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Finite time estimation for time-varying systems with delay in the measurements

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ABSTRACT

We build finite time observers for time-varying nonlinear systems with delays in the outputs, using a dynamic extension that computes fundamental matrices. Our observers achieve finite time convergence when no disturbances are present. When disturbances are present, we provide approximate values for the solutions, which lead to an upper bound on the approximation error after a suitable finite time. We illustrate our work in a class of systems arising in the study of vibrating membranes, where time-varying coefficients can be used to represent intermittent measurements.

1. Introduction

Constructing asymptotic observers for nonlinear systems is an important topic that is motivated by the difficulty of measuring state variables of systems; see [1] for an overview. Other key results in this direction include the finite dimensional systems in [2–4]. For asymptotic observers with delayed measurements, see [5,6].

However, less attention has been paid to constructing finite time observers, whose objective is to find values for states of the system after a predetermined finite time. Such problems are important in engineering processes with deadlines. The pioneering work by Engel and Kreisselmeier [7] on finite time observer design has been built upon by significant results such as [8] (which provides finite time observers with guaranteed bounds for solutions, including systems with disturbances), [9–11]. See also [12] for finite time continuous–discrete observers for systems with temporary loss of measurements. However, these works do not allow delayed measurements, which can limit their applicability in engineering contexts where delays occur; see [13–15]. Moreover, sliding mode finite time convergence methods (such as [16]) generally do not apply to the systems with nonlinearities and intermittent outputs with time varying (and possibly uncertain) delays that we study here. See also the notable works by Karafyllis and Jiang [17],Tsinias and Kitsos [18],Van Assche et al. [19] for observer designs under triangular structure, strong observability, or solvability conditions that we do not require here.

Here we help overcome the preceding challenges, by building a new class of finite time observers for a class of time-varying nonlinear systems with uncertainties and delayed output values. Although we allow nonlinear systems, our systems lead us to the problem of finding formulas for fundamental solutions for time-varying linear systems. Fundamental solutions provide an analog of the matrix exponential and are applicable to time-varying linear systems, and can be written as Peano–Baker formulas, but it is not generally possible to write them in closed form. We overcome this challenge by interconnecting our observers with dynamic extensions that compute the fundamental solutions. Due to the uncertainties, we provide an approximate method to reconstruct solutions, which provides an exact finite time reconstruction when there are no uncertainties.

We show how the difference between the value of the state and its estimation is bounded by a function of the past output value, the input, and the disturbances. We illustrate our observer design using a class of dynamics that includes Mathieu's equation from the study of vibrating membranes, which was studied in [12] for the case where there are no measurement delays.

Throughout this paper, the dimensions of our Euclidean spaces are arbitrary, unless otherwise noted. The standard Euclidean norm in $\mathbb{R}^a$, and the induced norm of matrices, are denoted by $|\cdot|$, and we assume that the initial times for our solutions are $t_0 = 0$, and that the initial and delay functions are constant at all times $t \leq 0$. Let I denote an identity matrix. Let $|\cdot|_\infty$ denote the sup norm of any matrix valued function over its entire domain.

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We use basic properties of fundamental matrices, e.g., from [20, Appendix C]. For bounded piecewise continuous matrix valued functions $M : \mathbb{R} \to \mathbb{R}^{n\times n}$, let Φ_M be the fundamental matrix solution for $\dot{z}(t) = M(t)z(t)$, so

$$\frac{\partial}{\partial t}\Phi_M(t,s) = M(t)\Phi_M(t,s) \ \text{ and } \ \Phi_M(t,t) = I \tag{1}$$

hold for all real values s and t.

2. Main result

We construct an observer for this class of nonlinear systems having a delay in the output and disturbances:

$$\begin{cases} \dot{x}(t) = [A(t) + \epsilon(t,x(t))]x(t) + f(t,y(t),u(t)) \\ y(t) = C(t)\Phi_A(t,t-h(t))x(t-h(t)) \end{cases} \tag{2}$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $C : \mathbb{R} \to \mathbb{R}^{q\times n}$ are piecewise continuous bounded matrix valued functions (where C could be 0 at some times and therefore can represent intermittent output observations), the state x is valued in $\mathbb{R}^n$, the output y is valued in $\mathbb{R}^q$, the possible known input u is valued in $\mathbb{R}^p$, the nonnegative valued bounded piecewise continuous function h represents a measurement delay, and the unknown bounded function $\epsilon : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ represents a disturbance. We assume that $f(t,y(t),u(t))$ and $\epsilon(t,x(t))$ in (2) are locally integrable functions of t, which provides the standard forward completeness property that we assume to hold in the sequel, which is that for each constant initial function, the corresponding solution of the system (2) is defined at all nonnegative times; sufficient (but not necessary) conditions for this integrability are provided by standard Lipschitzness conditions in the state and continuity in t for f and ϵ, and continuity of h. For simplicity, we also assume in our theorem that h is known, but see Remark 3 for the case where h contains uncertainties.

Systems of the form (2) occur in numerous engineering contexts; see our illustration below. While the output in (2) can be written as $y(t) = C_h(t)x(t-h(t))$ where $C_h(t) = C(t)\Phi_A(t,t-h(t))$, writing the output as in (2) will ease the observer design and its convergence proof below. Moreover, the usual setting in practical applications where the output is $y(t) = C_0x(t-h(t))$ for a constant matrix C_0 is covered by (2) by choosing $C(t) = C_0\Phi_A(t-h(t),t)$ because (by standard semigroup properties of fundamental solutions) this choice of C gives $C_0x(t-h(t)) = C(t)\Phi_A(t,t-h(t))x(t-h(t))$ for all $t \geq 0$.

In the proof of our theorem, a key formula that is needed to compute our observer will first be expressed as a solution of an implicit relation, rather than as an explicit formula. In order to solve this implicit relation to obtain the observer, it will be necessary to invert a matrix coefficient. The invertibility of the coefficient matrix will be ensured by the following two assumptions, which therefore play a role that is analogous to the invertibility of a suitable Jacobian in the well known multivariable implicit function theorem:

Assumption 1. There exist a bounded piecewise continuous matrix valued function $L : \mathbb{R} \to \mathbb{R}^{n\times q}$ and a constant $\tau > 0$ such that with the choice $F = A + LC$, the matrix valued function $E_\tau(t) = \Phi_A(t-\tau,t) - \Phi_F(t-\tau,t)$ is invertible for all $t \in \mathbb{R}$ and satisfies $|E_\tau^{-1}|_\infty < \infty$. □

Assumption 2. The constant $\tau > 0$ and the functions L and E_τ from Assumption 1 are such that with the choices

$$\overline{\upsilon}_h = |C\Phi_A(t,t-h(t))|_\infty \bar{\epsilon}\bar{h}e^{(|A|_\infty+\bar{\epsilon})\bar{h}} \tag{3}$$

and $\overline{E}(\tau) = \left|E_\tau^{-1}\right|_\infty$, we have

$$\begin{aligned} &\tau\overline{E}(\tau)\left[\bar{\epsilon}e^{(|A|_\infty+\bar{\epsilon})\tau} + (|L|_\infty\overline{\upsilon}_h + \bar{\epsilon})e^{(|F|_\infty+|L|_\infty\overline{\upsilon}_h+\bar{\epsilon})\tau}\right] \\ &< 1, \end{aligned} \tag{4}$$

where $\bar{\epsilon}$ and $\bar{h}$ are bounds on ϵ and h in (2), respectively. □

See Section 3 for ways to check our assumptions. For pedagogical purposes, we next introduce equations that are used in our main theorem, before stating the theorem. Our theorem will use the solutions of the initial value problems

$$\begin{cases} \dot{\alpha}_A(t) = A(t)\alpha_A(t), \quad \alpha_A(0) = I \\ \dot{\alpha}_F(t) = F(t)\alpha_F(t), \quad \alpha_F(0) = I, \end{cases} \tag{5}$$

where A is from (2) and F is from Assumption 1. We will see in Lemma 3 that α_A and α_F are invertible, and that $\Phi_A(t,s) = \alpha_A(t)\alpha_A^{-1}(s)$ for all real s and t, and similarly for F. Therefore, we can rewrite E_τ from Assumption 1 as

$$E_\tau^*(t) = \alpha_A(t-\tau)\alpha_A^{-1}(t) - \alpha_F(t-\tau)\alpha_F^{-1}(t). \tag{6}$$

While not standard in the observers literature, the expression (6) is useful because it provides a computable formula for E_τ, which contrasts with the original formula for E_τ that is expressed in terms of fundamental solutions that are generally not available in explicit closed form when A is time varying. Assuming h is known and recalling that f and u are assumed to be known as well, it follows that knowledge of the output y implies that we also know

$$y_\sharp(t) = y(t) + C(t)\alpha_A(t)\int_{t-h(t)}^{t}\alpha_A^{-1}(\ell)f(\ell,y(\ell),u(\ell))\mathrm{d}\ell, \tag{7}$$

which can be computed in practice using known quantities (or in terms of additional dynamical extensions without using integrals, as we show in Remark 2 below). Then in terms of the function L from Assumption 1, the function

$$g(\ell) = f(\ell,y(\ell),u(\ell)) - L(\ell)y_\sharp(\ell) \tag{8}$$

is also known. In terms of (5)–(8), our theorem is as follows (but see Remark 2 for a way to express the observer in our theorem without using integrals):

Theorem 1. *Let Assumptions 1–2 hold. Let $\alpha_A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\alpha_F : \mathbb{R} \to \mathbb{R}^{n\times n}$ be the solutions of (5). Then, with the notation of Assumptions 1–2 and (5)–(8) and $\bar{\epsilon}_3 = |L|_\infty\overline{\upsilon}_h + \bar{\epsilon}$ and $\bar{\epsilon} = |\epsilon|_\infty$ and*

$$\bar{\epsilon}_4 = \tau\left[\bar{\epsilon}e^{(|A|_\infty+\bar{\epsilon})\tau} + (|L|_\infty\overline{\upsilon}_h + \bar{\epsilon})e^{(|F|_\infty+|L|_\infty\overline{\upsilon}_h+\bar{\epsilon})\tau}\right], \tag{9}$$

$$\bar{\epsilon}_5 = \frac{\bar{h}\bar{\epsilon}\overline{E}(\tau)\left(e^{|F|_\infty\tau} + \bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau}\right)|L|_\infty|C|_\infty e^{(2|A|_\infty+\bar{\epsilon})\bar{h}}}{1-\overline{E}(\tau)\bar{\epsilon}_4}, \tag{10}$$

$$\begin{aligned} \gamma(t,\tau) = \ &\frac{\tau\overline{E}(\tau)}{1-\overline{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}\left[\bar{\epsilon}e^{(|A|_\infty+\bar{\epsilon})\tau}|f(\ell,y(\ell),u(\ell))|\right. \\ &\left.+\bar{\epsilon}_3e^{(|F|_\infty+\bar{\epsilon}_3)\tau}|g(\ell)|\right]\mathrm{d}\ell \\ &+\frac{\overline{E}(\tau)^2\bar{\epsilon}_4}{1-\overline{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}[e^{|A|_\infty\tau}|f(\ell,y(\ell),u(\ell))| \\ &+e^{|F|_\infty\tau}|g(\ell)|]\mathrm{d}\ell \\ &+\bar{\epsilon}_5\int_{t-\tau}^{t}\int_{\ell-h(\ell)}^{\ell}|f(m,y(m),u(m))|\,\mathrm{d}m\mathrm{d}\ell, \end{aligned} \tag{11}$$

where $\overline{\upsilon}_h = |C\Phi_A(t,t-h(t))|_\infty\bar{\epsilon}\bar{h}e^{(|A|_\infty+\bar{\epsilon})\bar{h}}$, the following is true: $E_\tau^*(t)$ is invertible for all $t \in \mathbb{R}$ and the observer

$$\begin{aligned} \hat{x}(t) = \ &E_\tau^*(t)^{-1}\int_{t-\tau}^{t}\left[\alpha_A(t-\tau)\alpha_A^{-1}(\ell)f(\ell,y(\ell),u(\ell))\right. \\ &\left.-\ \alpha_F(t-\tau)\alpha_F^{-1}(\ell)g(\ell)\right]\mathrm{d}\ell \end{aligned} \tag{12}$$

is such that the error estimate $\tilde{x}(t) = \hat{x}(t) - x(t)$ *satisfies*

$$|\tilde{x}(t)| \le \gamma(t, \tau) \tag{13}$$

for all solutions of (2) *for all* $t \ge \tau + |h|_\infty$. □

Remark 1. The bound γ only depends on known functions. When $\bar{\epsilon}$ converges to zero, then for fixed t and τ, $\gamma(t, \tau)$ converges to zero. When $\bar{\epsilon} = 0$, we get a finite time observer, and Assumption 2 is satisfied for all $\bar{h} \ge 0$, because all coefficients in (11) and in the left side of (4) have the factor $\bar{\epsilon}$, so $\gamma(t, \tau)=0$ for all t when $\bar{\epsilon} = 0$. Also, $\gamma(t, \tau)$ is bounded if f and y are bounded. Our reason for starting the index for $\bar{\epsilon}_i$ at $i = 3$ in Theorem 1 will become clear in Lemma 4. □

Remark 2. We can express $\hat{x}(t)$ without integrals. To see how, notice that for the equations in the system

$$\begin{cases} \dot{H}_1(t) = A(t)H_1(t) + f(t, y(t), u(t)) \\ \dot{H}_2(t) = F(t)H_2(t) + g(t) \end{cases} \tag{14}$$

and all $t \ge 0$, we can apply variation of parameters to obtain

$$\begin{aligned} &\int_{t-\tau}^{t} \alpha_A(t)\alpha_A^{-1}(\ell)f(\ell, y(\ell), u(\ell))\mathrm{d}\ell \\ &= H_1(t) - \alpha_A(t)\alpha_A(t-\tau)^{-1}H_1(t-\tau), \\ &\int_{t-\tau}^{t} \alpha_F(t)\alpha_F^{-1}(\ell)g(\ell)\mathrm{d}\ell \\ &= H_2(t) - \alpha_F(t)\alpha_F(t-\tau)^{-1}H_2(t-\tau) \text{ and} \\ &\int_{t-h(t)}^{t} \alpha_A(t)\alpha_A^{-1}(\ell)f(\ell, y(\ell), u(\ell))\mathrm{d}\ell \\ &= H_1(t) - \alpha_A(t)\alpha_A(t-h(t))^{-1}H_1(t-h(t)). \end{aligned} \tag{15}$$

Then the semigroup property of fundamental solutions gives

$$\begin{aligned} \hat{x}(t) &= E_\tau^*(t)^{-1}\big[\alpha_A(t-\tau)\alpha_A^{-1}(t)\big(H_1(t) \\ &\quad - \alpha_A(t)\alpha_A(t-\tau)^{-1}H_1(t-\tau)\big) \\ &\quad - \alpha_F(t-\tau)\alpha_F^{-1}(t)\big(H_2(t) \\ &\quad + \alpha_F(t)\alpha_F(t-\tau)^{-1}H_2(t-\tau)\big)\big] \\ y_\sharp(t) &= y(t) + C(t)[H_1(t) \\ &\quad - \alpha_A(t)\alpha_A^{-1}(t-h(t))H_1(t-h(t))], \end{aligned} \tag{16}$$

where g in (14) is computed using (8) and the $y_\sharp$ formula in (16), and $E_\tau^*(t)$ is computed from (6), in terms of the α_A and α_F values from (5). We can also compute α_A^{-1} and α_F^{-1} by solving additional dynamical extensions; see Lemma 3.

Theorem 1 covers significant special cases where h is a sawtooth shaped delay representing sampling. Then $h(t) = t - t_i$ for all $t \in [t_i, t_{i+1})$ and all sampling times t_i, and we can pick the delay bound $\bar{h} = \sup_{i\ge 0}(t_{i+1} - t_i)$, when this sup is finite. This is notable because sampling commonly occurs in engineering applications. The property ensuring finite time convergence is the integral structure of the observer which has a history of information about the nonlinearity f. □

Remark 3. Due to our linearity based analysis, we can straightforwardly generalize Theorem 1 to allow delays that contain uncertainty, by adding an uncertainty $\Delta_h(t)$ to the delay in the output in (2) and adding a term $\gamma_*(t, \tau, \bar{\Delta}_h)$ to the observation error bound $\gamma(t, \tau)$, where $\bar{\Delta}_h$ is a bound on Δ_h. This is done by replacing h by h_0 in the observer formulas, where $h(t) = h_0(t) + \Delta_h(t)$ is the true delay and h_0 is known, under suitable bounds on the piecewise continuous uncertainty Δ_h, as follows. We assume that the given output measurements are $y(t) = C_0(t)x(t-h(t))$ for a known continuous bounded matrix valued function C_0, which we write as $y(t) = C(t)\Phi_A(t, t-h(t))x(t-h(t))$ using the semigroup property as before, where $C(t) = C_0(t)\Phi_A(t-h(t), t)$. We also assume that Assumption 1–2 are satisfied with h and C in the assumptions replaced by h_0 and $C_h(t) = C_0(t)\Phi_A(t-h_0(t), t)$ respectively. Then we can use the bound on A to find a constant $\bar{\Delta}_h > 0$ such that Assumption 1–2 as stated above hold with $h(t) = h_0(t) + \Delta_h(t)$ and $C(t)$ when $|\Delta_h|_\infty \le \bar{\Delta}_h$ (without changing L or τ).

This ensures that when $|\Delta_h|_\infty \le \bar{\Delta}_h$, the conclusions of Theorem 1 hold, but when h is not known, the observer (12) cannot be implemented. Hence, when h is not known, we instead implement the approximating observer

$$\begin{aligned} &\hat{x}_{\text{new}}(t) = \\ &\hat{E}_\tau^*(t)^{-1}\int_{t-\tau}^{t}\Big[\alpha_A(t-\tau)\alpha_A^{-1}(\ell)f(\ell, y(\ell), u(\ell)) \\ &- \alpha_{F_0}(t-\tau)\alpha_{F_0}^{-1}(\ell)g_0(\ell)\Big]\mathrm{d}\ell, \text{ where } F_0 = A + LC_h \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\hat{E}_\tau^*(t) = \alpha_A(t-\tau)\alpha_A^{-1}(t) - \alpha_{F_0}(t-\tau)\alpha_{F_0}^{-1}(t), \\ &g_0(\ell) = f(\ell, y(\ell), u(\ell)) - L(\ell)\hat{y}_\sharp(\ell), \text{ and} \\ &\hat{y}_\sharp(t) = y(t) + C_h(t)\alpha_A(t)\int_{t-h_0(t)}^{t}\alpha_A^{-1}(\ell)f(\ell, y(\ell), u(\ell))\mathrm{d}\ell, \end{aligned} \tag{18}$$

and (17)–(18) can be computed from the available y, h_0, and L values (e.g., using the approach from Remark 2 with h replaced by h_0). Then our requirements are met by the choice $\gamma_*(t, \tau, \bar{\Delta}_h) = \sup\{|\hat{x}_{\text{new}}(t) - \hat{x}(t)| : |\Delta_h|_\infty \le \bar{\Delta}_h\}$ that takes the supremum over all possible values of the uncertainty Δ_h that satisfy $|\Delta_h|_\infty \le \bar{\Delta}_h$. Then $\gamma_*(t, \tau, 0) = 0$ for all $t \ge 0$, and we can use the conclusion of Theorem 1 and the triangle inequality to obtain the desired error bound

$$|x(t) - \hat{x}_{\text{new}}(t)| \le \gamma(t, \tau) + \gamma_*(t, \tau, \bar{\Delta}_h) \tag{19}$$

for all $t \ge \tau + |h|_\infty$. □

3. Checking Assumptions 1–2

Assumption 2 holds if $\bar{\epsilon}$ is a sufficiently small positive constant, and so can be regarded as a smallness condition on $\bar{\epsilon}$. When $\bar{\epsilon} = 0$, we get $\gamma(t, \tau) = 0$ for all t, and then $\hat{x}$ provides an exact reconstruction of x. As noted in [8], when (A, C) is a constant observable pair, the existence of the required matrix L (which in this case is constant) and constant $\tau > 0$ such that E_τ is invertible follows from [8, Lemma 1]. When A and H are time-varying with the same period τ, it follows from [20, Appendix C] that E_τ in Assumption 1 takes the constant value $E_\tau(0) = \Phi_A(-\tau, 0) - \Phi_F(-\tau, 0)$, and in that case we can use Lemma 3 to check that $E_\tau(0) = E_\tau^*(t) = \alpha_A(-\tau) - \alpha_F(-\tau)$ for all t, whose invertibility can be checked by computing its determinant. Hence, we use the rest of this section to develop sufficient conditions for Assumption 1 when A and F are not necessarily periodic.

To this end, we first recall the following lemma from [12]:

Lemma 1. *Let* $M_c \in \mathbb{R}^{n\times n}$ *be an invertible matrix. Let* $N_c \in \mathbb{R}^{n\times n}$ *be a matrix. Let* $\bar{n}$ *and* $\bar{m}$ *be two constants such that* $|M_c^{-1}| \le \bar{m}$ *and* $|N_c| \le \bar{n}$. *Assume that*

$$\bar{m}\bar{n} < 1. \tag{20}$$

Then $M_c + N_c$ *is invertible and the inequality*

$$\left|(M_c + N_c)^{-1} - M_c^{-1}\right| \le \frac{\bar{m}^2\bar{n}}{1 - \bar{m}\bar{n}} \tag{21}$$

is satisfied. □

We also use the following slightly more general version of a lemma from [12]:

Lemma 2. *Let $\mathcal{A}:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $\mathcal{E}:\mathbb{R}\to\mathbb{R}^{n\times n}$ be bounded piecewise continuous matrix valued functions. Let ϕ denote the fundamental solution of*

$$\dot{\zeta}(t)=[\mathcal{A}(t)+\mathcal{E}(t)]\,\zeta(t). \tag{22}$$

Then for all $t\in\mathbb{R}$ and $s\in\mathbb{R}$, the inequality

$$|\phi(t,s)-\Phi_{\mathcal{A}}(t,s)|\le e^{|\mathcal{A}|_\infty|t-s|}\left(e^{|\mathcal{E}|_\infty|t-s|}-1\right) \tag{23}$$

is satisfied. □

Proof. For all s and t, $z(t,s)=\phi(t,s)-\Phi_{\mathcal{A}}(t,s)$ satisfies $\frac{\partial}{\partial t}z(t,s)=(\mathcal{A}(t)+\mathcal{E}(t))\phi(t,s)-\mathcal{A}(t)\Phi_{\mathcal{A}}(t,s)=\mathcal{A}(t)z(t,s)+\mathcal{E}(t)\phi(t,s)$ and $z(s,s)=0$, so

$$z(t,s)=\int_s^t \Phi_{\mathcal{A}}(t,r)\mathcal{E}(r)\phi(r,s)\mathrm{d}r, \tag{24}$$

by a variation of parameters. Also, the Peano–Baker formula for $\phi(r,s)$ (e.g., from [20, p. 489]) gives

$$|\phi(r,s)|\le e^{|\mathcal{A}+\mathcal{E}|_\infty|r-s|} \text{ and } |\Phi_{\mathcal{A}}(r,s)|\le e^{|\mathcal{A}|_\infty|r-s|} \tag{25}$$

for all $r\in\mathbb{R}$. We can combine (25) with (24) to get

$$\begin{aligned}|z(t,s)| &\le \int_{\underline{s}}^{\bar{s}} e^{|\mathcal{A}|_\infty|t-r|}e^{(|\mathcal{A}|_\infty+|\mathcal{E}|_\infty)|r-s|}\mathrm{d}r|\mathcal{E}|_\infty\\ &\le e^{|t-s||\mathcal{A}|_\infty}|\mathcal{E}|_\infty\int_{\underline{s}}^{\bar{s}} e^{|\mathcal{E}|_\infty|r-s|}\mathrm{d}r,\end{aligned} \tag{26}$$

where $\underline{s}=\min\{s,t\}$ and $\bar{s}=\max\{s,t\}$ (by separately considering the cases $s\le t$ and $s>t$). The lemma now follows by upper bounding the last integral in (26). □

We can now provide a way to check Assumption 1, under bounds on the time-varying parts of $A(t)=A_0+\Delta_A(t)$ and $C(t)=C_0+\Delta_C(t)$ which allow the sup norm of the time varying parts to be at least as large as the norms of the corresponding constant parts A_0 and C_0 (as we illustrate in Section 6). In the following proposition, the required L_0 and τ are found using [8, Lemma 1]:

Proposition 1. *Let $(A_0,C_0)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{q\times n}$ be an observable pair, and choose any constant matrix L_0 and any constant $\tau>0$ such that $\Lambda_0=e^{-A_0\tau}-e^{-F_0\tau}$ is invertible, where $F_0=A_0+L_0C_0$ is Hurwitz. Let $\Delta_A:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $\Delta_C:\mathbb{R}\to\mathbb{R}^{q\times n}$ be piecewise continuous bounded functions. Assume that $\bar{N}|\Lambda_0^{-1}|<1$, where*

$$\bar{N}=e^{\tau|A_0|}(e^{\tau|\Delta_A|_\infty}-1)+e^{\tau|F_0|}(e^{\tau|\Delta_A+L_0\Delta_C|_\infty}-1). \tag{27}$$

Then Assumption 1 is satisfied by the functions $A(t)=A_0+\Delta_A(t)$ and $C(t)=C_0+\Delta_C(t)$ and $L(t)=L_0$ and the preceding choice of $\tau>0$. □

Proof. Write $E_\tau(t)=\Lambda_0+N_c(t)$ where $N_c=N_a-N_b$, $N_a(t)=\Phi_A(t-\tau,t)-e^{-A_0\tau}$, $N_b(t)=\Phi_F(t-\tau,t)-e^{-F_0\tau}$, and $F=A+L_0C$. Using Lemma 2 (with $\mathcal{A}=A_0$ and $\mathcal{E}=\Delta_A$ to bound N_a, and then with $\mathcal{A}=F_0$ and $\mathcal{E}=\Delta_A+L_c\Delta_C$ to bound N_b), we obtain $|N_a(t)|\le e^{|A_0|\tau}(e^{|\Delta_A|_\infty\tau}-1)$ and $|N_b(t)|\le e^{|F_0|\tau}(e^{|\Delta_A+L_0\Delta_C|_\infty\tau}-1)$ and therefore also $|N_c(t)|\le\bar{N}<1/|\Lambda_0^{-1}|$ for all real t, by and our condition $\bar{N}|\Lambda_0^{-1}|<1$. The proposition follows from applying Lemma 1 with $M_c=\Lambda_0$ and $N_c=N(t)$. □

4. Main lemmas

The following lemma explains how (5) can be used to construct the fundamental matrices Φ_A and Φ_F in Assumption 1, and ensures that the inverses of α_A and α_F in our theorem can also be obtained from dynamical extensions:

Lemma 3. *Let $M:\mathbb{R}\to\mathbb{R}^{n\times n}$ be a bounded piecewise continuous function. Let α_M and β_M be the solutions of*

$$\begin{cases}\dot{\alpha}_M(t)=M(t)\alpha_M(t), & \alpha_M(0)=I\\ \dot{\beta}_M(t)=-\beta_M(t)M(t), & \beta_M(0)=I,\end{cases} \tag{28}$$

respectively that are defined on $\mathbb{R}$. Then $\Phi_M(t,s)=\alpha_M(t)\beta_M(s)$ and $\Phi_M(s,t)=\Phi_M^{-1}(t,s)=\alpha_M(s)\beta_M(t)$ hold for all real s and t. Moreover, $\beta_M=\alpha_M^{-1}$. □

Proof. The function $\omega(t)=\alpha_M(t)\beta_M(t)$ satisfies $\dot{\omega}(t)=\dot{\alpha}_M(t)\beta_M(t)+\alpha_M(t)\dot{\beta}_M(t)=M(t)\alpha_M(t)\beta_M(t)-\alpha_M(t)\beta_M(t)M(t)=M(t)\omega(t)-\omega(t)M(t)$ for all $t\neq 0$ and $\omega(0)=I$. By standard existence and uniqueness properties for solutions of differential equations, this gives $\omega(t)=I$ for all $t\in\mathbb{R}$. Also, integrating the α_M subsystem of (28) gives $\alpha_M(t)=\Phi_M(t,0)$. Hence, the semigroup property (e.g., from [20, Appendix C]) gives $\alpha_M(t)=\Phi_M(t,s)\Phi_M(s,0)=\Phi_M(t,s)\alpha_M(s)$ for all real s and t. Also, $\omega(t)=I$ gives $\alpha_M(t)=\beta_M^{-1}(t)$ for all $t\in\mathbb{R}$, so $\Phi_M(t,s)=\alpha(t)\beta(s)$. The lemma now follows because $\Phi_M(s,t)=\Phi_M^{-1}(t,s)$. □

In the rest of this paper, we use the notation $\beta_M=\alpha_M^{-1}$ for matrix value functions M to make our notation concise. The next lemmas will be used in our proof of Theorem 1.

Lemma 4. *Consider the system*

$$\begin{cases}\dot{x}(t)=[A(t)+\epsilon_1(t)]x(t)+f_1(t)\\ y(t)=[C(t)+\epsilon_2(t)]x(t)+g(t)\end{cases} \tag{29}$$

where the bounded piecewise continuous matrix valued functions $A:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $C:\mathbb{R}\to\mathbb{R}^{q\times n}$ satisfy Assumption 1 for some function L and some constant $\tau>0$, x is valued in $\mathbb{R}^n$, the output y is valued in $\mathbb{R}^q$, and the ϵ_i's, f_1 and g are piecewise continuous. Assume that the ϵ_i's are bounded and let $\bar{\epsilon}_i$ be a bound on ϵ_i for $i=1,2$. Assume that

$$\begin{aligned}&\tau\left|E_\tau^{-1}\right|_\infty\left[\bar{\epsilon}_1e^{(|A|_\infty+\bar{\epsilon}_1)\tau}+\bar{L}e^{(|F|_\infty+|L|_\infty\bar{\epsilon}_2+\bar{\epsilon}_1)\tau}\right]\\ &<1,\quad\text{where}\quad F=A+LC\quad\text{and}\quad\bar{L}=|L|_\infty\bar{\epsilon}_2+\bar{\epsilon}_1.\end{aligned} \tag{30}$$

Set $\epsilon_3(t)=L(t)\epsilon_2(t)+\epsilon_1(t)$, and let ϕ_1 and ϕ_2 be the fundamental solutions of $\dot{\xi}_1(t)=[A(t)+\epsilon_1(t)]\xi_1(t)$ and $\dot{\xi}_2(t)=[F(t)+\epsilon_3(t)]\xi_2(t)$, respectively, and set $\zeta(t)=\phi_1(t-\tau,t)-\phi_2(t-\tau,t)$. Then, $\zeta(t)$ is invertible for all $t\in\mathbb{R}$, and with the choices

$$\begin{aligned}x_\dagger(t)&=-\zeta(t)^{-1}\int_{t-\tau}^t\phi_2(t-\tau,\ell)L(\ell)g(\ell)\mathrm{d}\ell\\ &+\zeta(t)^{-1}\int_{t-\tau}^t[\phi_1(t-\tau,\ell)f_1(\ell)-\phi_2(t-\tau,\ell)f_2(\ell)]\mathrm{d}\ell\end{aligned} \tag{31}$$

and $f_2=f_1-Ly$, we have

$$x(t)=x_\dagger(t) \tag{32}$$

for all $t\ge\tau$. □

Proof. From the definition of F, we deduce that the x dynamics from (29) admits the representation

$$\dot{x}(t)=[F(t)+\epsilon_3(t)]x(t)+f_2(t)+L(t)g(t). \tag{33}$$

Let $\psi_1(t,s)=\phi_1(t,s)^{-1}$ and $\psi_2(t,s)=\phi_2(t,s)^{-1}$. Then

$$\begin{aligned}&\frac{\partial\psi_1^\top}{\partial t}(t,0)=-[A(t)+\epsilon_1(t)]^\top\psi_1(t,0)^\top\text{ and}\\ &\frac{\partial\psi_2^\top}{\partial t}(t,0)=-[F(t)+\epsilon_3(t)]^\top\psi_2(t,0)^\top\end{aligned} \tag{34}$$

hold for all t (e.g., from [20, Appendix C]), and $\psi_i(t,s)=\phi_i(s,t)$ for all real values s and t and $i=1,2$. This gives $\zeta(t)=\psi_1(t,t-\tau)-\psi_2(t,t-\tau)$ for all real t.

Let $z_i(t) = \psi_i(t,0)x(t)$ for $i = 1,2$. Then we can apply the product rule to z_1 and apply (34) to get

$$\begin{aligned}&\dot{z}_1(t) = \psi_1(t,0)[A(t)+\epsilon_1(t)]x(t)\\&+\psi_1(t,0)f_1(t) + \frac{\partial \psi_1}{\partial t}(t,0)x(t) \;=\; \psi_1(t,0)f_1(t)\end{aligned} \tag{35}$$

for all t, which we then integrate to obtain

$$z_1(t) = z_1(t-\tau) + \int_{t-\tau}^{t} \psi_1(\ell,0)f_1(\ell)\mathrm{d}\ell. \tag{36}$$

Then the definition of z_1 gives

$$\begin{aligned}&\psi_1(t,0)x(t) =\\&\psi_1(t-\tau,0)x(t-\tau) + \int_{t-\tau}^{t} \psi_1(\ell,0)f_1(\ell)\mathrm{d}\ell\ ,\end{aligned} \tag{37}$$

which we can left multiply by $\phi_1(t-\tau,0)$ to obtain

$$\begin{aligned}&\phi_1(t-\tau,0)\phi_1(t,0)^{-1}x(t) =\\&x(t-\tau) + \int_{t-\tau}^{t} \phi_1(t-\tau,0)\phi_1(\ell,0)^{-1}f_1(\ell)\mathrm{d}\ell.\end{aligned} \tag{38}$$

By the semigroup property of the flow map ϕ_1, we obtain $\phi_1(t,t-\tau) = \phi_1(t,0)\phi_1^{-1}(t-\tau,0)$ and therefore also $\phi_1(t-\tau,0)\phi_1(t,0)^{-1} = \phi_1(t,t-\tau)^{-1}$, and $\phi_1(t-\tau,0)\phi_1^{-1}(\ell,0) = \phi_1(t-\tau,\ell)$ for all $\ell \in [t-\tau,t]$. It follows from (38) that

$$\begin{aligned}&\psi_1(t,t-\tau)x(t) =\\&x(t-\tau) + \int_{t-\tau}^{t} \phi_1(t-\tau,\ell)f_1(\ell)\mathrm{d}\ell.\end{aligned} \tag{39}$$

In the same way, we can use (33) to show that since

$$\dot{z}_2(t) = \psi_2(t,0)[f_2(t) + Lg(t)], \tag{40}$$

we obtain

$$\begin{aligned}&\psi_2(t,t-\tau)x(t) =\\&x(t-\tau) + \int_{t-\tau}^{t} \phi_2(t-\tau,\ell)[f_2(\ell)+L(\ell)g(\ell)]\mathrm{d}\ell\ ,\end{aligned} \tag{41}$$

by replacing z_1, ψ_1, and f_1 in (36)-(39) by z_2, ψ_2, and $f_2 + Lg$, respectively. This immediately gives

$$\begin{aligned}\zeta(t)x(t) \;=\; &\int_{t-\tau}^{t} \phi_1(t-\tau,\ell)f_1(\ell)\mathrm{d}\ell\\&-\int_{t-\tau}^{t} \phi_2(t-\tau,\ell)[f_2(\ell)+L(\ell)g(\ell)]\mathrm{d}\ell,\end{aligned} \tag{42}$$

by subtracting (41) from (39).

Let us next prove that ζ is invertible. We set

$$\begin{aligned}\epsilon_4(t) \;=\; &\psi_1(t,t-\tau) - \Phi_A(t-\tau,t)\\&+\Phi_F(t-\tau,t) - \psi_2(t,t-\tau).\end{aligned} \tag{43}$$

By using Lemma 2 and the relation $\psi_1(\ell,t-\tau) = \phi_1(t-\tau,\ell)$, we conclude that the four inequalities

$$\begin{aligned}&|\psi_1(\ell,t-\tau) - \Phi_A(t-\tau,\ell)|\\&\le \bar{\epsilon}_1(\ell-t+\tau)e^{(|A|_\infty+\bar{\epsilon}_1)(\ell-t+\tau)},\\&|\psi_2(\ell,t-\tau) - \Phi_F(t-\tau,\ell)|\\&\le \bar{\epsilon}_3(\ell-t+\tau)e^{(|F|_\infty+\bar{\epsilon}_3)(\ell-t+\tau)},\\&|\phi_1(t-\tau,\ell) - \Phi_A(t-\tau,\ell)| \le \bar{\epsilon}_1\tau e^{(|A|_\infty+\bar{\epsilon}_1)\tau},\ \text{and}\\&|\phi_2(t-\tau,\ell) - \Phi_F(t-\tau,\ell)| \le \bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau}\end{aligned} \tag{44}$$

hold for all $\ell \in [t-\tau,t]$, where $\bar{\epsilon}_3 = |\epsilon_3|_\infty$. It follows that

$$|\epsilon_4(t)| \le \bar{\epsilon}_1\tau e^{(|A|_\infty+\bar{\epsilon}_1)\tau} + \bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau} \tag{45}$$

for all $t \in \mathbb{R}$. With the choice $\bar{\epsilon}_3 = |L|_\infty\bar{\epsilon}_2 + \bar{\epsilon}_1$, we therefore conclude that (43) is bounded by

$$\begin{aligned}&\bar{\epsilon}_4 =\\&\tau\left[\bar{\epsilon}_1 e^{(|A|_\infty+\bar{\epsilon}_1)\tau} + (|L|_\infty\bar{\epsilon}_2+\bar{\epsilon}_1)e^{(|F|_\infty+|L|_\infty\bar{\epsilon}_2+\bar{\epsilon}_1)\tau}\right].\end{aligned} \tag{46}$$

Since $\zeta = E_\tau + \epsilon_4$, we can use Lemma 1 with $M_c = E_\tau(t)$ and $N_c = \epsilon_4(t)$ and (30) to conclude that $\zeta(t)$ is invertible for all t, which we combine with (42) to get (32). □

While useful from the theory point of view, the observer $x_\dagger(t)$ from Lemma 4 is not implementable, because the ϵ_i's are not assumed to be known, and because of the fundamental solutions in (31). This motivates the next lemma, which we prove as a corollary of Lemma 4, and which we later use to prove Theorem 1:

Lemma 5. *Let the requirements of Lemma 4 hold. Then, in terms of the notation f_1, τ, E_τ, ϵ_3, L, $F = A + LC$, and $\bar{\epsilon}_i$ for $i = 1,2$ from the statement of Lemma 4, and with the choices $\bar{\epsilon}_3 = |\epsilon_3|_\infty$, $\bar{E}(\tau) = |E_\tau^{-1}|_\infty$, $f_2 = f_1 - Ly$,*

$$\begin{aligned}\beta(\tau,t) = &\frac{\tau\bar{E}(\tau)}{1-\bar{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}\left[\bar{\epsilon}_1 e^{(|A|_\infty+\bar{\epsilon}_1)\tau}|f_1(\ell)|\right.\\&\left.+\bar{\epsilon}_3 e^{(|F|_\infty+\bar{\epsilon}_3)\tau}|f_2(\ell)|\right]\mathrm{d}\ell\\&+\frac{\bar{E}(\tau)^2\bar{\epsilon}_4}{1-\bar{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}[e^{|A|_\infty\tau}|f_1(\ell)| + e^{|F|_\infty\tau}|f_2(\ell)|]\mathrm{d}\ell\\&+\frac{|L|_\infty\bar{E}(\tau)}{1-\bar{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}\left(e^{|F|_\infty\tau} + \bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau}\right)|g(\ell)|\mathrm{d}\ell\end{aligned} \tag{47}$$

and $\bar{\epsilon}_4$ as defined by (46), the estimate

$$\hat{x}(t) = E(\tau)^{-1}\int_{t-\tau}^{t}[\Phi_A(t-\tau,\ell)f_1(\ell) - \Phi_F(t-\tau,\ell)f_2(\ell)]\,\mathrm{d}\ell \tag{48}$$

is such that the error

$$\tilde{x}(t) = \hat{x}(t) - x(t) \tag{49}$$

satisfies

$$|\tilde{x}(t)| \le \beta(t,\tau) \tag{50}$$

along all solutions of (29) for all $t \ge \tau$. □

Proof. We use calculations and notation from the proof of Lemma 4. We first deduce from (32) and (48) that

$$\begin{aligned}\tilde{x}(t) = &E(\tau)^{-1}\int_{t-\tau}^{t}[(\Phi_A(t-\tau,\ell) - \phi_1(t-\tau,\ell))f_1(\ell)\\&+(\phi_2(t-\tau,\ell) - \Phi_F(t-\tau,\ell))\,f_2(\ell)]\,\mathrm{d}\ell\\&+\left(E(\tau)^{-1} - \zeta(t)^{-1}\right)\int_{t-\tau}^{t}[\phi_1(t-\tau,\ell)f_1(\ell)\\&-\phi_2(t-\tau,\ell)f_2(\ell)]\mathrm{d}\ell + \zeta(t)^{-1}\int_{t-\tau}^{t}\phi_2(t-\tau,\ell)L(\ell)g(\ell)\mathrm{d}\ell\end{aligned}$$

for all $t \ge \tau$, where

$$\zeta(t) = \psi_1(t,t-\tau) - \psi_2(t,t-\tau) \tag{51}$$

as before. Then our choice $\bar{E}(\tau) = \left|E_\tau^{-1}\right|_\infty$ gives

$$\begin{aligned}|\tilde{x}(t)| \le &\bar{E}(\tau)\int_{t-\tau}^{t}[|\Phi_A(t-\tau,\ell) - \phi_1(t-\tau,\ell)|\,|f_1(\ell)|\\&+|\phi_2(t-\tau,\ell) - \Phi_F(t-\tau,\ell)|\,|f_2(\ell)|]\,\mathrm{d}\ell\\&+\left|E_\tau(t)^{-1} - \zeta(t)^{-1}\right|\int_{t-\tau}^{t}[|\phi_1(t-\tau,\ell)||f_1(\ell)|\\&+|\phi_2(t-\tau,\ell)||f_2(\ell)|]\mathrm{d}\ell\\&+|\zeta(t)^{-1}|\int_{t-\tau}^{t}|\phi_2(t-\tau,\ell)||L(\ell)||g(\ell)|\mathrm{d}\ell.\end{aligned}$$

Since

$$\zeta(t) = E_\tau(t) + \epsilon_4(t), \tag{52}$$

we deduce from Lemma 1 (with $M_c = E_\tau(t)$ and $N_c = \epsilon_4(t)$) that

$$\left|E_\tau(t)^{-1} - \zeta(t)^{-1}\right| \le \frac{\bar{E}(\tau)^2\bar{\epsilon}_4}{1-\bar{E}(\tau)\bar{\epsilon}_4}, \tag{53}$$

for all $t \in \mathbb{R}$. Setting

$$\bar{E}^\sharp(\tau) = \bar{E}(\tau) + \tfrac{\bar{E}(\tau)^2\bar{\epsilon}_4}{1-\bar{E}(\tau)\bar{\epsilon}_4} \tag{54}$$

it follows from (44) that

$$\begin{aligned}
&|\tilde{x}(t)| \le \bar{E}(\tau)\int_{t-\tau}^{t}\Big[\bar{\epsilon}_1\tau e^{(|A|_\infty+\bar{\epsilon}_1)\tau}|f_1(\ell)| \\
&+\bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau}|f_2(\ell)|\Big]\mathrm{d}\ell \\
&+\frac{\bar{E}(\tau)^2\bar{\epsilon}_4}{1-\bar{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}[|\phi_1(t-\tau,\ell)||f_1(\ell)| \\
&+|\phi_2(t-\tau,\ell)||f_2(\ell)|]\mathrm{d}\ell \\
&+|L|_\infty|\zeta(t)^{-1}|\int_{t-\tau}^{t}|\phi_2(t-\tau,\ell)||g(\ell)|\mathrm{d}\ell \\
&\le \bar{E}(\tau)\tau\int_{t-\tau}^{t}\Big[\bar{\epsilon}_1 e^{(|A|_\infty+\bar{\epsilon}_1)\tau}|f_1(\ell)| \\
&+\bar{\epsilon}_3 e^{(|F|_\infty+\bar{\epsilon}_3)\tau}|f_2(\ell)|\Big]\mathrm{d}\ell \\
&+\frac{\bar{E}(\tau)^2\bar{\epsilon}_4}{1-\bar{E}(\tau)\bar{\epsilon}_4}\int_{t-\tau}^{t}[\left(e^{|A|_\infty\tau}+\bar{\epsilon}_1\tau e^{(|A|_\infty+\bar{\epsilon}_1)\tau}\right)|f_1(\ell)| \\
&+\left(e^{|F|_\infty\tau}+\bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau}\right)|f_2(\ell)|]\mathrm{d}\ell \\
&+|L|_\infty\bar{E}^\sharp(\tau)\int_{t-\tau}^{t}\left(e^{|F|_\infty\tau}+\bar{\epsilon}_3\tau e^{(|F|_\infty+\bar{\epsilon}_3)\tau}\right)|g(\ell)|\mathrm{d}\ell,
\end{aligned}$$

and then the lemma follows from our choice (47) of β. □

Remark 4. By the formulas $\bar{\epsilon}_3 = |L|_\infty\bar{\epsilon}_2 + \bar{\epsilon}_1$ and (46), it follows that the upper bound (50) is independent of x. Also, when g is not present, the smaller $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ are, the smaller this upper bound is. Also, if $\bar{\epsilon}_1 = \bar{\epsilon}_2 = 0$ and g is not present, then $\hat{x}(t)$ gives the exact $x(t)$ value. □

5. Proof of Theorem 1

Let ϕ_1 be the fundamental solution for $\dot{\xi}(t) = [A(t)+\epsilon(t,x(t))]\xi(t)$, and set $\psi_1(t,s) = \phi_1^{-1}(t,s)$ for all real t and s. Then through the integration of the first equation in (2) over $[t-h(t),t]$, the same calculations that gave (37) (except with f_1 replaced by f, ϵ_1 replaced by ϵ, and τ replaced by $h(t)$) give

$$\begin{aligned}\psi_1(t,0)x(t) &= \psi_1(t-h(t),0)x(t-h(t)) \\ &+\int_{t-h(t)}^{t}\psi_1(\ell,0)f(\ell,y(\ell),u(\ell))\mathrm{d}\ell\end{aligned} \tag{55}$$

and so also

$$\begin{aligned}&\psi_1(t-h(t),0)^{-1}\psi_1(t,0)x(t) = x(t-h(t)) \\ &+\int_{t-h(t)}^{t}\psi_1(t-h(t),0)^{-1}\psi_1(\ell,0)f(\ell,y(\ell),u(\ell))\mathrm{d}\ell\end{aligned} \tag{56}$$

for all $t \ge \tau + |h|_\infty$. For all $t \ge \tau + |h|_\infty$, this gives

$$\begin{aligned}&y(t) = C(t)\Phi_A(t,t-h(t))x(t-h(t)) \\ &= C(t)\Phi_A(t,t-h(t))\psi_1(t-h(t),0)^{-1}\psi_1(t,0)x(t) \\ &-\int_{t-h(t)}^{t}C(t)\mathcal{G}(t,h(t),\ell)f(\ell,y(\ell),u(\ell))\mathrm{d}\ell,\end{aligned}$$

where $\mathcal{G}(t,h,\ell) = \Phi_A(t,t-h)\psi_1(t-h,0)^{-1}\psi_1(\ell,0)$. This equality and the fact that $\phi_1 = \psi_1^{-1}$ give

$$\begin{aligned}&y(t)+\int_{t-h(t)}^{t}C(t)\Phi_A(t,\ell)f(\ell,y(\ell),u(\ell))\mathrm{d}\ell \\ &= C(t)\Phi_A(t,t-h(t))\psi_1(t,t-h(t))x(t)+ \\ &\int_{t-h(t)}^{t}C(t)J(t,\ell,h(t))f(\ell,y(\ell),u(\ell))\mathrm{d}\ell,\ \text{ where}\end{aligned} \tag{57}$$

$$J(t,\ell,h) = \Phi_A(t,\ell) - \Phi_A(t,t-h)\phi_1(t-h,\ell), \tag{58}$$

by using the semigroup property of ϕ_1. Then (57) ensures that the function $y_\sharp$ as defined in (7) can be written as

$$\begin{aligned}&y_\sharp(t) = C(t)\Phi_A(t,t-h(t))\psi_1(t,t-h(t))x(t)+ \\ &\int_{t-h(t)}^{t}C(t)J(t,\ell,h(t))f(\ell,y(\ell),u(\ell))\mathrm{d}\ell,\end{aligned} \tag{59}$$

by the relation $\Phi_A(t,\ell) = \alpha_A(t)\beta_A(\ell)$ from Lemma 3.

We next represent the system (2) as

$$\left\{\begin{array}{rcl}\dot{x}(t) &=& [A(t)+\epsilon(t,x(t))]x(t)+f(t,y(t),u(t)) \\ y_\sharp(t) &=& [C(t)+\upsilon(t)]x(t) \\ && +\int_{t-h(t)}^{t}C(t)J(t,\ell,h(t))f(\ell,y(\ell),u(\ell))\mathrm{d}\ell\end{array}\right. \tag{60}$$

where

$$\upsilon(t) = C(t)\Phi_A(t,t-h(t))[\psi_1(t,t-h(t)) - \Phi_A(t-h(t),t)] \tag{61}$$

because $\Phi_A(r,s) = \Phi_A^{-1}(s,r)$ for all real values r and s. We now apply Lemma 5 to (60), with $\epsilon_1(t) = \epsilon(t,x(t))$, $f_1(t) = f(t,y(t),u(t))$, $\epsilon_2(t) = \upsilon(t)$, the output $y^\sharp$, and $g(t)$ being the integral in (60). Let us observe that from (44) (with τ replaced by $h(t)$), it follows that

$$|\psi_1(t,t-h(t)) - \Phi_A(t-h(t),t)| \le \bar{\epsilon}\bar{h}e^{(|A|_\infty+\bar{\epsilon})\bar{h}}$$

for all $t \ge 0$. We deduce that $|\upsilon(t)| \le \overline{\upsilon}_h$, where $\overline{\upsilon}_h$ is from Assumption 1, and Assumptions 1–2 ensure that the assumptions of Lemma 5 are satisfied. Moreover, the $\hat{x}$ from (12) in Theorem 1 agrees with $\hat{x}$ from (48) in Lemma 5 in this case. Also, the third inequality from (44) (with τ replaced by $h(t)$) implies that the function (58) satisfies

$$\begin{aligned}&|J(t,\ell,h(t))| \\ &= |\Phi_A(t,\ell) - \Phi_A(t,t-h(t))\phi_1(t-h(t),\ell)| \\ &= |\Phi_A(t,t-h(t))[\Phi_A(t-h(t),\ell) - \phi_1(t-h(t),\ell)]| \\ &\le e^{|A|_\infty\bar{h}}\bar{\epsilon}\bar{h}e^{(|A|_\infty+\bar{\epsilon})\bar{h}},\end{aligned} \tag{62}$$

where the second equality in (62) used the semigroup property of the fundamental solution Φ_A. The conclusion of Theorem 1 now follows by specializing the conclusion of Lemma 5 to the special case (60) and then applying Lemma 3 to A and F to express their fundamental matrix solutions Φ_A and Φ_F in terms of α_A, β_A, α_F, and β_F.

Remark 5. In the special case where $\mathcal{A}$ is a constant skew symmetric matrix, we can check that for any bounded piecewise continuous matrix valued function $\mathcal{E} : \mathbb{R} \to \mathbb{R}^{n\times n}$ and for the fundamental matrix solution ϕ of $\dot{\zeta}(t) = [\mathcal{A}+\mathcal{E}(t)]\zeta(t)$, we have $|\phi(r,s)| \le e^{|\mathcal{E}|_\infty|r-s|}$ and $|\phi(t,s) - e^{\mathcal{A}(t-s)}| \le e^{|\mathcal{E}|_\infty|t-s|} - 1$ for all real values r, s, and t. To check the first equality, it suffices to note that the time derivative of $V(\zeta) = |\zeta|^2/2$ along all solutions of $\dot{\zeta}(t) = [\mathcal{A}+\mathcal{E}(t)]\zeta(t)$ satisfies $\dot{V} \le 2|\mathcal{E}|_\infty V(\zeta(t))$, and then integrate the result (where we used the relation $\zeta^\top\mathcal{A}\zeta = \frac{1}{2}\zeta^\top\mathcal{A}\zeta + \frac{1}{2}\zeta^\top\mathcal{A}^\top\zeta = 0$). Then the second equality follows by computing the norm of $z(t,s)$ in (24), and then using the first equality to bound the $|\phi(r,s)|$ in the integrand and the fact that in this case, we have $|\phi_{\mathcal{A}}(t,r)| = |e^{\mathcal{A}(t-r)}| = 1$ for all values of t and r (using the orthogonality of the matrix $e^{\mathcal{A}s}$ for all choices of s, which follows because $\mathcal{A} = -\mathcal{A}^\top$ implies that $I = e^{s\mathcal{A}^\top + s\mathcal{A}} = e^{s\mathcal{A}^\top} e^{s\mathcal{A}}$). Hence, it follows from our proof of Theorem 1 that we can eliminate the $|A|_\infty$'s in the formula (3) for $\overline{\upsilon}_h$ and in the left side of the bound (4) when A is a constant skew symmetric matrix. We illustrate this point in Section 6. □

6. Illustration

Let us revisit the observer design for the Mathieu equation that we proposed in [12]. In [12], no delay in the output was allowed. Here, we consider the special case of (2) where

$$\begin{cases} \dot{x}(t) = [A(t) + \epsilon(t, x(t))]x(t) + f(t, y(t), u(t)) \\ y(t) = C(t)\Phi_A(t, t-h(t))x(t-h(t)) \end{cases} \tag{63}$$

where $f(t, y, u) = -e_2 u$, $A(t) = a_0(t)A_0$,

$$A_0 = \begin{bmatrix} 0 & 1 \\ -R_1 & 0 \end{bmatrix}, \quad \epsilon(t) = \begin{bmatrix} 0 & 0 \\ -R_2\cos(t) & 0 \end{bmatrix}, \tag{64}$$

e_i is the ith standard basis vector for $i = 1, 2$, h is piecewise continuous and bounded, the function $a_0 : \mathbb{R} \to [1, \bar{a}]$ is continuous, and the constant $\bar{a} \in [1, 2)$ will be specified. The Mathieu equation from [12] is the special case where a_0 is identically equal to 1. As in [12], we choose $L = [0\ \ 2R_1]^\top$, $R_1 = 1$, and a constant $R_2 \geq 0$. In [12], we studied the case where $C(t)$ was the constant matrix $C = e_1^\top$, a_0 was the constant 1, and the constant τ in the time invariant version of Assumption 1 from above was $\tau = \pi/2$. Here, we compare the performance of our observer in the $C = e_1^\top$ case with the performance for different choices of τ and $C(t) = [\max\{\cos(4t), 0\}\ 0]$, which can represent the effects of intermittent observations. In both cases, we choose $h(t) = 0.3\sin(t)$. In the latter case, $C(t) = C_0 + \Delta_C(t)$, where $C_0 = e_1^\top$ and $\Delta_C(t) = [\max\{\cos(4t), 0\} - 1\ \ 0]$ so $|C_0| = |\Delta_C|_\infty = 1$ (so the time invariant part C_0 of C is not dominating the time varying part Δ_C of C).

We first study the $C = e_1^\top$ case. In this case, we have

$$\begin{aligned} &\Phi_A(t, s) = e^{A_0(\mathcal{M}(t) - \mathcal{M}(s))} \text{ and} \\ &\Phi_F(t, s) = e^{F_0(\mathcal{M}(t) - \mathcal{M}(s))} \end{aligned} \tag{65}$$

for all real values s and t, where $\mathcal{M}(\ell) = \int_0^\ell a_0(r)\mathrm{d}r$, $F_0 = A_0 + Le_1^\top$, and $F = A + LC$ as before. Therefore,

$$\begin{aligned} E_\tau(t) &= \Phi_A(t-\tau, t) - \Phi_F(t-\tau, t) \\ &= e^{A_0\mathcal{L}(t)} - e^{F_0\mathcal{L}(t)}, \text{ where} \\ \mathcal{L}(t) &= \mathcal{M}(t-\tau) - \mathcal{M}(t) \\ &= -\int_{t-\tau}^t a_0(r)\mathrm{d}r \in [-\bar{a}\tau, -\tau] \text{ for all } t \in \mathbb{R}, \end{aligned} \tag{66}$$

because of our upper bound $\bar{a} \in [1, 2)$ for $a_0(t)$. Moreover, the matrix exponential in (65) can be written explicitly using the function $\mathcal{M}$ and the formulas

$$\begin{aligned} e^{A_0 t} &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \quad \text{and} \\ e^{F_0 t} &= \begin{bmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{bmatrix}. \end{aligned} \tag{67}$$

Also, $|A|_\infty = |a_0|_\infty$ and $|C| = 1$, and we can use MATLAB to check that $\det(e^{A_0 t} - e^{F_0 t}) \in [0.1, 2.9]$ for all $t \in [-1.1\pi/2, -\pi/4]$. It follows from (66) that Assumption 1 is satisfied for all choices of $\bar{a} \in [1, 1.1]$ and all $\tau \in [\pi/4, \pi/2]$. For simplicity, we choose $\tau = \pi/2$ and $\bar{a} = 1.1$ in the remainder of this section, but analogous reasoning applies for smaller values of $\tau \in [\pi/4, \pi/2]$ or larger values of $\bar{a} \in [1, 2)$. We now choose $R_2 = 0.02393$. Since $|C|$ is bounded by 1, our condition (4) from Assumption 2 is satisfied, because the preceding values give

$$\begin{aligned} &\tfrac{\pi}{2}\left|E_{\pi/2}^{-1}\right|_\infty \left[\bar{\epsilon}e^{\bar{\epsilon}\frac{\pi}{2}} + (2\overline{\upsilon}_h + \bar{\epsilon})e^{(1.1+2\overline{\upsilon}_h+\bar{\epsilon})\frac{\pi}{2}}\right] \\ &= 0.9994 < 1, \text{ where} \\ &\overline{\upsilon}_h \leq \bar{\epsilon}\bar{h}|C\Phi_A(t, t-h(t))|_\infty e^{\bar{\epsilon}\bar{h}} \leq 0.0072 \end{aligned} \tag{68}$$

and where $\bar{h} = 0.3$ is our bound on our delay $h(t) = 0.3\sin(t)$, because our choice of R_2 gives $\bar{\epsilon} = 0.02393$ as the bound for ϵ and we used skew symmetry of A; see Remark 5. Since Assumption 1–2 are satisfied, Theorem 1 produces the observer

$$\begin{aligned} \hat{x}(t) &= E_\tau^{-1}(t)\int_{t-\tau}^t G_\tau(t,\ell)f(\ell, y(\ell), u(\ell))\mathrm{d}\ell \\ &+ E_\tau^{-1}(t)\int_{t-\tau}^t \Phi_F(t-\tau, \ell)\Big\{Ly(\ell) \\ &+ LC\int_{\ell-h(\ell)}^{\ell} \Phi_A(\ell, m)f(m, y(m), u(m))\mathrm{d}m\Big\}\mathrm{d}\ell \end{aligned} \tag{69}$$

where $G_\tau(t,\ell) = \Phi_A(t-\tau,\ell) - \Phi_F(t-\tau,\ell)$, and we can also write (69) without integrations of $y(t)$ and without fundamental solution values using Remark 2.

We turn next to the case where $C(t) = [\max\{\cos(4t), 0\}\ 0]$ and $L = [0\ 2]^\top$. Although we used a time varying matrix $A(t)$ in the previous paragraph, in this case we choose $a_0(t) = 1$ for all t for simplicity, so $A(t) = A_0$ for all t, but analogous reasoning applies for time-varying A's in this case as well. Now the fundamental matrix for A is a matrix exponential, but the fundamental matrix for $F(t) = A + LC(t)$ does not admit a simple closed form, so we use the dynamic extension (5) to find the α_F and $\beta_F = \alpha_F^{-1}$ to form the expression $\Phi_F(t, s) = \alpha_F(t)\beta_F(s)$ for the fundamental matrix for F for the observer. However, we must first check that Assumption 1–2 are satisfied in this case. To this end, first notice that since F has period $\tau = \pi/2$, Assumption 1 will be satisfied if $E_\tau(0) = e^{-\tau A} - \Phi_F(-\tau, 0)$ is invertible. Moreover,

$$\Phi_F(-\tau, 0) = [\phi_F(-\tau, 0; e_1)\ \phi_F(-\tau, 0; e_2)], \tag{70}$$

where $\phi_F(-\tau, 0; e_i)$ is the solution of the final value problem $\dot{Z}(t) = F(t)Z(t)$, $Z(0) = e_i$ evaluated at $-\tau$ for $i = 1, 2$ (by the linearity of the dynamics $\dot{Z}(t) = F(t)Z(t)$). Using MATLAB to solve these initial value problems gives

$$\begin{aligned} &E_\tau(0) = e^{-\tau A} - \Phi_F(-\tau, 0) = \\ &\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0.5471 & -1.4462 \\ 0.6122 & 0.5471 \end{bmatrix} \end{aligned} \tag{71}$$

whose determinant is 0.2432. Hence, Assumption 1 is satisfied. To check Assumption 2, we add the assumption that $R_2 \in [0, 0.0204]$. Since $|A| = |C|_\infty = 1$, our condition (4) from Assumption 2 is satisfied, because

$$\begin{aligned} &\tfrac{\pi}{2}\left|E_{\pi/2}^{-1}\right|\left[\bar{\epsilon}e^{\bar{\epsilon}\frac{\pi}{2}} + (2\overline{\upsilon}_h + \bar{\epsilon})e^{(1+2\overline{\upsilon}_h+\bar{\epsilon})\frac{\pi}{2}}\right] \\ &\leq 0.9970 < 1, \text{ where } \overline{\upsilon}_h \leq \bar{\epsilon}\bar{h}|e^{A\bar{h}}|e^{\bar{\epsilon}\bar{h}} = 0.0062 \end{aligned} \tag{72}$$

where $\bar{h} = 0.3$ is a bound on our delay $h(t) = 0.3\sin(t)$, because our choice of R_2 gives $\bar{\epsilon} = 0.0204$ as the bound $|\epsilon|_\infty$ and where we again used skew symmetry of A. Moreover, 0.0204 is the upper bound on the possible R_2 values such that Assumption 2 is satisfied. Since Assumption 1–2 are satisfied, Theorem 1 provides the observer

$$\begin{aligned} \hat{x}(t) &= E_\tau^{-1}(0)\int_{t-\tau}^t J_1(t,\tau,\ell)f(\ell, y(\ell), u(\ell))\mathrm{d}\ell \\ &+ E_\tau^{-1}(0)\int_{t-\tau}^t \alpha_F(t-\tau)\beta_F(\ell)\Big\{Ly(\ell) \\ &+ L\int_{\ell-h(\ell)}^{\ell} C(\ell)e^{A(\ell-m)}f(m, y(m), u(m))\mathrm{d}m\Big\}\mathrm{d}\ell \end{aligned} \tag{73}$$

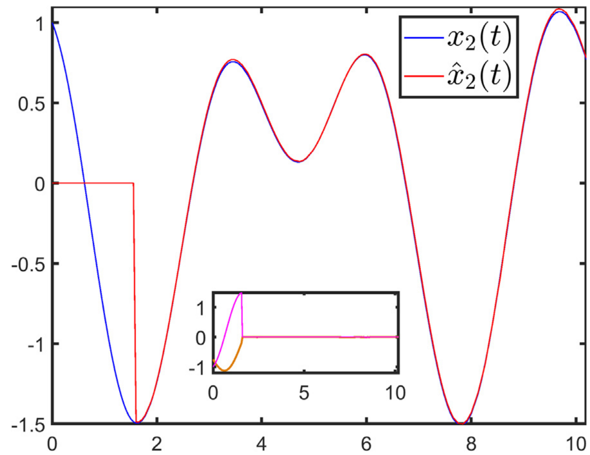

Fig. 1. Simulations of (69) with $a_0(t) = 1.05 + 0.05\sin(t)$ and $C = e_1^\top$. Main Figure: x_2 and its estimate $\hat{x}_2$ with $R_2 = 0.02393$. Inset: $\tilde{x}_1 = \hat{x}_1 - x_1$ (Orange) and $\tilde{x}_2 = \hat{x}_2 - x_2$ (Pink) with $R_2 = 0$. Time unit on horizontal axes is seconds. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

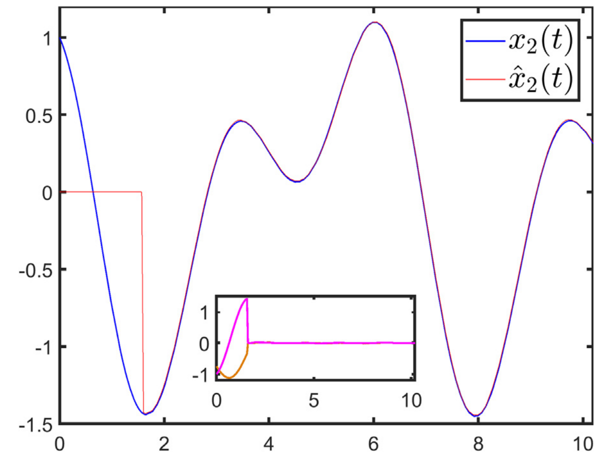

Fig. 2. Simulations of (73) with $a_0(t) = 1$ and $C(t) = [\max\{\cos(4t), 0\}\ 0]$. Main Figure: x_2 and its estimate $\hat{x}_2$ with $R_2 = 0.0204$. Inset: $\tilde{x}_1 = \hat{x}_1 - x_1$ (Orange) and $\tilde{x}_2 = \hat{x}_2 - x_2$ (Pink) with $R_2 = 0$. Time unit on horizontal axes is seconds. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where the constantness of E_τ^{-1} followed because A and F both have period $\pi/2$, and where

$$J_1(t, \tau, \ell) = e^{A(t-\tau-\ell)} - \alpha_F(t-\tau)\beta_F(\ell), \tag{74}$$

and with α_F being a solution of the dynamic extension (5) from our theorem and $\beta_F = \alpha_F^{-1}$ (which we can again write without integrals, using Remark 2). Notice that by including the intermittency in the output observations (and keeping the other parameters the same), we reduced the allowable maximum values of R_2 from 0.02393 to 0.0204. This is to be expected, because with only intermittent measurements, the observer has less information available.

In the simulations in Figs. 1–2, we compare the performances of the observer (69) for the case of constant C with the dynamic observer (73) for the case of intermittent observations. For all of our simulations, we choose the initial states $x(0) = [0.75\ 1]^\top$ of (63) and $\hat{x}(0) = [0\ 0]^\top$ and the delay $h(t) = 0.3\sin(t)$, and we used MATLAB and the SIMULINK Variable-Step ode45 Dormand–Prince solver. In Fig. 1, we applied our observer (69) to (63), with $C = e_1^\top$, $\tau = \pi/2$, $u(t) = \sin(2t)$, and $a_0(t) = 1.05 + 0.05\sin(t)$, which produces a time-varying coefficient matrix $A(t)$. In the main part of Fig. 1, we choose $R_2 = 0.02393$, and in the inset of Fig. 1, we show the corresponding observation error plots with $R_2 = 0$ (which corresponds to having $\bar{\epsilon} = 0$). In Fig. 2, we show the corresponding simulations using the dynamic observer (73), and with α_F computed using the dynamic extensions (5) and $C(t) = [\max\{\cos(4t), 0\}\ 0]$, and with the other parameters being the same as in the first simulation, except with $R_2 = 0.0204$ in the main part of Fig. 2 and $R_2 = 0$ in the inset of Fig. 2. In all cases, our simulations show rapid convergence of the observer values to the state values, and so help to illustrate our theorem in the special case of (63).

7. Conclusions

We provided a new class of finite time observers for a family of nonlinear systems with a pointwise delay. The novelty of our work included our allowing output delays (which can contain uncertainties), combined with a dynamic extension that computes fundamental solutions. By allowing time varying matrices in the output function, we can model temporary loss of measurements (which is motivated, e.g., by Parikh et al. [21]), which was more complicated than the intermittent observations problem from [12] because here we allow output delays that were not allowed in [12]. We conjecture that we can also design interval observers as was done in [8]. We also hope to design stabilizing output feedbacks based on the observers that we provided here.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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