Received: 13 February 2020 | Revised: 10 March 2021 | Accepted: 22 May 2021

DOI: 10.1002/nav.22009

RESEARCH ARTICLE

WILEY

Sequential selection for accelerated life testing via approximate Bayesian inference

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Funding information

US National Science Foundation, Grant/Award Number: CMMI-1855651.

Abstract

Accelerated life testing (ALT) is typically used to assess the reliability of material's lifetime under desired stress levels. Recent advances in material engineering have made a variety of material settings readily available. A critical question is how to efficiently conduct ALT to optimize reliability performance over different material settings. We propose a sequential selection approach to solve this problem. The proposed approach contains (1) a model updating mechanism to incorporate new experimental data in each step, and (2) a design criterion to guide new experiments that maximizes the potential to find the optimal material setting. To guarantee a tractable statistical mechanism for information collection, we develop explicit model parameter update formulas via approximate Bayesian inference. Theories show that our explicit update formulas give consistent parameter estimates. To guarantee that the design criterion in each step can make improvement on the identification of optimal material setting, this paper adopts an expected improvement-based design criterion for optimizing the material setting under target stress factor levels. We also give a heuristic on this design criterion to justify the statistical consistency of approximate Bayesian estimates. Simulation studies and a case study show that the proposed sequential selection approach can significantly improve the probability of identifying the material alternative with best reliability performance compared to other design approaches.

KEYWORDS

expected improvement, experimental design, log-normal model, optimum planning

1 | INTRODUCTION

1.1 | Motivation

Product reliability is often referred to as its ability of performing intended function under specific operating conditions. However, it might take months or years to observe a product failure under the desired operating conditions. Accelerated life test (ALT) is used to collect reliability information in a timely manner under accelerated operating conditions in the lab environment. Then the reliability information collected can be used to predict the lifetime under the normal operating conditions in field environment. Typically, ALT tests contain a given number of experimental units due to the availability of experimental resource. The classical problem of experimental design for ALT is to allocate the stress levels representing accelerated operating conditions to each test unit.

Recent advances in material engineering have made a variety of material settings readily available in lab testing. Among those different settings, the proportions of different elements in the material and mechanical procedures would greatly influence their reliability performance. For example, copper alloys are widely applied in various safety-/mission-critical industries, for example, aircraft bearings and bushings in aerospace industry, drilling and mining equipment in mining industry. Alloys can be formed under different mechanical procedures, which may alter their physical/chemical properties and further lead to different reliability performances. Thus, in addition to various environmental conditions of "load," "temperature," and "humidity" (Singh et al., 2007), the experimenter is also interested in finding the material with better reliability performance. This suggests that the selection of material settings (particularly, different mechanical procedure of copper alloys in this example; may also including material proportion, mechanical procedure, etc. in other situations) is quite critical to the product reliability. In this paper, we aim to select the material setting with the best reliability performance for ALT. To fulfill this aim, the experimenter of ALT needs to determine the stress levels and the material setting of each test unit.

As demonstrated in Lee et al. (2018), sequential design is often preferable compared to one-shot designs (i.e., allocating design points for all test units at the beginning stage of the experiments) in terms of improving the efficiency of test planning. The reasons are given as follows. First, testing labs are typically equipped with only a limited number of testing machines (e.g., one or two). Therefore, it is physically impossible to conduct all N experiments simultaneously. Second, efficient one-shot design relies on prior estimates of model parameter, and an accurate prior of model parameters is often difficult to obtain before conducting ALT. Particularly, this paper focuses on selecting the optimal material setting, and the advantage of the sequential test planning is to improve the efficiency in optimal decision-making. We propose a sequential selection approach to allocate experimental design settings to the test units. In each step of this sequential procedure, the experimental design setting for the new test unit is selected to maximize the expected gain on optimizing the reliability performance under a Bayesian log-normal model. For the computational convenience of sequential selection, we develop explicit model parameter update formulas via approximate Bayesian inference. Theories show that our explicit update formulas give consistent parameter estimates. In the next subsection, we point out the connection of our work to literature studies.

1.2 | Related literature

Our paper is closely related to the literature of experimental design for ALTs, as well as the literature on sequential experimental design and learning in simulation optimization. We review state-of-the-art approaches and recent advances from both communities and point out their connections to our paper.

The typical problem in designs for ALT is to allocate accelerated stress levels to experimental test units. The ASTM standard (Standard 2010) suggests balanced and equally spaced designs for ALT. Given a lower bound and an upper bound of a stress factor, equally spaced levels are chosen as design points. Each design point is applied to an equal number of experimental units. When multiple stress factors are incorporated, this standard design can be extended to factorial designs (Wu & Hamada, 2011) with all level combinations of the factors available. This standard design is developed to reduce the variance of parameter estimates or prediction. To further improve ALT, the optimal design is often carried out through minimizing or maximizing a function that involves the Fisher information derived under different model settings, see for examples, Meeker and Hahn (1977), Meeker and Escobar (2014), Pan and Yang (2014), and King et al. (2016). A recent review of this topic can be found in Nelson (2015). For example, Zhu and Elsayed (2013) propose an approach for the design of ALT plans for the model under the Weibull distribution assumption. Those optimal design approaches work well if the substituted parameter guesses in the model are accurate. This requirement is often impractical at the early stage of the experimentation. Bayesian methods with prior information can be developed to plan ALT under parameter uncertainty, see for examples, Zhang and Meeker (2006), Sha and Pan (2014), and Zhao et al. (2019). Along with more test results collected, it is also desired to update the prior information, and design ALT test in a sequential manner. Recently, Lee et al. (2018) developed a sequential Bayesian design approach for ALT to mitigate this drawback, and improve the efficiency in test planning. However, as noted earlier, most of existing experimental design approaches for ALT are developed to assess the reliability performance of a given product or material. In this paper, we focus on selecting the optimal material setting with the best reliability performance. The experimental design issue for this particular problem has not been discussed in the literature to the best of our knowledge.

Selecting the optimal design among different alternatives has been well known as the ranking and selection (R&S) problem in the simulation community, which can date back to Bechhofer (1954). In such problems, the experiment is usually under the limit of a fixed budget (e.g., time, materials), and the decision-maker wants to identify the optimal design correctly as much as possible. See Hong and Nelson (2009) and Chau et al. (2014) for more description. For the R&S problem, we say "correct selection" occurs if the selected alternative is truly the best design after the simulation budget is exhausted. The optimal budget allocation with respect to maximizing the probability of the correct selection is studied rigorously in Glynn and Juneja (2004). However, this optimal budget allocation requires certain knowledge of the designs and thus cannot be applied directly in practice; for more details, see the discussion in Chen and Ryzhov (2019). Therefore, modern researchers prefer to allocate their budget in a sequential manner, which is more practical and computationally tractable. In such sequential allocation algorithms, the decision-maker first spends part of the budget, observes the results, then determines how to allocate the remaining budget accordingly. There are many sequential allocation algorithms that have been proposed, including expected improvement (or EI; see Jones et al., 1998), optimal computing budget allocation (or OCBA; see Chen et al., 2000), indifference-zone method (Kim & Nelson, 2001), and top-two methods (Russo, 2017). The EI-type methods also include Chick et al. (2010), Powell and Ryzhov (2012), Qin et al. (2017), and Salemi et al. (2019). Other approaches include the reverse-engineering method with brutal force (Peng & Fu, 2017). Though various sequential allocation algorithms have been proposed, there is no previous work that applies them to material selection in ALT, where usually we encounter censored observations from experiments, as discussed later in Section 2. To overcome the inconvenience brought by the incomplete information, our work builds an approximate Bayesian model to learn the reliability performance of the materials, which allows us to apply the sequential allocation algorithms more efficiently in ALT.

1.3 | Overview of our paper

Our paper proposes a sequential selection approach to find the optimal material setting. Beyond assessing the reliability performance of a single type of material, the proposed approach incorporates the material setting as a decision variable, and is able to find the material with the best reliability performance efficiently. The proposed approach contains a model updating mechanism to incorporate new experimental data in each step, and a design criterion to guide new experiments that maximize the potential to find the optimal material setting.

For the model updating mechanism, this paper develops an approximate Bayesian inference approach for the log-normal model. This approach can be used to update the parameters of the log-normal model for sequential-ALT with censored observations. Our development makes it possible to efficiently implement the model parameter update with censored observation for sequential experimentation. We also provide theoretical results to show the statistical consistency of parameter estimation via the proposed sequential selection approach. Our update algorithms and corresponding consistency theory are not available in existing literature.

For the design criterion, this paper adopts an EI-based design criterion to select design points for both material setting and the target stress factor, and the goal of this design criterion is to maximize the material reliability performance. Under the log-normal model with a linear interaction term between material setting and the target stress, we simplify the expression of the design criterion, and give a heuristic on the proposed design to guarantee the statistical consistency of approximate Bayesian estimates.

The rest of the article is organized as follows. Section 2 provides detail description of our problem. Section 3 investigates the approximate Bayesian inference approach for the log-normal model and its corresponding theoretical properties. Section 4 introduces the design criterion for sequential selection. Section 5 compares the proposed approach with other test planning approaches using numerical examples. Section 6 concludes the paper with discussions and future directions.

2 | PROBLEM DESCRIPTION

ALT mostly considers different levels of the stress factors in testing and validating the reliability performance of a given product or material, which is often characterized by a lifetime model. In our problem, both stress factors and material features of the product are included in the test planning stage. The stress factors are denoted by a d dimensional vector $\boldsymbol{v}$, whereas the material features are denoted by a p dimensional vector $\boldsymbol{z}$. The stress factors are usually numerical variables providing the accelerated stress levels, such as temperature and humidity. The entries of the material feature vector $\boldsymbol{z}$ can be continuous variables indicating the key metrics of material characteristics, and they can also be categorical variables referring to different material types. For example, the material features may include the composition percentage of different elements in an alloy, as well as different types of metallurgical procedures (e.g., annealing, tempering, electroplating, etc.) used to process materials.

We assume that the mean performance of material reliability can be expressed by $\mu(\boldsymbol{z}, \boldsymbol{v}; \boldsymbol{\beta})$ as a function of stress factors $\boldsymbol{v}$ and material features $\boldsymbol{z}$ with an unknown parameter vector $\boldsymbol{\beta}$. A higher value of $\mu(\boldsymbol{z}, \boldsymbol{v}; \boldsymbol{\beta})$ indicates that the corresponding material setting $\boldsymbol{z}$ leads longer material lifetime in average under the stress level combination $\boldsymbol{v}$. Therefore, the goal of our problem is to find the material alternative $\boldsymbol{z}$ which leads the best mean reliability performance under the target stress levels $\boldsymbol{v}^*$:

$$\boldsymbol{z}^*(\boldsymbol{v}^*) \in \underset{\boldsymbol{z}\in\mathcal{Z}}{\operatorname{argmax}} \mu(\boldsymbol{z}, \boldsymbol{v}^*; \boldsymbol{\beta}), \tag{1}$$

where $\mathcal{Z}$ is a set of candidate material settings in our experiments.

Since the testing process (e.g., the material wear process as in Section 5.2) can be extremely complex, it is almost impossible to develop an accurate mathematical model for the mean material lifetime under multiple stress factors and material features. To solve this problem, a log-normal model is often used to surrogate the material lifetime (Meeker & Escobar, 2014):

$$\log(T) = \boldsymbol{x}(\boldsymbol{z}, \boldsymbol{v})^\top \boldsymbol{\beta} + \varepsilon, \tag{2}$$

where T is a random variable representing the lifetime of a test unit with experimental setting $\boldsymbol{x}(\boldsymbol{z}, \boldsymbol{v})$, ε is the error term following a normal distribution with mean zero and variance σ^2, and $\boldsymbol{x}(\boldsymbol{z}, \boldsymbol{v})$ collects the intercept, the stress factors $\boldsymbol{v}$, the material features $\boldsymbol{z}$, and the interactions between material features and stress factors. In particular,

$$\boldsymbol{x}(\boldsymbol{z}, \boldsymbol{v}) = (1, \boldsymbol{v}^\top, \boldsymbol{z}^\top, (\boldsymbol{z} \otimes \boldsymbol{v})^\top)^\top, \tag{3}$$

where $\boldsymbol{z} \otimes \boldsymbol{v}$ denotes the Kronecker product of $\boldsymbol{z}$ and $\boldsymbol{v}$, which is a $d \times p$ dimensional vector representing the interaction between material features and stress factors. To simplify the notation, we reduce $\boldsymbol{x}(\boldsymbol{z}, \boldsymbol{v})$ to $\boldsymbol{x}$ when there is no confusion. The linear coefficient $\boldsymbol{\beta}$ is a $(p+1)\times(d+1)$ dimension vector. After collecting life times T_i's from test units $i = 1, \dots, N$, the model parameters can be estimated via the maximum likelihood method.

In reliability studies, the lifetime T_i's are often given as the censored observations. Even under accelerated stress levels, the lifetime of a test unit can be as long as weeks or months. Thus, in the experimental stage, the tests will be terminated after a given observation time τ_i, even if the failure has not been observed. In additional to T_i, the failure of the i-th test is often recorded by a binary variable δ_i. If $\delta_i = 1$, failure is observed, and T_i is the lifetime of the i-th test unit. If $\delta_i = 0$, we only know that the lifetime T_i is greater than τ_i. Under the assumption of the log-normal model in (2), the likelihood function of $\boldsymbol{\beta}$ and σ^2 is

$$L\left(\boldsymbol{\beta}, \sigma^2 \mid \{T_i, \tau_i, \delta_i, \boldsymbol{x}_i\}_{i=1}^N\right) = \prod_{i=1}^N \left\{ \frac{1}{\sigma T_i} \phi\left(\frac{\log T_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\sigma} \right) \right\}^{\delta_i} \cdot \left\{ 1 - \Phi\left(\frac{\log \tau_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}}{\sigma} \right) \right\}^{1-\delta_i}, \tag{4}$$

where ϕ and Φ are the probability density function and the cumulative distribution function of the standard normal random variable, respectively.

Under the linear model setting, it is critically important to develop efficient experimental design approach to solve the optimization problem in (1). Since our goal is to find the optimal material setting more efficiently, we develop a sequential selection approach for ALT. Without loss of generality, we assume that the test lab is only equipped with one set of test machine. Thus, in each step of this sequential procedure, we only select one design point and allocate it to one test unit. The collected reliability information is used to update our belief regarding to the mean lifetime, and our belief regarding to the mean reliability performance of different material settings is used to determine the design for the next test unit. There are two main tasks under this development: (1) how to update the beliefs regarding the mean reliability performance of different material settings under the linear model setting with censored observations; (2) how to develop experimental design criterion to select new design points at each step. In this paper, we first develop the updating formula for our belief of the mean lifetime in Section 3, and then develop a policy to allocate experimental setting based on the updated belief in Section 4.

3 | APPROXIMATE BAYESIAN INFERENCE FOR LOG-NORMAL MODEL WITH INCOMPLETE OBSERVATIONS

In this section, we develop Bayesian update formulas for the log-normal model in (2). Under the linear model setting in (2), we assume that the prior of the linear coefficients $\boldsymbol{\beta}$ is a multivariate normal distribution with mean $\boldsymbol{\theta}_0$ and variance matrix Σ_0. If the lifetime T_i is not censored, the conjugacy property of the multivariate normal distribution also leads to a multivariate normal posterior distribution of $\boldsymbol{\beta}$. For $n = 1, \dots, N$, we denote $\boldsymbol{\theta}_n$ and Σ_n as the mean vector and variance matrix of the posterior distribution of $\boldsymbol{\beta}$ after including observations from the first n test units. It is straightforward to derive that

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} \Sigma_n \boldsymbol{x}_{n+1} \tag{5}$$

and

$$\Sigma_{n+1} = \Sigma_n - \frac{\Sigma_n \boldsymbol{x}_{n+1} \boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}. \tag{6}$$

where $\boldsymbol{x}_{n+1}$ is the design point of the $(n+1)$-st test unit, $y_{n+1} = \log T_{n+1}$ is the logarithm lifetime observation, and σ^2 is the variance of the error term in (2). In our development, we assume that σ^2 is known for notational convenience.

Notice that, the conjugacy property gives closed-form parameter update formulas, which further enables convenience in the development of sequential experimental policies. See for examples in Frazier et al. (2008) and Frazier et al. (2009). However, the conjugacy property does not hold if we have censored responses. An alternative method of constructing closed-form parameter update formulas under this situation is the moment-matching based approximate Bayesian inference. The technique of moment-matching has a long history in Bayesian statistics (Carlin & Louis, 2008). This technique has recently been used to develop closed-form Bayesian update equations for different problems, see, for examples, Dangauthier et al. (2008) and Chhabra and Das (2011). Particularly, based on this moment-matching technique, Zhang and Song (2017) investigate the Bayesian R&S approaches under a multivariate normal setting with unknown variance, and its statistical consistency has recently been demonstrated by Chen and Ryzhov (2020). Note that it is the log-normal model for which we use approximate Bayesian inference to derive closed-form parameter update in this paper, which is essentially different from the multivariate normal model in Zhang and Song (2017). Moreover, the purpose of using approximate Bayesian inference in Zhang and Song (2017) is to solve the challenge of unknown covariance in multivariate normal model, while the purpose of using approximate Bayesian inference in this paper is to tackle the incomplete observations. Due to these major differences, the theory developed in Zhang and Song (2017) is not applicable to this paper.

In this paper, the idea of approximate Bayesian inference is to approximate the posterior distribution of $\boldsymbol{\beta}$ as a multivariate normal distribution with mean $\boldsymbol{\theta}_{n+1}$ and variance Σ_{n+1}, which are the first and second moments of the posterior distribution of $\boldsymbol{\beta}$ given that $\delta_{n+1} = 0$, that is, $y_{n+1} > \log \tau_{n+1}$. The approximate Bayesian update formula is given in Proposition 1.

Proposition 1 *Assume that, at the $(n+1)$-st step, we observe $\delta_{n+1} = 0$ and $y_{n+1} > \log \tau_{n+1}$.*

Under the log-normal model, and the multivariate normal prior $\boldsymbol{\beta} \sim \mathrm{MVN}(\boldsymbol{\theta}_n, \Sigma_n)$, *the approximation Bayesian inference gives closed-form update formulas*:

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + \frac{\phi(\eta_n)}{(1-\Phi(\eta_n))\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}} \Sigma_n \boldsymbol{x}_{n+1}, \quad (7)$$

and

$$\Sigma_{n+1} = \Sigma_n - \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} + \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} \times \left(1 - \eta \frac{\phi(\eta_n)}{\Phi(\eta_n)} - \frac{\phi(\eta_n)^2}{\Phi(\eta_n)^2}\right)^2, \quad (8)$$

where

$$\eta_n = \frac{\log \tau_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}}, \quad (9)$$

and $\boldsymbol{\theta}_{n+1}$ *and* Σ_{n+1} *are the first and second moments of the posterior distribution of* $\boldsymbol{\beta}$ *given that* $\delta_{n+1} = 0$.

If material failure is observed, (6) indicates that the variance reduction is $\frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}$. Also, if there is a censored response, the amount of variance reduction will be reduced by $\frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}\left(1 - \eta_n \frac{\phi(\eta_n)}{\Phi(\eta_n)} - \frac{\phi(\eta_n)^2}{\Phi(\eta_n)^2}\right)^2$ as in (8). However, in sequential update, the effects of this additional term to the variance reduction are usually negligible. This is because that the variance Σ_n is small when n is large enough. Our numerical results often show that the variance update formulas in (6) and (8) lead to approximately equal variances. Therefore, in terms of the variance update, we adopt (6) for both complete and censored responses. As a result, the update formulation at step n can be summarized by

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + \delta_{n+1} \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} \Sigma_n \boldsymbol{x}_{n+1} + (1-\delta_{n+1}) \frac{\phi(\eta_n)}{(1-\Phi(\eta_n))\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}} \Sigma_n \boldsymbol{x}_{n+1},,$$

$$\Sigma_{n+1} = \Sigma_n - \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}, \quad (10)$$

with η_n given in (9).

We now discuss the consistency property of the proposed approximate Bayesian inference under incomplete observations. In the following context, we demonstrate the convergence of the sequence $(\boldsymbol{\theta}_n)_{n=0}^\infty$ based on the generic framework established in Chen and Ryzhov (2020). We make the following assumptions on the design vectors $(\boldsymbol{x}_n)_{n=0}^\infty$ from two different perspectives. Assumptions 1 and 2 ensure that $(\boldsymbol{x}_n)_{n=0}^\infty$ come from a nondegenerate distribution when they are considered as i.i.d. samples; Assumption 3 guarantees that each component of the regression coefficient $\boldsymbol{\beta}$ is learnt for a nonzero proportion of the total time in the long run, when $(\boldsymbol{x}_n)_{n=0}^\infty$ are considered as fixed vectors chosen from a finite deterministic set. In short, Assumptions 1–2 and Assumption 3 specify two completely different scenarios depending on the experimenter's view of the design vectors. Nevertheless, we can show the convergence of $(\boldsymbol{\theta}_n)_{n=0}^\infty$ in both scenarios.

Assumption 1 The design vectors $(\boldsymbol{x}_n)_{n=0}^\infty$ are drawn i.i.d. from a common distribution satisfying $\mathrm{E}(\boldsymbol{x}_n \boldsymbol{x}_n^\top) = \mathbf{A}$, where $\mathbf{A}$ is a positive-definite symmetric matrix.

Assumption 2 The sequence $(\boldsymbol{x}_n)_{n=0}^\infty$ satisfies $0 < \inf_n \|\boldsymbol{x}_n\|_1 \leq \sup_n \|\boldsymbol{x}_n\|_1 < \infty$ almost surely.

Assumption 3 All the design vectors $(\boldsymbol{x}_n)_{n=0}^\infty$ are chosen from a finite deterministic set $\mathcal{X} = \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K\}$, where $K \geq (p+1)(d+1)$ is a fixed positive integer and $\boldsymbol{\lambda}_j \neq 0$ for all $1 \leq j \leq K$. The optimal design is unique. The matrix $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K)$ is full rank. For all j, there exists some fixed constant $\alpha_j \in (0,1)$ such that $|N_{\boldsymbol{\lambda}_j,n}/n - \alpha_j| = O\left(n^{-\frac{1}{8}-\epsilon}\right)$, where ϵ is a fixed positive constant and $N_{\boldsymbol{\lambda}_j,n}$ is the number of times that design $\boldsymbol{\lambda}_j$ has been sampled by time n.

Theorem 1 *Suppose Assumptions* 1–2 *or Assumption* 3 *holds and the sequence* $(\log \tau_n)_{n=0}^\infty$ *is bounded, and suppose that* $\boldsymbol{\theta}_n$ *and* Σ_n *are updated using* (10). *Then,* $\boldsymbol{\theta}_n \to \boldsymbol{\beta}$ *almost surely.*

The proof of Theorem 1 is deferred to the Appendix. This theorem indicates that although we approximate the posterior distribution to a multivariate normal under censored observations, the approximation can be asymptotically accurate, since the updated parameter sequence $(\boldsymbol{\theta}_n)_{n=0}^\infty$ converges to the true model parameters. Note that Theorem 1 proves the convergence of $(\boldsymbol{\theta}_n)_{n=0}^\infty$ under two different sets of assumptions (i.e., Assumptions 1–2 or Assumption 3). Assumptions 1 and 2 state the requirement for the situation that the design points are selected from a continuous space. Assumption 3 requires that the design points are chosen from a finite space. In the field of ALTs, although the variables (e.g., temperature, humidity) are often continuous, practitioners usually consider that the design points take values at a finite number of levels, as it is easier to control in physical experiments.

We would like to remark the connection and difference between our theory and the theory in Chen and Ryzhov (2020). Theorem 1 is obtained under the framework of Chen and Ryzhov (2020), which considers approximate

Bayesian algorithms as a general stochastic approximation algorithm (Robbins & Monro, 1951). That being said, showing the consistency of approximate Bayesian algorithms that are built with different distribution families and approximation criteria still requires substantial effort, which can be seen from the demonstration examples of Chen and Ryzhov (2020). As (7)–(8) are derived from an approximate Bayesian model that has never been specifically studied before, the demonstration of its consistency is quite a substantive work.

4 | SEQUENTIAL SELECTION FOR RELIABILITY IMPROVEMENT

This section discusses how to select design points in a sequential manner. As mentioned earlier, we investigate a fully sequential procedure, and assume that only one experimental unit will be allocated in each step of the sequential procedure. Recall that our goal is to determine the material feature combination $z^*(\boldsymbol{v}^*)$ such that it has the best reliability performance under the target stress factor levels $\boldsymbol{v}^*$. At the n-th step of the sequential procedure, the optimal material setting based on the collected information can be expressed by

$$z^n(\boldsymbol{v}^*) \in \underset{z\in\mathcal{Z}}{\operatorname{argmax}} \mathrm{E}^n \mu(z, \boldsymbol{v}^*; \boldsymbol{\beta}), \tag{11}$$

where E^n represents that the expectation is taken with respect to the prior distribution of $\boldsymbol{\beta}$ at the n-th step. Under the log-normal model setting in (2), the objective in (11) can be simplified to

$$\mathrm{E}^n \mu(z, \boldsymbol{v}^*; \boldsymbol{\beta}) = \mathrm{E}^n \left[\boldsymbol{x}(z, \boldsymbol{v}^*)^\top \boldsymbol{\beta}\right] = \boldsymbol{x}(z, \boldsymbol{v}^*)^\top \boldsymbol{\theta}_n$$

with $\boldsymbol{x}(z, \boldsymbol{v}^*)$ given in (3). To meet the requirement of our goal in (1), new design points in each step should be determined to maximize the improvement the target optimization problem. The improvement of the objective in (1) by adding new design points in the $(n+1)$-st step can be quantified by

$$\begin{aligned} &\max_{z\in\mathcal{Z}} \mathrm{E}^{n+1} \mu(z, \boldsymbol{v}^*; \boldsymbol{\beta}) - \max_{z\in\mathcal{Z}} \mathrm{E}^n \mu(z, \boldsymbol{v}^*; \boldsymbol{\beta}) \\ &= \max_{z\in\mathcal{Z}} \left[\boldsymbol{x}(z, \boldsymbol{v}^*)^\top \boldsymbol{\theta}_{n+1}\right] - \max_{z\in\mathcal{Z}} \left[\boldsymbol{x}(z, \boldsymbol{v}^*)^\top \boldsymbol{\theta}_n\right]. \end{aligned} \tag{12}$$

Since $\boldsymbol{\theta}_{n+1}$ is a random vector that depends on the selected design points $\boldsymbol{x}_{n+1} = \boldsymbol{x}(z_{n+1}, \boldsymbol{v}_{n+1})$, the $(n+1)$-st design point should be chosen to maximize the expectation of the value of improvement given that $(z, \boldsymbol{v})$ is the design point at the $(n+1)$-st step. Therefore, the acquisition function to select the new design point can be expressed by

$$\begin{aligned} \mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*) = \mathrm{E}\left\{ \max_{z'\in\mathcal{Z}} \left[\boldsymbol{x}(z', \boldsymbol{v}^*)^\top \boldsymbol{\theta}_{n+1}\right] | z_{n+1} = z, \boldsymbol{v}_{n+1} = \boldsymbol{v} \right\} \\ - \max_{z'\in\mathcal{Z}} \left[\boldsymbol{x}(z', \boldsymbol{v}^*)^\top \boldsymbol{\theta}_n\right], \end{aligned} \tag{13}$$

where the expectation is taken with respect to the posterior predictive distribution of y_{n+1} given that $z_{n+1} = z$ and $\boldsymbol{v}_{n+1} = \boldsymbol{v}$ are the $(n+1)$-st design point. This EI-type acquisition function is typically used in selecting design points for optimization problem in a sequential manner, see Powell and Ryzhov (2012) for examples of the EI-type design criterion under different developments.

For our problem, (13) can be further reduced to

$$\begin{aligned} &\mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*) \\ &= \mathrm{E}_G \left\{ \max_{z'\in\mathcal{Z}} \left[(z' \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} + \frac{\boldsymbol{x}(z', \boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}(z, \boldsymbol{v})}{\sqrt{\sigma^2 + \boldsymbol{x}^\top(z, \boldsymbol{v}) \Sigma_n \boldsymbol{x}(z, \boldsymbol{v})}} G \right] \right\} \\ &\quad - \max_{z'\in\mathcal{Z}} \left[(z' \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} \right], \end{aligned} \tag{14}$$

where the expectation E_G is taken with respect to the standard normal random variable G, $\widetilde{\boldsymbol{v}}^* = (1, (\boldsymbol{v}^*)^\top)^\top$, and $\boldsymbol{\theta}_n = (\boldsymbol{\theta}_{n,0}^\top, \boldsymbol{\theta}_{n,1}^\top)^\top$ with $\boldsymbol{\theta}_{n,0}$ and $\boldsymbol{\theta}_{n,1}$ being vectors of size $d+1$ and $p(d+1)$, respectively. The new design point $\boldsymbol{x}_{n+1} = \boldsymbol{x}(z_{n+1}, \boldsymbol{v}_{n+1})$ is selected to maximize this design criterion. Some details on how to obtain (14) are given in Appendix D.

For our problem, the number of candidate material settings in $\mathcal{Z}$ is often finite, say, $\mathcal{Z} = \{z^1, \dots, z^K\}$. Under this situation, $\mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*)$ has a closed-form expression according to Frazier et al. (2009). Let

$$b_n^k(z, \boldsymbol{v}; \boldsymbol{v}^*) = \frac{\boldsymbol{x}(z^k, \boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}(z, \boldsymbol{v})}{\sqrt{\sigma^2 + \boldsymbol{x}^\top(z, \boldsymbol{v}) \Sigma_n \boldsymbol{x}(z, \boldsymbol{v})}}$$

for $k = 1, \dots, K$. For notational convenience, we assume that $b_n^k(z, \boldsymbol{v}; \boldsymbol{v}^*) < b_n^{k+1}(z, \boldsymbol{v}; \boldsymbol{v}^*)$ for $k = 1, \dots, K-1$. Following Frazier et al. (2009), we have that

$$\begin{aligned} \mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*) &= \sum_{k=1}^{K} \left[b_n^{k+1}(z, \boldsymbol{v}; \boldsymbol{v}^*) - b_n^k(z, \boldsymbol{v}; \boldsymbol{v}^*) \right] \\ &\quad \cdot g\left\{ -\frac{|(z^{k+1} \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} - (z^k \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1}|}{b_n^{k+1}(z, \boldsymbol{v}; \boldsymbol{v}^*) - b_n^k(z, \boldsymbol{v}; \boldsymbol{v}^*)} \right\}, \end{aligned} \tag{15}$$

where $g(u) = u\Phi(u) + \phi(u)$. To maximize $\mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*)$, we can compute its gradient with regard to $\boldsymbol{v}$ according to Zhang and Hwang (2019), and use gradient based optimization approaches to find the maximum of $\mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*)$ for each given $z \in \mathcal{Z}$.

Note that the EI-type design criterion in (13) may not lead to a closed-form expression as in (15) if the posterior of the coefficient $\boldsymbol{\beta}$ is not a multivariate normal distribution in each step. The proposed approximation Bayesian update in Section 3 guarantees that the multivariate normal posterior distribution holds. Besides convenient and efficient model update, the proposed Bayesian approximation also plays an important role in simplifying the computation of sequential design selection.

In the field of accelerated tests, the design vectors $(\boldsymbol{x}_n)_{n=0}^\infty$ are usually considered fixed and chosen from a finite deterministic set $\mathcal{X} = \{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_K\}$. In this scenario, one needs to make sure that the sequential selection algorithm satisfies Assumption 3 to ensure the consistency of $(\boldsymbol{\theta}_n)_{n=0}^\infty$. However, the asymptotic behavior of EI-type criteria as well as other sequential selection strategies under linear model is not fully addressed in the literature. That being said, recent work (Ryzhov, 2016) has shown that under independent beliefs, classic EI-type criteria tend to allocate almost all the resource to the optimal design asymptotically, even though they can sample each design infinitely often. Consequently, they fail to satisfy the requirements of Assumption 3. Nonetheless, Chen and Ryzhov (2019) shows that under independent beliefs, the conditions in Assumption 3 can be easily satisfied by introducing a preselection step to balance the resource allocated to the optimal design and suboptimal designs. Therefore, we adopt this technique to modify the proposed EI approach (15) as a heuristic to address Assumption 3. In particular, if $\boldsymbol{x}_n^*$ is the selected design at the n-th step, the preselection step checks

$$N^2_{\boldsymbol{x}_n^*,n} < \sum_{\lambda \neq \boldsymbol{x}_n^*, \lambda \in \mathcal{X}} N^2_{\lambda,n}, \tag{16}$$

where $N_{\lambda,n}$ is the number of replication allocated to design points $\lambda \in \mathcal{X}$. If (16) holds, assign $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n^*$. If (16) does not hold, assign $\boldsymbol{x}_{n+1} = \arg\max_{\boldsymbol{x} \neq \boldsymbol{x}_n^*} \mathrm{EI}^n(\boldsymbol{x}; \boldsymbol{v}^*)$, where $\mathrm{EI}^n(\boldsymbol{x}; \boldsymbol{v}^*) = \mathrm{EI}^n(\boldsymbol{z}, \boldsymbol{v}; \boldsymbol{v}^*)$ is given by (15). Rigorous study of the asymptotic behavior of this heuristic under correlated beliefs would be a substantial contribution, but it is clearly out of scope of the current paper, which focuses on developing performance-guaranteed sequential selection algorithm for ALT.

5 | NUMERICAL STUDY

This section provides some synthetic examples and a case study on accelerated wear testing to compare the numerical performances of different model updates and experimental design approaches. In terms of model updates, we compare the proposed approximation Bayesian update formulas in (10) with the exact update, that is, refitting the log-normal model using all the data points, which does not possess tractable parameter updating formulas. This is the standard approach used in the literature, for example, Lee et al. (2018). Those two alternative approaches are denoted by "approx" and "exact," respectively. In terms of experimental design, the proposed EI-based design procedure described in Section 4 (specifically, (15) with modification (16)) is denoted by **SeqEI** in the comparison. We also compare our approach with two existing design methods in the literature:

1. (**Design:**) Full factorial design, see for example, Wu and Hamada (2011). For each stress factor, we choose equally spaced design points within a lower bound and an upper bound. As noted in the literature review section, this is the standard design following ASTM standard (Standard 2010).
2. (**SeqD:**) Sequential Bayesian D-optimal design in Lee et al. (2018). Numerical results in Lee et al. (2018) have demonstrated the advantage of sequential design over a number of alternative design approaches in the literature.

We consider all possible combinations of the two model update approaches and the three experimental design approaches. The six alternatives involved in our numerical comparison are denoted by "Design approx," "Design exact," "SeqD approx," "SeqD exact," "SeqEI approx," and "SeqEI exact," respectively.

Notice that, the EI-type sequential design criterion in (13) may not lead to a closed-form expression as in (15) if the posterior of the coefficients $\boldsymbol{\beta}$ is not a multivariate normal distribution. For "SeqEI approx," our model (the posterior distribution of $\boldsymbol{\beta}$) can be represented by a multivariate normal distribution completely, based on the proposed approximation Bayesian update in Section 3. Thus, the proposed Bayesian approximation also plays an important role in simplifying the computation of sequential design selection. However, under the exact model update, the implementation of this EI-type sequential design criterion is impractical, since it may require Markov chain Monte Carlo (MCMC) to approximate the value of (13) for each candidate $\boldsymbol{v}$ and $\boldsymbol{z}$ at each step. In our implementation of "SeqEI exact," we process the model and the experimental design selection under two separate tracks: the design criterion in (15) is obtained under the proposed approximate model update (the same as in "SeqEI approx"), whereas the collected data points are used to refit the exact model and determine the optimal material setting according to (11) at each step. In this way, we can evaluate the effects of model update and sequential design separately.

The full factorial designs are one-shot designs, which are not originally developed for a sequential experimentation. To compare the full factorial design under a sequential manner, we make it adaptable for a sequential procedure. First, we generate a full factorial design with respect to the number of levels of the material feature factors and the stress factors. Since the total number of steps N is usually greater than the run size of this full factorial design, we replicate the runs in the full factorial design one by one to make total run size equal to N (i.e., the runs in original full factorial design may not have exact equal number of replications). Finally, we randomize the order of the design within the N runs, and let them enter the sequential procedure one by one.

The goal of our problem is to choose the material setting with the best reliability performance. In practice, we often consider a finite number of material settings. Thus, we consider discrete levels of the material factors, and use probability of correct selection at the target stress level $\boldsymbol{v}^*$ to evaluate different approaches. According to (1) and (11), the probability of correct selection can be expressed by $\mathrm{P}(\boldsymbol{z}^n(\boldsymbol{v}^*) = \boldsymbol{z}^*(\boldsymbol{v}^*))$, where the probability is taken with respect to $\boldsymbol{z}^n(\boldsymbol{v}^*)$, which is a random variable due to the randomness of collected responses. In our numerical study, the probability of correct selection is estimated empirically by

$$\hat{P}(\boldsymbol{z}^n(\boldsymbol{v}^*) = \boldsymbol{z}^*(\boldsymbol{v}^*)) = \frac{1}{R}\sum_{r=1}^{R} I(\boldsymbol{z}_r^n(\boldsymbol{v}^*) = \boldsymbol{z}^*(\boldsymbol{v}^*)), \tag{17}$$

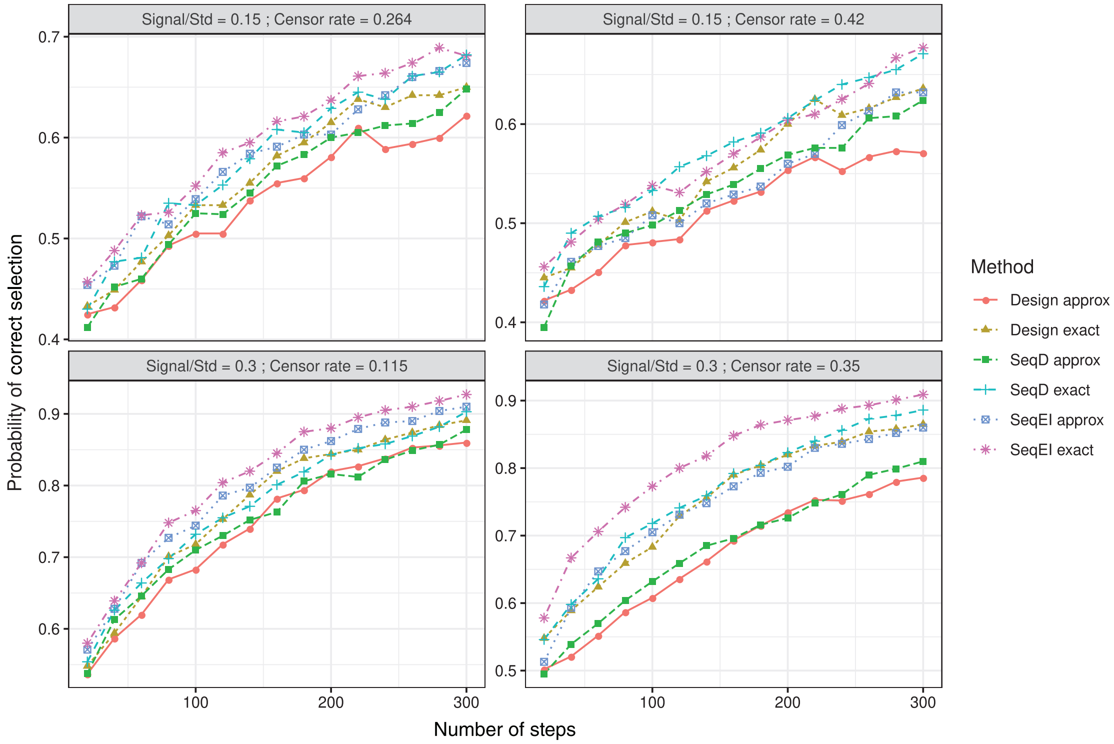

FIGURE 1 The estimated probability of correct selection for different settings with $K = 3$

where R is the total number of replications, $I(\cdot)$ is an indicator function, and $z_r^n(\boldsymbol{v}^*)$ is the selected optimal material setting at the n-th step from the r-th replication. In the synthetic examples and the case study, we use $R = 100$ to compute the estimated probability of correct selection. In all of our numerical examples, we set the observation time τ_i in (4) to be a constant.

5.1 | Synthetic examples

In this study, we directly generate data from the log-normal model in (2). The stress factor $\boldsymbol{v}$ contains three dimensions. For each dimension, the design points of the accelerated lab experiments are taken value from $\{0.5, 1\}$, whereas the targeted environmental condition is specified to be 0.1. For the material factors, we generate one factor with K levels. The first level of this material factor is specified to be optimal with the best reliability performance in average. We generate four random variables from uniform distribution $U(-1/30, 0)$ to be the linear coefficients corresponding to the intercept and each of three stress factors. The generated four-dimensional linear coefficients are denoted by a vector $\boldsymbol{\beta}_1$. The linear coefficients of each remaining material level are generated by $\boldsymbol{\beta}_1 + \boldsymbol{\beta}_k$ for $k = 2, \ldots, K$, where each component of $\boldsymbol{\beta}_k$ is a uniform random variable from $-1/30$ to 0. This setting guarantees that the first level of the material factor has the best reliability performance in average, and the average lifetime decreases as stress factor levels increase. A total number of 100 replications is used to estimate the probability of correct selection as in (17). For each replication, we generate 15 data points for each material setting to obtain the prior distributions for the linear coefficients.

In Figure 1, we consider a case with three material settings, that is, $K = 3$. We generate the responses under different signal to noise ratios. The signal level (i.e., the value of coefficients) is fixed as described earlier. The value of standard deviation σ in (2) is set to be 0.2 or 0.1, and resulted value of "Signal/Std" is 0.15 as in the top panel of Figure 1 or 0.3 as in the bottom panel of Figure 1. The value of the constant observational time τ_i in (4) is set to be 1.1 or 1.2 to generate different levels of response censor rates. As shown in Figure 1, the censoring rate varies from 10% to 26% if $\tau_i = 1.2$ (left panel), whereas the censoring rate is above 30% if $\tau_i = 1$ (right panel). Under a similar setting, we show the results of a scenario with four and six material settings (i.e., $K = 4$ and 6) in Figures 2 and 3.

The results in Figures 1–3 show that "SeqEI" based approaches give the highest probability of correct selection. Since the design criterion of "SeqEI" is developed to improve the optimization problem in (1), it outperforms "Design" and "SeqD," both of which aim for reducing the variances of model coefficients. We also see that, "approx" approach does not perform well if the censoring rate is high (say, above 30%). It demonstrates that the efficiency of the proposed approximate model updating approach can deteriorate if there is a significant large portion of censored observations. However, as shown in the literature of ALT, the censor rate in practice is often below 15%, for example, Han and Balakrishnan (2010), Elsayed et al. (2006), and Lee et al. (2018). Therefore, the numerical results show that "approx" approach

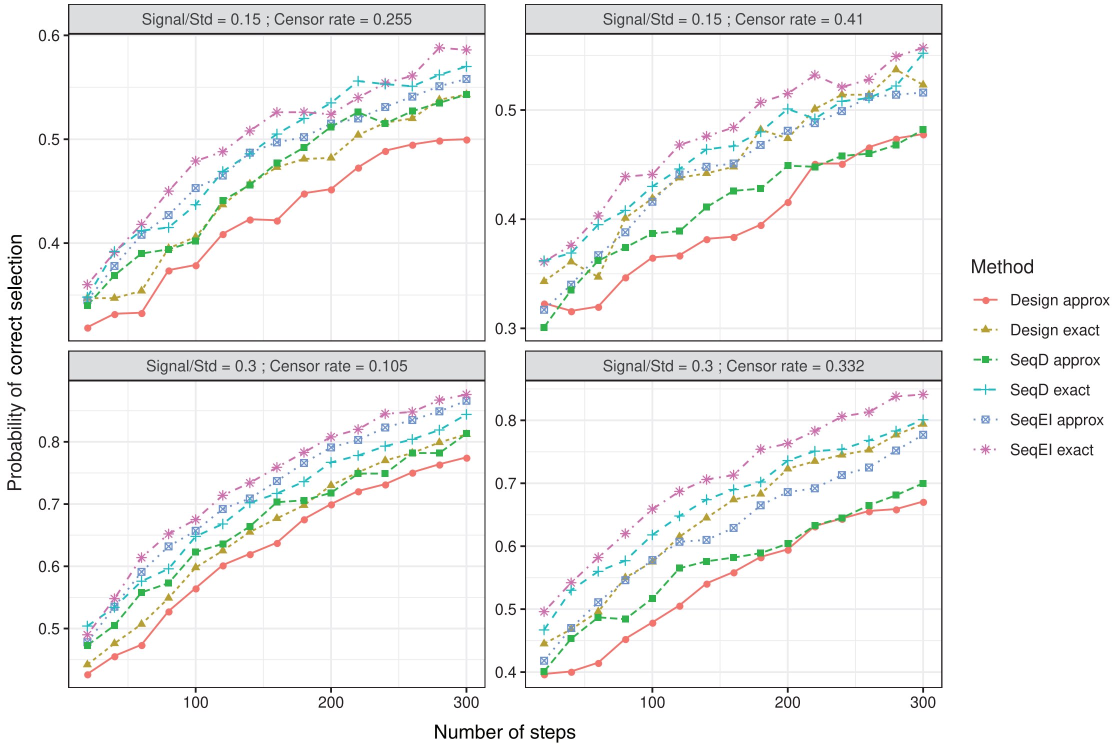

FIGURE 2 The estimated probability of correct selection for different settings with $K = 4$

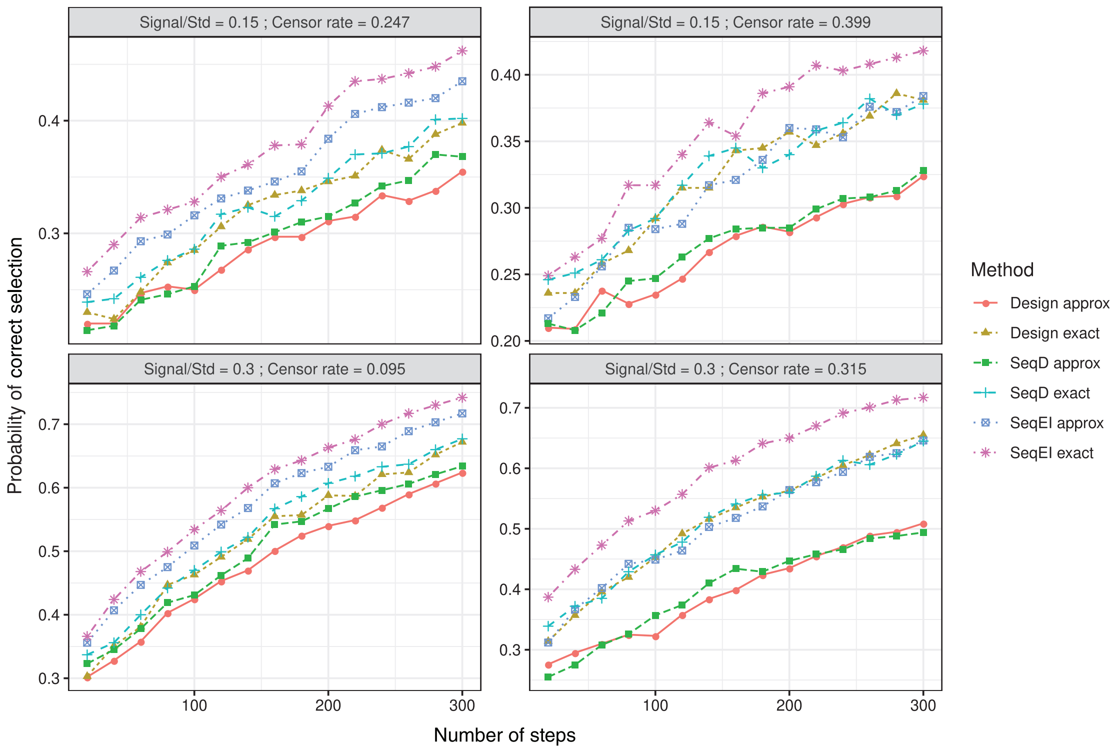

FIGURE 3 The estimated probability of correct selection for different settings with $K = 6$

is powerful under the censor rate for most common ALT cases in practice. Overall, in the three examples, "SeqEI exact" gives the best performance, and the performance of "SeqEI approx" is competitive to the best when the censoring rate is below 30%. For challenging scenarios, for example, lower signal to noise ratio, higher censor rate, and larger number of alternative materials, "SeqEI" based approaches demonstrate obvious advantages compared to other design approaches in identifying the optimal material setting.

5.2 | A case study on accelerated wear tests

We consider a material wear test of copper alloys as an example to demonstrate the performance of the proposed

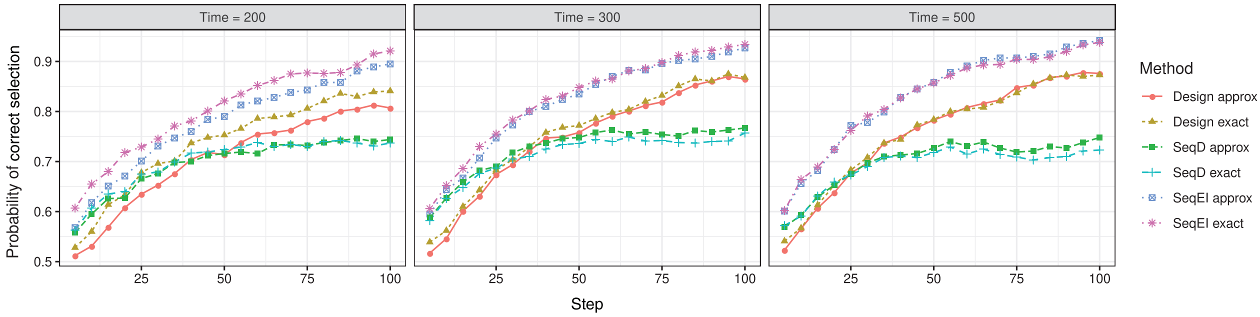

FIGURE 4 The estimated probability of correct selection for the case study

sequential selection approach. Because of high strength and exceptional bearing properties of copper alloys, they are widely considered in various safety-/mission-critical industries, for example, aircraft bearings and bushings in aerospace industry, drilling and mining equipment in mining industry. This case study considers the reliability performance of Cu–Ni–Sn alloys in the accelerated wear tests. This study investigates two types of material specimens, namely as-received Cu–Ni–Sn and annealed Cu–Ni–Sn specimens. Due to the annealing process, the microstructures as well as physical/chemical properties of annealed Cu–Ni–Sn specimens will be altered as compared to the as-received ones. Thus, their reliability performances may differ accordingly. The experimenter is interested in finding the material with better reliability performance. Wear tests were carried out using a Koehler K93500 pin-on-disc tester under various environmental conditions of "load," "temperature," and "humidity" (Singh et al., 2007). For each testing unit of Cu alloy specimens, in-situ monitoring outputs of wear performance (e.g., wear depth in μm) are measured over time by a linear variable displacement transducer. A material failure is recorded if the material weight loss is above a given threshold value. Historical data contains the information of the wearing processes of 18 experimental units.

The experimental observations of all 18 experimental units are provided for our study. Unfortunately, follow-up experiments are not available to further validate the proposed approach. Therefore, to implement the sequential selection procedure, we develop a pseudo simulator to model the historical data. This pseudo simulator is built on a Gaussian process model, which enables generating replications of data under each design approach. Under this pseudo simulator, the log response is not a linear function of the material factor and stress factors. We are able to investigate the robustness of the proposed approach under this nonlinear setting. The goal of this case study is to choose the materiel option that maximizes the reliability performance. According to the evidence shown from the data and domain knowledge, we identify that as-received Cu–Ni–Sn alloy is more reliable than annealed Cu–Ni–Sn alloy. With this information, we are able to estimate the probability of correct selection as in (17). In this study, we consider that the observation times τ_i equal to 200, 300, and 500 to generate different censoring rate of the responses.

The results of different approaches are shown in Figure 4. The censoring rates corresponding to observational times 200, 300, and 500 are 44.8%, 34.5%, and 29.6%, respectively. Similar to the results from Section 5.1, "SeqEI exact" gives the best performance in general, and the performance of "SeqEI approx" is competitive to the best when the censoring rate is low.

6 | CONCLUSION

In this paper, we consider the problem of determining the optimal material setting in accelerated lab experiments. First, the major challenge of this problem is that the observations of the lifetime of the materials in accelerated lab experiment are censored and sequential. To assess the reliability of different material settings, we model the lifetime of materials by log-normal distributions and build a sequential learning model via approximate Bayesian inference to estimate the regression coefficient. Under this model, we derive closed-form updating equations for the estimates of the unknown parameters under the log-normal distribution family, which greatly improves the efficiency of the learning process of the unknown parameter. Second, we consider selecting the optimal material design as an R&S problem and adopt an EI-type criterion to select design points for both material settings and environmental factors in a sequential manner at each stage of the experiment. Based on the sequential estimator obtained from our approximate Bayesian model for the regression coefficient, we derive a closed-form expression for our EI-type acquisition function, which makes our sequential selection strategy of design points computationally tractable. Third, we prove the consistency of our approximate Bayesian model by considering it as a generalized stochastic approximation algorithm. This consistency analysis not only provides the theoretical foundation of our proposed approach for determining the optimal material design in accelerated life experiments, but also demonstrates its general applicability to other sequential learning and selection schemes where the uncertainty can be modeled by log-normal distributions.

Our convergence proof is established under mild and practical conditions, which can help other researchers develop theoretical analysis for future approximate Bayesian models. Finally, we show the empirical performance of our approach by conducting two numerical studies, one of which is a synthetic example and the other is a case study on accelerated wear test. For both examples, our approach has consistently achieved a high probability of successfully identifying the optimal material design.

We would like to point out the limitations of our current work and future directions to overcome them. First, our case study considers a simple situation with only two material settings, and the benefit of using our procedure for this situation is limited. As demonstrated in the numerical study section, the proposed approach can be used to optimize the material setting with multiple level combinations. Therefore, it will be interesting to further investigate the real performance of the proposed approach under the situation of choosing the optimal material setting with multiple material design factors. Second, our case study is developed based on simulation experiments generated by real data. It is interesting to apply our approach to guide real experiments. Third, the observation time for each experimental unit is assumed to be given in this paper. However, it would be more beneficial to adaptively control the observational times for the test units based on the results of past experiments, as an optimized time allocation strategy should be able to reduce the censor rate of experimental units and improve the learning efficiency consequently. This should be especially critical for ALT with high censor rate.

ACKNOWLEDGMENT

Wenjun Cai would like to acknowledge funding from the US National Science Foundation under grant number CMMI-1855651.

DATA AVAILABILITY STATEMENT

The data (i.e., code for simulation study and case study) that support the findings of this study are available on request from the corresponding author. The data are not publicly available due to privacy or ethical restrictions.

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How to cite this article: Chen, Y., Zhang, Q., Li, M., & Cai, W. (2021). Sequential selection for accelerated life testing via approximate Bayesian inference. *Naval Research Logistics (NRL)*, 1–16. https://doi.org/10.1002/nav.22009

APPENDIX A

PROOF OF PROPOSITION 1

First of all, according to the assumption of the log-normal model, we have that

$$y_{n+1} \sim N(\boldsymbol{x}_{n+1}^{\top}\boldsymbol{\theta}_n, \sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1}).$$

Then $y_{n+1} \mid y_{n+1} > \log\tau_{n+1}$ follows a truncated normal distribution (see e.g., Johnson et al., 1970), and its mean and variance are given by

$$\begin{aligned}&\mathrm{E}(y_{n+1} \mid y_{n+1} > \log\tau_{n+1})\\ &= \boldsymbol{x}_{n+1}^{\top}\boldsymbol{\theta}_n + \frac{\phi(\eta_n)}{1-\Phi(\eta_n)}\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1}}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Var}(y_{n+1} \mid y_{n+1} > \log\tau_{n+1})\\ &= (\sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1})\left(1 - \eta\frac{\phi(\eta_n)}{\Phi(\eta_n)} - \frac{\phi(\eta_n)^2}{\Phi(\eta_n)^2}\right)^2.\end{aligned}$$

According to (5) and (6), we have that

$$\begin{aligned}\boldsymbol{\beta} \mid y_{n+1} \sim \mathrm{MVN}\Bigg(&\boldsymbol{\theta}_n + \frac{y_{n+1} - \boldsymbol{x}_{n+1}^{\top}\boldsymbol{\theta}_n}{\sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1}}\Sigma_n\boldsymbol{x}_{n+1},\\ &\times \Sigma_n - \frac{\Sigma_n\boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^{\top}\Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1}}\Bigg),\end{aligned}$$

Therefore, the posterior mean and variance of $\boldsymbol{\beta}$ given $y_{n+1} > \log\tau_{n+1}$ can be derived by

$$\begin{aligned}\mathrm{E}(\boldsymbol{\beta} \mid y_{n+1} > \log\tau_{n+1}) &= \mathrm{E}\left[\mathrm{E}(\boldsymbol{\beta}|y_{n+1}) \mid y_{n+1} > \log\tau_{n+1}\right]\\ &= \boldsymbol{\theta}_n + \frac{\mathrm{E}(y_{n+1} \mid y_{n+1} > \log\tau_{n+1}) - \boldsymbol{x}_{n+1}^{\top}\boldsymbol{\theta}_n}{\sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1}}\Sigma_n\boldsymbol{x}_{n+1}\\ &= \boldsymbol{\theta}_n + \frac{\phi(\eta_n)}{(1-\Phi(\eta_n))\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^{\top}\Sigma_n\boldsymbol{x}_{n+1}}}\Sigma_n\boldsymbol{x}_{n+1}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Var}(\boldsymbol{\beta} \mid y_{n+1} > \log\tau_{n+1})\\ &\quad= \mathrm{E}(\mathrm{Var}(\boldsymbol{\beta} \mid y_{n+1}) \mid y_{n+1} > \log\tau_{n+1})\\ &\qquad+ \mathrm{Var}(\mathrm{E}(\boldsymbol{\beta} \mid y_{n+1}) \mid y_{n+1} > \log\tau_{n+1})\end{aligned}$$

$$= \Sigma_n - \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} + \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{(\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1})^2} \times \mathrm{Var}(y_{n+1} \mid y_{n+1} > \log \tau_{n+1})$$

$$= \Sigma_n - \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} + \frac{\Sigma_n \boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top \Sigma_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} \times \left(1 - \eta_n \frac{\phi(\eta_n)}{\Phi(\eta_n)} - \frac{\phi(\eta_n)^2}{\Phi(\eta_n)^2}\right)^2.$$

APPENDIX B

PROOF OF THEOREM 1

B.1 | Proof of Theorem 1 under Assumptions 1 and 2

From law of large number, Assumptions 1 and 2 lead to $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{\infty} \boldsymbol{x}_k \boldsymbol{x}_k^\top = \mathbf{A}$ almost surely. Furthermore, denote $\mathbf{B} = \frac{1}{\sigma^2}\mathbf{A}$, then by Lemma EC.2 in Chen and Ryzhov (2020), we have the following result on the convergence rate of Σ_n.

Lemma 1 *Suppose Assumptions* 1 *and* 2 *hold*, *then*, *with probability* 1,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{3}{4}}} \left\| \frac{1}{n+1}\Sigma_{n+1}^{-1} - \mathbf{B} \right\|_2^2 < \infty.$$

This lemma will be used in the proof of Theorem 1.

In the remaining of this proof, we assume that a suitable set of measure 0 is discarded, so we do not have to repeat the qualification "almost surely." Notice that, according to the Woodbury matrix identity (Woodbury, 1950), the updating formulas in (10) can be expressed by

$$\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + \delta_{n+1} \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2} \Sigma_{n+1}\boldsymbol{x}_{n+1} + (1 - \delta_{n+1}) \frac{\phi(\eta_n)\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}}{(1 - \Phi(\eta_n))\sigma^2} \Sigma_{n+1}\boldsymbol{x}_{n+1}, \tag{B.1}$$

$$\Sigma_{n+1}^{-1} = \Sigma_n^{-1} + \frac{1}{\sigma^2}\boldsymbol{x}_{n+1}\boldsymbol{x}_{n+1}^\top, \tag{B.2}$$

where η_n is expressed in (9). The development of the proof will be based on the expressions above.

Without loss of generality, let $\boldsymbol{\beta} = 0$. Denote

$$\xi_n = \frac{\log \tau_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma},$$

$$Q_n = -\delta_{n+1} \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2} \mathbf{B}^{-\frac{1}{2}} \boldsymbol{x}_{n+1} - (1 - \delta_{n+1}) \frac{\phi(\xi_n)}{(1 - \Phi(\xi_n))\sigma} \mathbf{B}^{-\frac{1}{2}} \boldsymbol{x}_{n+1},$$

$$\begin{aligned}
b_n = &-\delta_{n+1}\mathbf{B}^{\frac{1}{2}} \left(\frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2}(n+1)\Sigma_{n+1}\boldsymbol{x}_{n+1} - \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2}\mathbf{B}^{-1}\boldsymbol{x}_{n+1} \right) \\
&- (1 - \delta_{n+1})\mathbf{B}^{\frac{1}{2}} \left(\frac{\phi(\eta_n)\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}}{(1 - \Phi(\eta_n))\sigma^2}(n+1)\Sigma_{n+1} \boldsymbol{x}_{n+1} - \frac{\phi(\xi_n)}{(1 - \Phi(\xi_n))\sigma}\mathbf{B}^{-1}\boldsymbol{x}_{n+1} \right) \\
= &-\delta_{n+1}\mathbf{B}^{\frac{1}{2}} \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2}(n+1)\Sigma_{n+1}\boldsymbol{x}_{n+1} \\
&- (1 - \delta_{n+1})\mathbf{B}^{\frac{1}{2}} \frac{\phi(\eta_n)\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}}{(1 - \Phi(\eta_n))\sigma^2} \times (n+1)\Sigma_{n+1}\boldsymbol{x}_{n+1} \\
&- Q_n.
\end{aligned}$$

Then, (B.1) is equivalent to

$$\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_{n+1} = \mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n - \frac{1}{n+1}(Q_n + b_n).$$

Taking the ℓ^2-norm, we have

$$\begin{aligned}
\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_{n+1}\right\|_2^2 &= \boldsymbol{\theta}_{n+1}^\top \mathbf{B}\boldsymbol{\theta}_{n+1} \\
&= \left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2 + \frac{1}{(n+1)^2}\|Q_n\|_2^2 + \frac{1}{(n+1)^2}\|b_n\|_2^2 \\
&\quad - \frac{2}{n+1}Q_n^\top \mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n - \frac{2}{n+1}b_n^\top \mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n + \frac{2}{(n+1)^2}Q_n^\top b_n.
\end{aligned} \tag{B.3}$$

From (B.2), we have

$$\lim_{n\to\infty} (n+1)\Sigma_{n+1} = \mathbf{B}^{-1}. \tag{B.4}$$

Define the Borel sigma-algebra

$$\mathcal{F}_n \triangleq \mathcal{B}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n+1}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_n, \tau_1, \dots, \tau_{n+1}, y_1, \dots, y_n, \delta_1, \dots, \delta_n, \Sigma_1, \dots, \Sigma_n).$$

Since y_{n+1} is normally distributed and $\sup_{\boldsymbol{x}} \left| \frac{d}{dx}\frac{\phi(x)}{1-\Phi(x)} \right| \le 1$, by (B.4) and Assumptions 1 and 2, there must exist a positive constant C_1 such that for all n,

$$\mathrm{E}\left(\|Q_n\|_2^2 | \mathcal{F}_n\right) \le C_1 \left(1 + \left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\right). \tag{B.5}$$

Similarly, with triangular inequality, there must also be a constant C_2 such that

$$\mathrm{E}\left(\|b_n\|_2^2 | \mathcal{F}_n\right) \le C_2 \left(1 + \left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\right). \tag{B.6}$$

By Cauchy–Schwarz inequality, from (B.5) and (B.6), we have

$$\begin{aligned}
\mathrm{E}\left(\left|2Q_n^\top b_n\right| | \mathcal{F}_n\right) &\le \mathrm{E}\left(2\|Q_n\|_2 \|b_n\|_2 | \mathcal{F}_n\right) \\
&\le \mathrm{E}\left(\|Q_n\|_2^2 + \|b_n\|_2^2 | \mathcal{F}_n\right) \\
&\le (C_1 + C_2)\left(1 + \left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\right).
\end{aligned} \tag{B.7}$$

We can also find that

$$\mathrm{E}\left(\left|\frac{2}{n+1}b_n^\top \mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right| | \mathcal{F}_n\right)$$

$$
\begin{aligned}
&\le \mathrm{E}\left(2\left\|\frac{1}{(n+1)^{3/8}}b_n\right\|_2\left\|\frac{1}{(n+1)^{5/8}}\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2|\mathcal{F}_n\right)\\
&\le \mathrm{E}\left(\left\|\frac{1}{(n+1)^{3/8}}b_n\right\|_2^2+\left\|\frac{1}{(n+1)^{5/8}}\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2|\mathcal{F}_n\right)\\
&\le \frac{1}{(n+1)^{3/4}}\mathrm{E}\left(\|b_n\|_2^2|\mathcal{F}_n\right)+\frac{1}{(n+1)^{5/4}}\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2,
\end{aligned}
$$

where the first inequality holds by Cauchy-Schwarz inequality. Since y_{n+1} is normally distributed and $\sup_x\left|\frac{d}{dx}\frac{\phi(x)}{1-\Phi(x)}\right|\le 1$, by (B.4) and Assumptions 1 and 2, there must exist two positive constants C_3 and C_4 such that

$$
\begin{aligned}
\mathrm{E}\left(\|b_n\|_2^2|\mathcal{F}_n\right) &\le C_3\left\|(n+1)\Sigma_{n+1}-\mathbf{B}^{-1}\right\|_2^2\\
&\quad\times\left(1+\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\right)\\
&= C_3\left\|(n+1)\Sigma_{n+1}\left(\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right)\mathbf{B}^{-1}\right\|_2^2\\
&\quad\times\left(1+\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\right)\\
&\le C_4\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2\left(1+\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\right),
\end{aligned}
$$

where the last inequality holds due to (B.4) and the submultiplicativity of the norm $\|\cdot\|_2$. Thus, we have

$$
\begin{aligned}
&\mathrm{E}\left(\left|\frac{2}{n+1}b_n^\top\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right||\mathcal{F}_n\right)\\
&\le\left(\frac{C_4}{(n+1)^{3/4}}\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2+\frac{1}{(n+1)^{5/4}}\right)\\
&\quad\times\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\\
&\quad+\frac{C_4}{(n+1)^{3/4}}\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2.
\end{aligned}\tag{B.8}
$$

Finally, for any $\boldsymbol{\zeta}$, we have

$$
\begin{aligned}
&\mathrm{E}\left(Q_n^\top\mathbf{B}^{\frac{1}{2}}\boldsymbol{\zeta}|\mathcal{F}_n\right)\\
&=\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}\left(\frac{\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}}{\sigma^2}\Phi\left(\frac{\log\tau_{n+1}}{\sigma}\right)+\frac{1}{\sigma}\phi\left(\frac{\log\tau_{n+1}}{\sigma}\right)\right.\\
&\quad\left.-\frac{1-\Phi\left(\frac{\log\tau_{n+1}}{\sigma}\right)}{\sigma}\frac{\phi\left(\frac{\log\tau_{n+1}-\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}}{\sigma}\right)}{1-\Phi\left(\frac{\log\tau_{n+1}-\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}}{\sigma}\right)}\right).
\end{aligned}
$$

Denote

$$
\begin{aligned}
R_n(\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta})&\triangleq\frac{\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}}{\sigma^2}\Phi\left(\frac{\log\tau_{n+1}}{\sigma}\right)+\frac{1}{\sigma}\phi\left(\frac{\log\tau_{n+1}}{\sigma}\right)\\
&\quad-\frac{1-\Phi\left(\frac{\log\tau_{n+1}}{\sigma}\right)}{\sigma}\frac{\phi\left(\frac{\log\tau_{n+1}-\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}}{\sigma}\right)}{1-\Phi\left(\frac{\log\tau_{n+1}-\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}}{\sigma}\right)},
\end{aligned}
$$

then we have

$$
\mathrm{E}\left(Q_n^\top\mathbf{B}^{\frac{1}{2}}\boldsymbol{\zeta}|\mathcal{F}_n\right)=\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}R_n(\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}).\tag{B.9}
$$

Since $(\log\tau_n)_{n=0}^\infty$ is bounded and $\frac{d}{du}R_n(u)>0$, we can see that $R_n(u)=0$ if and only if $u=0$, and for all $\epsilon>0$,

$$
\inf_{(\boldsymbol{x}_n^\top\boldsymbol{\zeta})^2>\epsilon,n\in\mathbb{N}}\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta}R_n(\boldsymbol{x}_{n+1}^\top\boldsymbol{\zeta})>0.
$$

Now, combining (B.3) with (B.5)–(B.9), we have

$$
\begin{aligned}
&\mathrm{E}\left(\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_{n+1}\right\|_2^2|\mathcal{F}_n\right)\\
&\le\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2\left(1+\frac{2(C_1+C_2)}{(n+1)^2}\right.\\
&\quad\left.+\frac{C_4}{(n+1)^{3/4}}\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2+\frac{1}{(n+1)^{5/4}}\right)\\
&\quad+\frac{2(C_1+C_2)}{(n+1)^2}+\frac{C_4}{(n+1)^{3/4}}\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2\\
&\quad-\frac{2}{n+1}\boldsymbol{x}_{n+1}^\top\boldsymbol{\theta}_nR_n(\boldsymbol{x}_{n+1}^\top\boldsymbol{\theta}_n).
\end{aligned}
$$

From Lemma 1, we have

$$
\begin{aligned}
&\sum_{n=0}^\infty\frac{2(C_1+C_2)}{(n+1)^2}+\frac{C_4}{(n+1)^{3/4}}\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2\\
&\quad+\frac{1}{(n+1)^{5/4}}<\infty,\\
&\sum_{n=0}^\infty\frac{2(C_1+C_2)}{(n+1)^2}+\frac{C_4}{(n+1)^{3/4}}\left\|\frac{1}{n+1}\Sigma_{n+1}^{-1}-\mathbf{B}\right\|_2^2<\infty.
\end{aligned}
$$

Then, by Theorem 1 in Robbins and Siegmund (1985), $\lim_{n\to\infty}\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2$ exists and

$$
\sum_{n=0}^\infty\frac{1}{n+1}\boldsymbol{x}_{n+1}^\top\boldsymbol{\theta}_nR_n(\boldsymbol{x}_{n+1}^\top\boldsymbol{\theta}_n)<\infty
$$

almost surely. Therefore, for every sample path, there must exist a subsequence $(\boldsymbol{x}_{n_k+1}^\top\boldsymbol{\theta}_{n_k})$ of $(\boldsymbol{x}_{n+1}^\top\boldsymbol{\theta}_n)$ such that as $k\to\infty$,

$$
\boldsymbol{x}_{n_k+1}^\top\boldsymbol{\theta}_{n_k}\to 0.
$$

On the other hand, since $\lim_{n\to\infty}\left\|\mathbf{B}^{\frac{1}{2}}\boldsymbol{\theta}_n\right\|_2^2$ exists, then for one sample path, the sequence $(\boldsymbol{\theta}_n)$ is bounded. Therefore, there must exist a subsequence $(\boldsymbol{\theta}_{n_{k_j}})$ of $(\boldsymbol{\theta}_{n_k})$ such that as $j\to\infty$,

$$
\boldsymbol{\theta}_{n_{k_j}}\to\nu,
$$

where ν is a fixed vector. Then by Assumption 2, we have

$$
\begin{aligned}
\lim_{j\to\infty}\left|\boldsymbol{x}_{n_{k_j}+1}^\top\nu\right|&=\lim_{j\to\infty}\left|\boldsymbol{x}_{n_{k_j}+1}^\top(\nu-\boldsymbol{\theta}_{n_{k_j}}+\boldsymbol{\theta}_{n_{k_j}})\right|\\
&\le\lim_{j\to\infty}\left|\boldsymbol{x}_{n_{k_j}+1}^\top(\nu-\boldsymbol{\theta}_{n_{k_j}})\right|+\lim_{j\to\infty}\left|\boldsymbol{x}_{n_{k_j}+1}^\top\boldsymbol{\theta}_{n_{k_j}}\right|\\
&=0.
\end{aligned}
$$

Thus, for any arbitrary $\epsilon>0$, there exists an integer J such that for all $j\ge J$,

$$
\left|\boldsymbol{x}_{n_{k_j}+1}^\top\nu\right|<\epsilon.\tag{B.10}
$$

However, since $(\boldsymbol{x}_{n_{k_j}+1})_{j=J}^\infty$ is also an infinite sequence of i.i.d. samples from a common distribution, there must exist M linearly independent vectors $\boldsymbol{x}_{n_{k_{j_1}}+1},\ldots,\boldsymbol{x}_{n_{k_{j_M}}+1}$ from $(\boldsymbol{x}_{n_{kj}+1})_{j=J}^\infty$

that can be a basis of $\mathbb{R}^M$, where $M = (p+1)(d+1)$; otherwise, suppose all $(\boldsymbol{x}_{n_{k_j}+1})_{j=J}^{\infty}$ come from a subspace $\boldsymbol{v} \subsetneq \mathbb{R}^M$, then there must be a nonzero vector $\boldsymbol{\gamma} \in \boldsymbol{v}^{\perp}$ such that

$$\begin{aligned}\boldsymbol{\gamma}^\top \mathbf{A}\boldsymbol{\gamma} &= \boldsymbol{\gamma}^\top \left(\lim_{J'\to\infty} \frac{1}{J'} \sum_{j=J}^{J'} \boldsymbol{x}_{n_{k_j}+1}\boldsymbol{x}_{n_{k_j}+1}^\top \right)\boldsymbol{\gamma} \\ &= \lim_{J'\to\infty} \frac{1}{J'} \sum_{j=J}^{J'} (\boldsymbol{x}_{n_{k_j}+1}^\top \boldsymbol{\gamma})^2 \\ &= 0,\end{aligned}$$

where the first equality holds by Assumptions 1 and 2, but this contradicts that $\mathbf{A}$ is positive-definite.

Then, to satisfy (B.10), since ϵ can be arbitrarily small, by Assumption 2, ν has to be the zero vector. Thus, $\boldsymbol{\theta}_{n_{k_j}} \to 0$, so $\lim_{j\to\infty} \left\| \mathbf{B}^{\frac{1}{2}} \boldsymbol{\theta}_{n_{k_j}} \right\|_2^2 = 0$, but $(\boldsymbol{\theta}_{n_{k_j}})$ is a subsequence of $(\boldsymbol{\theta}_n)$ and $\lim_{n\to\infty} \left\| \mathbf{B}^{\frac{1}{2}} \boldsymbol{\theta}_n \right\|_2^2$ exists; therefore, $\lim_{n\to\infty} \left\| \mathbf{B}^{\frac{1}{2}} \boldsymbol{\theta}_n \right\|_2^2 = 0$, so we have $\boldsymbol{\theta}_n \to 0$ for every sample path, thus $\boldsymbol{\theta}_n \to 0$ almost surely.

B.2 | Proof of Theorem 1 under Assumption 3

First, we introduce the following lemma, which gives a similar result as Lemma 1 but does not require Assumption 1. The proof can be found in Appendix C.

Lemma 2 *Suppose Assumption* 3 *holds. Then, with probability* 1,

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{3}{4}}} \left\| \frac{1}{n+1} \Sigma_{n+1}^{-1} - \mathbf{B} \right\|_2^2 < \infty,$$

where $\mathbf{B} = \frac{1}{\sigma^2} \sum_{j=1}^{K} \alpha_j \boldsymbol{\lambda}_j \boldsymbol{\lambda}_j^\top$ *is a positive definite symmetric matrix.*

Denote $\mathbf{A} = \sigma^2 \mathbf{B}$. Note that by Assumption 3, in the long run, design $\boldsymbol{\lambda}_j$ is sampled a nonzero proportion α_j of the total time. Thus, for any infinite sequence of selected designs, with probability 1, each design $\boldsymbol{\lambda}_j$ should be selected at least once. Together with Assumption 3, this allows us to find a finite number of design vectors from this sequence that can form a basis for $\mathbb{R}^M$, without assuming the design vectors are i.i.d. samples. Then, as Assumption 3 leads to Assumption 2 and Lemma 2 provides the same convergence rate result as Lemma 1, we can simply repeat the proof in Appendix B (Proof of Theorem 1 under Assumptions 1 and 2) to obtain the desired convergence of $(\boldsymbol{\theta}_n)_{n=0}^{\infty}$.

APPENDIX C

PROOF OF LEMMA 2

Note that Assumption 3 leads to Assumption 2. From the structure of $\mathbf{B}$, it is obvious that $\mathbf{B}$ is positive-definite and symmetric under Assumption 3. From (B.2), by Assumption 3, we can see that

$$\begin{aligned}&\left\| \frac{1}{n+1} \Sigma_{n+1}^{-1} - \mathbf{B} \right\|_2^2 \\ &= \left\| \frac{1}{n+1} \Sigma_0^{-1} + \frac{1}{\sigma^2} \sum_{j=1}^{K} \left(\frac{N_{\beta\lambda_j,n+1}}{n+1} - \alpha_j \right) \boldsymbol{\lambda}_j \boldsymbol{\lambda}_j^\top \right\|_2^2 \\ &= O\left(\frac{1}{n^{\frac{1}{4}+\epsilon}} \right),\end{aligned}$$

where ϵ is a fixed positive constant. This leads to the desired convergence result.

APPENDIX D

SOME DETAILS ON DERIVING THE EI CRITERION

We provide some details on deriving the EI criterion in Section 4. Since $\boldsymbol{\theta}_{n+1}$ with non-censored response is given by (5), we have that

$$\begin{aligned}\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_{n+1} &= \boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_n \\ &+ \frac{y_{n+1} - \boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n}{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}} \boldsymbol{x}(z,\boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}_{n+1}.\end{aligned}$$

Under the log-normal model and the prior distribution of $\boldsymbol{\beta} \sim \mathrm{MVN}(\boldsymbol{\theta}_n, \Sigma_n)$, it is straightforward to derive that the posterior predictive distribution of y_{n+1} is a normal distribution with mean $\boldsymbol{x}_{n+1}^\top \boldsymbol{\theta}_n$ and variance $\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}$. Therefore, we can express

$$\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_{n+1} = \boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_n + \frac{\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}_{n+1}}{\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}} G, \quad \text{(B.11)}$$

where G is a standard normal random variable.

Note that the log-normal model in (2) contains the interaction between material settings $\mathbf{z}$ and stress factors $\mathbf{v}$, which makes the EI-type acquisition functions in Powell and Ryzhov (2012) not directly applicable to this situation. Therefore, we derive the EI criterion for our model as follows. Let $\widetilde{\boldsymbol{v}}^* = (1, (\boldsymbol{v}^*)^\top)^\top$. Then, $\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_n = (\widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,0} + (z \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1}$, where $\boldsymbol{\theta}_n = (\boldsymbol{\theta}_{n,0}^\top, \boldsymbol{\theta}_{n,1}^\top)^\top$ with $\boldsymbol{\theta}_{n,0}$ and $\boldsymbol{\theta}_{n,1}$ being vectors of size $d+1$ and $p(d+1)$, respectively. Accordingly,

$$\max_{z\in\mathcal{Z}} \left[\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_n \right] = (\widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,0} + \max_{z\in\mathcal{Z}} \left[(z \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} \right], \quad \text{(B.12)}$$

and

$$\begin{aligned}&\max_{z\in\mathcal{Z}} \left[\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_{n+1} \right] \\ &= \max_{z\in\mathcal{Z}} \left\{ \boldsymbol{x}(z,\boldsymbol{v}^*)^\top \boldsymbol{\theta}_n + \frac{\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}_{n+1}}{\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}} G \right\} \\ &= (\widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,0} \\ &+ \max_{z\in\mathcal{Z}} \left\{ (z \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} + \frac{\boldsymbol{x}(z,\boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}_{n+1}}{\sqrt{\sigma^2 + \boldsymbol{x}_{n+1}^\top \Sigma_n \boldsymbol{x}_{n+1}}} G \right\}. \quad \text{(B.13)}\end{aligned}$$

Plugging (B.12) and (B.13) into (13), we obtain that

$$\mathrm{EI}^n(z, \boldsymbol{v}; \boldsymbol{v}^*) = \mathrm{E}_G \left\{ \max_{z' \in \mathcal{Z}} \left[(z' \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} \right] + \frac{\boldsymbol{x}(z', \boldsymbol{v}^*)^\top \Sigma_n \boldsymbol{x}(z, \boldsymbol{v})}{\sqrt{\sigma^2 + \boldsymbol{x}^\top(z, \boldsymbol{v}) \Sigma_n \boldsymbol{x}(z, \boldsymbol{v})}} G \right\} - \max_{z' \in \mathcal{Z}} \left[(z' \otimes \widetilde{\boldsymbol{v}}^*)^\top \boldsymbol{\theta}_{n,1} \right], \tag{B.14}$$

where the expectation E_G is taken with respect to the random variable G. The new design point $\boldsymbol{x}_{n+1} = \boldsymbol{x}(z_{n+1}, \boldsymbol{v}_{n+1})$ is selected to maximize this acquisition function.