TROPICAL CONVEX HULLS OF POLYHEDRAL SETS

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Abstract. In this paper we focus on the tropical convex hull of convex sets and polyhedral complexes. We give a vertex description of the tropical convex hull of a line segment and a ray. Next we show that tropical convex hull and ordinary convex hull commute in two dimensions and characterize tropically convex polyhedra in any dimension. Finally we show that the dimension of a tropically convex fan depends on the coordinates of its rays and give a lower bound on the degree of a fan tropical curve using only tropical techniques.

Introduction

Tropical convexity is the analog of classical convexity in the *tropical semiring* $(\mathbb{R}, \oplus, \odot)$ where $a \oplus b = \min(a, b)$, and $a \odot b = a + b$. The goal of this paper is to explore the interplay between tropical convexity and its classical counterpart. Our aim is to describe the tropical convex hull of polyhedra, polyhedral complexes, and in particular, tropical curves.

The primary focus of tropical convexity is the study of *tropical polytopes*: the tropical convex hull of finite sets. These are widely studied [DS04, CGQS05, CGQ04, GS07, Jos05, GM10, AGG10] and find applications in various areas of mathematics. Recently, techniques from tropical convexity have been applied to mechanism design [CT16], optimization [AGG12], and maximum likelihood estimation [RSTU18]. Some specific applications are the resolution of monomial ideals [DY07], and discrete event dynamic systems [BCOQ92]. Moreover, computational tools exist to aid in further study of tropical polytopes [Jos09, AGG10].

A tropical polytope is not always classically convex, but does have an explicit description as the finite union of some ordinary polytopes [DS04]. Tropical polytopes which are also ordinary polytopes are called *polytropes* as discussed in [JK10]. However, there exist ordinary polytopes which are tropically convex, but are not finitely generated (for an example, see Figure 3). Here we further examine this relationship between classical convexity and tropical convexity by studying the structure of the tropical convex hull of polyhedral sets. Our first result is the following:

Theorem (Theorems 1.4 and 1.10)**.** *If $a, b \in \mathbb{R}^n$ and $U \subset \mathbb{R}^2$, then*

(i) $\operatorname{tconv}\operatorname{conv}(a, b) = \operatorname{conv}\operatorname{tconv}(a, b)$;
(ii) $\operatorname{tconv}\operatorname{pos}(a) = \operatorname{pos}\operatorname{tconv}(0, a)$;
(iii) $\operatorname{tconv}\operatorname{conv} U = \operatorname{conv}\operatorname{tconv} U$.

Ordinary and tropical convex hull do not commute as in part (i) even for small examples (e.g., triangles) in dimension 3. However, the tropical convex hull of an ordinary polyhedron is itself an ordinary polyhedron. We characterize which ordinary polyhedra are tropically convex.

Theorem (Theorem 2.6)**.** *A full-dimensional ordinary polyhedron is tropically convex if and only if all of its defining halfspaces are tropically convex.*

Many properties and theorems valid in classical convexity are also valid in the tropical setting; for example, separation of convex sets [CGQ04, GS07], Minkowski-Weyl Theorem [GK07, GK11, Jos05], Carathéodory and Helly Theorems [DS04, GM10], and Farkas Lemma [DS04].

Here we consider the classical result in algebraic geometry (see for example [EH87]) which bounds the degree of a projective variety X from below by

$$\dim \operatorname{span} X - \dim X + 1 \leq \deg X. \tag{1}$$

Our first aforementioned result describing the tropical convex hull of line segments and rays provides some information on the dimension of tropical convex hulls. Using this result we study a tropical analogue of (1) in the case of tropical curves. We can substitute $\operatorname{span} X$ either with the tropical convex hull of a tropical curve Γ or with a tropical linear space of smallest dimension containing Γ. The latter may not be unique and it is not easy to determine. Thus, we choose to replace $\operatorname{span} X$ with $\operatorname{tconv} \Gamma$. The tropical analogue of (1) we consider is

$$\dim \operatorname{tconv} \Gamma \leq \deg \Gamma. \tag{2}$$

If Γ is realizable, then this follows immediately from the classical inequality (1). In Section 3 we give a proof of (2) for fan tropical curves that relies entirely on tropical techniques.

The structure of this paper is as follows. In Section 1 we recall basic definitions of tropical convexity. Then we describe the tropical convex hull of a line segment and a ray as ordinary polyhedra. Using this result we show the dimensions are easily calculable using coordinates of the respective endpoints. We also prove that ordinary and tropical convex hull commute in two dimensions. In Section 2 we prove that convexity and polyhedrality are preserved after taking the tropical convex hull. Next we classify tropically convex ordinary halfspaces, linear spaces, and polyhedra. Finally, in Section 3, we use our results to prove the inequality (2) in the case of fan tropical curves.

1. Line segments, rays, and sets in $\mathbb{R}^2$

Key definitions from tropical convexity are presented in the first part of this section. A description of the tropical convex hull of any arbitrary set is given in Proposition 1.1. In Theorem 1.4 we show that ordinary and tropical convex hull commute in any dimension in the case of two points. This allows us to find the dimension of the tropical convex hull of a line segment or a ray using the coordinates of its endpoints in Corollary 1.7. We also prove ordinary and tropical convex hull always commutes in two dimensions in Theorem 1.10.

A set $U \subset \mathbb{R}^n$ is *tropically convex* if $(a \odot x) \oplus (b \odot y)$ is in U for any $x, y \in U$ and $a, b \in \mathbb{R}$ with $a \oplus b = 0$. The *tropical convex hull* of $U \subset \mathbb{R}^n$ is the smallest tropically convex set that contains U. This is defined equivalently in [GK07] by

$$\operatorname{tconv} U = \bigcup_{V \subset U : |V| < \infty} \operatorname{tconv} V. \tag{3}$$

If $V = \{v_1, \ldots, v_k\}$ is a finite set, then by [GK07, Definition 2.1] its tropical convex hull is given by

$$\operatorname{tconv} V = \left\{ a_1 \odot v_1 \oplus \cdots \oplus a_k \odot v_k \mid a_i \in \mathbb{R}, \quad \bigoplus_{i=1}^{k} a_i = 0 \right\}.$$

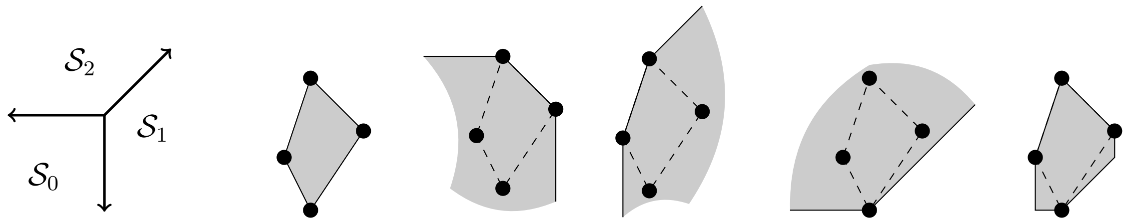

Figure 1. Illustration of Proposition 1.1 in $\mathbb{PT}^2$. From left to right: The three sectors, a polytope P, the Minkowski sums $P+\mathcal{S}_0, P+\mathcal{S}_1$, $P+\mathcal{S}_2$, and $\operatorname{tconv} P$.

Furthermore, points in $\operatorname{tconv} V$ can be characterized by types as defined in [DS04]. Let $[n] = \{1,\dots,n\}$ and $[n]_0 = \{0,1,\dots,n\}$. Given a point $x \in \mathbb{R}^n$, the *type of x relative to V*, or *covector* in [FR15, LS19], is the n-tuple $T_x = (T_1,\dots,T_n)$ such that $T_j \subseteq [k]$ for all j, and $i \in T_j$ if $\min(v_i - x)$ is obtained in the jth coordinate. This is equivalent to saying that $i \in T_j$ if $x \in v_i + \mathcal{S}_j$, where $\mathcal{S}_j$ is a *sector* of $\mathbb{R}^n$ spanned by $\{-e_i : i \in [n]\}$ for $j = 0$, and $\{e_0, -e_i : i \in [n], i \neq j\}$ for $j \in [n]$. Here $e_1,\dots,e_n$ represent the standard unit vectors in $\mathbb{R}^n$ with $e_{ij} = 1$ if $i = j$ and $e_{ij} = 0$ otherwise. We denote the vector $\sum_{i=1}^n e_i$ by e_0. The cone $\mathcal{S}_j$ is the closure of one of the $n+1$ connected components of $\mathbb{R}^n \setminus L_{n-1}$. Here L_{n-1} denotes the max-standard tropical hyperplane, or the tropicalization of $V(x_1 + \ldots + x_n + 1)$ with the max convention, whose cones are $\operatorname{pos}(-e_{i_1},\dots,-e_{i_n})$.

The proof of the Tropical Farkas Lemma [DS04] states that $x \in \operatorname{tconv} V$ if and only if the jth entry of T_x is nonempty for all j, meaning there exists at least one v_i such that $x \in v_i + \mathcal{S}_j$ [JL16, Lemma 28]. As a consequence, we have the following proposition which also holds true in the case of $U \subset (\mathbb{R} \cup \{\infty\})^n$ [LS19, Proposition 7.3]. We give here a proof for completeness. Figure 1 gives an example of (4) in $\mathbb{R}^2$.

Proposition 1.1. *If $U \subset \mathbb{R}^n$, then the tropical convex hull of U is equal to the intersection of the Minkowski sums of U with each of the sectors. That is*

$$\operatorname{tconv} U = \bigcap_{j=0}^{n} (U + \mathcal{S}_j). \tag{4}$$

Proof. If $x \in \operatorname{tconv} U$, then (3) implies that $x \in \operatorname{tconv} V$ for some finite set $V \subset U$. By the Tropical Farkas Lemma [DS04] we obtain $x \in \bigcap_{j=0}^n (V + \mathcal{S}_j)$, hence $x \in \bigcap_{j=0}^n (U + \mathcal{S}_j)$. On the other hand, if $x \in \bigcap_{j=0}^n (U + \mathcal{S}_j)$, then there exist $u_1,\dots,u_n \in U$ such that $x \in u_j + \mathcal{S}_j$ for every j. For $V = \{u_1,\dots,u_n\}$ it follows that $x \in \bigcap_{j=0}^n (V + \mathcal{S}_j) = \operatorname{tconv} V \subset \operatorname{tconv} U$. $\square$

As a direct consequence of Proposition 1.1 we obtain Corollary 1.2. Note that it can also be proven directly by using the definition of tropical convex hull. Lemma 1.3 shows that repeatedly taking the convex hull and tropical convex hull of a set stabilizes after one step.

Corollary 1.2. *If $P \subset \mathbb{R}^n$ is convex, then* $\operatorname{tconv} P$ *is convex.*

Corollary 1.3. *If $U \subset \mathbb{R}^n$, then* $\operatorname{tconv} \operatorname{conv} U = \operatorname{tconv}(\operatorname{conv} \operatorname{tconv} U)$.

Proof. The forward direction is immediate since $\operatorname{conv} U \subset \operatorname{conv} \operatorname{tconv} U$. The containment $\operatorname{tconv} U \subseteq \operatorname{tconv} \operatorname{conv} U$ and Corollary 1.2 imply $\operatorname{conv} \operatorname{tconv} U \subseteq \operatorname{tconv} \operatorname{conv} U$, so its tropical convex hull is also contained in $\operatorname{tconv} \operatorname{conv} U$. $\square$

Let a and b be points in $\mathbb{R}^n$. For the remainder of the section we assume that

$$a = (0,\dots,0) \text{ and } 0 < b_1 < \cdots < b_n. \tag{5}$$

In this case, using [DS04, Proposition 4], the tropical line segment $\operatorname{tconv}(a,b)$ is a concatenation of line segments with $n+1$ *pseudovertices* in $\mathbb{R}^n$ given by $p_0 = a$ and

$$p_j = (b_1, \ldots, b_{j-1}, b_j, \ldots, b_j) \text{ for } j \in [n]. \tag{6}$$

If a and b do not satisfy (5), we can apply first a linear transformation which translates a to the origin and then another that relabels coordinates so that $0 \leq b_1 \leq \ldots \leq b_n$. If $b_i = b_j$ for some $i \neq j$ or $b_j = 0$ for some j, then the pseudovertices of $\operatorname{tconv}(a,b)$ lie in the tropically convex hyperplane $x_i - x_j = 0$ or $x_j = 0$ and the same holds for $\operatorname{conv}\operatorname{tconv}(a,b)$ [DS04, Theorem 2]. Thus $\operatorname{tconv}\operatorname{conv}(a,b)$ and $\operatorname{conv}\operatorname{tconv}(a,b)$ lie in the hyperplane $x_i - x_j = 0$ or $x_j = 0$. Each of these hyperplanes is isomorphic to $\mathbb{R}^{n-1}$. We can repeat this process until the appropriate projection of b has distinct positive coordinates.

The following theorem shows that the tropical convex hull and convex hull commute for two points in $\mathbb{R}^n$ for all n.

Theorem 1.4. *If a, b are points in $\mathbb{R}^n$, then*

(i) $\operatorname{tconv}\operatorname{conv}(a,b) = \operatorname{conv}\operatorname{tconv}(a,b)$;
(ii) $\operatorname{tconv}\operatorname{pos}(a) = \operatorname{pos}\operatorname{tconv}(0,a)$.

Corollary 1.2 implies the forward containment of Theorem 1.4(i). For the converse, we use an explicit description of $\operatorname{conv}\operatorname{tconv}(a,b)$ given in the following lemma.

Lemma 1.5. *If $a, b \in \mathbb{R}^n$ satisfy $a = (0, \ldots, 0)$ and $0 < b_1 < \cdots < b_n$, then* $\operatorname{conv}\operatorname{tconv}(a,b)$ *is a full-dimensional simplex whose $\mathcal{H}-$representation is given by*

$$\begin{aligned} b_1 - x_1 &\geq 0 \\ -(b_{j+1} - b_j)x_{j-1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} &\geq 0 \qquad \textit{for } j \in [n-1]. \\ -x_{n-1} + x_n &\geq 0 \end{aligned} \tag{7}$$

Proof. Observe that the vertices of $\operatorname{conv}\operatorname{tconv}(a,b)$ are the pseudovertices $p_0, \ldots, p_n$ of $\operatorname{tconv}(a,b)$ as described in (6). These are $n+1$ affinely independent points of $\mathbb{R}^n$ since the vectors $p_1 - a = p_1, \ldots, p_{n-1} - a = p_{n-1}, b - a = b$ are linearly independent. This implies $\operatorname{conv}\operatorname{tconv}(a,b)$ is a simplex. Hence, each of its $n+1$ facets is the convex hull of n vertices. To show that (7) is the $\mathcal{H}-$representation of $\operatorname{conv}\operatorname{tconv}(a,b)$ we will show that the corresponding equation of each one of the $n+1$ inequalities is the hyperplane supporting one of the facets of $\operatorname{conv}\operatorname{tconv}(a,b)$.

Let $x = (x_1, \ldots, x_n)$ be a point in $\operatorname{conv}\operatorname{tconv}(a,b) = \operatorname{conv}(a, p_1, \ldots, p_{n-1}, b)$. The jth coordinate of x is given by

$$x_j = \lambda_1 b_1 + \ldots + \lambda_{j-1} b_{j-1} + (\lambda_j + \lambda_{j+1} + \ldots + \lambda_n)\, b_j$$

where $\lambda_1 + \ldots + \lambda_n \leq 1$ and $\lambda_i \geq 0$ for every i. Substituting the coordinates of x into the first linear form of (7) we obtain $(1 - \lambda_1 - \cdots - \lambda_n)b_1$. Since $\lambda_1 + \ldots + \lambda_n \leq 1$ and $b_1 \geq 0$ it follows that $b_1 - x_1 \geq 0$. Note that equality occurs if and only if x is in the facet $\operatorname{conv}(p_1, \ldots, p_{n-1}, b)$. Thus, $b_1 - x_1 = 0$ defines this facet of $\operatorname{conv}\operatorname{tconv}(a,b)$, that is $\{b_1 - x_1 = 0\} \cap \operatorname{conv}\operatorname{tconv}(a,b) = \operatorname{conv}(p_1, \ldots, p_{n-1}, b)$.

After substituting into the second linear form of (7) we have that

$$-(b_{j+1} - b_j)x_{j-1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} = \lambda_j (b_{j-1} - b_j)(b_j - b_{j+1}).$$

Since $\lambda_j \geq 0$ and $b_j \geq b_{j-1}$ for each j, we know x satisfies the second inequality. Here equality occurs if and only if x is in the facet $\operatorname{conv}(a, p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_{n-1}, b)$, so

$$-(b_{j+1} - b_j)x_{j-1} + (b_{j+1} - b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} = 0$$

defines this facet of $\operatorname{conv}\operatorname{tconv}(a,b)$ for each $j \in [n-1]$.

Lastly, we have that $-x_{n-1}+x_n = \lambda_n(b_n - b_{n-1}) \geq 0$. Equality holds if and only if x is in the facet $\operatorname{conv}(a, p_1, \ldots, p_{n-1})$, and hence this facet is defined by $-x_{n-1}+x_n = 0$. □

Lemma 1.6. *If $a, b \in \mathbb{R}^n$ and V is a finite subset of $\operatorname{conv}(a,b)$, then*

$$\operatorname{tconv}(V) \subset \operatorname{conv}\operatorname{tconv}(a,b).$$

Proof. Without loss of generality, assume $a = (0,\ldots,0)$ and $0 < b_1 < \ldots < b_n$. Let $V = \{\lambda_1 b, \lambda_2 b, \ldots, \lambda_r b\} \subset \operatorname{conv}(a,b)$ for some parameters $\lambda_i \in [0,1]$. Assume the parameters are ordered $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r \leq 1$. Take $x \in \operatorname{tconv} V$ and let T_x be the type of x relative to V. By [DS04, Lemma 10], the point x satisfies

$$x_k - x_j \leq \lambda_i(b_k - b_j) \text{ for } j,k \in [n] \text{ with } i \in T_j. \tag{8}$$

We will show that x satisfies the $\mathcal{H}$-representation of $\operatorname{conv}\operatorname{tconv}(a,b)$ given in Lemma 1.5. Since the union of all coordinates T_j of T_x covers $[r]$, (8) implies that

$$0 \leq \frac{x_{j+1}-x_j}{b_{j+1}-b_j} \leq \frac{x_j - x_{j-1}}{b_j - b_{j-1}} \leq 1 \quad \text{for all } j \in [n-1].$$

For $j = 1$, this implies $\frac{x_1}{b_1} \leq 1$, so $b_1 - x_1 \geq 0$. For $j \in [n-1]$, rewriting the inequality $\frac{x_{j+1}-x_j}{b_{j+1}-b_j} \leq \frac{x_j - x_{j-1}}{b_j - b_{j-1}}$ shows that $-(b_{j+1}-b_j)x_{j-1} + (b_{j+1}-b_{j-1})x_j - (b_j - b_{j-1})x_{j+1} \geq 0$. Lastly, if $j = n-1$, then $0 \leq \frac{x_n - x_{n-1}}{b_n - b_{n-1}}$, so $-x_{n-1}+x_n \geq 0$. □

Proof of Theorem 1.4. For part (i), assume without loss of generality that $a = (0,\ldots,0)$ and $0 < b_1 < \cdots < b_n$. Corollary 1.2 and the containment $\operatorname{tconv}(a,b) \subset \operatorname{tconv}\operatorname{conv}(a,b)$ imply that $\operatorname{conv}\operatorname{tconv}(a,b) \subseteq \operatorname{tconv}\operatorname{conv}(a,b)$. Now take $x \in \operatorname{tconv}\operatorname{conv}(a,b)$. Since the tropical convex hull of a set is the union of the tropical convex hulls of all its subsets, it follows that there is a finite set $V \subset \operatorname{conv}(a,b)$ such that $x \in \operatorname{tconv}(V)$. Lemma 1.6 implies $\operatorname{tconv}(V) \subset \operatorname{conv}\operatorname{tconv}(a,b)$, so $x \in \operatorname{conv}\operatorname{tconv}(a,b)$.

To show part (ii), take $x \in \operatorname{tconv}\operatorname{pos}(a)$. There exist scalars $\lambda_0, \ldots, \lambda_n \geq 0$ such that $\lambda_j a \in \operatorname{pos}(a)$ for each $j \in [n]_0$ and $x \in \operatorname{tconv}(0, \lambda_0 a, \ldots, \lambda_n a)$. Assume the scalars are ordered $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ so $x \in \operatorname{tconv}\operatorname{conv}(0, \lambda_n a)$. By Theorem 1.4(*i*) it follows that $x \in \operatorname{conv}\operatorname{tconv}(0, \lambda_n a)$. Furthermore, this means $x \in \operatorname{pos}\operatorname{tconv}(0, \lambda_n a)$. The pseudovertices of $\operatorname{tconv}(0, \lambda_n a)$ and $\operatorname{tconv}(0,a)$ are scalar multiples of one another meaning $x \in \operatorname{pos}\operatorname{tconv}(0,a)$. The other inclusion $\operatorname{pos}\operatorname{tconv}(0,a) \subset \operatorname{tconv}\operatorname{pos}(0,a)$ follows from Corollary 1.2. □

Corollary 1.7. *If a and b are points in $\mathbb{R}^n$, then*

(i) $\dim \operatorname{tconv}\operatorname{conv}(a,b)$ *is the number of nonzero distinct coordinates of $a - b$;*
(ii) $\dim \operatorname{tconv}\operatorname{pos}(a)$ *is the number of nonzero distinct coordinates of a.*

Proof. Part (i) follows from the proof of Lemma 1.5 since $\operatorname{tconv}\operatorname{conv}(a,b)$ is a full-dimensional simplex in $\mathbb{R}^d$ where d is the number of nonzero distinct coordinates in $a - b$. For part (ii) observe that the generators of $\operatorname{pos}\operatorname{tconv}(0,a)$ are the pseudovertices of $\operatorname{tconv}(0,a)$ which are vertices of $\operatorname{tconv}\operatorname{conv}(0,a)$. □

As a consequence of Corollary 1.7 we have the following result for tropically convex fans. An application of this lemma appears in Section 3.

Lemma 1.8. *If F is a tropically convex fan in $\mathbb{R}^n$, then $\dim F$ is equal to the maximum number of nonzero distinct coordinates of a point in F.*

Proof. Let d be the maximum number of nonzero distinct coordinates of any point in F, and let x be one such point in F. Since F is a tropically convex fan it contains $\operatorname{tconv}\operatorname{pos}(x)$. Corollary 1.7 implies that $\dim \operatorname{tconv}\operatorname{pos}(x) = d$, hence $\dim\ F \geq d$. Suppose that $\dim F > d$. Let C be a cone contained in F such that $\dim C = \dim F$. By hypothesis, each point in C has at most d nonzero distinct coordinates. This implies that C is contained in the union of finitely many linear spaces in $\mathbb{R}^n$ of dimension at most d. This contradicts the assumption that $\dim C = \dim F > d$. □

Now we consider arbitrary sets in $\mathbb{R}^2$ and give a generalization of Theorem 1.4.

Lemma 1.9. *If $V \subset \mathbb{R}^2$ is finite, then* $\operatorname{tconv}\operatorname{conv} V = \operatorname{conv}\operatorname{tconv} V$.

Proof. We prove the lemma by showing that each vertex of $\operatorname{tconv}\operatorname{conv} V$ is either a point in V or a pseudovertex of $\operatorname{tconv} V$.

By Proposition 1.1 we know $\operatorname{tconv}\operatorname{conv} V = \bigcap_{j=0}^{2}(\mathcal{S}_j + \operatorname{conv} V)$. A face of a Minkowski sum of polyhedra is a Minkowski sum of a face from each summand. Since $\mathcal{S}_j$ has only one vertex, namely the origin, it follows that the vertices of $\mathcal{S}_j + \operatorname{conv} V$ are precisely the vertices of $\operatorname{conv} V$. The facets of $\mathcal{S}_j + \operatorname{conv} V$ arise as either the sum of the vertex of $\mathcal{S}_j$ and an edge of $\operatorname{conv} V$, or as the sum of a vertex of $\operatorname{conv} V$ and a ray of $\mathcal{S}_j$. In the former case, these are simply the edges of $\operatorname{conv} V$. In the latter case, these are the unbounded edges parallel to a ray of $\mathcal{S}_j$ and the vertex of each of them is a vertex $v \in V$.

From this description of the facets and vertices of $\mathcal{S}_j + \operatorname{conv} V$ we deduce that a vertex of $\operatorname{tconv}\operatorname{conv} V$ is either a vertex of $\operatorname{conv} V$ or it is the intersection of a facet of $\mathcal{S}_i + \operatorname{conv} V$ and a facet of $\mathcal{S}_j + \operatorname{conv} V$ for some $i, j \in [2]_0$. Suppose that neither of the facets is an edge of $\operatorname{conv} V$ (Otherwise we would get a vertex of $\operatorname{conv} V$.), then the intersection point is a pseudovertex of $\operatorname{tconv}(v, w)$ and a vertex of $\operatorname{conv}\operatorname{tconv} V$. Suppose that only one of the facets is an edge of $\operatorname{conv} V$. This intersection point must be a vertex of $\operatorname{conv} V$. Otherwise it is in the interior of the edge of $\operatorname{conv} V$, which implies that the ray intersecting the edge also intersects the interior of $\operatorname{conv} V$ and hence is not a facet. □

Theorem 1.10. *If $U \subset \mathbb{R}^2$, then* $\operatorname{tconv}\operatorname{conv} U = \operatorname{conv}\operatorname{tconv} U$.

Proof. The forward containment is implied by the fact that $\operatorname{tconv}\operatorname{conv} U$ is convex by Corollary 1.2.

For backward containment, suppose that $x \in \operatorname{tconv}\operatorname{conv} U$. Then by (3) it follows that there exists a finite set $V \subset \operatorname{conv} U$, such that $x \in \operatorname{tconv} V$. The classical Carathéodory Theorem implies that each point $v_i \in V$ can be written as a convex combination of finitely many points in U. Call this set $A_i \subset U$. Since V is finite, it follows that $A = \bigcup_i A_i$ is a finite subset of U and $V \subset \operatorname{conv} A$. Now we have $x \in \operatorname{tconv} V \subset \operatorname{tconv}\operatorname{conv} A$. It follows $x \in \operatorname{conv}\operatorname{tconv} A$ by Lemma 1.9. Since $A \subset U$, this implies $x \in \operatorname{conv}\operatorname{tconv} U$. □

Theorem 1.10 does not hold in general when $n \geq 3$. It is not difficult to find examples for which $\operatorname{conv}(\operatorname{tconv} V)$ is not tropically convex.

Example 1.11. Let $P \subset \mathbb{R}^3$ be the triangle in Figure 2 with vertices $v_1 = (0, 0, 0)$, $v_2 = (1, 2, 3)$, and $v_3 = (4, 1, 7)$. The convex hull of $\operatorname{tconv}(v_1, v_2, v_3)$ has 7 vertices and is not tropically convex. In fact, it is possible to find a point x in the classical line segment v_1v_3 such that the tropical convex hull of x and the midpoint of the line segment v_2v_3 is not

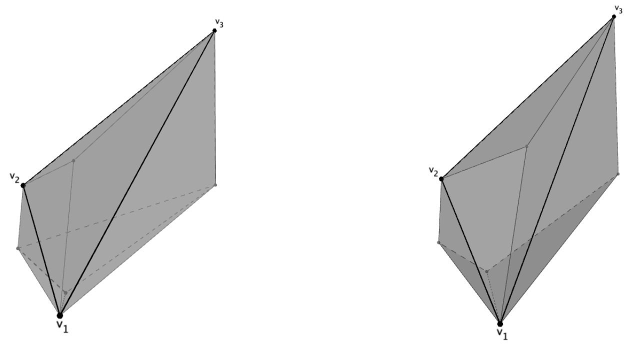

Figure 2. Illustration of Example 1.11. Left: Convex hull of $\operatorname{tconv}(v_1, v_2, v_3)$ with P in bold. Right: Tropical convex hull of P with P in bold. The polytope on the left is strictly contained in the polytope on the right.

contained in $\operatorname{tconv}(v_1, v_2, v_3)$. Using Proposition 1.1 we compute the tropical convex hull of P which is a polytope with 7 vertices strictly containing $\operatorname{conv}(\operatorname{tconv}(v_1, v_2, v_3))$. △

2. Polyhedral sets

In this section we examine the tropical convex hull of arbitrary polyhedral sets, halfspaces, and linear spaces. The main result of this section is Theorem 2.6 which classifies all tropically convex ordinary polyhedra in $\mathbb{R}^n$.

Lemma 2.1. *If $P \subset \mathbb{R}^n$ is a polyhedron (resp. cone, polyhedral complex, fan, polytope), then $\operatorname{tconv} P$ is a polyhedron (resp. cone, polyhedral complex, fan, polytope).*

Proof. If P is a polyhedron then $\operatorname{tconv} P$ is a polyhedron since it is the intersection of the finitely many polyhedra $P + \mathcal{S}_j$. If P is a cone then $P + \mathcal{S}_j$ is a cone for every j and (4) implies that $\operatorname{tconv} P$ is also a cone.

Now let P be a polyhedral complex, so $P = \cup_{i=1}^N P_i$ where each P_i is a polyhedron. By (4) it follows that

$$\operatorname{tconv} P = \operatorname{tconv}\left(\bigcup_{i=1}^N P_i\right) = \bigcap_{j=0}^n \bigcup_{i=1}^N (P_i + \mathcal{S}_j).$$

Observe that by distributing the intersection over the union of Minkowski sums we obtain the union of N^{n+1} sets. Each set in the union is an intersection of $n+1$ Minkowski sums of the form $(P_{i_0} + \mathcal{S}_0) \cap \ldots \cap (P_{i_n} + \mathcal{S}_n)$, where $(i_0, \ldots, i_n) \in \{N\}^{n+1}$, so

$$\operatorname{tconv} P = \bigcup_{(i_0,\ldots,i_n)\in\{N\}^{n+1}} ((P_{i_0} + \mathcal{S}_0) \cap \cdots \cap (P_{i_n} + \mathcal{S}_n)).$$

It follows that $\operatorname{tconv} P$ is a polyhedral complex since the finite intersection of polyhedra is a polyhedron. In fact, the polyhedral structure may be given by a refinement of the polyhedral complex whose polyhedra are $\{(P_{i_0} + \mathcal{S}_0) \cap \cdots \cap (P_{i_n} + \mathcal{S}_n)\}_{(i_0,\ldots,i_n)\in\{N\}^{n+1}}$. If P is a fan, the results on polyhedral complexes and cones imply $\operatorname{tconv} P$ is also a fan.

Lastly, let P be a polytope. To show $\operatorname{tconv} P$ is a polytope it suffices to show it is bounded. Suppose $\operatorname{tconv} P$ is not bounded. Hence it contains a ray $w + \operatorname{pos}(v)$. Since P is bounded, again (4) implies that $\operatorname{pos}(v)$ is contained in each sector $\mathcal{S}_j$. This is not possible since the intersection of all sectors is the origin. □

Using the following lemma, we classify all tropically convex ordinary halfpaces in Proposition 2.3.

Lemma 2.2. *Let $\mathcal{H}$ be a halfspace in $\mathbb{R}^n$. If $\mathcal{S}_j$ is one of the standard sectors in $\mathbb{R}^n$ for $j \in [n]_0$, then either $\mathcal{H} + \mathcal{S}_j = \mathcal{H}$ or $\mathcal{H} + \mathcal{S}_j = \mathbb{R}^n$.*

Proof. Let $\mathcal{H}$ be defined by $\{\sum_{k=1}^n a_k x_k \geq 0\}$ and let $\mathcal{S}_j$ be one of the standard sectors in $\mathbb{R}^n$ for $j \in [n]_0$. If $\mathcal{S}_j \subset \mathcal{H}$, then it follows immediately that $\mathcal{H} + \mathcal{S}_j = \mathcal{H}$.

Suppose that $\mathcal{S}_j \not\subset \mathcal{H}$. This means that at least one of the rays $\operatorname{pos} e_i, i \neq j$, generating $\mathcal{S}_j$ is contained in $\mathcal{H}^c$; equivalently $\sum_{k=1}^n a_k e_{ik} < 0$. We will consider two cases. First, suppose that $i = 0$, and recall that $e_0 = (1, \ldots, 1)$. It follows that $\operatorname{pos} e_0 \not\subset \mathcal{H}$, and hence $\sum_{k=1}^n a_k < 0$. If $y \in \mathcal{H}^c$, then $\sum_{k=1}^n a_k y_k < 0$. Let $\lambda \in \mathbb{R}$ such that

$$\lambda \geq \frac{\sum_{k=1}^n a_k y_k}{\sum_{k=1}^n a_k} > 0,$$

which implies $\lambda \sum_{k=1}^n a_k \leq \sum_{k=1}^n a_k y_k$. Hence, $0 \leq \sum_{k=1}^n a_k(y_k - \lambda)$, implying that for any $y \in \mathcal{H}^c$, the point $y - \lambda e_0 \in \mathcal{H}$ for the choice of λ specified above.

For the second case let e_i be the vector containing a -1 in position i and 0 otherwise. Suppose that $\operatorname{pos} e_i \not\subset \mathcal{H}$ and let $y \in \mathcal{H}^c$. Then we have that $\sum_{k=1}^n a_k e_{ik} = -a_i < 0$ and $\sum_{k=1}^n a_k y_k < 0$. Let $\lambda \in \mathbb{R}$ be such that

$$\lambda \geq -\frac{\sum_{k=1}^n a_k y_k}{a_i} > 0.$$

Hence, $-\lambda \sum_{k=1}^n a_k e_{ik} + \sum_{k=1}^n a_k y_k \geq 0$ and $\sum_{k=1}^n a_k(y - \lambda e_{ik}) \geq 0$. It follows that $y - \lambda e_i \in \mathcal{H}$.

This shows that if $\mathcal{S}_j \not\subset \mathcal{H}$, then any point in $\mathcal{H}^c$ can be written as $(y - \lambda e_i) + \lambda e_i, i \neq j$, with $y - \lambda e_i \in \mathcal{H}$. Hence, $\mathcal{H} + \mathcal{S}_j = \mathbb{R}^n$. □

Proposition 2.3. *If $\mathcal{H}$ is a halfspace in $\mathbb{R}^n$, then either* $\operatorname{tconv} \mathcal{H} = \mathcal{H}$ *or* $\operatorname{tconv} \mathcal{H} = \mathbb{R}^n$.

Proof. By Proposition 1.1 we know $\operatorname{tconv} \mathcal{H} = \bigcap_{j=0}^n (\mathcal{S}_j + \mathcal{H})$. Using Lemma 2.2, if there exists $j \in [n]_0$ such that $\mathcal{S}_j \subset \mathcal{H}$, then $\operatorname{tconv} \mathcal{H} = \mathcal{H}$. Otherwise $\operatorname{tconv} \mathcal{H} = \mathbb{R}^n$. □

The following proposition shows that a halfspace is tropically convex if and only if either all of the entries of its inner normal vector are nonpositive, or it contains at most one positive entry such that the sum of all entries is nonegative.

Proposition 2.4. *A halfspace $\mathcal{H} = \{\sum_{k=1}^n a_k x_k \geq 0\}$ in $\mathbb{R}^n$ is tropically convex if and only if there exists a $j \in [n]_0$ such that $S_j \subset \mathcal{H}$. This happens if and only if exactly one of the following conditions is satisfied.*

(i) *If $a_k \leq 0$ for every $k \in [n]$, then $\mathcal{S}_0 \subset \mathcal{H}$.*
(ii) *If $a_j \geq 0$, $a_k \leq 0$ for every $k \neq j$, and $a_j + \sum a_k \geq 0$, then $\mathcal{S}_j \subset \mathcal{H}$.*

Proof. The first statement follows immediately from Proposition 2.3. The sector $\mathcal{S}_j$ is contained in $\mathcal{H}$ if and only if the spanning rays e_i of $\mathcal{S}_j$ for $i \neq j$ satisfy the inequality $\sum_{k=1}^n a_k e_{ik} \geq 0$. This inequality is satisfied precisely in cases (i) and (ii) listed above. □

Lemma 2.5. *A linear space is tropically convex if and only if it is an intersection of hyperplanes of the form $\{x_i - x_j = 0 \mid i \neq j\}$ or $\{x_k = 0\}$.*

Proof. By [DS04, Theorem 2], hyperplanes of the form $\{x_i - x_j = 0\}$ and $\{x_i = 0\}$ are tropically convex. Hence, the intersection of any hyperplanes of this form is also tropically convex.

Conversely, let $L \subset \mathbb{R}^n$ be a linear space and suppose L is tropically convex. Consider $\operatorname{conv}(0, x)$ for some $x \in L$. By Corollary 1.7, the dimension of the tropical convex hull of $\operatorname{conv}(0, x)$ is equal to the number of distinct nonzero coordinates of x. Since L is tropically convex, x has at most $\dim L$ distinct nonzero coordinates. This implies L is contained in the union of the intersections of some hyperplanes $\{x_i - x_j = 0\}$ and $\{x_k = 0\}$. Since L is convex, it follows that L is just an intersection of $\{x_i - x_j = 0\}$ and $\{x_k = 0\}$ for some $i \neq j$ and k. □

The following theorem is the main result of this section.

Theorem 2.6. *A full-dimensional ordinary polyhedron is tropically convex if and only if all of its defining halfspaces are tropically convex.*

Proof. Let $P \subset \mathbb{R}^n$ be a full-dimensional, ordinary polyhedron. Since P is full-dimensional, it has a unique, irredundant hyperplane representation. If all defining halfspaces of P are tropically convex, then P is tropically convex.

Suppose that P is tropically convex and there exists a defining halfspace $\mathcal{H}$ of P that is not tropically convex. Let H be the hyperplane at the boundary of $\mathcal{H}$. Since $\mathcal{H}$ is not tropically convex, it follows that H is not tropically convex. Otherwise, by Lemma 2.5 H is parallel to one of the facets of the standard tropical hyperplane, so both $\mathcal{H}$ and $-\mathcal{H}$ are tropically convex. Let $x', y' \in \mathcal{H}$ such that $\operatorname{tconv}(x', y') \not\subset \mathcal{H}$. This implies that there exist $x, y \in \operatorname{tconv}(x', y') \cap H$ such that $(\operatorname{tconv}(x, y) \setminus \{x, y\}) \subset \mathcal{H}^c$. Hence, at least one pseudovertex p of $\operatorname{tconv}(x, y)$ is in $\mathcal{H}^c$. After relabeling, we assume the coordinates of $y - x$ are ordered

$$y_1 - x_1 \leq \cdots \leq y_s - x_s \leq 0 \leq y_{s+1} - x_{s+1} \leq \cdots \leq y_n - x_n.$$

Generalizing the result [DS04, Proposition 3] there are two forms for the pseudovertices of $\operatorname{tconv}(x, y)$ in $\mathbb{R}^n$ based on the signs of the coordinates of the difference $y - x$. For any $s < j \leq n$ the pseudovertex is

$$p = (y_1, y_2, \dots, y_j, y_j - x_j + x_{j+1}, \dots, y_j - x_j + x_n)$$

and for $j \leq s$ the pseudovertex is $(y_1 - y_j + x_j, \dots, y_{j-1} - y_j + x_j, x_j, x_{j+1}, \dots, x_n)$. We provide the computation for the former and omit it for the latter as the proof is analogous. Since $p \in \mathcal{H}^c$, it follows that $\sum_{k=1}^n a_k p_k < 0$.

Using a translation T along H we can translate x and y so that at least one of the points Tx or Ty is contained in P. Without loss of generality, we may assume that $Tx \in P$. If $Ty \in P$, then we are done. Suppose that $Ty \notin P$. Consider the line segment $\operatorname{conv}(Tx, Ty) \subset H$, which must intersect the boundary of P. Let the point of intersection be z, which can be written as $z = \lambda Tx + (1 - \lambda)Ty$, for $0 < \lambda < 1$. We claim that $\operatorname{tconv}(Tx, z) \not\subset \mathcal{H}$.

Note that one of the pseudovertices of $\operatorname{tconv}(Tx, z)$ is $p' = (z_1, z_2, \dots, z_j, z_j - Tx_j + Tx_{j+1}, \dots, z_j - Tx_j + Tx_n)$. We will show that $p' \notin \mathcal{H}$. Note that

$$\begin{aligned}\sum_{k=1}^{n} a_k p_k &= a_1 y_1 + \cdots + a_j y_j + a_{j+1}(y_j - x_j + x_{j+1}) + \cdots + a_n(y_j - x_j + x_n) \\ &= \sum_{k=1}^{j} a_k y_k + \sum_{k=j+1}^{n} a_k (y_j - x_j) + \sum_{k=j+1}^{n} a_k x_k < 0\end{aligned}$$

where the inequality is preserved under the translation T. That is, $\sum_{k=1}^n a_k T p_k < 0$.

We compute the following:

$$\begin{aligned}
\sum_{k=1}^{n} a_k p'_k &= a_1 z_1 + a_2 z_2 + \cdots + a_j z_j + a_{j+1}(z_j - Tx_j + Tx_{j+1}) + \cdots + a_n(z_j - Tx_j + Tx_n) \\
&= \sum_{k=1}^{j} a_k z_k + \sum_{k=j+1}^{n} a_k(z_j - Tx_j) + \sum_{k=j+1}^{n} a_k Tx_k \\
&= \sum_{k=1}^{j} a_k(\lambda Tx_k + (1-\lambda)Ty_k) + \sum_{k=j+1}^{n} a_k(\lambda Tx_j + (1-\lambda)Ty_j - Tx_j) + \sum_{k=j+1}^{n} a_k Tx_k \\
&= (1-\lambda)\sum_{k=1}^{j} a_k Ty_k + (1-\lambda)\sum_{k=j+1}^{n} a_k(Ty_j - Tx_j) + (1-\lambda)\sum_{k=j+1}^{n} a_k Tx_k.
\end{aligned}$$

Hence, $\sum_{k=1}^{n} a_k p'_k = (1-\lambda)\sum_{k=1}^{n} a_k Tp_k < 0$. This implies that there are two points in P, namely Tx and z, whose tropical convex hull is not in P. This contradicts the assumption that P is tropically convex. □

Corollary 2.7. *Let $P \subset \mathbb{R}^n$ be a polyhedron of dimension $d < n$. P is tropically convex if and only if it is contained in a tropically convex linear space L of dimension d and its $\mathcal{H}$-representation in L is given only by tropically convex halfspaces.*

Proof. After translation, we may assume that P contains the origin. Hence, P is contained in a unique, d-dimensional linear subspace L. If L is tropically convex, then by Lemma 2.5 P is contained in the intersection of finitely many hyperplanes of the form $\{x_k = 0\}$ for $k \in [n]$, and $\{x_i - x_j = 0 \mid i \neq j\}$ for $i, j \in [n]$. Now we can work in L by deleting the x_k and x_i coordinates. Note that the restriction of this projection map to P is an isomorphism. We now consider P in the d-dimensional linear subspace L. Equivalently, we can work in $\mathbb{R}^d$ where P is full-dimensional and has a unique, irredundant halfspace representation. By Theorem 2.6 it follows that P is tropically convex in L if and only if the halfspaces defining P in L are tropically convex. Hence, the inner normal vectors of the defining halfspaces satisfy Proposition 2.4. The lift of each of these hyperplanes to $\mathbb{R}^n$ will have the same equation, hence it still satisfies the conditions of Proposition 2.4. Therefore, each halfspace in L is tropically convex in L if and only if it is tropically convex in $\mathbb{R}^n$.

Suppose that L is not tropically convex. Then there exist two points $x, y \in L$ such that $\operatorname{tconv}(x, y) \not\subseteq L$. Using a translation argument similar to that in the proof of Theorem 2.6, we can find two points $Tx, z \in P$ whose tropical convex hull is not contained in P. Hence, P is not tropically convex. □

Remark 2.8. The authors of [FK11] characterize distributive polyhedra. Any such polyhedron P has the property that $\min(x, y)$ and $\max(x, y)$ are contained in P. Note that only polytropes are distributive polytopes. This is not true for tropically convex polyhedra. For example, consider the triangle $P \subset \mathbb{R}^2$ in Figure 3 whose vertices are the origin, $(3, 1)$, and $(1, 3)$. This is a tropically convex polytope by Theorem 1.10, but not a distributive polytope. In particular, it is not max-closed since $\max(B, C) \notin P$.

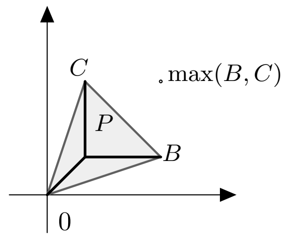

Figure 3. A tropically convex triangle that is not distributive since the point $\max(B, C)$ is not contained in it. In black the tropical convex hull of the vertices.

3. Lower bound on the degree of a tropical curve

For the remainder of the paper we use an alternative definition of the tropical convex hull from [DS04, Proposition 4]. We work in the tropical projective torus $\mathbb{PT}^n \cong \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ which is isomorphic to $\mathbb{R}^n$ as follows. Given a set $U \subset \mathbb{R}^{n+1}$, its tropical convex hull is the set of all possible tropical linear combinations $a_1 \odot u_1 \oplus \ldots \oplus a_k \odot u_k$ with $u_i \in U$ and $a_i \in \mathbb{R}$. With this definition we have $\operatorname{tconv} U + \mathbb{R}\mathbf{1} = \operatorname{tconv} U$. Taking the quotient with $\mathbb{R}\mathbf{1}$ we obtain

$$\operatorname{tconv} U = \operatorname{tconv} \{u \in \mathbb{R}^n : (0, u) + \mathbb{R}\mathbf{1} \subset U\}$$

computed using (3). It follows that the results obtained in Section 1 also hold in this case.

Let Γ be a tropical curve. This is a weighted balanced rational polyhedral complex of dimension one in $\mathbb{PT}^n$. The degree of Γ is defined to be the multiplicity at the origin of the stable intersection between Γ and the standard tropical hyperplane [MS15, Definition 3.6.5]. For realizable curves, this is equal to the degree of any classical curve which tropicalizes to Γ [MS15, Corollary 3.6.16]. Let $r_1, \ldots, r_k$ be the rays of a tropical curve Γ where $r_i = w + \operatorname{pos}(v_i)$ for some $w \in \mathbb{PT}^n$. Since $\Gamma \subset \mathbb{PT}^n$ we can choose each $v_i \in \mathbb{PT}^n$ to be the minimal nonnegative integer vector representative that generates r_i. If the multiplicity of the ray r_i in Γ is m_i, then by [BGS17, Lemma 2.9] we have

$$(\deg \Gamma)\mathbf{1} = \sum_{i=1}^{k} m_i v_i. \tag{9}$$

The main result of this section is Theorem 3.4, which states that a tropical ﬀan curve Γ satisfies the inequality

$$\dim \operatorname{tconv} \Gamma \le \deg \Gamma. \tag{2}$$

The proof relies entirely on tropical and combinatorial techniques and uses results from Sections 1 and 2. Here we state the following two results we reference within the subsequent proofs.

Theorem 3.1. [DSS05, Theorem 4.2] *The tropical rank of a $k \times n$ matrix M is equal to one plus the dimension of the tropical convex hull of the columns of M in $\mathbb{R}^k/\mathbb{R}\mathbf{1}$.*

Lemma 3.2. [RGST05, Lemma 5.1] *An $n \times n$ matrix M is singular if and only if its rows lie on a tropical hyperplane in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.*

As a first step towards proving (2), we prove the following lemma.

Lemma 3.3. *If $\Gamma \subset \mathbb{PT}^n$ is a fan tropical curve and $W \subset \Gamma$ is finite, then*

$$\dim \operatorname{tconv} W \leq \deg \Gamma.$$

Proof. Let $\deg \Gamma = d$ and Γ be given by rays $r_1 = \operatorname{pos}(v_1), \ldots, r_k = \operatorname{pos}(v_k)$ with minimal nonnegative vectors $v_1, \ldots, v_k$. Let $W \subset \Gamma$ be a finite set of points and $\operatorname{Supp} W$ denote the set of minimal nonnegative vectors of rays which contain a point of W. That is,

$$\operatorname{Supp} W = \{v_i \mid w \in \operatorname{pos}(v_i) \text{ for some } w \in W\}.$$

First suppose $|\operatorname{Supp} W| = 1$, so $W \subset r_i$ for some $i \in [k]$ and $\dim \operatorname{tconv} W \leq \dim \operatorname{tconv} r_i$. Each ray of Γ has at most d nonzero distinct entries since $\deg \Gamma = d$. By Lemma 1.8 this means $\dim \operatorname{tconv} r_i \leq d$ for all $i \in [k]$ and $\dim \operatorname{tconv} W \leq d$.

Let M be the $(n+1) \times k$ matrix whose columns are $v_1, \ldots, v_k$. We also assume $n+1$, $k \geq d+2$. Otherwise, the result is trivially true. We will show that the tropical rank of M is at most $d+1$, implying that $\operatorname{tconv}(v_1, \ldots, v_k) \leq d$. Let D be any $(d+2) \times (d+2)$ submatrix of M. Each row of D has all nonnegative entries and must have at least two zeros because $\deg \Gamma = d$. Hence, the rows of D lie in the tropicalization of the ordinary hyperplane $V(x_0 + \ldots + x_{d+1})$ in $\mathbb{PT}^{d+1}$. By Lemma 3.2 this implies D is tropically singular, so the tropical rank of M is at most $d+1$. Using Theorem 3.1 we deduce that the dimension of the tropical convex hull of the columns of M is at most d.

Now suppose $|\operatorname{Supp} W| = |W|$, so each point of W is on a distinct ray of Γ. More specifically, each point of W is a classical scalar multiple of some distinct v_i. The tropical convex hull of any $d+2$ columns of M has dimension at most d and the same holds if each column is scaled since the location of the zero entries is not affected.

Next suppose $1 < |\operatorname{Supp} W| < |W|$ and let $W = \{w_1, \ldots, w_s\}$. Let M' be the $(n+1) \times s$ matrix whose columns are $w_1, \ldots, w_s$. More specifically, its columns are classical scalar multiples of some v_is in $\operatorname{Supp} W$. We know from the previous case that M is tropically singular and the tropical rank is at most $d+1$. By Lemma 3.2 we have that the columns of any $(d+2) \times (d+2)$ submatrix of M are contained in some hyperplane in $\mathbb{PT}^{d+1}$. If a point is contained in a tropical hyperplane, so is any classical scalar multiple of that point since any tropical hyperplane is a fan. For this reason, the columns of any $(d+2) \times (d+2)$ submatrix of M' must also be contained in at least one of these hyperplanes of $\mathbb{PT}^{d+1}$ from before. Therefore, M' has tropical rank at most $d+1$ and $\dim \operatorname{tconv} W \leq d$. ∎

Theorem 3.4. *If $\Gamma \subset \mathbb{PT}^n$ is a fan tropical curve, then* $\dim \operatorname{tconv} \Gamma \leq \deg \Gamma$.

Proof. Let $\deg \Gamma = d$ and suppose $\dim \operatorname{tconv} \Gamma = d+1$. Since $\operatorname{tconv} \Gamma$ is a fan, there exists a point p with $d+2$ distinct coordinates by Lemma 1.8. Moreover, Γ contains the ray $\operatorname{pos}(p)$. Note that we can choose p to be the minimal nonnegative integer vector that generates this ray. Since p has $d+2$ distinct coordinates, we may assume that $0 = p_0 < p_1 < \cdots < p_{d+1}$. Let $\lambda_i p$ be $d+2$ distinct points on the ray $\operatorname{pos}(p)$ and assume $\lambda_1 < \lambda_2 < \cdots < \lambda_{d+2}$. Let M_p be the $(n+1) \times (d+2)$ matrix whose columns are $\lambda_i p$ for $i \in [d+2]$. Then, up to permutation of rows, M_p contains the $(d+2) \times (d+2)$ submatrix

$$D = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \lambda_1 p_1 & \lambda_2 p_1 & \ldots & \lambda_{d+2} p_1 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 p_d & \lambda_2 p_d & \ldots & \lambda_{d+2} p_d \\ \lambda_1 p_{d+1} & \lambda_2 p_{d+1} & \ldots & \lambda_{d+2} p_{d+1} \end{pmatrix}.$$

We will show that D has tropical rank $d+2$ by showing that the tropical determinant of D has a unique minimum attained on its antidiagonal. Using Laplace expansion along the first row, we write the tropical determinant of D as

$$\operatorname{tropDet}(D) = \min_{i\in[d+2]} 0 + \operatorname{tropDet}(D_i)$$

where D_i is the $(d+1)\times(d+1)$ submatrix of D obtained by deleting its first row and ith column. We first claim that $\operatorname{tropDet}(D_i) = m_i$ for any $i \in [d+2]$ where

$$m_i = \lambda_1 p_{d+1} + \lambda_2 p_d + \lambda_3 p_{d-1} + \cdots + \lambda_{i-1} p_{d-i+3} + \lambda_{i+1} p_{d-i+2} + \cdots + \lambda_{d+1} p_2 + \lambda_{d+2} p_1.$$

Recall that for a $(d+1)\times(d+1)$ matrix X, its tropical determinant can be written

$$\operatorname{tropDet}(X) = \bigoplus_{\sigma\in S_{d+1}} x_{1\sigma(1)} \odot x_{2\sigma(2)} \odot \cdots \odot x_{d+1,\sigma(d+1)}.$$

Let

$$\begin{aligned}\sigma(m_i) = \lambda_1 p_{\sigma(d+1)} + \lambda_2 p_{\sigma(d)} + \lambda_3 p_{\sigma(d-1)} + \cdots + \lambda_{i-1} p_{\sigma(d-i+3)} \\ + \lambda_{i+1} p_{\sigma(d-i+2)} + \cdots + \lambda_{d+1} p_{\sigma(2)} + \lambda_{d+2} p_{\sigma(1)}.\end{aligned}$$

Any permutation σ can be decomposed into adjacent transpositions of the form $\tau = (j, j+1)$. It suffices to show that $m_i < \tau(m_i)$ to conclude $m_i < \sigma(m_i)$ for any permutation $\sigma \in S_{d+1}$. Let $\tau(m_i)$ represent the expression m_i where p_j and p_{j+1} have been exchanged. First, suppose that $j > d-i+2$, which implies that

$$m_i - \tau(m_i) = (\lambda_{d-j+2} - \lambda_{d-j+1})(p_j - p_{j+1}) < 0.$$

Similarly, if $j < d-i+2$, then

$$m_i - \tau(m_i) = (\lambda_{d-j+3} - \lambda_{d-j+2})(p_j - p_{j+1}) < 0.$$

If $j = d-i+2$, then

$$m_i - \tau(m_i) = (\lambda_{i+1} - \lambda_{i-1})(p_{d-i+2} - p_{d-i+3}) < 0.$$

It follows that $m_i < \tau(m_i)$ for any transposition $\tau = (j, j+1)$.

Finally, we have $\operatorname{tropDet}(D) = \min_{i\in[d+2]} m_i$. For any $i \in [d+1]$

$$m_{i+1} - m_i = (a_i - a_{i+1})p_{d-i+2} < 0.$$

meaning $m_{i+1} < m_i$. Hence the unique minimum is obtained for $i = d+2$. This implies D has tropical rank at least $d+2$, so by Theorem 3.1 the dimension of the tropical convex hull of the columns of D is at least $d+1$ which contradicts Lemma 3.3. □

The following proposition shows that (2) holds for some special types of tropical curves which are not fans.

Proposition 3.5. *Let Γ be a tropical curve in $\mathbb{PT}^n$ with rays $r_1, \ldots, r_k$. If $\dim \operatorname{tconv} \Gamma = \max_{i\in[k]}\{\dim \operatorname{tconv} r_i\}$, then $\dim \operatorname{tconv} \Gamma \le \deg \Gamma$.*

Proof. Let $\dim \operatorname{tconv} \Gamma = \max_{i\in[k]}\{\dim \operatorname{tconv} r_i\} = d$ and $v_1, \ldots, v_k \in \mathbb{PT}^n$ be the minimal nonnegative integer vectors such that $r_i = w_i + \operatorname{pos}(v_i) \subset \mathbb{PT}^n$ for $i \in [k]$. Then there exists some $j \in [k]$ such that $\dim \operatorname{tconv}\ r_j = d$. By Corollary 1.7 v_j has $d+1$ distinct entries. Hence the maximum component of v_j is at most d. By (9) we have that $\dim \operatorname{tconv} \Gamma = d \le \deg \Gamma$. □

However, Proposition 3.5 does not hold for all tropical curves.

Example 3.6. Let Γ be the fan tropical curve in $\mathbb{PT}^2$ with rays spanned by $(0,1,0)$, $(0,0,1)$, $(0,0,-1)$, and $(0,-1,0)$ emanating from the origin. Each ray $r \subset \Gamma$ is tropically convex so $\max_{r\in\Gamma}\{\dim \operatorname{tconv} r\} = 1$. However, $\dim \operatorname{tconv} \Gamma = 2$. In fact, $\operatorname{tconv}(\operatorname{pos}(0,-1,0), \operatorname{pos}(0,0,1))$ is the 2-dimensional cone spanned by $(0,-1,0)$ and $(0,0,1)$. △

Finally, we give an example of a tropical curve where the smallest dimension of a linear space containing it is larger than the dimension of the tropical convex hull of the curve.

Example 3.7. Consider the tropical curve Γ_F over the field of Puiseux series $\mathbb{C}\{\{t\}\}$ given by the fan whose rays are the columns of M_F :

$$M_F = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The curve Γ_F has degree 3 and there is no 2 dimensional tropical linear space containing it [MS15, Section 5.3]. We now prove that $\dim \operatorname{tconv} \Gamma_F = 2$.

Let $v_1, v_2, \dots, v_7 \in \mathbb{PT}^6$ denote the columns of M_F. Using Macaulay2 [GS02] we compute that the tropical rank of M_F is 3. By Theorem 3.1 $\dim \operatorname{tconv}(v_1, \dots, v_7) = 2$ hence $\dim \operatorname{tconv} \Gamma_F \geq 2$. We will show that $\dim \operatorname{tconv} V \leq 2$ for any finite $V \subset \Gamma_F$. Note that this is not implied by Lemma 3.3.

For a finite set $V \subset \Gamma_F$ we can consider $\operatorname{Supp} V$ as in the proof of Lemma 3.3. Suppose that $|\operatorname{Supp} V| = 7$, implying that each point of $V \subset \Gamma_F$ is on a distinct ray. The tropical rank of M_F is 3 and is invariant under positive scaling of the columns of M_F, which implies $\dim \operatorname{tconv}(\lambda_1 v_1, \dots, \lambda_7 v_7) \leq 2$ for any $\lambda_i > 0$. If all 7 points are on the same ray we have that $\dim \operatorname{tconv} \operatorname{pos}(v_i) = 1$ for each $i \in [7]$, since each ray is tropically convex. Hence, $\dim \operatorname{tconv} V = 1$. For the last case, suppose $V \subset \Gamma_F$ is such that $|\operatorname{Supp} V| < 7$. For each $i \in [7]$ let $V_i = \{\lambda_{i1} v_i, \dots, \lambda_{ik_i} v_i\} \subset V$ and $\lambda_{\max_i} = \max\{\lambda_{i1}, \dots, \lambda_{ik_i}\}$. Since each V_i lies on a tropically convex ray, it follows that $V_i \subseteq \operatorname{tconv}(0, \lambda_{\max_i} v_i) \subset \operatorname{tconv}(\lambda_{\max_1} v_1, \dots, \lambda_{\max_7} v_7)$. Hence, $\operatorname{tconv} V \subset \operatorname{tconv}(\lambda_{\max_1} v_1, \dots, \lambda_{\max_7} v_7)$. The dimension of the tropical convex hull of any choice of the columns of M_F is at most 2, hence $\dim \operatorname{tconv} V \leq 2$.

In order to prove that $\dim \operatorname{tconv} \Gamma_F \leq 2$ we use a similar argument to the one in the proof of Theorem 3.4. Suppose that $\dim \operatorname{tconv} \Gamma_F = 3$. By Corollary 1.7, $\operatorname{tconv} \Gamma_F$ contains a point p with 4 distinct coordinates. Since Γ_F is a fan, Corollary 2.1 implies that $\operatorname{tconv} \Gamma_F$ contains the ray $\operatorname{pos}(p)$, and we can choose p to be the minimal nonnegative integer vector generating the ray. We may assume that $0 = p_0 < p_1 < p_2 < p_3$. Let $a_1 p, a_2 p, a_3 p$, and $a_4 p$ be four distinct points on $\operatorname{pos}(p)$ with $0 < a_1 < a_2 < a_3 < a_4$. Let M_p be the matrix with columns $a_i p$ for $i \in [4]$. Up to permutation of the rows, M_p contains the 4×4 submatrix

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_1 p_1 & a_2 p_1 & a_3 p_1 & a_4 p_1 \\ a_1 p_2 & a_2 p_2 & a_3 p_2 & a_4 p_2 \\ a_1 p_3 & a_2 p_3 & a_3 p_3 & a_4 p_3 \end{pmatrix}.$$

The tropical determinant of D is $a_1 p_3 + a_2 p_2 + a_3 p_1$, and D is tropically nonsingular. Hence, the tropical rank of M_p is at least 4 and $\dim \operatorname{tconv}(a_1 p, \dots, a_4 p) \geq 3$. Each $a_i p \in$

$\operatorname{tconv}\Gamma_F$ can be written as a tropical linear combination of a finite number of points on Γ_F. Hence, $\operatorname{tconv}(a_1p,\dots,a_4p)\subset\operatorname{tconv}W$ for a finite $W\subset\Gamma_F$. This is a contradiction because $\dim\operatorname{tconv}W\leq 2$ for all finite $W\subset\Gamma_F$. Thus $\dim\operatorname{tconv}\Gamma_F=2$. $\triangle$

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