NON-COMMUTATIVE CI OPERATORS

DAVID EISENBUD, IRENA PEEVA, AND FRANK-OLAF SCHREYER

ABSTRACT. We provide a counterexample to the 1980 conjecture that the CI operators can be chosen to commute on a sufficiently high truncation of the minimal free resolution of a module over a complete intersection.

CI operators (Complete Intersection operators, sometimes called Eisenbud operators) are used in the study of free resolutions over complete intersections. In this note, we provide a minimal counterexample to the 1980 conjecture that the CI operators commute on a sufficiently high truncation of the minimal free resolution of any module over a local complete intersection.

Eisenbud showed in [Ei] that if S is a regular local ring and $0 \neq f \in S$, then every minimal free resolution over the hypersurface $S/(f)$ is eventually periodic of period at most 2. In proving this result and trying to find its analogue for complete intersections of higher codimension, he defined CI operators as follows: Suppose that S is a local ring and $f_1, \dots, f_c \in S$. Set $R = S/(f_1, \dots, f_c)$. If

$$(\mathbf{F}, \partial) : \cdots \longrightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots$$

is a complex of finitely generated free R-modules then, lifting the maps ∂_{i+1} to maps $\widetilde{\partial}_{i+1} : \widetilde{F}_{i+1} \longrightarrow \widetilde{F}_i$ of free S-modules, it is possible to write

$$\widetilde{\partial}^2 = \sum_{j=1}^{c} f_j \widetilde{t}_j \tag{1.1}$$

for some maps $\widetilde{t}_j : \widetilde{F}_{i+1} \longrightarrow \widetilde{F}_{i-1}$. Tensoring these maps with R we get maps t_j, called *CI operators* on $\mathbf{F}$; a different construction of operators was previously introduced by Gulliksen [Gu]. When $f_1, \dots, f_c$ is a regular sequence, the CI operators t_j are endomorphisms of the complex $\mathbf{F}$ having homological degree -2. They are also independent of the choice of the expression (1.1),

2010 *Mathematics Subject Classification.* Primary: 13D02.

Key words and phrases. Syzygies, Complete intersections.

Eisenbud is partially supported by NSF grant DMS-1001867, and Peeva is partially supported by NSF grants DMS-1702125 and DMS-1406062. All authors are partially supported by NSF grant 0932078000 while in residence at MSRI.

and, up to homotopy, they are functorial and independent of the choice of $\widetilde{\partial}$ by [Ei, 1.2 and 1.5].

Let $R = S/(f_1, \dots, f_c)$ be a complete intersection of codimension c in a regular local ring S. We consider the CI operators on a minimal R-free resolution $\mathbf{F}$. Having established the homotopy commutativity of the CI operators in [Ei], Eisenbud wrote:

> **Conjecture** (1980): *The minimal free resolution of any R-module is a subcomplex of a standard resolution, in such a way that the maps t_i may be chosen to be induced by the standard t_i. In particular, the maps t_i may be chosen to commute.*
>
> *In the spirit of this paper, it would be interesting to prove this conjecture just for some truncation of any minimal free resolution.*

Here the term "standard resolution" refers to the Eisenbud-Shamash construction. We will call the full conjecture above the Embedding Conjecture, and refer to the statement that the t_i commute as the *Commutativity Conjecture*.

Consider a finitely generated R-module M. In [AGP, Section 9], Avramov, Gasharov and Peeva showed that the degeneration of a certain spectral sequence is an obstruction to the Embedding Conjecture, and observed that the embeddability of the minimal resolution of M implies that $\mathrm{Ext}_R(M, k)$ is generated as a $k[\chi_1, \dots, \chi_c]$-module in degrees at most the projective dimension of M as an S-module, where the action of the $\chi_1, \dots, \chi_c$ is induced by the CI operators. This gives easy counterexamples to embeddability over any complete intersection of codimension ≥ 2. For example, suppose that $R = k[[x, y]]/(x^2, y^2)$ and M is the second cosyzygy of k as an R-module. If the embeddability conjecture held, then $\mathrm{Ext}^*_R(M, k)$ would be generated by $\mathrm{Ext}^{\leq 2}_R(M, k)$, and in particular, $\mathrm{Ext}^{odd}_R(M, k)$ would be generated by $\mathrm{Ext}^1_R(M, k)$. However, as one easily computes, the CI operators act trivially on $\mathrm{Ext}^1_R(M, k)$ but $\mathrm{Ext}^3_R(M, k) \neq 0$.

On the other hand Avramov and Buchweitz showed in [AB, Theorem 5.3] that when $c = 2$ and $\mathrm{Ext}_R(M, k)$ is generated in degrees ≤ 2, then the minimal resolution of M is indeed embeddable in the Eisenbud-Shamash construction and, in particular, commutativity holds in this case. Thus, the hope in the last sentence quoted above is realized in the codimension 2 case. Also, [EP, Theorem 5.1.4] proves a weaker commutativity property in any codimension. Moreover, the obstruction defined in [AGP] vanishes for any sufficiently high truncation of the minimal resolution of M no matter what the codimension ([AGP, Theorem 9.4]).

These results leave open the question whether the Commutativity Conjecture holds for high truncations in codimension ≥ 3. We settle this in the negative:

Theorem 1.2. *Let k be any field, let S be the localization of $k[a,b,c]$ at the maximal ideal (a,b,c) and let $R = S/(a^2,b^2,c^2)$. Let M be the module $R/(a,bc)$. The CI operators cannot be chosen to be commutative on any truncation of the minimal R-free resolution of M.*

Proof. Set $R' = k[a]/(a^2)$ and $R'' = k[b,c]/(b^2,c^2)$. Consider the R'-module $M' := R'/(a)$ and the R''-module $M'' := R''/(bc)$. Note that $M \cong M' \otimes_k M''$ as $R = R' \otimes_k R''$-modules, and thus the minimal R-free resolution of M can be written $\mathbf{F} := \mathbf{F}' \otimes_k \mathbf{F}''$, where $\mathbf{F}'$ and $\mathbf{F}''$ are the minimal resolutions of M' and M'' over R' and R'', respectively. We will make use of this description to understand the possible CI operators on $\mathbf{F}$.

To begin, we may write $\mathbf{F}'$ as the polynomial ring $R'[A]$ as graded free R'-modules, with differential $\partial(A^i) = aA^{i-1}$.

In the same spirit, we write the underlying graded module of the truncation $\mathbf{F}''_{\geq 1}$ as $R''[B,C]$, and $F''_0 = R''$. To avoid confusion, we write ν instead of 1 for the generator of $F''_1 = R''$. We define the differential on F''_1 and F''_2 by $\partial(\nu) = bc$, $\partial(B) = b\nu$, $\partial(C) = c\nu$. For $i \geq 2$ we define $\partial(B^i) = bB^{i-1}$, $\partial(C^i) = cC^{i-1}$, and, if $i,j \geq 1$, then $\partial(B^iC^j) = bB^{i-1}C^j + (-1)^i cB^iC^{j-1}$. It is easy to check that $\partial^2 = 0$ and that the resulting complex is indeed a minimal free resolution of M''.

The first 4 terms $\mathbf{F}''_{\leq 4}$ of the complex $\mathbf{F}''$, with respect to the ordered bases $B^i, B^{i-1}C, \ldots, C^i$ for F''_{i+1}, have the form

$$R''^4 \xrightarrow{\begin{pmatrix} b & c & 0 & 0 \\ 0 & b & -c & 0 \\ 0 & 0 & b & c \end{pmatrix}} R''^3 \xrightarrow{\begin{pmatrix} b & -c & 0 \\ 0 & b & c \end{pmatrix}} R''^2 \xrightarrow{\begin{pmatrix} b & c \end{pmatrix}} R'' \xrightarrow{bc} R''\,.$$

To compute CI operators t_b, t_c for the regular sequence b^2, c^2 on $\mathbf{F}''_{\leq 4}$ we denote by $\widetilde{\partial}$ the choice of lifting of ∂ to matrices over $k[b,c]$ by the formulas above. For example,

$$\widetilde{\partial}^2(B) = \widetilde{\partial}(b\nu) = b^2c\,.$$

Thus we can choose

$$t_b(B) = c$$
$$t_c(B) = 0.$$

Similarly, we can choose

$$t_c(C) = b$$
$$t_b(C) = 0\,.$$

Furthermore,

$$\widetilde{\partial}^2(BC^2) = \widetilde{\partial}(-cBC + bC^2) = -cbC + c^2B + bcC = c^2B,$$

and thus we can choose

$$t_b(BC^2) = 0$$
$$t_c(BC^2) = B\,.$$

We have

$$(t_bt_c - t_ct_b)(BC^2) = c \neq 0\,,$$

so these CI operators do not commute. We will see that they cannot be made to commute by a different choice of lifting. The idea in our proof is that this non-commutativity is propelled to higher homological degrees by multiplying by A^i.

The differential of the complex $\mathbf{F}$ is

$$\partial_p = \sum_{m+i=p} \partial_i \otimes \mathrm{Id} + (-1)^i \,\mathrm{Id} \otimes \partial_m\,;$$

thus for $f \in F''_m$ we get

$$\partial(A^if) = aA^{i-1}f + (-1)^iA^i\partial(f)\,.$$

Hence we may take

$$\widetilde{\partial}^2(A^if) = a^2A^{i-2}f + A^i\widetilde{\partial}^2(f)\,.$$

Therefore, we can define a CI operator t_a corresponding to a^2 and we can extend the CI operators t_b and t_c constructed above to CI operators on $\mathbf{F}' \otimes (\mathbf{F}''_{\leq 4})$ as follows:

$$t_a(A^if) = A^{i-2}f$$
$$t_b(A^if) = A^it_b(f)$$
$$t_c(A^if) = A^it_c(f)\,.$$

We could define the operators on all of $\mathbf{F}$ similarly, but we will not need to use these formulas.

We are now ready to prove the non-commutativity. Fix an $i \geq 0$. Consider the element $A^iBC^2 \in F_{i+4}$. By the formulas above we have

$$(t_bt_c - t_ct_b)(A^iBC^2) = cA^i \neq 0\,.$$

Thus the CI operators t_b and t_c do not commute.

Let t'_b and t'_c be another choice of CI operators with respect to the elements b^2 and c^2. By [Ei], they differ from the CI operators chosen above by homotopies, that, is, there exist, for each index j, maps $h_b, h_c : F_j \longrightarrow F_{j-1}$ such that

$$t'_b - t_b = h_b\partial + \partial h_b \qquad \text{and} \qquad t'_c - t_c = h_c\partial + \partial h_c\,.$$

To complete the proof we will show that

$$(t'_bt'_c - t'_ct'_b)A^iBC^2 \neq 0.$$

We have the following equality, where we have labeled the lines so that we can refer to them:

$$
\begin{aligned}
&(1)\quad (t'_bt'_c - t'_ct'_b)A^iBC^2\\
&(2)\quad\ \ = (t_bt_c - t_ct_b)A^iBC^2\\
&(3)\quad\ \ + (h_b\partial h_c\partial + \partial h_bh_c\partial + \partial h_b\partial h_c - h_c\partial h_b\partial - \partial h_ch_b\partial - \partial h_c\partial h_b)A^iBC^2\\
&(4)\quad\ \ + (h_b\partial^2h_c - h_c\partial^2h_b)A^iBC^2\\
&(5)\quad\ \ + \partial(h_bt_c - h_ct_b)A^iBC^2\\
&(6)\quad\ \ + (t_bh_c\partial + t_b\partial h_c - t_ch_b\partial - t_c\partial h_b)A^iBC^2\\
&(7)\quad\ \ - (h_c\partial t_b)A^iBC^2\\
&(8)\quad\ \ + (h_b\partial t_c)A^iBC^2\,.
\end{aligned}
$$

As computed above, the entry $(t_bt_c - t_ct_b)A^iBC^2$ in the second line is equal to $cA^i \in F'_i \otimes F''_0$. We will show that all other terms on the right have components in $F'_i \otimes F''_0$ that are contained in $(a, b, c^2)F'_i \otimes F''_0$, and thus the sum is nonzero.

The terms in lines (3) and (4) are contained in $(a, b, c)^2\mathbf{F}$ since $\partial(\mathbf{F}) \subset (a, b, c)\mathbf{F}$. (In fact the terms in the forth line vanish because $\partial^2 = 0$.)

Consider the terms in line (5). Their components in $F'_i \otimes F''_0$ are contained in

$$\big(\partial(F'_{i+1}) \otimes F''_0\big) \oplus \big(F'_i \otimes \partial(F''_1)\big) \subset (a, bc)F'_i \otimes F''_0.$$

For the terms in line (6), note that $t_b(A^i f) = A^i t_b(f)$, $t_c(A^i f) = A^i t_c(f)$, and also in the resolution $\mathbf{F}''$ we have $t_b(F_2'') = cF_0''$ and $t_c(F_2'') = bF_0''$. Since $\partial(\mathbf{F}) \subset (a, b, c)\mathbf{F}$, it follows that the component in $F_i' \otimes F_0''$ is in $(b, c)(a, b, c)F_i' \otimes F_0''$.

The term in line (7) is 0 because $t_b(A^i BC^2) = A^i t_b(BC^2) = 0$.

Since $t_c(A^i BC^2) = A^i t_c(BC^2) = A^i B$, the last term is

$$(h_b \partial t_c)A^i BC^2 = h_b \partial(A^i B) = h_b\big(aA^{i-1}B + (-1)^i A^i b\nu\big) \in (a, b)\mathbf{F}\,.$$

This completes the proof that the CI operators cannot be made to commute on any truncation of the resolution $\mathbf{F}$. □

References

[AB] L. Avramov and R.-O. Buchweitz: Homological algebra modulo a regular sequence with special attention to codimension two, *J. Algebra* **230** (2000), 24–67.

[AGP] L. Avramov, V. Gasharov, and I. Peeva: Complete intersection dimension, *Publ. Math. IHES* **86** (1997), 67–114.

[Ei] D. Eisenbud: Homological algebra on a complete intersection, with an application to group representations, *Trans. Amer. Math. Soc.* **260** (1980), 35–64.

[EP] D. Eisenbud, I. Peeva: Minimal Free Resolutions over Complete Intersections, research monograph, *Springer Lecture Notes in Math.* **2152**, Springer, 2016.

[Gu] T. Gulliksen: A change of ring theorem with applications to Poincaré series and intersection multiplicity, *Math. Scand.* **34** (1974), 167–183.

Mathematics Department, University of California at Berkeley, Berkeley, CA 94720, USA

Mathematics Department, Cornell University, Ithaca, NY 14853, USA

Mathematics Department, Universität des Saarlandes, Campus E2 4, D-66123, Saarbrücken, Germany