of $\tilde{f}(t)$ at $k\omega_0$ for $k \in \mathcal{K}_{n_1}$. Let $n \in [N] \setminus \{n_1, n_2\}$. Since $\mathcal{K}_n \subset \mathcal{K}_{n_1}$, it follows from (9) and (15) that

$$\frac{Y_n(k\omega_0)}{\tilde{F}(k\omega_0)} = \hat{H}_n(k\omega_0), \quad n = n_1, n_2, \quad k \in \mathcal{K}_n, \tag{16}$$

where the left-hand side is known. Therefore, by the assumption $|\mathcal{K}_n| \geq 2L$, Theorem 2 implies that $\hat{h}_n \in \mathcal{S}_L$ is uniquely determined. In other words, we have

$$\left(\tilde{f}(t), \{\hat{h}_{n_1}(t), \hat{h}_n(t)\}\right) \in \mathcal{C}\left(f(t), \{h_{n_1}(t), h_n(t)\}\right).$$

By repeating this for all $n \in [N] \backslash \{n_1, n_2\}$, we obtain that all sparse filters are identified up to the fundamental ambiguity. □

Theorem 4 implies that two channels are sufficient to uniquely identify the filters and from those two channels $2L^2$ Fourier measurements are sufficient. If there are more than 2 channels, then $2L$ Fourier measurements from the rest of the channel are necessary and sufficient. Hence, for $N > 2$, all the channels except any two operate at minimum number of measurements.

Since the sampling rate at each channel is proportional to the number of desired Fourier measurements, the overall sampling rate is $\left(4L^2 + 2(N-2)L\right)\omega_0$. The sampling rate neither depends on the bandwidth of the signal nor on the resolution of the filters. Furthermore, we require an overall $4L^2 + 2(N-2)L$ time-domain measurements to identify the filters.

The sufficient number of Fourier measurements and the number of channels for the filter identifiability scale similarly to the analogous results for the discrete-time case derived in [13]. A natural question arises here is whether the results in [13] can be applied to the problem considered in this paper. To apply the finite-dimensional results in [13] to the CT-S-MBD problem in this paper, two additional conditions need to be satisfied. First, the time-delays are required to be on a grid. Second, the sampling should be on the grid, which results in a very high sampling rate. In contrast, the results in this paper do not require the time-delays to be on a grid and the signal is sampled at a sub-Nyquist rate.

4. CONCLUSION

In this paper, we showed that a CT-S-MBD problem can be identified from sub-Nyquist samples. The sampling rate in each channel is independent of the bandwidth of the output signals and only a function of sparsity of the filters. Additionally, we show that only two channels are sufficient. As the cost of hardware of a multi-receiver system is determined by the overall sampling rate, the results are pivotal in designing low-cost receivers.

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