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WEIGHTED RESTRICTION ESTIMATES AND APPLICATION TO FALCONER DISTANCE SET PROBLEM

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Abstract. We prove some weighted Fourier restriction estimates using polynomial partitioning and refined Strichartz estimates. As application we obtain improved spherical average decay rates of the Fourier transform of fractal measures, and therefore improve the results for the Falconer distance set conjecture in three and higher dimensions.

1. Introduction. In this article we prove improved partial result for Falconer distance set conjecture in dimension three and higher. Let $E \subset \mathbb{R}^d$ be a compact subset; its distance set $\Delta(E)$ is defined by

$$\Delta(E) := \big\{|x-y| : x, y \in E\big\}.$$

In [8], Falconer conjectured that:

CONJECTURE 1.1. (Falconer) *Let $d \geq 2$ and $E \subset \mathbb{R}^d$ be a compact set. Then*

$$\dim(E) > \frac{d}{2} \Longrightarrow \big|\Delta(E)\big| > 0.$$

Here $|\cdot|$ denotes the Lebesgue measure and $\dim(\cdot)$ *is the Hausdorff dimension.*

Falconer distance problem is a very natural and important question in harmonic analysis and geometric measure theory. It is known to be intimately related to a number of other important open questions, some of which will be mentioned later in the paper. Open in every dimension, Falconer's conjecture has been studied by several authors: see for instance Falconer [8], Mattila [15], Bourgain [1], Wolff [27] and Erdoğan [5, 6, 7]. The previously best known results are that $\dim(E) > \frac{d}{2} + \frac{1}{3}$ implies $|\Delta(E)| > 0$, due to Wolff [27] in dimension two and Erdoğan [6] in dimension three and higher. Our main result is the following improvement:

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THEOREM 1.2. *Let $d \geq 3$ and $E \subset \mathbb{R}^d$ be a compact set with*

$$\dim(E) > \alpha, \quad \alpha := \begin{cases} 1.8, & d = 3, \\ \frac{d}{2} + \frac{1}{4} + \frac{d+1}{4(2d+1)(d-1)}, & d \geq 4. \end{cases}$$

Then $|\Delta(E)| > 0$.

It is well known (see [6, 15, 27] for example) that Falconer's problem can be approached by weighted Fourier restriction (extension) estimates, which is the route we take in our proof. Consider the Fourier extension operator for the paraboloid

$$Ef(x) := \int_{B^{d-1}} e^{i(x'\cdot\omega + x_d|\omega|^2)} f(\omega)\, d\omega$$

where B^{d-1} denotes the unit ball in $\mathbb{R}^{d-1}$ and $x = (x', x_d) \in \mathbb{R}^d$.

Let $\mathcal{F}_{\alpha,d}$ denote the collection of non-negative measurable functions $H : \mathbb{R}^d \to \mathbb{R}$ satisfying that

$$\int_{B(x_0,r)} \big|H(x)\big|\, dx \leq r^\alpha, \quad \forall x_0 \in \mathbb{R}^d,\ \forall r \geq 1. \tag{1.1}$$

Remark 1.3. Note that for $p \geq 1$, $H \in \mathcal{F}_{\alpha,d}$, and function F with $\operatorname{supp}\widehat{F} \subset C \cdot B^d$, a constant dilation of the unit ball, there holds

$$\int |F|^p H\, dx \leq C_\alpha \int |F|^p\, dx,$$

where C_α denotes an absolute constant that depends on α and constant C. We defer the justification of this observation to Subsection 2.3.

We write $A \lessapprox B$ if $A \leq C_\varepsilon R^\varepsilon B$ for any $\varepsilon > 0$, $R > 1$, where C_ε is an absolute constant depending on ε. Let B_R^d denote an arbitrary ball of radius R in $\mathbb{R}^d$. When it is clear from the context what the dimension is, we sometimes abbreviate B_R^d as B_R. We also use B^d to denote the unit ball in $\mathbb{R}^d$ centered at the origin. By a standard argument, Theorem 1.2 follows from the weighted restriction estimates below.

THEOREM 1.4. *Let $d \geq 3$ and $\alpha \in (0, d]$. Then,*

$$\|Ef\|_{L^{2d/(d-1)}(B_R^d; H dx)} \lessapprox R^{\gamma_d^0(\alpha)} \|f\|_{L^2(B^{d-1})}$$

holds for all $f \in L^2(B^{d-1})$*, all* $R > 1$ *and all* $H \in \mathcal{F}_{\alpha,d}$*, where*

$$\gamma_3^0(\alpha) := \begin{cases} 0, & \alpha \in (0,2], \\ \frac{\alpha}{3} - \frac{2}{3}, & \alpha \in (2,3], \end{cases}$$

$$\gamma_d^0(\alpha) := \begin{cases} \frac{(d-1)(\alpha+1-d)}{2d}, & \alpha \in \big(\#_d, d\big], \\ \frac{\big(1+2S_4^d\big)\alpha}{4d} - \frac{1}{2d} - \frac{S_4^d}{2}, & \alpha \in \big(d-1, \#_d\big], \\ \frac{S_4^d\alpha}{2d} + \frac{1}{4} - \frac{3}{4d} - \frac{S_4^d}{2}, & \alpha \in (d-2, d-1], \\ \frac{S_\ell^d\alpha}{2d} + \frac{1}{4} - \frac{\ell-1}{4d} - \frac{S_\ell^d}{2}, & \alpha \in \left(d-\frac{\ell}{2}, d-\frac{\ell}{2}+\frac{1}{2}\right], \ \forall 5 \le \ell \le d, \\ 0, & \alpha \in \left(0, \frac{d}{2}\right], \end{cases} \qquad (d \ge 4)$$

with $S_\ell^d := \sum_{i=\ell}^d \frac{1}{i}$ *and* $\#_d := \frac{2d(d-2-S_4^d)}{2d-3-2S_4^d}$.

It is known that weighted restriction estimates in the vein of the above can be used to prove partial results towards Falconer's problem via a famous scheme due to Mattila. Briefly speaking, Theorem 1.4 implies improved estimates for spherical average decay rates of the Fourier transform of fractal measures, from which improved results for the Falconer distance set conjecture follow. We provide a detailed discussion of Mattila's approach in Section 2. In fact, only the range $\alpha \in [\frac{d}{2}, \frac{d+1}{2}]$ of Theorem 1.4 is relevant for the sake of proving our main Theorem 1.2. However, understanding the sharp spherical average Fourier decay rates of fractal measures for all $\alpha \in (0,d]$ is of its own interest, which for instance is related to the divergence set of solutions to Schrödinger and the wave equations (see for example [14] and the references therein). Moreover, it would be interesting to see whether our method can be extended to study sharp Fourier decay rates of measures with respect to geometric objects other than the sphere, such as the cone, which is known to have an application to projection problems in geometric measure theory [18].

To give the precise description of our improved estimates for spherical average decay rates of the Fourier transform of fractal measures, we make the following preliminary definition.

Definition 1.5. A compactly supported probability measure μ is called α-dimensional if it satisfies

$$\mu\big(B(x,r)\big) \le C_\mu r^\alpha, \quad \forall r > 0, \, \forall x \in \mathbb{R}^d. \tag{1.2}$$

Let $\beta_d(\alpha)$ denote the supremum of the numbers β for which

$$\|\widehat{\mu}(R\cdot)\|^2_{L^2(S^{d-1};d\sigma)} \leq C_{\alpha,\mu} R^{-\beta} \tag{1.3}$$

whenever $R > 1$ and μ is α-dimensional. Here S^{d-1} denotes the unit sphere in $\mathbb{R}^d$ and $d\sigma$ denotes its surface measure.

The problem of identifying the precise value of $\beta_d(\alpha)$ was proposed by Mattila [16]. In two dimensions, the sharp decay rates are known:

$$\beta_2(\alpha) = \begin{cases} \alpha, & \alpha \in (0, 1/2], \quad \text{(Mattila [15])} \\ 1/2, & \alpha \in [1/2, 1], \quad \text{(Mattila [15])} \\ \alpha/2, & \alpha \in [1, 2], \quad \text{(Wolff [27]).} \end{cases}$$

The problem remains open when $d \geq 3$ and $\alpha > \frac{d-1}{2}$. See Lucà-Rogers [14] and the references therein, for example, for a discussion of various partial results. In higher dimensions, the previously best known lower bounds are

$$\beta_d(\alpha) \geq \begin{cases} \alpha, & \alpha \in \left(0, \dfrac{d-1}{2}\right], \quad \text{(Mattila [15])} \\ \dfrac{d-1}{2}, & \alpha \in \left[\dfrac{d-1}{2}, \dfrac{d}{2}\right], \quad \text{(Mattila [15])} \\ \alpha - 1 + \dfrac{d+2-2\alpha}{4}, & \alpha \in \left[\dfrac{d}{2}, \dfrac{d}{2} + \dfrac{2}{3} + \dfrac{1}{d}\right], \quad \text{(Erdoğan [6])} \\ \alpha - 1 + \dfrac{(d-\alpha)^2}{(d-1)(2d-\alpha-1)}, & \alpha \in \left[\dfrac{d}{2} + \dfrac{2}{3} + \dfrac{1}{d}, d\right], \quad \text{(Lucà-Rogers [14]).} \end{cases}$$

For $d \geq 3$, we obtain the following lower bound of $\beta_d(\alpha)$, $\alpha \in (0, d]$, which improves the previously best known results above for all $\alpha \in (d/2, d)$:

THEOREM 1.6. *Let $d \geq 3$ and $\alpha \in (0, d]$. If $d = 3$, then*

$$\beta_3(\alpha) \geq \begin{cases} \dfrac{2\alpha}{3}, & \alpha \in (0, 2], \\ \dfrac{4}{3}, & \alpha \in \left(2, \dfrac{19}{9}\right], \\ \dfrac{3}{4}\alpha - \frac{1}{4}, & \alpha \in \left(\dfrac{19}{9}, 3\right]. \end{cases}$$

If $d \geq 4$, then

$$\beta_d(\alpha) \geq \max\left(\beta_d^0(\alpha), \alpha - 1 + \frac{d-\alpha}{d+1}\right),$$

with $\beta_d^0(\alpha)$ defined as

$$\beta_d^0(\alpha) := \begin{cases} \dfrac{(d-1)^2}{d}, & \alpha \in \big(\#_d, d\big], \\ \dfrac{\big(2d-3-2S_4^d\big)\alpha}{2d} + \dfrac{1}{d} + S_4^d, & \alpha \in \big(d-1, \#_d\big], \\ \dfrac{\big(d-1-S_4^d\big)\alpha}{d} - \dfrac{1}{2} + \dfrac{3}{2d} + S_4^d, & \alpha \in (d-2, d-1], \\ \dfrac{\big(d-1-S_\ell^d\big)\alpha}{d} - \dfrac{1}{2} + \dfrac{\ell-1}{2d} + S_\ell^d, & \alpha \in \left(d-\dfrac{\ell}{2}, d-\dfrac{\ell}{2}+\dfrac{1}{2}\right], \ \forall 5 \le \ell \le d, \\ \dfrac{(d-1)\alpha}{d}, & \alpha \in \left(0, \dfrac{d}{2}\right], \end{cases}$$

where S_ℓ^d, $\#_d$ are defined as in Theorem 1.4 above.

In the higher dimensional case of Theorem 1.6 above, $\alpha - 1 + \frac{d-\alpha}{d+1}$ gives the better lower bound if α is large while $\beta_d^0(\alpha)$ is better if α is small. We also point out that for $\alpha \in (0, d/2)$, the lower bound in Theorem 1.6 is not as good as the result of Mattila [15].

Remark 1.7. Note that by Stein-Tomas restriction theorem and Hölder's inequality, for $d \ge 3$ we have

$$\|Ef\|_{L^{2d/(d-1)}(B_R;Hdx)} \lesssim R^{\frac{\alpha(d-1)}{2d(d+1)}} \|f\|_{L^2(B^{d-1})}$$

for all $f \in L^2(B^{d-1})$, all $R > 1$ and all $H \in \mathcal{F}_{\alpha,d}$. This estimate is better than Theorem 1.4 when $\alpha \ge d - \frac{1}{d}$. In our approach, when α is large, the exponent $\gamma_d^0(\alpha) = \frac{(d-1)(\alpha+1-d)}{2d}$ comes from a technical constraint in the proof. More precisely, it arises in the step of applying parabolic rescaling (as introduced in Lemma 2.1 below) when reducing the linear estimate to a (weak) bilinear one, and that estimate is not good enough. While for application to the average decay rates for large α, instead of applying Theorem 1.4, we prove a linear L^2 estimate using directly the so-called refined Strichartz estimate (see Theorem 3.1 below), which gives the decay rates $\alpha - 1 + \frac{d-\alpha}{d+1}$ in Theorem 1.6 and improves previously best known results when α is large (see Section 3 for details).

Remark 1.8. It follows from the variable-coefficient generalization as discussed in [10, 11], that the same weighted restriction estimates in Theorem 1.4 above still hold true if one replaces the paraboloid by sphere or other positively curved hypersurfaces. In particular, to deduce Theorems 1.2 and 1.6 from Theorem 1.4 using Mattila's approach, as described in Subsection 2.2 below, we will replace the paraboloid in the weighted restriction estimates by the sphere. For the sake of simplicity, we choose to present the proof of Theorem 1.4 using the paraboloid, where computations such as ones arising in parabolic rescaling (see Lemma 2.1) can be made much cleaner.

The estimates of the Fourier decay rate of fractal measures in Theorem 1.6 also imply the following improved result for the pinned distance set problem, by applying Theorem 1.4 of a very recent work of Liu [13].

COROLLARY 1.9. *Let $d \geq 3$ and $E \subset \mathbb{R}^d$ be a compact set with*

$$\dim(E) > \alpha, \quad \alpha := \begin{cases} 1.8, & d = 3, \\ \frac{d}{2} + \frac{1}{4} + \frac{d+1}{4(2d+1)(d-1)}, & d \geq 4. \end{cases}$$

Then there exists $x \in E$ such that its pinned distance set

$$\Delta_x(E) := \big\{ |x-y| : y \in E \big\}$$

has positive Lebesgue measure.

In addition, Theorem 1.6 implies directly improved upper bounds of the Hausdorff dimension of divergence sets of solutions to wave equations, by applying [14, Proposition 1.5]. We omit the details.

The key ingredients in our proofs are the method of polynomial partitioning developed by the second author [9, 10] and (linear and bilinear) refined Strichartz estimates obtained by Li and the first two authors in [4]. Polynomial partitioning has proved to be extremely powerful in the study of restriction type problems such as the restriction estimates for the paraboloid [9, 10], the cone [22] and Hörmander-type oscillatory integral operators [11]. The sharp Schrödinger maximal estimate in $\mathbb{R}^2$ [4] was also recently derived via the polynomial partitioning scheme, combined with the aforementioned refined Strichartz estimates.

Compared to [6], where the previously best known result for Falconer's problem in $d \geq 3$ was proved via a similar route through weighted restriction estimates, our argument has the following advantages. First, the use of polynomial partitioning enables one to obtain a more delicate estimate by inducting on dimensions and extracting information from every intermediate dimension. Second, in every fixed intermediate dimension, compared to Hölder's inequality that is used in [6], the (linear and bilinear) refined Strichartz estimates provide much finer estimates. The latter advantage is particularly important for deriving the three-dimensional case of Theorem 1.4, where there is not much information available from lower dimensions while the bilinear refined Strichartz estimate plays a key role. In fact, in a recent work of Shayya [23], polynomial partitioning alone was applied to obtain some improved weighted restriction estimate in $\mathbb{R}^3$, which doesn't seem to imply improved partial result towards Falconer's conjecture without the help of the refined Strichartz estimate.

Other than weighted restriction estimates and decay of spherical means of Fourier transform of fractal measures, other approaches (with a heavier geometric flavor) have been tried as well for Falconer's problem. For instance, in two dimensions the full Falconer conjecture has been established by Orponen [20] under the additional assumption that the set considered is Ahlfors-David regular, with a slightly weaker conclusion that the upper Minkowski dimension of its distance set is equal to 1. This assumption was later relaxed by Shmerkin [25] to the one that the packing dimension of the set coincides with its Hausdorff dimension. We also point the interested reader to [12, 19, 21, 24] and the references therein for more results along this line of research.

The structure of the paper is as follows. In Section 2, we review preliminaries including parabolic rescaling, wave packet decomposition and Matilla's approach, and explain the connection between Theorems 1.4 and 1.6 and how they imply Theorem 1.2. In Section 3, we review linear and bilinear refined Strichartz estimates, and obtain some partial improvements towards Falconer's distance set problem and average decay rates. In Section 4, we prove Theorem 1.4 in the case $d=3$, using polynomial partitioning and bilinear refined Strichartz. The proof of the $d\geq 4$ case of Theorem 1.4 in the restricted range $\alpha\in[\frac{d}{2},\frac{d+1}{2}]$ is presented in Section 5, and additional ingredients that are needed in generalizing it to the full range of α are discussed in Section 6.

List of notation. We write $A\lessapprox B$ if $A\leq C_\varepsilon R^\varepsilon B$ for any $\varepsilon>0$, $R>1$; $A\lesssim_\varepsilon B$ if $A\leq C_\varepsilon B$, $A\lesssim_{K,\varepsilon} B$ if $A\leq C_{K,\varepsilon}B$, etc; $A\lesssim B$ if $A\leq CB$ for a constant C which only depends on some unimportant fixed variables such as d,α and sometimes ε too.

Let m be a dimension in the range $1\leq m\leq d$. Denote

$$r_m:=\frac{2(m+1)}{m}=p_{m+1}<p_m:=\frac{2m}{m-1}<q_m:=\frac{2(m+1)}{m-1}.$$

Let B_R^m stand for a ball of radius R in $\mathbb{R}^m$, B^m denote the unit ball in $\mathbb{R}^m$ and B_R abbreviate B_R^d for simplicity.

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2. Preliminaries.

2.1. Parabolic rescaling.

LEMMA 2.1. *There exists an absolute constant C so that the following holds true. Let $p\geq 1$, $\alpha\in(0,d]$ and $\tilde{R}$ be a sufficiently large constant. Suppose that*

$$\|Ef\|_{L^p(B_R;Hdx)}\leq \tilde{C}R^\gamma\|f\|_{L^2(B^{d-1})} \tag{2.1}$$

holds for all $f \in L^2(B^{d-1})$*, all* $1 < R \le \tilde{R}/2$ *and all* $H \in \mathcal{F}_{\alpha,d}$*. Then*

$$\|Ef\|_{L^p(B_R;Hdx)} \le C\tilde{C}K^{\frac{\alpha+1}{p}-\frac{d-1}{2}-\gamma}R^{\gamma}\|f\|_{L^2(B^{d-1})}$$

holds for all $H \in \mathcal{F}_{\alpha,d}$*, all* $1 < R \le \tilde{R}$ *and all* $f \in L^2$ *with support in some ball of radius* $1/K$ *inside* B^{d-1}*, where* K *is any large constant* $< \tilde{R}$*.*

Proof. Let $H \in \mathcal{F}_{\alpha,d}$ and $f \in L^2$ with $\operatorname{supp} f \subset B(\omega_0, 1/K) \subset B^{d-1}$. We write $\omega = \omega_0 + \frac{1}{K}\xi \in B(\omega_0, 1/K)$, then by change of variables,

$$\left|Ef\left(x', x_d\right)\right| = \frac{1}{K^{(d-1)/2}}\left|Eg\left(y', y_d\right)\right|,$$

where $g \in L^2(B^{d-1})$ with $\|g\|_2 = \|f\|_2$, more precisely,

$$g(\xi) = \frac{1}{K^{(d-1)/2}} f\left(\omega_0 + \frac{1}{K}\xi\right),$$

and the new coordinates (y', y_d) are related to the old coordinates (x', x_d) by

$$\begin{cases} y' = \dfrac{1}{K}x' + \dfrac{2x_d}{K}\omega_0, \\ y_d = \dfrac{x_d}{K^2}. \end{cases}$$

For simplicity, we denote the relation above by $y = T(x)$. Therefore,

$$\left\|Ef(x)\right\|_{L^p(B_R;H(x)dx)} = K^{\frac{d+1}{p}-\frac{d-1}{2}+\frac{\alpha-d}{p}}\left\|Eg(y)\right\|_{L^p(\tilde{B};H^*(y)dy)},$$

where $\tilde{B} = T(B_R)$ is contained in a box of dimensions $\sim \frac{R}{K} \times \cdots \times \frac{R}{K} \times \frac{R}{K^2}$, and the function H^* is given by

$$H^*(y) = K^{d-\alpha}H\left(T^{-1}y\right).$$

Note that for any $x_0 \in \mathbb{R}^d$ and any $r \ge 1$,

$$\int_{B(x_0,r)} H^*(y)\,dy = K^{d-\alpha}K^{-(d+1)}\int_{\bar{B}} H(x)\,dx,$$

where $\bar{B} = T^{-1}(B(x_0,r))$ is contained in K balls of radius $\sim Kr$, hence it follows from $H \in \mathcal{F}_{\alpha,d}$ that

$$\int_{B(x_0,r)} H^*(y)\,dy \lesssim K^{-\alpha-1}K(Kr)^{\alpha} = r^{\alpha},$$

i.e., a constant multiple of H^* lies in $\mathcal{F}_{\alpha,d}$. Applying (2.1) to functions g and H^* with radius R/K we obtain

$$\|Ef\|_{L^p(B_R;Hdx)} \lesssim \tilde{C} K^{\frac{\alpha+1}{p}-\frac{d-1}{2}-\gamma} R^\gamma \|f\|_{L^2}.$$

This completes the proof. □

2.2. Mattila's approach. Our study of Falconer's distance set problem follows a scheme that goes back to Mattila [15]. We briefly recall this approach here. See also for example Lemma 2.1 in [7].

Let $d\sigma$ be the $(d-1)$-dimensional surface measure on S^{d-1} and $E_{S^{d-1}}$ stand for the extension operator from the unit sphere S^{d-1}.

THEOREM 2.2. (Mattila [15]) *Fix $\alpha \in (d/2, d)$. Assume that for all α-dimensional compactly supported probability measure μ there holds*

$$\left\|\widehat{\mu}(R\cdot)\right\|_{L^2(S^{d-1};d\sigma)} \leq C_\mu R^{\frac{\alpha-d}{2}}, \quad \forall R > 1. \tag{2.2}$$

Then Falconer's conjecture holds for α, i.e., for any compact subset E of $\mathbb{R}^d$,

$$\dim(E) > \alpha \Longrightarrow \left|\Delta(E)\right| > 0.$$

Sketch of the proof of Theorem 2.2. If E is a compact subset of $\mathbb{R}^d$ with $\dim E > \alpha$, then by Frostman's lemma (see for instance [17, Theorem 2.7]) E supports an $(\alpha+\varepsilon_0)$-dimensional measure μ, for some $\varepsilon_0 > 0$. In particular, μ is also α-dimensional and the α-dimensional energy of μ is finite:

$$I_\alpha(\mu) := \int\int |x-y|^{-\alpha} d\mu(x) d\mu(y) = c_\alpha \int |\widehat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi < \infty.$$

We refer the reader to [28, Chapter 8] for more details of α-dimensional energy and the justification of the equality above. We then have by assumption (2.2)

$$\begin{aligned} &\int_1^\infty \left(\int_{S^{d-1}} \left|\widehat{\mu}(Rx)\right|^2 d\sigma(x)\right)^2 R^{d-1} dR \\ &\qquad \lesssim \int_1^\infty \left(\int_{S^{d-1}} |\widehat{\mu}(Rx)|^2 d\sigma(x)\right) R^{\alpha-d} \cdot R^{d-1} dR \sim I_\alpha(\mu) < \infty, \end{aligned} \tag{2.3}$$

where the last equivalence follows from the Fourier representation of the energy.

Matilla proved that the quantity on the left-hand side of (2.3) can be related to $\Delta(E)$. More precisely, let $\nu = \Delta(\mu \times \mu)$, i.e.,

$$\int f\, d\nu = \int f(|x-y|)\, d\mu(x) d\mu(y).$$

It is easy to see that $\widehat{\nu} \in L^2$ would imply $|\Delta(E)| > 0$. Via a fairly straightforward calculation, one can reduce $\|\widehat{\nu}\|_{L^2}$ to the left-hand side of (2.3), thus the desired result follows immediately from (2.3). See [28, Chapter 9, Section B] or [13, Appendix] for more details. □

Remark 2.3. As noted in Wolff [27] and Erdoğan [6], Mattila's approach described above cannot be used to prove the full Falconer's conjecture in dimension 2 or 3. The reason is due to counter-examples which restrict the range of the Fourier decay rates $\beta_d(\alpha)$ when $\alpha \geq d/2$:

$$\beta_d(\alpha) \leq \begin{cases} \dfrac{\alpha}{2}, & d=2, \quad [15, 26] \\ \dfrac{\alpha+1}{2}, & d=3, \quad [17, \text{Chapter } 15.2] \\ \alpha - 1 + \dfrac{2(d-\alpha)}{d}, & d \geq 4, \quad [14]. \end{cases}$$

Therefore, the best possible thresholds towards Falconer's conjecture that Theorem 2.2 would imply are $\frac{4}{3}$ when $d=2$ and $\frac{5}{3}$ when $d=3$. However, one might be able to prove Falconer's conjecture in dimension $d \geq 4$ using this method.

PROPOSITION 2.4. *Let $\alpha \in (0,d)$, $p \geq 1$, and $E_{S^{d-1}} f := (f d\sigma)^{\vee}$, the inverse Fourier transform of $f d\sigma$. Suppose that*

$$\big\|E_{S^{d-1}} f\big\|_{L^p(B_R;H dx)} \lesssim R^{\gamma+\varepsilon} \|f\|_{L^2(S^{d-1})}, \quad \forall R > 1 \tag{2.4}$$

holds for all $f \in L^2(S^{d-1})$ and all $H \in \mathcal{F}_{\alpha,d}$, and that

$$\gamma \leq \alpha\left(\frac{1}{p} + \frac{1}{2}\right) - \frac{d}{2}. \tag{2.5}$$

Then, Falconer's conjecture holds for α, i.e.,

$$\dim(E) > \alpha \Longrightarrow \big|\Delta(E)\big| > 0.$$

Proof. The proof is essentially contained in Wolff [27] and Erdoğan [7]. We follow their treatment here.

Given (2.4), it suffices to verify the averaged decay estimate (2.2) and apply Theorem 2.2. Without loss of generality we assume μ is supported in the unit ball. We use a duality argument. Take an arbitrary function $f \in L^2(S^{d-1})$. By (2.4), we have for all $H \in \mathcal{F}_{\alpha,d}$,

$$\left(\int_{B_R} |(f d\sigma)^{\vee}(x)|^p H(x) dx\right)^{\frac{1}{p}} \lesssim R^{\gamma+\varepsilon} \|f\|_{L^2(S^{d-1};d\sigma)}. \tag{2.6}$$

Now take a radial Schwartz bump function ψ such that $\psi(x) = 1$ for all $|x| = 1$, and such that $\widehat{\psi}$ has compact support. Notice that a constant multiple of $R^{\alpha} \cdot \mu(\frac{\cdot}{R}) *$

$|\widehat{\psi}|$ is a function in $\mathcal{F}_{\alpha,d}$, where the dilated measure $\mu(\frac{\cdot}{R})$ is defined as

$$\int f(x)\,d\mu\Big(\frac{\cdot}{R}\Big) := \int f(Rx)\,d\mu.$$

Indeed, for any $r \geq 1$, the measure of any ball of radius r with respect to $R^\alpha \cdot \mu(\frac{\cdot}{R}) * |\widehat{\psi}|$ is $\lesssim R^\alpha(\frac{r}{R})^\alpha = r^\alpha$. Moreover, $R^\alpha \cdot \mu(\frac{\cdot}{R}) * |\widehat{\psi}|$ has its L^1 norm $\lesssim R^\alpha$ times the total measure of μ, which is in turn bounded by a constant times R^α.

Applying (2.6) to the function $H = R^\alpha \cdot \mu(\frac{\cdot}{R}) * |\hat{\psi}|$ and using Hölder's inequality, one obtains

$$\int_{B_R} |(f d\sigma)^\vee(x)| H(x) dx \lesssim R^{\gamma+\frac{\alpha}{p'}+\varepsilon} \|f\|_{L^2(S^{d-1};d\sigma)} \tag{2.7}$$

or

$$\int_{B_R} \big|(f d\sigma)^\vee(x)\big| \left(R^\alpha \cdot \mu\Big(\frac{x}{R}\Big) * |\widehat{\psi}|\right) dx \lesssim R^{\gamma+\frac{\alpha}{p'}+\varepsilon} \|f\|_{L^2(S^{d-1};d\sigma)}. \tag{2.8}$$

Since $\psi = 1$ on the unit sphere, $(f d\sigma)^\vee * \widehat{\psi} = (f d\sigma)^\vee$. Hence,

$$\int_{B_R} \big|(f d\sigma)^\vee(x)\big| d\mu\Big(\frac{x}{R}\Big) \lesssim R^{\gamma-\frac{\alpha}{p}+\varepsilon} \|f\|_{L^2(S^{d-1};d\sigma)}. \tag{2.9}$$

Note that the measure $d\mu(\frac{\cdot}{R})$ has Fourier transform $\widehat{\mu}(R\cdot)$. By duality and Matilla's Theorem 2.2, Falconer's conjecture holds for α as long as $\gamma - \frac{\alpha}{p} \leq \frac{\alpha-d}{2}$ (we removed the ϵ here because when $\dim(E) > \alpha$, it is also $>$ some $\alpha+\varepsilon$). This is equivalent to $\gamma \leq \alpha(\frac{1}{p}+\frac{1}{2}) - \frac{d}{2}$, as claimed in (2.5). □

It is now clear, according to Proposition 2.4, that Theorem 1.2 follows directly from Theorem 1.4.

Remark 2.5. From the proof of the proposition above, one concludes directly that under the assumption of Proposition 2.4 except for (2.5), there holds the lower bound estimate for the Fourier decay rates of fractal measures

$$\beta_d(\alpha) \geq 2\left(\frac{\alpha}{p} - \gamma\right), \tag{2.10}$$

where $\beta_d(\alpha)$ is as defined in (1.3). From this we see that Theorem 1.6 follows from Theorem 1.4 and (3.7) below.

2.3. Proof of Remark 1.3. For the sake of completeness, we give a justification of Remark 1.3 in this subsection.

Let ψ be a Schwartz bump function such that $\psi = 1$ on the ball $C \cdot B^d$, hence $F = F * \check{\psi}$ as $\operatorname{supp} \widehat{F} \subset B^d$. Therefore for all $p \geq 1$, by Hölder's inequality,

$$\begin{aligned}\int |F|^p H\, dx &= \int \left| \int F(y) \check{\psi}(x-y)\, dy \right|^p H(x)\, dx \\ &\leq \int \left(\int |F(y)|^p |\check{\psi}(x-y)|\, dy \right) \left(\int |\check{\psi}(x-y)|\, dy \right)^{p-1} H(x)\, dx \\ &\lesssim \int |F(y)|^p \left(\int |\check{\psi}(x-y)| H(x)\, dx \right) dy.\end{aligned}$$

Observe that for any $y \in \mathbb{R}^d$ and sufficiently large $M = M(\alpha) > 0$,

$$\begin{aligned}\int |\check{\psi}(x-y)| H(x)\, dx &\leq C_M \sum_{j=0}^{\infty} \int \chi_{B(y,2^j)}(x) 2^{-jM} H(x)\, dx \\ &\lesssim C_M \sum_{j=0}^{\infty} 2^{j(\alpha - M)} < \infty,\end{aligned}$$

where we have used the fact that $H \in \mathcal{F}_{\alpha,d}$. Hence the desired estimate follows.

2.4. Wave packet decomposition. We use the same setup as in Section 3 of [10], which we briefly recall here. Let f be a function on B^{d-1}, we break it up into pieces $f_{\theta,\nu}$ that are essentially localized in both position and frequency. Cover B^{d-1} by finitely overlapping balls θ of radius $R^{-1/2}$ and cover $\mathbb{R}^{d-1}$ by finitely overlapping balls of radius $R^{\frac{1+\delta}{2}}$, centered at $\nu \in R^{\frac{1+\delta}{2}} \mathbb{Z}^{d-1}$. Using partition of unity, we have a decomposition

$$f = \sum_{(\theta,\nu) \in \mathbb{T}} f_{\theta,\nu} + \mathrm{RapDec}(R) \|f\|_{L^2},$$

where $f_{\theta,\nu}$ is supported in θ and has Fourier transform roughly supported in a ball of radius $R^{1/2+\delta}$ around ν. The functions $f_{\theta,\nu}$ are approximately orthogonal. In other words, for any set $\mathbb{T}' \subset \mathbb{T}$ of pairs (θ, ν), we have

$$\left\| \sum_{(\theta,\nu) \in \mathbb{T}'} f_{\theta,\nu} \right\|_{L^2}^2 \sim \sum_{(\theta,\nu) \in \mathbb{T}'} \|f_{\theta,\nu}\|_{L^2}^2. \tag{2.11}$$

For each pair (θ, ν), the restriction of $Ef_{\theta,\nu}$ to B_R is roughly supported on a tube $T_{\theta,\nu}$ and rapidly decays away from it. $T_{\theta,\nu}$ has radius $R^{1/2+\delta}$ and length R, with direction $G(\theta) \in S^{d-1}$ determined by θ and location determined by ν: more precisely,

$$T_{\theta,\nu} := \left\{ (x', x_d) \in B_R : |x' + 2x_d \omega_\theta - \nu| \leq R^{1/2+\delta} \right\}.$$

Here $\omega_\theta \in B^{d-1}$ is the center of θ, and

$$G(\theta) = \frac{\big(-2\omega_\theta, 1\big)}{\big|\big(-2\omega_\theta, 1\big)\big|}.$$

In our proof, a key concept is a wave packet being *tangent* to an algebraic variety. We write $Z(P_1, \ldots, P_{d-m})$ for the set of common zeros of the polynomials $P_1, \ldots, P_{d-m}$. The variety $Z(P_1, \ldots, P_{d-m})$ is called a *transverse complete intersection* if

$$\nabla P_1(x) \wedge \cdots \wedge \nabla P_{d-m}(x) \neq 0 \quad \text{for all } x \in Z\big(P_1, \ldots, P_{d-m}\big).$$

For each $\varepsilon > 0$, there is a sequence of small parameters

$$\delta_{\deg} \ll \delta \ll \delta_{d-1} \ll \delta_{d-2} \ll \cdots \ll \delta_1 \ll \delta_0 \ll \varepsilon.$$

For $Z = Z(P_1, \ldots, P_{d-m})$, D_Z denotes an upper bound of the degrees of $P_1, \ldots, P_{d-m}$. Usually $D_Z \le R^{\delta_{\deg}}$ unless noted otherwise.

Let Z be an algebraic variety and M be a positive number. For any $(\theta, \nu) \in \mathbb{T}$, we say that $T_{\theta,\nu}$ is $MR^{-1/2}$-*tangent* to Z if

$$T_{\theta,\nu} \subset N_{MR^{1/2}} Z \cap B_R, \quad \text{and} \quad \text{Angle}\big(G(\theta), T_z Z\big) \le MR^{-1/2}$$

for any non-singular point $z \in N_{2MR^{1/2}}(T_{\theta,\nu}) \cap 2B_R \cap Z$.

Let

$$\mathbb{T}_Z(M) := \big\{(\theta,\nu) \in \mathbb{T} \mid T_{\theta,\nu} \text{ is } MR^{-1/2}\text{-tangent to } Z\big\},$$

and we say that f is concentrated in wave packets from $\mathbb{T}_Z(M)$ if

$$\sum_{(\theta,\nu) \notin \mathbb{T}_Z(M)} \big\|f_{\theta,\nu}\big\|_{L^2} \le \text{RapDec}(R)\|f\|_{L^2}.$$

Since the radius of $T_{\theta,\nu}$ is $R^{1/2+\delta}$, R^δ is the smallest interesting value of M.

3. Linear and bilinear refined Strichartz estimates in higher dimensions. One of the key ingredients in our proof is the linear and bilinear refined Strichartz estimates established in [4]. Below by "dyadically a constant" we mean "a constant up to a factor of 2".

THEOREM 3.1. (Linear refined Strichartz for m-variety in d dimensions) *Let $d \ge 2$ and m be a dimension in the range $2 \le m \le d$. Let $q_m = 2(m+1)/(m-1)$. Suppose that $Z = Z(P_1, \ldots, P_{d-m})$ is a transverse complete intersection where* $\text{Deg}\, P_i \le D_Z$. *Suppose that $f \in L^2(B^{d-1})$ is concentrated in wave packets from $\mathbb{T}_Z(M)$. Suppose that $Q_1, Q_2, \ldots$ are lattice $R^{1/2}$-cubes in B_R, so that*

$$\|Ef\|_{L^{q_m}(Q_j)} \textit{ is dyadically a constant in } j.$$

Suppose that these cubes are arranged in horizontal strips of the form $\mathbb{R} \times \cdots \times \mathbb{R} \times \{t_0, t_0 + R^{1/2}\}$, *and that each such strip contains* $\sim \sigma$ *cubes* Q_j. *Let* Y *denote* $\bigcup_j Q_j$. *Then*

$$\|Ef\|_{L^{q_m}(Y)} \lessapprox M^{O(1)} \sigma^{-\frac{1}{m+1}} R^{-\frac{d-m}{2(m+1)}} \|f\|_{L^2(B^{d-1})}. \tag{3.1}$$

THEOREM 3.2. (Bilinear refined Strichartz for m-variety in d dimensions) *Let* $d \geq 2$ *and* m *be a dimension in the range* $2 \leq m \leq d$. *Let* $q_m = 2(m+1)/(m-1)$. *For functions* f_1 *and* f_2 *in* $L^2(B^{d-1})$, *with supports separated by* ~ 1, *suppose that* f_1 *and* f_2 *are concentrated in wave packets from* $\mathbb{T}_Z(M)$, *where* $Z = Z(P_1, \ldots, P_{d-m})$ *is a transverse complete intersection with* $\operatorname{Deg} P_i \leq D_Z$. *Suppose that* $Q_1, Q_2, \cdots, Q_N$ *are lattice* $R^{1/2}$*-cubes in* B_R, *so that for each* i,

$$\big\|Ef_i\big\|_{L^{q_m}(Q_j)} \text{ is dyadically a constant in } j.$$

Let Y *denote* $\bigcup_{j=1}^N Q_j$. *Then*

$$\begin{aligned} &\Big\| |Ef_1|^{1/2} |Ef_2|^{1/2} \Big\|_{L^{q_m}(Y)} \\ &\lessapprox M^{O(1)} R^{-\frac{d-m}{2(m+1)}} N^{-\frac{1}{2(m+1)}} \big\|f_1\big\|_{L^2(B^{d-1})}^{1/2} \big\|f_2\big\|_{L^2(B^{d-1})}^{1/2}. \end{aligned} \tag{3.2}$$

Theorem 3.1 and Theorem 3.2 were proved in [4] in the case $m = 2, d = 3$, via the Bourgain-Demeter ℓ^2-decoupling theorem [2] and induction on scales. The proof for general m and d follows from exactly the same lines, with only changes in numerology, thus we skip the proof and refer interested readers to Section 7 in [4].

The following weighted linear and bilinear restriction estimates are immediate consequences of Theorem 3.1 and Theorem 3.2.

COROLLARY 3.3. (Linear weighted L^2 estimate) *Let* $d \geq 2$ *and* $\alpha \in (0, d]$. *Let* m *be a dimension in the range* $2 \leq m \leq d$. *Suppose that* $Z = Z(P_1, \ldots, P_{d-m})$ *is a transverse complete intersection where* $\operatorname{Deg} P_i \leq D_Z$, *and that* $f \in L^2(B^{d-1})$ *is concentrated in wave packets from* $\mathbb{T}_Z(M)$ *and* $H \in \mathcal{F}_{\alpha,d}$. *Then*

$$\|Ef\|_{L^2(B_R; H dx)} \lessapprox M^{O(1)} R^{\frac{1}{2} - \frac{d-\alpha}{2(m+1)}} \|f\|_{L^2(B^{d-1})}. \tag{3.3}$$

Proof. Without loss of generality, assume that $\|f\|_{L^2} = 1$. We break B_R into $R^{1/2}$-cubes Q_j. Let $\mathcal{Y}_{\gamma,\sigma}$ denote the collection of those Q_j's such that

- $\|Ef\|_{L^{q_m}(Q_j)} \sim \gamma$,
- the horizontal $R^{1/2}$-strip containing Q_j contains $\sim \sigma$ $R^{1/2}$-cubes satisfying the above condition.

Define $Y_{\gamma,\sigma} := \bigcup_{Q_j \in \mathcal{Y}_{\gamma,\sigma}} Q_j$. Note that we can assume $1 \leq \sigma \leq R^{\frac{d-1}{2}}$ and $R^{-C} \leq \gamma \leq R^C$, where C is a large constant. More precisely, the upper bound of γ follows from the trivial estimate $\|Ef\|_{L^\infty} \lesssim 1$, and the lower bound of γ can be assumed since the contribution to $\|Ef\|_{L^2(B_R;Hdx)}$ from those subsets $Y_{\gamma,\sigma}$ with sufficiently small γ is negligible. Hence there are only $\sim (\log R)^2$ relevant dyadic scales (γ,σ), and

$$\|Ef\|_{L^2(B_R;Hdx)} \lesssim (\log R)^2 \|Ef\|_{L^2(Y;Hdx)},$$

where $Y = Y_{\gamma,\sigma}$ for some (γ,σ). Therefore by Hölder's inequality and Theorem 3.1 one has

$$\begin{aligned}\|Ef\|_{L^2(B_R;Hdx)} &\lessapprox \|Ef\|_{L^{2(m+1)/(m-1)}(Y)} \left(\int_Y H\,dx\right)^{1/(m+1)} \\ &\lessapprox M^{O(1)} \sigma^{-1/(m+1)} R^{-(d-m)/2(m+1)} \|f\|_{L^2} \left(NR^{\alpha/2}\right)^{1/(m+1)},\end{aligned}$$

where N is the number of $R^{1/2}$-cubes in Y. Note that $N \lesssim \sigma R^{1/2}$, the above is thus further bounded by

$$\lessapprox M^{O(1)} R^{\frac{1}{2}-\frac{d-\alpha}{2(m+1)}},$$

as desired. □

COROLLARY 3.4. (Bilinear weighted L^{r_m} estimate) *Let $d \geq 2$ and $\alpha \in (0,d]$. Let m be a dimension in the range $2 \leq m \leq d$. For functions f_1 and f_2 in $L^2(B^{d-1})$, with supports separated by ~ 1, suppose that f_1 and f_2 are concentrated in wave packets from $\mathbb{T}_Z(M)$, where $Z = Z(P_1,\ldots,P_{d-m})$ is a transverse complete intersection with* $\operatorname{Deg} P_i \leq D_Z$. *Let $r_m := \frac{2(m+1)}{m}$ and $H \in \mathcal{F}_{\alpha,d}$. Then,*

$$\begin{aligned}\left\| |Ef_1|^{1/2} |Ef_2|^{1/2} \right\|_{L^{r_m}(B_R;Hdx)} \\ \lessapprox M^{O(1)} R^{-\frac{2d-2m-\alpha}{4(m+1)}} \left\|f_1\right\|_{L^2(B^{d-1})}^{1/2} \left\|f_2\right\|_{L^2(B^{d-1})}^{1/2}.\end{aligned} \tag{3.4}$$

Proof. Without loss of generality, assume that $\|f_1\|_{L^2} = \|f_2\|_{L^2} = 1$. We break B_R into $R^{1/2}$-cubes Q_j. Let $\mathcal{Y}_{\gamma_1,\gamma_2}$ denote the collection of those Q_j's such that

$$\left\|Ef_i\right\|_{L^{q_m}(Q_j)} \sim \gamma_i, \quad i = 1,2.$$

Define $Y_{\gamma_1,\gamma_2} := \bigcup_{Q_j \in \mathcal{Y}_{\gamma_1,\gamma_2}} Q_j$. Note that there are only $\sim (\log R)^2$ relevant dyadic scales (γ_1,γ_2), similarly as in the proof of Corollary 3.3, hence

$$\left\| |Ef_1|^{1/2} |Ef_2|^{1/2} \right\|_{L^{r_m}(B_R;Hdx)} \lesssim (\log R)^2 \left\| |Ef_1|^{1/2} |Ef_2|^{1/2} \right\|_{L^{r_m}(Y;Hdx)},$$

where $Y = Y_{\gamma_1,\gamma_2} := \bigcup_{j=1}^{N} Q_j$ for some (γ_1,γ_2). Therefore by Hölder's inequality and Theorem 3.2 one obtains

$$\begin{aligned}
&\left\| |Ef_1|^{1/2} |Ef_2|^{1/2} \right\|_{L^{r_m}(B_R;H dx)} \\
&\lessapprox \left\| |Ef_1|^{1/2} |Ef_2|^{1/2} \right\|_{L^{q_m}(Y)} \left(\int_Y H\, dx \right)^{\frac{1}{2(m+1)}} \\
&\lessapprox M^{O(1)} R^{-\frac{d-m}{2(m+1)}} N^{-\frac{1}{2(m+1)}} \|f_1\|_{L^2}^{1/2} \|f_2\|_{L^2}^{1/2} \left(N R^{\frac{\alpha}{2}} \right)^{\frac{1}{2(m+1)}} = M^{O(1)} R^{-\frac{2d-2m-\alpha}{4(m+1)}},
\end{aligned}$$

as desired. □

Remark 3.5. The linear weighted L^2 estimate in Corollary 3.3 in the case $m = d$ says that for all $d \geq 2$, $\alpha \in (0,d]$, $f \in L^2(B^{d-1})$, $H \in \mathcal{F}_{\alpha,d}$,

$$\|Ef\|_{L^2(B_R;H dx)} \lessapprox R^{\frac{1}{2}-\frac{d-\alpha}{2(d+1)}} \|f\|_{L^2}. \tag{3.5}$$

Therefore it follows from Proposition 2.4 that for a compact subset E of $\mathbb{R}^d$,

$$\dim(E) > \frac{d}{2} + \frac{1}{4} + \frac{3}{8d+4} \Longrightarrow \left|\Delta(E)\right| > 0. \tag{3.6}$$

This already improves Erdoğan's result [6] for $d > 4$. In addition, by Remark 2.5 one obtains

$$\beta_d(\alpha) \geq \alpha - 1 + \frac{d-\alpha}{d+1}. \tag{3.7}$$

This improves Erdoğan's result [6] in the range $\alpha > \frac{d}{2} + \frac{1}{d-1}$ and Lucà-Rogers' result [14] in the full range.

The results (3.6) on Falconer's problem and (3.7) on average decay rates in the above can be further improved to Theorem 1.4, by combining refined Strichartz estimates and the method of polynomial partitioning developed by the second author [9, 10]. This will be the content of the rest of the paper.

4. Weighted extension estimates in three dimensions: proof of Theorem 1.4 for $d = 3$. In this section, we prove the case $d = 3$ of Theorem 1.4 using polynomial partitioning and bilinear refined Strichartz estimates, building on the work of [4, 9].

For any $\alpha \in (0,3]$ and $p > 3$, we will prove that

$$\|Ef\|_{L^p(B_R;H dx)} \lessapprox R^{\gamma_3^0(\alpha)} \|f\|_{L^2(B^2)} \tag{4.1}$$

holds for all $f \in L^2(B^2)$, all $R > 1$ and all $H \in \mathcal{F}_{\alpha,3}$, where

$$\gamma_3^0(\alpha) = \begin{cases} 0, & \alpha \in (0,2], \\ \dfrac{\alpha}{3} - \dfrac{2}{3}, & \alpha \in (2,3]. \end{cases}$$

The weighted L^3 estimate in Theorem 1.4 follows from Hölder's inequality and the estimate (4.1) by taking $p \to 3^+$. Note that one can assume R is sufficiently large, as otherwise the bound (4.1) becomes trivial. The proof uses induction on the radius R. More precisely, we will show that the desired estimate follows if it holds true with a smaller R value.

4.1. Polynomial partitioning and cell contributions. We pick a degree $D = R^{\delta_{\text{deg}}}$, where $\delta_{\text{deg}} \ll \delta\varepsilon$. We first recall the following polynomial partitioning theorem (cf. Theorem 1.4 in [9]):

THEOREM 4.1. *(Guth [9]) Suppose that $W \geq 0$ is a (non-zero) L^1 function on $\mathbb{R}^n$. Then for each D there is a non-zero polynomial P of degree at most D so that $\mathbb{R}^n \setminus Z(P)$ is a union of $\sim D^n$ disjoint open sets O_i, and the integrals $\int_{O_i} W$ are all equal.*

According to the theorem above, there exists a non-zero polynomial P of degree at most D such that $\mathbb{R}^3 \setminus Z(P)$ is a union of $\sim D^3$ disjoint open sets O_i and for each i there holds

$$\|Ef\|^p_{L^p(B_R;Hdx)} \sim D^3 \|Ef\|^p_{L^p(B_R \cap O_i;Hdx)}. \tag{4.2}$$

Moreover, the polynomial P is a product of distinct non-singular polynomials.

Define the *wall*

$$W := N_{R^{1/2+\delta}} Z(P) \cap B_R, \tag{4.3}$$

where $\delta \ll \varepsilon$ and $N_{R^{1/2+\delta}} Z(P)$ stands for the $R^{1/2+\delta}$-neighborhood of the variety $Z(P)$ in $\mathbb{R}^3$. For each cell O_i, set

$$O_i' := \big[O_i \cap B_R\big] \setminus W \quad \text{and} \quad \mathbb{T}_i := \big\{(\theta,\nu) \in \mathbb{T} : T_{\theta,\nu} \cap O_i' \neq \emptyset\big\}. \tag{4.4}$$

For each function f, define

$$f_i := \sum_{(\theta,\nu) \in \mathbb{T}_i} f_{\theta,\nu}. \tag{4.5}$$

Then on each cell O_i', up to a rapidly decaying tail,

$$Ef(x) \sim Ef_i(x), \quad x \in O_i', \tag{4.6}$$

since $Ef_{\theta,\nu}$ is roughly supported on the tube $T_{\theta,\nu}$. As the number of roots of any non-zero polynomial over a field is at most its degree, we have a simple geometric observation: for each (θ,ν),

$$\#\{i : T_{\theta,\nu} \cap O_i' \neq \emptyset\} \leq D+1. \tag{4.7}$$

This geometric observation and orthogonality (2.11) allow us to control the L^2 norms of f_i's:

$$\sum_i \|f_i\|_{L^2}^2 \lesssim D\|f\|_{L^2}^2. \tag{4.8}$$

Break $\|Ef\|_{L^p(B_R;Hdx)}^p$ into

$$\sum_i \|Ef\|_{L^p(O_i';Hdx)}^p + \|Ef\|_{L^p(W;Hdx)}^p,$$

and call it the *algebraic* case if the wall contribution $\|Ef\|_{L^p(W;Hdx)}^p$ dominates, i.e.,

$$\|Ef\|_{L^p(B_R;Hdx)}^p \sim \|Ef\|_{L^p(W;Hdx)}^p.$$

We first consider the *non-algebraic* case, where the main contribution to $\|Ef\|_{L^p(B_R;Hdx)}^p$ comes from the cells O_i'. In the non-algebraic case,

$$\|Ef\|_{L^p(B_R;Hdx)}^p \sim D^3\|Ef\|_{L^p(O_i';Hdx)}^p \tag{4.9}$$

still holds for $\sim D^3$ indices i's, and among which by pigeonholing one can pick an i_0 such that

$$\|f_{i_0}\|_{L^2}^2 \lesssim D^{-2}\|f\|_{L^2}^2. \tag{4.10}$$

Now the non-algebraic case can be handled by induction:

$$\begin{aligned}\|Ef\|_{L^p(B_R;Hdx)}^p \sim D^3\|Ef\|_{L^p(O_{i_0}';Hdx)}^p &\lesssim D^3\|Ef_{i_0}\|_{L^p(B_R;Hdx)}^p \\ &\lesssim D^3\left(R^{\varepsilon+\gamma_3^0(\alpha)}\|f_{i_0}\|_{L^2}\right)^p \\ &\lesssim D^{3-p}\left(R^{\varepsilon+\gamma_3^0(\alpha)}\|f\|_{L^2}\right)^p.\end{aligned}$$

Recall that $D = R^{\delta_{\deg}}$ and R is assumed to be sufficiently large compared to any constant depending on ε, therefore $D^{3-p} \ll 1$ provided that $p > 3$, thus the induction closes.

4.2. Wall contribution. To deal with the wall term $\|Ef\|^p_{L^p(W;Hdx)}$, we break B_R into $\sim R^{3\delta}$ balls B_j of radius $R^{1-\delta}$.

For any $(\theta,\nu)\in\mathbb{T}$, $T_{\theta,\nu}$ is said to be *tangent* to the wall W in a given ball B_j if it satisfies that $T_{\theta,\nu}\cap B_j\cap W\neq\emptyset$ and

$$\text{Angle}\big(G(\theta),T_z\big[Z(P)\big]\big)\le R^{-1/2+2\delta} \tag{4.11}$$

for any non-singular point $z\in 10T_{\theta,\nu}\cap 2B_j\cap Z(P)$. Recall that $G(\theta)\in S^2$ is the direction of the tube $T_{\theta,\nu}$. Here $T_z[Z(P)]$ stands for the tangent space to the variety $Z(P)$ at the point z, and by a non-singular point we mean a point z in $Z(P)$ with $\nabla P(z)\neq 0$. Since P is a product of distinct non-singular polynomials, the non-singular points are dense in $Z(P)$. We note that if $T_{\theta,\nu}$ is tangent to W in B_j, then $T_{\theta,\nu}\cap B_j$ is contained in the $R^{1/2+\delta}$-neighborhood of $Z(P)\cap 2B_j$.

We say that $T_{\theta,\nu}$ is *transverse* to the wall W in the ball B_j if it enjoys the property that $T_{\theta,\nu}\cap B_j\cap W\neq\emptyset$ and

$$\text{Angle}\big(G(\theta),T_z\big[Z(P)\big]\big)> R^{-1/2+2\delta} \tag{4.12}$$

for some non-singular point $z\in 10T_{\theta,\nu}\cap 2B_j\cap Z(P)$.

Let $\mathbb{T}_{j,\text{tang}}$ represent the collection of all $(\theta,\nu)\in\mathbb{T}$ such that $T_{\theta,\nu}$'s are tangent to the wall W in B_j, and $\mathbb{T}_{j,\text{trans}}$ denote the collection of all $(\theta,\nu)\in\mathbb{T}$ such that $T_{\theta,\nu}$'s are transverse to the wall W in B_j.

Define $f_{j,\text{tang}}:=\sum_{(\theta,\nu)\in\mathbb{T}_{j,\text{tang}}}f_{\theta,\nu}$ and $f_{j,\text{trans}}:=\sum_{(\theta,\nu)\in\mathbb{T}_{j,\text{trans}}}f_{\theta,\nu}$. Since non-singular points are dense in $Z(P)$, any $T_{\theta,\nu}$ intersecting W inside B_j is either tangential or transverse. Therefore on $B_j\cap W$, up to a rapidly decaying tail, $Ef(x)$ can be split into a transverse term and a tangential term:

$$Ef(x)\sim Ef_{j,\text{tang}}(x)+Ef_{j,\text{trans}}(x). \tag{4.13}$$

However, since we will need to use a bilinear structure when analyzing the tangent contribution, here we use a more refined decomposition instead: breaking $Ef(x)$ into a linear transverse term and a bilinear tangential term.

More precisely, decompose the unit ball B^2 into balls τ of radius $1/K$, where $K=K(\varepsilon)\ll R$ is a large parameter. Decompose $f=\sum_\tau f_\tau$, where $\operatorname{supp} f_\tau\subseteq\tau$.

Let $S_\varepsilon:=\{x\in B_R:\exists\tau \text{ s.t. } |Ef_\tau(x)|>K^{-\varepsilon^4}|Ef(x)|\}$. We will show by parabolic rescaling that the contribution from S_ε is acceptable. In fact, by the definition of S_ε,

$$\big\|Ef(x)\big\|^p_{L^p(S_\varepsilon;Hdx)}\le K^{\varepsilon^4 p}\sum_\tau\big\|Ef_\tau(x)\big\|^p_{L^p(B_R;Hdx)}.$$

By parabolic rescaling and induction on scales (Lemma 2.1), the right-hand side is bounded by

$$\lesssim K^{\varepsilon^4 p}\sum_{\tau}\left[K^{\frac{\alpha+1}{p}-1-\varepsilon-\gamma_3^0}R^{\varepsilon+\gamma_3^0(\alpha)}\|f_\tau\|_{L^2}\right]^p$$
$$\lesssim K^{(\varepsilon^4+\frac{\alpha+1}{p}-1-\varepsilon-\gamma_3^0)p}\left[R^{\varepsilon+\gamma_3^0(\alpha)}\|f\|_{L^2}\right]^p.$$

Note that $\frac{\alpha+1}{p}-1-\gamma_3^0\le 0$ (this is the reason why we set $\gamma_3^0=\frac{\alpha-2}{3}$ for $\alpha>2$). By choosing $K=K(\varepsilon)$ large enough so that

$$K^{(\varepsilon^4+\frac{\alpha+1}{p}-1-\varepsilon-\gamma_3^0)}\ll 1,$$

the induction closes and therefore the term involving S_ε plays an unimportant role.

For points not in S_ε, we have the following decomposition into a transverse term and a bilinear tangential term (cf. [4, Lemma 6.2]), which follows quickly from a standard argument that reduces linear estimate to multilinear ones.

LEMMA 4.2. *For each point $x\in B_j\cap W$ satisfying $\max_\tau|Ef_\tau(x)|\le K^{-\varepsilon^4}|Ef(x)|$, there exists a sub-collection I of the collection of all possible $1/K$-balls τ, such that*

$$\big|Ef(x)\big|\lesssim\big|Ef_{I,j,\text{trans}}(x)\big|+K^{10}\,\text{Bil}\big(Ef_{j,\text{tang}}(x)\big), \tag{4.14}$$

where

$$f_{I,j,\text{trans}}:=\sum_{\tau\in I}f_{\tau,j,\text{trans}},$$

and the bilinear tangent term is given by

$$\text{Bil}(Ef_{j,\text{tang}}(x)):=\max_{\substack{\tau_1,\tau_2\\ \text{dist}(\tau_1,\tau_2)\ge 1/K}}\big|Ef_{\tau_1,j,\text{tang}}(x)\big|^{1/2}\big|Ef_{\tau_2,j,\text{tang}}(x)\big|^{1/2}.$$

By Lemma 4.2 we bound the wall term $\|Ef\|^p_{L^p(W;Hdx)}$ by

$$\lesssim\sum_j\big\|\max_I\big|Ef_{I,j,\text{trans}}(x)\big|\big\|^p_{L^p(B_j\cap W;Hdx)} \tag{4.15}$$
$$+K^{10p}\sum_j\big\|\text{Bil}\big(Ef_{j,\text{tang}}(x)\big)\big\|^p_{L^p(B_j\cap W;Hdx)}. \tag{4.16}$$

We handle the transverse term by induction on the radius of the physical ball and control the L^2 norms of $f_{j,\text{trans}}$ using the following lemma, which says that $T_{\theta,\nu}$ crosses the wall W transversely in at most $R^{O(\delta_{\text{deg}})}$ many balls B_j.

LEMMA 4.3. (Lemma 3.5 in [9]) *For each $(\theta,\nu)\in\mathbb{T}$, the number of $R^{1-\delta}$-balls B_j for which $(\theta,\nu)\in\mathbb{T}_{j,\text{trans}}$ is at most* $\text{Poly}(D)=R^{O(\delta_{\text{deg}})}$.

Note that the geometric lemma above is essentially the tube version of the fact that a line can transversely intersect $Z(P)$ in at most D points. Lemma 4.3 and orthogonality (2.11) imply the bound:

$$\sum_j \|f_{j,\text{trans}}\|_{L^2}^2 \le R^{O(\delta_{\text{deg}})}\|f\|_{L^2}^2.$$

We now estimate the linear transverse term (4.15). The term (4.15) is dominated by

$$\sum_j \sum_{I\subseteq \mathcal{T}} \left\|Ef_{I,j,\text{trans}}(x)\right\|^p_{L^p(B_j\cap W;Hdx)}, \tag{4.17}$$

where $\mathcal{T}$ is the collection of all possible $1/K$-balls in B^2, and the sum is taken over all subsets of $\mathcal{T}$. Since there are at most 2^{K^2} I's, we apply (4.1) with radius $R^{1-\delta}$ to obtain

$$\begin{aligned}\sum_j \sum_{I\subseteq \mathcal{T}} \left\|Ef_{I,j,\text{trans}}(x)\right\|^p_{L^p(B_j\cap W;Hdx)} &\le \sum_j 2^{K^2}\left[C_\varepsilon R^{(1-\delta)(\varepsilon+\gamma_3^0)}\|f_{j,\text{trans}}\|_{L^2}\right]^p\\ &\lesssim 2^{K^2}R^{O(\delta_{\text{deg}})-\delta\varepsilon p-\delta\gamma_3^0 p}\left[C_\varepsilon R^{\varepsilon+\gamma_3^0}\|f\|_{L^2}\right]^p.\end{aligned}$$

Since $\delta_{\text{deg}} \ll \delta\varepsilon$, it follows that $2^{K^2}R^{O(\delta_{\text{deg}})-\delta\varepsilon p-\delta\gamma_3^0 p} \ll 1$, thus the induction on the transverse term closes.

It remains to estimate the bilinear tangent term (4.16). The proof uses the bilinear refined Strichartz. By Corollary 3.4 in the case $m = 2$ and $d = 3$, we have the following:

Let $\alpha \in (0,3]$. For functions f_1 and f_2 in $L^2(B^2)$, with supports separated by ~ 1, suppose that f_1 and f_2 are concentrated in wave packets from $\mathbb{T}_Z(M)$, where $Z = Z(P)$ and P is a product of distinct non-singular polynomials. Then for any $H \in \mathcal{F}_{\alpha,3}$,

$$\left\|\left|Ef_1\right|^{1/2}\left|Ef_2\right|^{1/2}\right\|_{L^3(B_R;Hdx)} \lessapprox M^{O(1)}R^{(\alpha-2)/12}\left\|f_1\right\|_{L^2}^{1/2}\left\|f_2\right\|_{L^2}^{1/2}. \tag{4.18}$$

Now we estimate the bilinear tangent term (4.16).

$$\begin{aligned}&K^{10p}\sum_j \left\|\text{Bil}\left(Ef_{j,\text{tang}}(x)\right)\right\|^p_{L^p(B_j\cap W;Hdx)}\\ &\quad\le K^{10p}\sum_j \sum_{\substack{\tau_1,\tau_2\\ \text{dist}\left(\tau_1,\tau_2\right)\ge 1/K}} \left\|\left|Ef_{\tau_1,j,\text{tang}}\right|^{1/2}\left|Ef_{\tau_2,j,\text{tang}}\right|^{1/2}\right\|^p_{L^p(B_j;Hdx)}.\end{aligned}$$

To finish the proof of the estimate (4.1), it suffices to show

$$\left\|\left|Ef_{\tau_1,j,\text{tang}}\right|^{1/2}\left|Ef_{\tau_2,j,\text{tang}}\right|^{1/2}\right\|_{L^3(B_j;Hdx)} \lessapprox R^{\gamma_3^0}\left\|f_{\tau_1}\right\|_{L^2}^{1/2}\left\|f_{\tau_2}\right\|_{L^2}^{1/2}, \tag{4.19}$$

for each pair (τ_1,τ_2) with $\mathrm{dist}(\tau_1,\tau_2)\geq 1/K$. We will do so by applying (4.18) to $f_{\tau_i,j,\mathrm{tang}}$ on each ball B_j.

Expand $f_{\tau_i,j,\mathrm{tang}}$ into smaller wave packets at the scale $\rho=R^{1-\delta}$ on the ball B_j, i.e., cover B^2 with caps $\tilde\theta$ of radius $\rho^{-1/2}$ and cover $\mathbb{R}^2$ by finitely overlapping balls of radius $\sim\rho^{\frac{1+\delta}{2}}$, centered at vectors $\tilde\nu\in\rho^{\frac{1+\delta}{2}}\mathbb{Z}^2$. Each new wave packet $Ef_{\tilde\theta,\tilde\nu}$ is roughly supported on a tube $T_{\tilde\theta,\tilde\nu}$ of radius $\rho^{\frac12+\delta}$ and has length ρ. For a detailed description of the wave packet decomposition of $f_{\tau_i,j,\mathrm{tang}}$ on a smaller ball, see [10, Section 7].

By definition of $f_{\tau_i,j,\mathrm{tang}}$, each new wave packet lies in the $\sim R^{1/2+\delta}$-neighborhood of Z and the angles between the wave packets and the tangent space of Z are bounded by $R^{-1/2+2\delta}$. In order to apply (4.18), one needs to verify that $f_{\tau_i,j,\mathrm{tang}}$ is indeed concentrated in *new* wave packets (at the scale ρ) from $T_z(M)$ for some M. Define M so that $\rho^{1/2}M=R^{1/2+\delta}$. Since $\rho=R^{1-\delta}$, it follows $M=R^{(3/2)\delta}$, thus $M\rho^{-1/2}=R^{-1/2+2\delta}$. Each new wave packet lies in the $M\rho^{1/2}$-neighborhood of Z, and the angles between the wave packets and the tangent space of Z are bounded by $M\rho^{-1/2}$. Therefore, the new wave packets are concentrated in $\mathbb{T}_Z(M)$, which enables one to apply (4.18). Now since $M^{O(1)}=R^{O(\delta)}$ and $(\alpha-2)/12\leq\gamma_3^0$, the bound from (4.18) implies (4.19). The proof is complete.

5. Weighted extension estimates in higher dimensions: proof of Theorem 1.4 for $d\geq 4$, $\alpha\in[\frac{d}{2},\frac{d+1}{2}]$. In this section we prove Theorem 1.4 in the case $d\geq 4$ and $\alpha\in[\frac{d}{2},\frac{d+1}{2}]$ using polynomial partitioning. Our goal is to show that

$$\|Ef\|_{L^{2d/(d-1)}(B_R^d;Hdx)}\lessapprox R^{\frac{\alpha}{2d^2}-\frac{1}{4d}}\|f\|_{L^2(B^{d-1})} \tag{5.1}$$

holds for all $f\in L^2(B^{d-1})$, all $R>1$ and all $H\in\mathcal{F}_{\alpha,d}$.

Roughly speaking, we will iterate the argument in Section 4 in each dimension. Because of the complexity of the iteration scheme and some technical issues, we present the argument using the notion of *narrow* and *broad* part of Ef. The broad part, which is the main body of the proof, is estimated by Theorem 5.1 below, and the narrow part is handled by Lemma 5.2 based on parabolic rescaling.

To start with, fixing a large constant K, we decompose B^{d-1} into balls τ of radius K^{-1} and B_R into balls B_{K^2} of radius K^2. One naturally has $f=\sum_\tau f_\tau:=\sum_\tau f\chi_\tau$, and $G(\tau)$ denotes the set of directions of wave packets of f_τ. We use $\mathrm{Angle}(G(\tau),V)$ to denote the smallest angle between $v\in G(\tau)$ and $v'\in V\setminus\{0\}$ whenever V is a vector space in $\mathbb{R}^d$. We are now ready to define the following *broad* norm of Ef. Fix $H\in\mathcal{F}_{\alpha,d}$,

$$\|Ef\|^p_{BL^p_A(B_R;Hdx)}:=\sum_{B_{K^2}\subset B_R}\mu_{Ef}(B_{K^2}), \tag{5.2}$$

where

$$\mu_{Ef}(B_{K^2}) := \min_{V_1,\dots,V_A:1-\text{subspace of }\mathbb{R}^d} \left(\max_{\tau:\text{Angle}\left(G(\tau),V_a\right)>K^{-1}\text{ for all }a} \int_{B_{K^2}} \left|Ef_\tau\right|^p H\,dx \right).$$

Here $A \ll K$ is a large constant to be determined later. Note that the exact value of A is not very important, which is only included in the definition to ensure that certain versions of triangle inequality and Hölder's inequality hold true for the broad norm (which, strictly speaking, is still not a *norm*). Therefore we usually write $\|\cdot\|_{BL^p_A}$ as $\|\cdot\|_{BL^p}$ for short. We also point out that μ_{Ef} can be extended to be a measure on B_R, by making it a constant multiple of the Lebesgue measure on each ball B_{K^2}. More precisely, one can define

$$\mu_{Ef}(Q) = \frac{|Q|}{|B_{K^2}|}\mu_{Ef}(B_{K^2}), \quad \forall Q \subset B_{K^2},$$

and let

$$\mu_{Ef}(U) = \sum_{B_{K^2}} \mu_{Ef}(U \cap B_{K^2})$$

for general set U.

The k-broad norm (a slightly different one, where there is no weight H in the integral on B_{K^2}) was first invented by the second author in [10, Section 1], where unweighted Fourier extension estimates with respect to the unweighted k-broad norm are obtained (see [10, Theorem 1.5]). As explained in [10], the significance of such k-broad estimates is that, it can be used as a formally weaker substitute for the (still open) k-linear restriction conjecture to obtain improved linear restriction estimate, following a fairly standard scheme originated in Bourgain-Guth [3] that converts multilinear estimates to linear ones. The broad norm we are using here is the weighted version of the one in [10] with $k = 2$.

The main chunk of the proof of (5.1) is the following weighted extension estimate with respect to our newly defined broad norm.

THEOREM 5.1. *Let $d \geq 4$, $\alpha \in [\frac{d}{2}, \frac{d+1}{2}]$ and $p_d := \frac{2d}{d-1}$. For all $\varepsilon > 0$, there is a large constant A so that the following holds for any value of K, $R > 1$, $H \in \mathcal{F}_{\alpha,d}$:*

$$\|Ef\|_{BL^{p_d}(B_R;Hdx)} \lesssim_{K,\varepsilon} R^{\varepsilon+\gamma_d}\|f\|_{L^2(B^{d-1})},$$

where $\gamma_d := \frac{\alpha}{2d^2} - \frac{1}{4d}$.

Note that Theorem 5.1 will be further generalized to Theorem 6.1 in Section 6. To see that Theorem 5.1 implies the desired estimate (5.1), in other words, to see that the leftover *narrow* part of $\|Ef\|_{L^p(B_R;Hdx)}$ that is not taken care of by the broad norm is harmless, it suffices to apply Lemma 5.2 below with $p = p_d$, and note that it is straightforward to check

$$\gamma_d \geq \frac{1-d}{2} + \frac{\alpha+1}{p_d}.$$

We point out that the terminology *narrow* here refers to the fact that if $\|Ef\|_{L^p(B_R;Hdx)}$ is not dominated by $\|Ef\|_{BL^p(B_R;Hdx)}$, then f must be concentrated in caps τ that are contained in a K^{-1}-angular neighborhood of one of the vector spaces $\{V_a\}$.

LEMMA 5.2. *Let $d \geq 3$, $p \geq 2$ and $\alpha \in (0,d]$. Assume that for all $\varepsilon > 0$, there exists large constant $A = A(\varepsilon)$ such that*

$$\|Ef\|_{BL^p_A(B_R;Hdx)} \lesssim_{K,\varepsilon} R^{\varepsilon+T_d}\|f\|_{L^2(B^{d-1})} \tag{5.3}$$

holds for all $K, R > 1, H \in \mathcal{F}_{\alpha,d}$, and that

$$T_d \geq \frac{1-d}{2} + \frac{\alpha+1}{p}. \tag{5.4}$$

Then, for all $\varepsilon > 0$, $R > 1$, $H \in \mathcal{F}_{\alpha,d}$, there holds

$$\|Ef\|_{L^p(B_R;Hdx)} \leq C_\varepsilon R^{\varepsilon+T_d}\|f\|_{L^2(B^{d-1})}. \tag{5.5}$$

Proof. The main tools that will be used here are induction on radius R and parabolic rescaling. It is easy to see that it suffices to consider the case that R is large. Suppose that the desired estimate holds true if one replaces R by $R/2$. We write $\|Ef\|_{L^p(B_R;Hdx)}$ as

$$\left(\sum_{B_{K^2}\subset B_R} \|Ef\|^p_{L^p(B_{K^2};Hdx)}\right)^{1/p},$$

and for each B_{K^2}, take 1-subspaces $V'_1,\ldots,V'_A$ of $\mathbb{R}^d$ depending on B_{K^2} and f to be the minimizers obeying

(5.6)

$$\max_{\tau\notin V'_a \text{ for all } a}\int_{B_{K^2}} |Ef_\tau|^p H\,dx = \min_{V_1,\ldots,V_A:1-\text{subspace}}\ \max_{\tau\notin V_a \text{ for all } a}\int_{B_{K^2}} |Ef_\tau|^p H\,dx,$$

where $\tau \notin V_a$ means that $\mathrm{Angle}(G(\tau),V_a) > K^{-1}$. Then on each B_{K^2}, by applying the Minkowski inequality to function

$$Ef = \sum_{\tau\notin V'_a \text{ for all } a} Ef_\tau + \sum_{\tau\in V'_a \text{ for some } a} Ef_\tau,$$

we bound $\|Ef\|_{L^p(B_R;Hdx)}$ by

$$\left(\sum_{B_{K^2}\subset B_R}\left\|\sum_{\tau\notin V_a' \text{ for all } a} Ef_\tau\right\|^p_{L^p(B_{K^2};Hdx)}\right)^{1/p} + \left(\sum_{B_{K^2}\subset B_R}\left\|\sum_{\tau\in V_a' \text{ for some } a} Ef_\tau\right\|^p_{L^p(B_{K^2};Hdx)}\right)^{1/p}.$$

By the choice as in (5.6) and assumption (5.3), the first term is bounded by

$$K^{O(1)}\|Ef\|_{BL^p_A(B_R;Hdx)} \lesssim_{K,\varepsilon} R^{\varepsilon+T_d}\|f\|_{L^2(B^{d-1})}.$$

Note that there are only $O(A)$ many τ's that are within an angle of K^{-1} to one of $V_1',\dots,V_A'$. We choose $K=K(\varepsilon)$ large enough so that $A=A(\varepsilon)\le K^\delta$. Hence, the second term is controlled by

$$\le K^{O(\delta)}\|\max_\tau |Ef_\tau|\|_{L^p(B_R;Hdx)} \le K^{O(\delta)}\left(\sum_\tau \|Ef_\tau\|^p_{L^p(B_R;Hdx)}\right)^{1/p}.$$

Note that to prove the desired estimate (5.5), one can induct on radius R. Therefore by applying Lemma 2.1 which is based on parabolic rescaling and induction on radius R, the narrow part above is further estimated by

$$\lesssim C_\varepsilon K^{O(\delta)}K^{\frac{\alpha+1}{p}-\frac{d-1}{2}-\varepsilon-T_d}R^{\varepsilon+T_d}\left(\sum_\tau \|f_\tau\|^p_{L^2}\right)^{1/p}.$$

Due to orthogonality and the fact $p\ge 2$, we have $\left(\sum_\tau \|f_\tau\|^p_{L^2}\right)^{1/p}\lesssim \|f\|_{L^2}$. Moreover by the assumption (5.4), $K^{O(\delta)+\frac{\alpha+1}{p}-\frac{d-1}{2}-\varepsilon-T_d}\ll 1$. Therefore, the narrow part can be estimated as desired by induction and the proof is complete. □

It remains to prove Theorem 5.1. As in the three-dimensional case treated in the previous section, we apply polynomial partitioning (but iteratively in different dimensions). To make use of induction on dimensions, we generalize Theorem 5.1 to the following main inductive proposition:

PROPOSITION 5.3. *Given $d\ge 4$, $\alpha\in[\frac{d}{2},\frac{d+1}{2}]$. For all $\varepsilon>0$, there exist a large constant $\bar{A}>1$ and small constants $0<\delta\ll\delta_{d-1}\ll\dots\ll\delta_1\ll\varepsilon$ so that the following holds. Let m be a dimension in the range $2\le m\le d$, and $p_m:=\frac{2m}{m-1}$. Suppose that $Z=Z(P_1,\dots,P_{d-m})$ is a transverse complete intersection with* $\operatorname{Deg} P_i\le D_Z$*, and that $f\in L^2(B^{d-1})$ is concentrated in wave packets from $\mathbb{T}_Z(R^{\delta_m})$. Then* *for any* $1 \le A \le \bar{A}$, $R \ge 1$ *and* $H \in \mathcal{F}_{\alpha,d}$,

$$\|Ef\|_{BL_A^{p_m}(B_R;Hdx)} \le C\big(K,\varepsilon,m,D_Z\big) R^{m\varepsilon} R^{\delta(\log \bar{A} - \log A)} R^{\gamma_m} \|f\|_{L^2}, \tag{5.7}$$

where

$$\gamma_m := \begin{cases} -\dfrac{d}{4m} + \dfrac{1}{4}, & 2 \le m \le d-1, \\ \dfrac{\alpha}{2d^2} - \dfrac{1}{4d}, & m = d. \end{cases}$$

In the proposition, the condition that f is concentrated in wave packets from $\mathbb{T}_Z(R^{\delta_m})$ is defined at the end of Subsection 2.4. It is easy to see that the case $m = d$, $Z = \mathbb{R}^d$, $A = \bar{A}$ in the proposition above is precisely the desired result of Theorem 5.1. Proposition 5.3 will be proven by induction (on dimension m, radius R, and on A) with the assistance of the linear refined Strichartz in each step (more precisely, the linear weighted L^2 estimate in Corollary 3.3 for each dimension m). The rest of this section is devoted to the proof of Proposition 5.3.

The base case $m = 2$ (for all R and A) follows immediately from the unweighted estimate (Proposition 8.1 of [10]):

$$\|Ef\|_{BL_A^4(B_R;dx)} \lesssim R^{\varepsilon} R^{\delta(\log \bar{A} - \log A)} R^{-\frac{d}{8}+\frac{1}{4}} \|f\|_{L^2}, \tag{5.8}$$

where the unweighted broad norm is defined as in (5.2) except that one replaces each $\int_{B_{K^2}} |Ef_\tau|^p H\, dx$ by $\int_{B_{K^2}} |Ef_\tau|^p\, dx$. Indeed, applying Remark 1.3 to each B_{K^2}, one can show that

$$\|Ef\|_{BL_A^4(B_R;Hdx)} \lesssim \|Ef\|_{BL_A^4(B_R;dx)},$$

thus the desired estimate follows.

If R is small, then choosing the implicit constant large enough will finish the proof. If $A = 1$, then by choosing $\bar{A}$ large enough, the desired estimate follows from the trivial $L^1 \to L^\infty$ estimate of E. Now fix $m \le d$ and assume that the desired estimates hold true if one decreases m or A, or decreases R by half.

We say we are in algebraic case if there is a transverse complete intersection $Y^{m-1} \subset Z^m$ of dimension $m-1$, defined using polynomials of degree $\le D(\varepsilon, D_Z)$ (a function to be determined later), such that

$$\mu_{Ef}\big(N_{R^{1/2+\delta_m}}(Y) \cap B_R\big) \gtrsim \mu_{Ef}\big(B_R\big).$$

Otherwise we say that we are in the non-algebraic (or *cellular*) case.

5.1. The non-algebraic case. In the non-algebraic case, we use polynomial partitioning and induction on radius R. Since the argument is exactly the same as in Subsection 8.1 of [10], here we just give a brief description. One would like to argue similarly as in the cellular case for $d = 3$ presented in Subsection 4.1. However, since f is concentrated near a sub-manifold Z whose dimension is smaller than d, one needs to be careful with the construction of the polynomial that is needed for the partitioning.

First by pigeonholing we can locate a significant piece of $N_{R^{1/2+\delta_m}}(Z)\cap B_R$ where at each point the angle between the tangent space of Z and a fixed m-plane V is within $1/100$. Then perform the regular polynomial partitioning in V and pull the polynomial on V back via the orthogonal projection $\pi:\mathbb{R}^d\to V$. We end up with a polynomial P on $\mathbb{R}^d$ of degree $\le D=D(\varepsilon,D_Z)$, for which $\mathbb{R}^d\backslash Z(P)$ is a union of $\sim D^m$ open sets O_i and the following properties hold. Define $W:=N_{R^{1/2+\delta}}Z(P)$, $O_i':=O_i\backslash W$ and $f_i=\sum_{(\theta,\nu)\in\mathbb{T}_i}f_{\theta,\nu}$, where

$$\mathbb{T}_i:=\big\{(\theta,\nu):T_{\theta,\nu}\cap O_i'\neq\emptyset\big\}.$$

Since we are in the non-algebraic case, for $\sim D^m$ cells O_i',

$$\|Ef\|^p_{BL^p_A(B_R;Hdx)}\lesssim D^m\|Ef\|^p_{BL^p_A(O_i';Hdx)}\lesssim D^m\big\|Ef_i\big\|^p_{BL^p_A(B_R;Hdx)}.$$

In addition, by orthogonality and the geometric observation that each (θ,ν) belongs to $\lesssim D$ collections $\mathbb{T}_i$, as mentioned in (4.7) above, we have

$$\sum_i\big\|f_i\big\|^2_{L^2}\lesssim D\|f\|^2_{L^2}.$$

Therefore, by the same argument as in three dimensions, the induction for the non-algebraic case closes provided that $p>p_m=\frac{2m}{m-1}$. Then, applying Hölder's inequality and choosing p close enough to p_m^+ justifies the same estimate for the endpoint $p=p_m$.

5.2. The algebraic case. In the algebraic case, there exists a transverse complete intersection Y of dimension $m-1$, defined using polynomials of degree $\le D(\varepsilon,D_Z)$ such that

$$\mu_{Ef}\big(N_{R^{1/2+\delta_m}}(Y)\cap B_R\big)\gtrsim\mu_{Ef}\big(B_R\big).$$

In this case, we first subdivide B_R into smaller balls B_j of radius ρ, chosen such that $\rho^{1/2+\delta_{m-1}}=R^{1/2+\delta_m}$. One has

$$\|Ef\|^{p_m}_{BL^{p_m}_A(B_R;Hdx)}\lesssim\sum_j\|Ef_j\|^{p_m}_{BL^{p_m}_A(B_j;Hdx)}+\text{RapDec}(R)\|f\|^{p_m}_{L^2},$$

where

$$f_j:=\sum_{(\theta,\nu)\in\mathbb{T}_j}f_{\theta,\nu},\quad \mathbb{T}_j:=\big\{(\theta,\nu)\colon T_{\theta,\nu}\cap N_{R^{1/2+\delta_m}}(Y)\cap B_j\neq\emptyset\big\}.$$

Similarly as in Section 4, we further subdivide $\mathbb{T}_j$ into tubes that are tangent to Y and tubes that are transverse to Y. We say that $T_{\theta,\nu} \in \mathbb{T}_j$ is *tangent* to Y in B_j if

$$T_{\theta,\nu} \cap 2B_j \subset N_{R^{1/2+\delta_m}}(Y) \cap 2B_j = N_{\rho^{1/2+\delta_{m-1}}}(Y) \cap 2B_j \tag{5.9}$$

and for any non-singular point $y \in Y \cap 2B_j \cap N_{10R^{1/2+\delta_m}} T_{\theta,\nu}$,

$$\text{Angle}\big(G(\theta), T_y Y\big) \le \rho^{-1/2+\delta_{m-1}}. \tag{5.10}$$

We denote the tangent and transverse wave packets by

$$\mathbb{T}_{j,\text{tang}} := \big\{(\theta,\nu) \in \mathbb{T}_j \colon T_{\theta,\nu} \text{ is tangent to } Y \text{ in } B_j\big\}, \quad \mathbb{T}_{j,\text{trans}} := \mathbb{T}_j \setminus \mathbb{T}_{j,\text{tang}},$$

and let

$$f_{j,\text{tang}} = \sum_{(\theta,\nu)\in\mathbb{T}_{j,\text{tang}}} f_{\theta,\nu}, \quad f_{j,\text{trans}} = \sum_{(\theta,\nu)\in\mathbb{T}_{j,\text{trans}}} f_{\theta,\nu}, \tag{5.11}$$

then

$$\sum_j \big\|Ef_j\big\|^{p_m}_{BL^{p_m}_A(B_j;Hdx)} \lesssim \sum_j \big\|Ef_{j,\text{tang}}\big\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)} + \sum_j \big\|Ef_{j,\text{trans}}\big\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)}.$$

We will control the contribution from the tangent wave packets by induction of the dimension m, and the one from the transverse wave packets by induction on the radius R.

5.3. The tangent sub-case. In this subsection, we control the tangent term

$$\sum_j \big\|Ef_{j,\text{tang}}\big\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)}$$

by induction on dimension m. In order to apply the induction hypotheses to $Ef_{j,\text{tang}}$ on B_j, one needs to first redo the wave packet decomposition at the scale ρ. By definition of $\mathbb{T}_{j,\text{tang}}$, it is easy to check, via (5.9) and (5.10), that such a wave packet $T_{\tilde\theta,\tilde\nu}$ of dimensions $\rho^{1/2+\delta} \times \cdots \times \rho^{1/2+\delta} \times \rho$ is $\rho^{-1/2+\delta_{m-1}}$-tangent to Y in B_j, in other words, $f_{j,\text{tang}}$ satisfies the hypotheses of Proposition 5.3 at scale ρ in dimension $m-1$. Therefore, by induction on the dimension one has

$$\begin{aligned} &\big\|Ef_{j,\text{tang}}\big\|_{BL^{p_{m-1}}_{A/2}(B_j;Hdx)} \\ &\quad \le C\big(K,\varepsilon,m-1,D(\varepsilon,D_Z)\big)\rho^{(m-1)\varepsilon}\rho^{\delta(\log \bar{A}-\log(A/2))}\rho^{\gamma_{m-1}}\big\|f_{j,\text{tang}}\big\|_{L^2}. \end{aligned} \tag{5.12}$$

On the other hand, it follows immediately from the definition of the broad norm and Corollary 3.3 that

$$\begin{aligned}\left\|Ef_{j,\text{tang}}\right\|^2_{BL^2(B_j;Hdx)} &\le \sum_\tau \left\|Ef_{\tau,j,\text{tang}}\right\|^2_{L^2(B_j;Hdx)}\\ &\le C_\varepsilon \rho^{O(\delta_{m-1})}\rho^{\varepsilon+1-\frac{d-\alpha}{m}}\sum_\tau \left\|f_{\tau,j,\text{tang}}\right\|^2_{L^2}\\ &\le C_\varepsilon \rho^{O(\delta_{m-1})}\rho^{\varepsilon+1-\frac{d-\alpha}{m}}\left\|f_{j,\text{tang}}\right\|^2_{L^2}.\end{aligned} \tag{5.13}$$

Observing that $2 < p_m = \frac{2m}{m-1} < \frac{2(m-1)}{m-2} = p_{m-1}$, one can interpolate estimates (5.12) and (5.13) above to obtain

$$\begin{aligned}&\left\|Ef_{j,\text{tang}}\right\|_{BL^{p_m}_{A/2}(B_j;Hdx)}\\ &\quad\le C_{K,\varepsilon,m-1,D(\varepsilon,D_Z)}\rho^{O(\delta_{m-1})+(m-1)\varepsilon+\delta\left(\log\bar{A}-\log(A/2)\right)+\gamma_{m,m-1}}\left\|f_{j,\text{tang}}\right\|_{L^2},\end{aligned}$$

where

$$\gamma_{m,m-1} = \left(\frac{1}{2}-\frac{d-\alpha}{2m}\right)\cdot\frac{1}{m}+\gamma_{m-1}\cdot\left(1-\frac{1}{m}\right).$$

Note that the number of balls B_j is $\lesssim R^{O(\delta_{m-1})}$, hence one can sum over the balls to obtain

$$\begin{aligned}&\left(\sum_j \left\|Ef_{j,\text{tang}}\right\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)}\right)^{1/p_m}\\ &\le C_{K,\varepsilon,m-1,D(\varepsilon,D_Z)}R^{O(\delta_{m-1})+(m-1)\varepsilon+\delta(\log\bar{A}-\log(A/2))}\rho^{\gamma_{m,m-1}}\|f\|_{L^2}\\ &\le C_{K,\varepsilon,m,D(\varepsilon,D_Z)}R^{O(\delta_{m-1})+(m-1)\varepsilon+(\log 2)\delta}R^{\delta(\log\bar{A}-\log A)+\gamma_{m,m-1}}\|f\|_{L^2}.\end{aligned} \tag{5.14}$$

In the last inequality above, even though $\gamma_{m,m-1}$ can be negative, one still has $\rho^{\gamma_{m,m-1}} \le R^{O(\delta_{m-1})}R^{\gamma_{m,m-1}}$. Since $\delta, \delta_{m-1} \ll \varepsilon$,

$$R^{O(\delta_{m-1})+(m-1)\varepsilon+(\log 2)\delta} \ll R^{m\varepsilon},$$

hence the inductive argument for the tangent term is done as long as

$$\gamma_m \ge \left(\frac{1}{2}-\frac{d-\alpha}{2m}\right)\cdot\frac{1}{m}+\gamma_{m-1}\cdot\left(1-\frac{1}{m}\right). \tag{5.15}$$

5.4. The transverse sub-case. In this subsection we deal with the transverse term

$$\sum_j \left\|Ef_{j,\text{trans}}\right\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)}$$

by induction on the radius R. The argument is exactly the same as in the Subsection 8.4 of [10], hence we omit the details and only briefly recall several essential steps.

As in the tangent sub-case, in order to apply induction on radius, we need to redo wave packet decomposition for $f_{j,\text{trans}}$ at scale ρ. Since the old relevant wave packets are in $\mathbb{T}_Z(R^{\delta_m})$, for a new relevant wave packet $T_{\tilde{\theta},\tilde{\nu}}$ of dimensions $\rho^{1/2+\delta} \times \cdots \times \rho^{1/2+\delta} \times \rho$, the angle between $G(\tilde{\theta})$ and the tangent spaces of Z near their intersection is $\lesssim R^{-1/2+\delta_m} + \rho^{-1/2} \lesssim \rho^{-1/2+\delta_m}$. However, in the transverse case, it is not necessarily true that the new smaller wave packets are contained in the $\rho^{1/2+\delta_m}$-neighborhood of Z, which prevents us from directly applying the induction hypothesis.

To overcome this, we decompose $N_{R^{1/2+\delta_m}}(Z) \cap B_j$ into translates of $N_{\rho^{1/2+\delta_m}}(Z) \cap B_j$, say $N_{\rho^{1/2+\delta_m}}(Z+b) \cap B_j$, $|b| \leq R^{1/2+\delta_m}$. Define $f_{j,\text{trans},b}$ using the new wave packets which intersect $N_{\rho^{1/2+\delta_m}}(Z+b) \cap B_j$. Because of the angle condition, $f_{j,\text{trans},b}$ is concentrated in new wave packets that are $\rho^{-1/2+\delta_m}$-tangent to $Z+b$ inside B_j. We can choose a set of translations $\{b\}$ such that

$$\left\|Ef_{j,\text{trans}}\right\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)} \lesssim (\log R) \sum_b \left\|Ef_{j,\text{trans},b}\right\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)}. \tag{5.16}$$

By orthogonality and Lemma 5.7 in [10] which controls the transverse intersections between a tube and an algebraic variety, one has

$$\sum_{j,b} \left\|f_{j,\text{trans},b}\right\|^2_{L^2} \lesssim \|f\|^2_{L^2}. \tag{5.17}$$

Moreover, there holds the equi-distribution estimate (cf. Section 7 of [10])

$$\max_b \left\|f_{j,\text{trans},b}\right\|^2_{L^2} \leq R^{O(\delta_m)} \left(\frac{R^{1/2}}{\rho^{1/2}}\right)^{-(d-m)} \left\|f_{j,\text{trans}}\right\|^2_{L^2}. \tag{5.18}$$

By inductive hypothesis we can apply (5.7) to $\|Ef_{j,\text{trans},b}\|_{BL^{p_m}_{A/2}(B_j;Hdx)}$ to obtain

$$\begin{aligned}\sum_j \left\|Ef_{j,\text{trans}}\right\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)} &\lesssim (\log R) \sum_{j,b} \|Ef_{j,\text{trans},b}\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)} \\ &\lesssim (\log R) \sum_{j,b} \left[\rho^{m\varepsilon} \rho^{\delta(\log \bar{A} - \log(A/2))} \rho^{\gamma_m} \left\|f_{j,\text{trans},b}\right\|_{L^2}\right]^{p_m}.\end{aligned}$$

It follows from (5.17) and (5.18) that

$$\sum_{j,b} \|f_{j,\text{trans},b}\|^{p_m}_{L^2} \leq R^{O(\delta_m)} \left(\frac{R^{1/2}}{\rho^{1/2}}\right)^{-(d-m)(\frac{p_m}{2}-1)} \|f\|^{p_m}_{L^2},$$

therefore,

$$\sum_j \|Ef_{j,\text{trans}}\|^{p_m}_{BL^{p_m}_{A/2}(B_j;Hdx)} \lesssim R^{O(\delta_m)}\left[\rho^{m\varepsilon}R^{\delta(\log\bar{A}-\log A)}\rho^{\gamma_m}\right]^{p_m}\left(\frac{R^{1/2}}{\rho^{1/2}}\right)^{-(d-m)(\frac{p_m}{2}-1)}\|f\|_2^{p_m}.$$

Choosing $\delta_m \ll \varepsilon\delta_{m-1}$, one has

$$R^{O(\delta_m)}\left(\frac{R}{\rho}\right)^{-m\varepsilon} = R^{O(\delta_m)}R^{-O(\varepsilon\delta_{m-1})} \ll 1.$$

Henceforth the induction closes as long as

$$\frac{1}{2}(d-m)\left(\frac{p_m}{2}-1\right)+p_m\gamma_m \geq 0,$$

that is,

$$\gamma_m \geq -\frac{d}{4m}+\frac{1}{4}. \tag{5.19}$$

5.5. Summary. Because of the inductive argument for the non-algebraic case, the exponent $p_m = \frac{2m}{m-1}$ is the smallest possible one can work with. Starting with

$$\gamma_2 = -\frac{d}{8}+\frac{1}{4},$$

the algebraic case gives the constraint

$$\gamma_m \geq \max\left\{-\frac{d}{4m}+\frac{1}{4}, \left(\frac{1}{2}-\frac{d-\alpha}{2m}\right)\cdot\frac{1}{m}+\gamma_{m-1}\cdot\left(1-\frac{1}{m}\right)\right\}.$$

It is straightforward to check that in the range $\alpha \in [\frac{d}{2}, \frac{d+1}{2}]$, one can take

$$\gamma_m = \begin{cases} -\dfrac{d}{4m}+\dfrac{1}{4}, & 2 \leq m \leq d-1, \\ \dfrac{\alpha}{2d^2}-\dfrac{1}{4d}, & m = d. \end{cases}$$

This completes the proof of Proposition 5.3.

6. Generalized weighted extension estimates in higher dimensions: proof of Theorem 1.4 for $d \geq 4$, $\alpha \in (0,d]$. In this section, we prove Theorem 1.4 for $d \geq 4$ and all $\alpha \in (0,d]$. As in Section 5, by Lemma 5.2, the desired estimate follows from the following broad extension estimate.

THEOREM 6.1. *Let $d \geq 4$, $\alpha \in (0,d]$ and $p_d = \frac{2d}{d-1}$. For all $\varepsilon > 0$, there is a large constant A so that the following holds for any value of $K, R > 1$ and any $H \in \mathcal{F}_{\alpha,d}$:*

$$\|Ef\|_{BL^{p_d}(B_R;Hdx)} \lesssim_{K,\varepsilon} R^{\varepsilon+\gamma_d(\alpha)} \|f\|_{L^2(B^{d-1})},$$

where

$$\gamma_d(\alpha) := \begin{cases} \frac{\left(1+2S_4^d\right)\alpha}{4d} - \frac{1}{2d} - \frac{S_4^d}{2}, & \alpha \in (d-1,d], \\ \frac{S_4^d \alpha}{2d} + \frac{1}{4} - \frac{3}{4d} - \frac{S_4^d}{2}, & \alpha \in (d-2,d-1], \\ \frac{S_\ell^d \alpha}{2d} + \frac{1}{4} - \frac{\ell-1}{4d} - \frac{S_\ell^d}{2}, & \alpha \in \left(d-\frac{\ell}{2}, d-\frac{\ell}{2}+\frac{1}{2}\right], \ \forall 5 \leq \ell \leq d, \\ 0, & \alpha \in \left(0, \frac{d}{2}\right], \end{cases}$$

and $S_\ell^d := \sum_{i=\ell}^d \frac{1}{i}$ if $\ell \leq d$, 0 otherwise.

To prove Theorem 1.4 in the general case, recall that according to Lemma 5.2, an estimate for the broad part implies the same estimate for the regular L^p norm as long as condition

$$\gamma_d(\alpha) \geq \frac{1-d}{2} + \frac{\alpha+1}{p_d}$$

is satisfied. It is straightforward to check that this is indeed the case when

$$\alpha \leq \#_d := \frac{2d\left(d-2-S_4^d\right)}{2d-3-2S_4^d}.$$

When $\#_d < \alpha \leq d$, in order for the narrow part to be controlled, the best bound one can get from the broad estimate above is

$$\|Ef\|_{L^{p_d}(B_R;Hdx)} \leq C(\varepsilon) R^{\varepsilon+\frac{1-d}{2}+\frac{\alpha+1}{p_d}} \|f\|_{L^2},$$

which is exactly the desired estimate for $\alpha \in (\#_d, d]$ in Theorem 1.4.

We also point out that when $\alpha \in (\frac{d}{2}, \frac{d+1}{2}]$, the estimate in Theorem 6.1 coincides with Theorem 5.1.

It remain to prove Theorem 6.1. The proof follows from the same strategy as Theorem 5.1, where the main tools are polynomial partitioning and induction on scales and dimensions. To make all inductions work, we formulate the following main inductive proposition in a more general setting:

PROPOSITION 6.2. *Given $d \geq 4$, $\alpha \in (0,d]$. For all $\varepsilon > 0$, there exist a large constant $\bar{A} > 1$ and small constants $0 < \delta \ll \delta_{d-1} \ll \ldots \ll \delta_1 \ll \varepsilon$ so that the following holds. Let m be a dimension in the range $3 \leq m \leq d$, and $p_m := \frac{2m}{m-1}$. Suppose that $Z = Z(P_1, \ldots, P_{d-m})$ is a transverse complete intersection with $\mathrm{Deg} P_i \leq D_Z$, and that $f \in L^2(B^{d-1})$ is concentrated in wave packets from $\mathbb{T}_Z(R^{\delta_m})$. Then for any $1 \leq A \leq \bar{A}$, $R \geq 1$ and $H \in \mathcal{F}_{\alpha,d}$,*

$$\|Ef\|_{BL_A^{p_m}(B_R;Hdx)} \leq C(K,\varepsilon,m,D_Z) R^{m\varepsilon} R^{\delta(\log \bar{A} - \log A)} R^{\gamma_m} \|f\|_{L^2}, \tag{6.1}$$

where

$$\gamma_m(\alpha) := \begin{cases} \frac{\alpha}{12} - \frac{d}{6} + \frac{1}{3}, & m = 3, \\ \frac{(1+2S_4^m)\alpha}{4m} + \frac{m-1}{2m} - \frac{(1+S_4^m)d}{2m}, & 4 \leq m \leq d, \end{cases} \quad \text{if } \alpha \in (d-1,d];$$

$$\gamma_m(\alpha) := \begin{cases} -\frac{d}{12} + \frac{1}{4}, & m = 3, \\ \frac{S_4^m \alpha}{2m} + \frac{2m-3}{4m} - \frac{(1+2S_4^m)d}{4m}, & 4 \leq m \leq d, \end{cases} \quad \text{if } \alpha \in (d-2,d-1];$$

$$\gamma_m(\alpha) := \begin{cases} -d\frac{d}{4m} + \frac{1}{4}, & 3 \leq m \leq \ell - 1, \\ \frac{S_\ell^m \alpha}{2m} + \frac{2m-\ell+1}{4m} - \frac{(1+2S_\ell^m)d}{4m}, & \ell \leq m \leq d, \end{cases}$$

$$\text{if } \alpha \in (d - \frac{\ell}{2}, d - \frac{\ell}{2} + \frac{1}{2}],\ \forall 5 \leq \ell \leq d;$$

$$\gamma_m(\alpha) := -\frac{d}{4m} + \frac{1}{4}, \quad 3 \leq m \leq d, \quad \text{if } \alpha \in \left(0, \frac{d}{2}\right].$$

Theorem 6.1 follows from Proposition 6.2 by taking $m = d$, $Z = \mathbb{R}^d$ and $A = \bar{A}$. And Proposition 6.2 coincides with Proposition 5.3 when $\alpha \in (\frac{d}{2}, \frac{d+1}{2}]$.

The proof of Proposition 6.2 proceeds very similarly as Proposition 5.3. To begin with, assume $m = 3$. To validate the inductive argument for the non-algebraic case, the exponent $p_3 = 3$ is the smallest possible one we can work with. The transverse case gives a constraint (5.19):

$$\gamma_3(\alpha) \geq -\frac{d}{12} + \frac{1}{4}.$$

As for the tangent sub-case, recall that by interpolating with an L^2 estimate which is based on linear refined Strichartz, we have an estimate with essential exponent

$$\gamma_{3,2} = \frac{\alpha}{18} - \frac{5d}{36} + \frac{1}{3}.$$

On the other hand, by the bilinear weighted L^3 estimate in Corollary 3.4 (it follows from a randomization argument that k-linear estimate is stronger than k-broad estimate, cf. [11]), we have another estimate for the tangent term with essential exponent

$$\gamma'_{3,2} = \frac{\alpha}{12} - \frac{d}{6} + \frac{1}{3}.$$

In summary, we have the estimate (6.1) when $m = 3$ with

$$\gamma_3(\alpha) = \max\left\{-\frac{d}{12} + \frac{1}{4}, \min\left\{\frac{\alpha}{18} - \frac{5d}{36} + \frac{1}{3}, \frac{\alpha}{12} - \frac{d}{6} + \frac{1}{3}\right\}\right\}.$$

Note that

$$\frac{\alpha}{18} - \frac{5d}{36} + \frac{1}{3} \geq \frac{\alpha}{12} - \frac{d}{6} + \frac{1}{3}$$

for all $\alpha \leq d$, meaning that the bilinear refined Strichartz works better than the linear refined Strichartz in this case. And

$$-\frac{d}{12} + \frac{1}{4} \geq \frac{\alpha}{12} - \frac{d}{6} + \frac{1}{3}$$

for $\alpha \leq d - 1$. This completes the proof for the base case $m = 3$.

Now, fix $4 \leq m \leq d$ and assume that the desired estimates hold true if one decreases m, R, or A. From the same argument as in the previous section, we have the desired estimate (6.1) with

$$\gamma_m(\alpha) = \max\left\{-\frac{d}{4m} + \frac{1}{4}, \left(\frac{1}{2} - \frac{d-\alpha}{2m}\right) \cdot \frac{1}{m} + \gamma_{m-1}(\alpha) \cdot \left(1 - \frac{1}{m}\right)\right\},$$

where the second exponent in the above is a consequence of interpolation with the L^2 estimate in Corollary 3.3 implied by the linear refined Strichartz estimate, $\forall \alpha \in (0, d]$. Note that even though for certain m, bilinear refined Strichartz would provide a better bound (i.e., a smaller exponent) for the tangent contribution, it would not translate into a better $\gamma_m(\alpha)$ due to the constraint from the transverse contribution (i.e., the first exponent in the above).

It remains to check that one can indeed take $\gamma_m(\alpha)$ as stated in Proposition 6.2, which follows from straightforward computation and is left to the reader.

6.1. Comparison of tools. There are various tools that have been used in the argument above and in Section 4 and 5, such as linear and bilinear refined Strichartz estimates, which we would like to discuss a bit more and compare in this subsection.

First, as pointed out in Remark 3.5, applying the linear refined Strichartz estimate directly, one can immediately obtain some result on Falconer's problem for $d > 4$, which is already better than the previously best known bounds but is not as good as Theorem 1.2. In fact, as shown in Theorem 1.6, the linear refined Strichartz estimate produces better result for estimating the spherical Fourier decay rates of fractal measures when α is close to d, while when α is close to $\frac{d}{2}$, the relevant value for Falconer's problem, it doesn't behave as well as desired without the help of polynomial partitioning. The reason here is that the strategy of combining refined Strichartz and polynomial partitioning becomes more and more effective as α decreases from d to $\frac{d}{2}$, which is because, roughly speaking, that as α decreases, it becomes more sensitive to information extracted from intermediate dimensions.

Second, in the proof of Proposition 5.3 and 6.2, we have studied the tangent sub-case using interpolation between the induction hypothesis from one dimension lower and the weighted L^2 estimate in Corollary 3.3 which is based on the linear refined Strichartz. Alternatively, one may instead apply directly Hölder's inequality or the bilinear weighted estimate in Corollary 3.4 which is based on the bilinear refined Strichartz.

More precisely, Corollary 3.4 can be applied for each m, similarly as in the proof of the base case $m = 3$ of Proposition 6.2, to obtain an estimate for the tangent term. Or, Hölder's inequality implies that

$$\begin{aligned} &\|Ef_{j,\text{tang}}\|_{BL^{p_m}(B_j;Hdx)} \\ &\lesssim \|Ef_{j,\text{tang}}\|_{BL^{p_{m-1}}(B_j;Hdx)} \left(\int_{N_{R^{1/2+\delta_{m-1}}}(Y)\cap B_j} H\,dx\right)^{\frac{1}{p_m}-\frac{1}{p_{m-1}}}, \end{aligned} \tag{6.2}$$

which, combined with the fact that $H \in \mathcal{F}_{\alpha,d}$ and the induction hypothesis on $(m-1)$-dimensional varieties, produces another estimate for the contribution from tangent wave packets.

Both estimates already yield improvement of previously best known results for Falconer's problem and the Fourier decay rates of fractal measures, but are weaker than Theorem 1.2 and Theorem 1.6. Roughly speaking, the method involving Hölder's inequality produces the weakest result among all options, the method via interpolation is the best when m is larger than $d/2$, otherwise the bilinear refined Strichartz approach behaves better. However, as already mentioned in the proof of Proposition 6.2, it turns out that it is unnecessary to apply the stronger bilinear refined Strichartz even if m is small, which is because in this case the constraint arising from the transverse sub-case is too strong.

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