SMOOTHNESS OF SCHUBERT VARIETIES IN TWISTED AFFINE GRASSMANNIANS

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To Michael Rapoport on his 70th birthday

Abstract
We give a complete list of smooth and rationally smooth normalized Schubert varieties in the twisted affine Grassmannian associated with a tamely ramified group and a special vertex of its Bruhat–Tits building. The particular case of the quasi-minuscule Schubert variety in the quasi-split but non-split form of Spin_8 *(ramified triality) provides an input needed in the article by He–Pappas–Rapoport classifying Shimura varieties with good or semi-stable reduction.*

1. Introduction

Let k be an algebraically closed field, and let G be a connected reductive group over the Laurent series field $F = k(\!(t)\!)$. Associated with any special vertex x of the Bruhat–Tits building is the twisted affine Grassmannian $\mathrm{Gr}_{G,x}$. Under the additional assumption that G splits over a tamely ramified extension of F, we give a complete answer to the question of whether a given (normalized) Schubert variety in $\mathrm{Gr}_{G,x}$ is smooth or singular (resp., rationally smooth or not rationally smooth).

If G is split and $\mathrm{char}(k) = 0$, then such a classification is known by the work of Evens and Mirković in [9] and Malkin, Ostrik, and Vybornov in [26]. The answer is strikingly simple: the Schubert variety $\mathrm{Gr}_{G,x}^{\leq\mu}$ corresponding to a cocharacter $\mu \in X_*(G)$ is smooth if and only if μ is minuscule.

If G is not split, then our classification has a similar flavor, but the phenomenon of *exotic smoothness* enters in: there are surprising additional cases of smoothness, where the group G is a ramified odd unitary group and μ is quasi-minuscule. Unlike the split case, the nature of the special vertex x now plays a pivotal role which was first observed by the second named author in [1, Proposition 4.16] (see Theorem 1.2 for a precise statement).

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Our work is intertwined with the work of He, Pappas, and Rapoport in [17] which classifies Shimura varieties with good or semi-stable reductions by giving a corresponding classification of (slight modifications of) Pappas–Zhu local models (see [32]). The connection between this article and [17] arises in the following way: in [32, Theorem 9.1], it is proved that the special fiber of any local model $\mathbb{M}_K^{\mathrm{loc}}(G, \{\mu\})$ is isomorphic to an explicit union of Schubert varieties in a (twisted) partial affine flag variety over k. The smooth local models are those whose special fiber is a single smooth Schubert variety. By [17, Theorem 1.2], this implies that the parahoric $K = K_x$ is a special maximal parahoric associated with some special vertex x and occurs in the following situations: either G is split so that x is hyperspecial, or G is non-split and the triple (G, μ, x) is of *exotic good reduction type*. Exotic good reduction comes in three kinds:

(1) even unitary exotic, discovered by Pappas and Rapoport in [31, Section 5.3];
(2) odd unitary exotic, discovered by the second named author in [1, Proposition 4.16];
(3) orthogonal exotic, discovered by He, Pappas, and Rapoport in [17, Section 5.1].

In cases (1) and (3), the corresponding Schubert variety $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ is minuscule (hence smooth), and the choice of special vertex x plays no role. In case (2), the Schubert variety $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ is quasi-minuscule, and the choice of special vertex plays a crucial role. This case relates to the phenomenon of exotic smoothness in twisted affine Grassmannians.

1.1. Statement of the results

Let k be an algebraically closed field, and let $F = k(\!(t)\!)$ be the formal Laurent series field, with absolute Galois group I. Let G be a connected reductive group over F which is adjoint, is absolutely simple, and splits over a tamely ramified extension of F. Associated to every special vertex x in the Bruhat–Tits building, we have the twisted affine Grassmannian $\mathrm{Gr}_{G,x}$. If G is split, then all special vertices are conjugate under $G_{\mathrm{ad}}(F)$. If G is not split, then this is no longer true (see [37, Section 2.5]). This fact plays an important role in the phenomenon of exotic smoothness of Schubert varieties.

We choose further a pair $T \subset B \subset G$ of a maximal torus and a Borel subgroup defined over F which are in good position with respect to x (see Section 2 below). Associated with each dominant $\bar{\mu} \in X_*(T)_I^+$ is the Schubert variety $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}} \subset \mathrm{Gr}_{G,x}$ which is an irreducible projective k-variety.

Let M denote the set of minimal elements of $X_*(T)_I^+ \backslash \{0\}$ with respect to the partial ordering $\leq$ defined by the échelonnage coroots $\breve{\Sigma}^\vee \subset X_*(T)_I$ (see [12]). Recall that $\bar{\mu} \in M$ is

- *minuscule* if $\langle \alpha, \bar{\mu} \rangle \in \{0, \pm 1\}$ for all roots $\alpha \in \breve{\Sigma}$,
- *quasi-minuscule* otherwise.

In the second case, there exists a unique root $\gamma \in \breve{\Sigma}$ with $\langle \gamma, \bar{\mu} \rangle \geq 2$, γ is necessarily a highest root, and $\bar{\mu} = \gamma^\vee$. Further, $\langle \alpha, \bar{\mu} \rangle \in \{0, \pm 1, \pm 2\}$ for all $\alpha \in \breve{\Sigma}$ (see [28, Lemma 1.1]). Conversely, if $\bar{\mu} \in X_*(T)_I^+ \backslash \{0\}$ belongs to the coroot lattice and if $|\langle \alpha, \bar{\mu} \rangle| \leq 2$, $\forall \alpha \in \breve{\Sigma}$, then $\bar{\mu} \in M$ and hence $\bar{\mu}$ is quasi-minuscule. Therefore, any irreducible root system possesses a *unique* quasi-minuscule coweight.

Geometrically, $\bar{\mu}$ being minuscule means that $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}} = \mathrm{Gr}_{G,x}^{\bar{\mu}}$ is a single stratum, whereas $\bar{\mu}$ being quasi-minuscule means that $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}} = \mathrm{Gr}_{G,x}^{\bar{\mu}} \amalg \{e\}$, where $e \in \mathrm{Gr}_{G,x}(k)$ is the basepoint.

Under the identification $X_*(T)_I = X^*((T^\vee)^I)$, the échelonnage coroots $\breve{\Sigma}^\vee$ correspond to the roots for $((G^\vee)^I, (T^\vee)^I)$ by [12, Section 5.1], where $(G^\vee)^I$ is a simple and semi-simple connected reductive group with maximal torus $(T^\vee)^I$ (see Proposition A.1). Note that $\bar{\mu} \in X_*(T)_I$ is (quasi-)minuscule with respect to $\breve{\Sigma}$ if and only if it is (quasi-)minuscule when viewed as a $(T^\vee)^I$-weight. Similarly, a fundamental $(T^\vee)^I$-weight ω_i can be viewed as an element $\omega_i \in X_*(T)_I$. Our main results are as follows.

THEOREM 1.1

Let $\bar{\mu} \in X_*(T)_I \backslash \{0\}$ be dominant. The Schubert variety $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}}$ is rationally smooth if and only if $x \in \mathscr{B}(G, F)$ is any special vertex and the pair $(G, \bar{\mu})$ belongs up to isomorphism to the following list:

- *any G, and $\bar{\mu}$ minuscule (for a complete list, see [17, Section 5.2]);*
- *split groups:*
  - $G = \mathrm{PGL}_2$, *and any $\bar{\mu}$;*
  - $G = \mathrm{PGL}_n$, *$n \geq 3$, and $\bar{\mu} = l \cdot \omega_i$, $i \in \{1, n-1\}$, and $l \geq 2$;*
  - $G = \mathrm{PSp}_{2n}$, *$n \geq 2$, and $\bar{\mu}$ quasi-minuscule;*
  - $G = \mathrm{SO}_7$, *and $\bar{\mu} = \omega_3$ (not quasi-minuscule);*
  - $G = G_2$, *and $\bar{\mu}$ quasi-minuscule;*
- *non-split groups:*
  - $G = \mathrm{PU}_3$, *and any $\bar{\mu}$;*
  - $G = \mathrm{PU}_{2n+1}$, *$n \geq 2$, and $\bar{\mu}$ quasi-minuscule;*
  - $G = \mathrm{PSO}_{2n+2}$, *$n \geq 2$, and $\bar{\mu}$ quasi-minuscule;*
  - $G = \mathrm{PU}_6$, *and $\bar{\mu} = \omega_3$ (not quasi-minuscule);*
  - $G = {}^3D_{4,2}$, *the ramified triality, and $\bar{\mu}$ quasi-minuscule.*

Note that PU_4 is isomorphic to the non-split PSO_6, and therefore, the quasi-minuscule Schubert variety for PU_4 is rationally smooth as well.

For the formulation of our next result, we introduce the following notion. The triple $(G, \bar{\mu}, x)$ is called of *exotic smoothness* if $G \simeq \mathrm{PU}_{2n+1}$ for some $n \geq 1$, the element $\bar{\mu} \in X_*(T)_I^+ \backslash \{0\}$ is quasi-minuscule, and x corresponds up to $G(F)$-conjugation to an *almost modular* lattice, that is, the lattice times a uniformizer is contained in the dual of the lattice, with colength 1. See Section 5 below for a more conceptual interpretation of the last condition in terms of the Bruhat–Tits building. (In the terminology of Section 5, the condition on x above amounts to requiring that x is special but not absolutely special.)

The following result verifies a conjectural classification which Rapoport postulated in conversations with the second author in 2010.

THEOREM 1.2
The normalization $\tilde{\mathrm{Gr}}_{G,x}^{\leq \bar{\mu}}$ *is smooth if and only if either* $\bar{\mu}$ *is minuscule or the triple* $(G, \bar{\mu}, x)$ *is of exotic smoothness.*

We note that Schubert varieties are normal if $\mathrm{char}(k) \nmid |\pi_1(G)|$ by [30, Theorem 6.1], for example, if the characteristic of k is zero or sufficiently large. However, there are non-normal Schubert varieties in general, for example, the Schubert variety for $G = \mathrm{PGL}_2$ and quasi-minuscule $\bar{\mu}$ is non-normal if $\mathrm{char}(k) = 2$ (see [13]).

If G is non-split, then the only pairs with minuscule coweights are $(\mathrm{PU}_{2n}, \omega_1)$ and $(\mathrm{PSO}_{2n+2}, \omega_n)$ (see Remark 4.3). These relate to the cases (1) and (3) of local models of exotic good reduction above. The remaining case (2) corresponds to the case of exotic smoothness.

Our approach to the classification is as follows. We first classify all rationally smooth Schubert varieties, and for this, the nature of x is unimportant. We prove that $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}}$ is rationally smooth if and only if the representation $V_{\bar{\mu}}$ of $(G^\vee)^I$ is weight-multiplicity-free (see Proposition 2.2). For this, we use the ramified geometric Satake correspondence (see [35], [38]). Next, we use Howe's classification of all weight-multiplicity-free representations of simple simply connected groups (Theorem 4.4). Together with our list of all possibilities for the reductive groups $(G^\vee)^I$ for G adjoint and absolutely simple (Lemma 4.2), we are able to establish the list in Theorem 1.1 of all such pairs $(G, \bar{\mu})$ such that $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}}$ is rationally smooth (see Section 4.1).

Since the normalization of Schubert varieties is a finite, birational, universal homeomorphism by [16, Proposition 3.1], the cohomological characterization of rational smoothness (Proposition 2.1) shows that the variety $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}}$ is rationally smooth if and only if its normalization $\tilde{\mathrm{Gr}}_{G,x}^{\leq \bar{\mu}}$ is rationally smooth. In particular, we obtain the same list of rationally smooth *normalized Schubert varieties*. The remaining work is to determine which $\tilde{\mathrm{Gr}}_{G,x}^{\leq \bar{\mu}}$ on this list are smooth. The proof is given in Section 5.1 below. For split groups, we only rely on the case of PGL_2, the

Levi lemma, and the quasi-minuscule cases of [26], and hence, we do not rely on computer-aided calculations. For the non-split case, we rely on a few calculations for classical groups from [1], [29], [31], and [17]. Here we use that $\tilde{\mathrm{Gr}}^{\le\bar\mu}_{G,x}$ is isomorphic to a Schubert variety for a suitable central extension $\tilde{G} \to G$ in order to apply these results.

The most difficult case in our proof is the quasi-minuscule Schubert variety for the *ramified triality*, that is, the non-split form of Spin_8. This case is studied in Section 8, and it is also used by [17, Theorem 1.2] to rule out the possibility of additional cases of exotic good reduction. The ramified triality plays a special role, in that it is not amenable to the methods in [17].

Let us note that the results in the split and $\mathrm{char}(k) = 0$ context (see [9], [26]) are stronger: the smooth locus of $\mathrm{Gr}^{\le\bar\mu}_{G,x}$ is exactly the open stratum $\mathrm{Gr}^{\bar\mu}_{G,x}$. Due to the phenomenon of exotic smoothness, this fails in the non-split case. In Section 5.2, we formulate a conjecture which describes the precise conditions on x needed to ensure that this description of the smooth locus holds.

In light of Theorem 1.2, in order to give a classification of smooth Schubert varieties, it suffices to understand which $\mathrm{Gr}^{\le\bar\mu}_{G,x}$ are normal. We plan to address this question in [13].

2. Rational smoothness of Schubert varieties

Let k be an algebraically closed field, and let $F = k(\!(t)\!)$ denote the Laurent series field. Let G be a connected reductive group over F which splits over a tamely ramified Galois extension F'/F. Denote $I = \mathrm{Gal}(F'/F)$. Let $x \in \mathscr{B}(G,F)$ be a special vertex in the Bruhat–Tits building, and denote by $\mathrm{Gr}_{G,x} := LG/L^+\mathcal{G}_x$ the twisted affine Grassmannian in the sense of [30]. Let $S \subset G$ be a maximal F-split torus such that x belongs to the apartment $\mathscr{A}(G,S,F)$ (see [6, Theorem 7.4.18(i)]). The centralizer $T = Z_G(S)$ is a maximal torus defined over F (because by Steinberg's theorem G is quasi-split). Let $B \subset G$ be a Borel subgroup containing T and defined over F.

We equip the coinvariants $X_*(T)_I$ with the dominance order $\le$ with respect to the échelonnage root system $\breve{\Sigma}$ (see [12]). We denote by $X_*(T)_I^+ \subset X_*(T)_I$ the submonoid of dominant elements. (One can show that $X_*(T) \to X_*(T)_I$ induces a *surjective* map of monoids $X_*(T)^+ \to X_*(T)_I^+$.) For each $\bar\mu \in X_*(T)_I^+$, we have the Schubert variety $\mathrm{Gr}^{\le\bar\mu}_{G,x} \subset \mathrm{Gr}_{G,x}$ and the open orbit embedding $j_{\bar\mu}\colon \mathrm{Gr}^{\bar\mu}_{G,x} \hookrightarrow \mathrm{Gr}^{\le\bar\mu}_{G,x}$. For $\bar\mu, \bar\lambda \in X_*(T)_I^+$, we have $\mathrm{Gr}^{\bar\lambda}_{G,x} \subset \mathrm{Gr}^{\le\bar\mu}_{G,x}$ if and only if $\bar\lambda \le \bar\mu$ in the dominance order (see [34, Corollary 1.8, Proposition 2.8]).

Since the I-action preserves a pinning of $G^\vee$, the group $(G^\vee)^I$ is a possibly disconnected reductive $\bar{\mathbb{Q}}_\ell$-group (see [11, Proposition 4.1(a)]). There is a unique (up to isomorphism) irreducible representation $V_{\bar\mu}$ of $(G^\vee)^I$ with highest $(T^\vee)^I$-weight

$\bar{\mu}$ (see [38, Lemma 4.10], [35, Corollary A.9], [12, Section 5.2]). For each $\bar{\lambda} \le \bar{\mu}$, we denote by $d_{\bar{\mu}}(\bar{\lambda})$ the dimension of the $\bar{\lambda}$-weight space $V_{\bar{\mu}}(\bar{\lambda})$.

Fix a prime number ℓ coprime to char(k). Denote $d_{\bar{\mu}} = \dim(\mathrm{Gr}_{G,x}^{\le\bar{\mu}})$. The intersection complex $\mathrm{IC}_{\bar{\mu}} = j_{\bar{\mu},!*}\bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}]$ corresponds under the ramified geometric Satake isomorphism (see [35], [38]) to the irreducible $\bar{\mathbb{Q}}_\ell$-representation $V_{\bar{\mu}}$ of $(G^\vee)^I$.

Recall that an irreducible variety Y of dimension d over k is called *ℓ-rationally smooth* if, for every point $y \in Y(k)$ with closed immersion $i_y\colon \mathrm{Spec}(k) \hookrightarrow Y$, there is an isomorphism

$$i_y^! \bar{\mathbb{Q}}_\ell \cong \bar{\mathbb{Q}}_\ell[-2d]$$

in the derived category $D_c^b(\{y\}, \bar{\mathbb{Q}}_\ell)$. This notion coincides with the one given in [19, Definition A.1], since $\mathbb{H}_y^n(Y, \bar{\mathbb{Q}}_\ell) := \mathbb{H}^n(Y, i_{y,*}i_y^!\bar{\mathbb{Q}}_\ell) = H^n(i_y^! \bar{\mathbb{Q}}_\ell)$. Further, we say $y \in Y(k)$ is an ℓ-rationally smooth point of Y if y is contained in an ℓ-rationally smooth Zariski-open subset of Y. Therefore, by definition the ℓ-rationally smooth locus is open in Y. It is clear that every smooth variety is ℓ-rationally smooth. But there exist many non-smooth, but ℓ-rationally smooth varieties.

We use the following characterization of ℓ-rational smoothness which was explained to us by David Hansen. Let $p : Y \to \mathrm{Spec}(k)$ be the structure morphism, and consider the Verdier dualizing complex $\omega_Y := p^!\bar{\mathbb{Q}}_\ell$. Denote by $\mathbb{D}_Y(\mathcal{F}) = R\mathcal{H}om_{D_c^b(Y)}(\mathcal{F}, \omega_Y)$, where $\mathcal{F}$ belongs to $D_c^b(Y)$, the derived category of bounded constructible $\bar{\mathbb{Q}}_\ell$-complexes on Y. It follows that $\omega_Y = \mathbb{D}_Y(\bar{\mathbb{Q}}_\ell)$.

PROPOSITION 2.1 (Hansen)
The following statements are equivalent:
(i) *Y is ℓ-rationally smooth,*
(ii) $\omega_Y \simeq \bar{\mathbb{Q}}_\ell[2d]$,
(iii) $\mathrm{IC}_Y \simeq \bar{\mathbb{Q}}_\ell[d]$.

Proof
The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are straightforward. We abbreviate by writing $A := \bar{\mathbb{Q}}_\ell$. For (i) $\Rightarrow$ (ii), using $\mathbb{D}_Y(A) = \omega_Y$ we note that, for any closed point y, the stalk $i_y^! A$ is the dual of $i_y^*\omega_Y$. Thus, by (i), the complex ω_Y is concentrated in degree $-2d$ and $(H^{-2d}(\omega_Y))_y \simeq A$. This forces any choice of non-zero map $A[2d] \to \omega_Y$ to be an isomorphism. Note that such non-zero maps exist because $\mathrm{Hom}_{D_c^b(Y)}(A[2d], \omega_Y) = \mathbb{H}^{-2d}(Y, \omega_Y)$ is dual to $\mathbb{H}_c^{2d}(Y, A) \simeq A$.

For (ii) $\Rightarrow$ (iii), we can choose maps $A[d] \to \mathrm{IC}_Y \to \omega_Y[-d]$ which are isomorphisms on a dense open subset. (Choose any non-zero map $A[d] \to \mathrm{IC}_Y$ using that $\mathrm{Hom}_{D_c^b(Y)}(A[d], \mathrm{IC}_Y) = \mathbb{H}^{-d}(Y, \mathrm{IC}_Y)$ is dual to $\mathbb{H}_c^d(Y, \mathrm{IC}_Y) \simeq A$,[1] and choose $\mathrm{IC}_Y \to \omega_Y[-d]$ by taking the Verdier dual of the first map.) Now (ii) guarantees that $A[d] \cong \omega_Y[-d]$ is a perverse sheaf, and the aforementioned maps split $A[d]$ off as a direct summand of IC_Y. Since IC_Y is a simple perverse sheaf, this implies (iii). ☐

The following proposition is proved using the ramified geometric Satake correspondence (see [35], [38]) as well as elaborations on it such as [38, Theorem 5.1].

Recall that the ramified geometric Satake equivalence provides an equivalence of Tannakian categories

$$\mathrm{Perv}_{L^+\mathcal{G}}(\mathrm{Gr}_{G,x}) \simeq \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}\big((G^\vee)^I\big),$$

under which the intersection complex $\mathrm{IC}_{\bar{\mu}}$ corresponds to the representation $V_{\bar{\mu}}$. The left-hand side is the category of $L^+\mathcal{G}$-equivariant perverse sheaves on $\mathrm{Gr}_{G,x}$ equipped with the tensor structure given by the convolution product and the fiber functor given by global cohomology. The ℓ-rational smoothness of $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ is related to the structure of the $(G^\vee)^I$-representation $V_{\bar{\mu}}$ as follows.

PROPOSITION 2.2

The following are equivalent.

(i) *The Schubert variety* $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ *is ℓ-rationally smooth.*

(ii) *The intersection complex* $\mathrm{IC}_{\bar{\mu}}$ *is isomorphic to the constant sheaf* $\bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}]$.

(iii) *One has* $d_{\bar{\mu}}(\bar{\lambda}) = 1$ *for all* $\bar{\lambda} \in X_*(T)_I^+$, $\bar{\lambda} \leq \bar{\mu}$.

In particular, the rational smoothness of $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ *is independent of the choice of ℓ and of the choice of special vertex x. Thus, we replace "ℓ-rationally smooth" from now on by "rationally smooth" for Schubert varieties.*

Proof

Denote by $\mathcal{G} = \mathcal{G}_x$ the special parahoric group scheme. Write $S_{\bar{\mu}} := \mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$, and write $C_{\bar{\lambda}} := \mathrm{Gr}_{G,x}^{\bar{\lambda}}$ for any $\bar{\lambda} \in X_*(T)_I^+$, $\bar{\lambda} \leq \bar{\mu}$. Let $i_{\bar{\lambda}}\colon \{x_{\bar{\lambda}}\} \hookrightarrow S_{\bar{\mu}}$ be the closed immersion of the basepoint $x_{\bar{\lambda}} \in C_{\bar{\lambda}}(k)$ corresponding to $\bar{\lambda}$.

(i) $\Rightarrow$ (ii): This is a special case of Proposition 2.1.

(ii) $\Rightarrow$ (iii): Assume that $\mathrm{IC}_{\bar{\mu}} = \bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}]$. By setting $q = 1$ in [38, Theorem 5.1], we see that the dimension of the total cohomology of $i_{\bar{\lambda}}^*\mathrm{IC}_{\bar{\mu}} = \bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}]$ is equal to $d_{\bar{\mu}}(\bar{\lambda})$ for all $\bar{\lambda} \in X_*(T)_I^+$, $\bar{\lambda} \leq \bar{\mu}$. This is equal to 1, which proves (iii).

[1]For any non-empty open subset $U \subset Y$ with closed complement $Z \subset Y$, the natural map $\mathbb{H}_c^d(U, \mathrm{IC}_Y|_U) \to \mathbb{H}_c^d(Y, \mathrm{IC}_Y)$ is an isomorphism. This follows from $\mathrm{IC}_Y|_Z \in {}^pD^{\leq -1}(Z)$ and the estimate of middle-perverse cohomological amplitude $p_! :\leq d-1$ for $p : Z \to \mathrm{Spec}(k)$ (see [2, Section 4.2.4]). We apply this to any non-empty open subset $U \subset Y$ with $\mathrm{IC}_Y|_U = A[d]$.

(iii) $\Rightarrow$ (i): Assume that $d_{\bar{\mu}}(\bar{\lambda}) = 1$ for all $\bar{\lambda} \in X_*(T)_I^+$ with $\bar{\lambda} \le \bar{\mu}$. By [38, Theorem 5.1, Proposition 5.4], we have

$$i_{\bar{\lambda}}^* \mathrm{IC}_{\bar{\mu}} = \bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}], \qquad i_{\bar{\lambda}}^! \mathrm{IC}_{\bar{\mu}} = \bar{\mathbb{Q}}_\ell[-d_{\bar{\mu}}]. \tag{2.1}$$

Indeed, [38, Theorem 5.1] implies that $\dim(H^*(i_{\bar{\lambda}}^* \mathrm{IC}_{\bar{\mu}})) = 1$, and then in conjunction with [38, Proposition 5.4] we see that $\dim(H^{-d_{\bar{\mu}}}(i_{\bar{\lambda}}^* \mathrm{IC}_{\bar{\mu}})) = 1$, which yields the first formula. The second formula follows from the first by applying Verdier duality.

Since any point in $S_{\bar{\mu}}(k)$ lies in the $L^+\mathcal{G}$-orbit of some basepoint $x_{\bar{\lambda}} \in C_{\bar{\lambda}}(k)$, this implies that $\mathrm{IC}_{\bar{\mu}} = \mathcal{F}[d_{\bar{\mu}}]$, where $\mathcal{F} := H^{-d_{\bar{\mu}}}(\mathrm{IC}_{\bar{\mu}})$ is an $L^+\mathcal{G}$-equivariant constructible $\bar{\mathbb{Q}}_\ell$-sheaf. Here we are using the principle that if a complex $K \in D_c^b(S_{\bar{\mu}}, \bar{\mathbb{Q}}_\ell)$ is cohomologically supported in degree $n \in \mathbb{Z}$, then $K = H^n K[-n]$ in $D_c^b(S_{\bar{\mu}}, \bar{\mathbb{Q}}_\ell)$.

Hence, to prove that $S_{\bar{\mu}}$ is rationally smooth it is by (2.1) enough to prove that

$$\mathrm{IC}_{\bar{\mu}} = \bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}] \tag{2.2}$$

or equivalently $\mathcal{F} = \bar{\mathbb{Q}}_\ell$. Let D denote the derived category $D_c^b(S_{\bar{\mu}}, \bar{\mathbb{Q}}_\ell)$, and write $\mathbb{H}(K)$ for the global cohomology of an object $K \in D$. We have

$$\mathrm{Hom}_D\big(\bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}], \mathrm{IC}_{\bar{\mu}}\big) = \mathbb{H}^{-d_{\bar{\mu}}}(\mathrm{IC}_{\bar{\mu}}) = \mathbb{H}^0(\mathcal{F}), \tag{2.3}$$

and this vector space corresponds under geometric Satake to the 1-dimensional lowest weight space of $V_{\bar{\mu}}$. We claim that any vector $v \in \mathbb{H}^{-d_{\bar{\mu}}}(\mathrm{IC}_{\bar{\mu}})\backslash\{0\}$ induces an isomorphism

$$\iota_v : \bar{\mathbb{Q}}_\ell[d_{\bar{\mu}}] \xrightarrow{\sim} \mathrm{IC}_{\bar{\mu}} = \mathcal{F}[d_{\bar{\mu}}]. \tag{2.4}$$

The map ι_v is necessarily an isomorphism on a dense open subset by (2.1). Hence, the kernel of $\iota_v[-d_{\bar{\mu}}]$ is a subsheaf of $\bar{\mathbb{Q}}_\ell$ which is supported on a nowhere-dense closed subset; any such subsheaf of $\bar{\mathbb{Q}}_\ell$ is zero. Therefore, ι_v is an injective morphism of constructible abelian sheaves, which is an isomorphism on the stalks at all closed points, since by (2.1) $\mathcal{F}$ has 1-dimensional stalks everywhere. This implies that ι_v is an isomorphism and completes the proof. $\square$

Remark 2.3

If G is split, there is a sharper stratum-by-stratum version of Proposition 2.2. Suppose $\lambda, \mu \in X_*(T)^+$ satisfy $\lambda \le \mu$. Then x_λ belongs to the rationally smooth locus of S_μ if and only if $d_\mu(\lambda) = 1$. This is well-known, but for completeness we explain the proof here. Let W (resp., W_x, resp., $\mathcal{F}\ell_G$) denote the Iwahori–Weyl group (resp., finite Weyl group, resp., affine flag variety) for G relative to the special vertex x and an alcove $\mathbf{a}$ containing x in its closure. Let $w_\lambda \in W$ denote the unique longest element in $W_x\lambda(t)W_x$. Let $\tilde{x}_\lambda \in \mathcal{F}\ell_G$ denote the basepoint in the Iwahori orbit corresponding to w_λ. Then as $\pi : \mathcal{F}\ell_G \to \mathrm{Gr}_{G,x}$ is represented by a smooth surjective $L^+\mathcal{G}$-equivariant morphism, x_λ belongs to the rationally smooth locus of S_μ if and only if $\tilde{x}_\lambda$ belongs to the rationally smooth locus of the Schubert variety $S_{w_\mu} := \pi^{-1}(S_\mu)$ in $\mathcal{F}\ell_G$. By [19, Theorem A.2], this is equivalent to the triviality of certain Kazhdan–Lusztig polynomials, namely, $P_{w',w_\mu}(q) = 1$ for all $w' \in W$ with $w_\lambda \le w' \le w_\mu$. (In [19, Theorem A.2], this is proved for Schubert varieties in the classical flag variety for a split group, but the proof carries over to the affine flag varieties.) This is equivalent to the single equality $P_{w_\lambda,w_\mu}(q) = 1$ (e.g., [4, Theorem 6.2.10], using that all $P_{u,v}(q)$ have non-negative coefficients for $u, v \in W$ by [20]). Since $P_{u,v}(0) = 1$ (e.g., [4, Lemma 6.1.9]), the equality is equivalent to $P_{w_\lambda,w_\mu}(1) = 1$. Finally, this is equivalent to $d_\mu(\lambda) = 1$ by Lusztig's multiplicity formula $P_{w_\lambda,w_\mu}(1) = d_\mu(\lambda)$ ([24, Theorem 6.1]). (Because Kazhdan–Lusztig polynomials $P_{x,y}(q^{1/2}) \in \mathbb{Z}[q^{1/2}]$ attached to Hecke algebras with unequal parameters are not known to belong to $\mathbb{Z}_{\ge 0}[q]$ (see [25]), it is not clear that the same argument can be used to handle quasi-split but non-split groups.) We remark that Berenstein and Zelevinsky in [3] have classified, for any connected reductive complex group, all pairs of weights (μ, λ) satisfying $d_\mu(\lambda) = 1$.

3. The classification for reductive groups: Passing to adjoint groups

We proceed with the notation of Section 2. Let $G \to G_{\mathrm{ad}}$ be the canonical map to the adjoint group, and denote by $T_{\mathrm{ad}} \subset B_{\mathrm{ad}}$ the image of $T \subset B$. The image of the special vertex x under $\mathscr{B}(G, F) \to \mathscr{B}(G_{\mathrm{ad}}, F)$ defines a special vertex x_{ad}. The map $G \to G_{\mathrm{ad}}$ extends to a map of parahoric $\mathcal{O}_F$-groups $\mathcal{G}_x \to \mathcal{G}_{x_{\mathrm{ad}}}$. By the functoriality of the loop group construction, we obtain a map $LG \to LG_{\mathrm{ad}}$ (resp., $L^+\mathcal{G}_x \to L^+\mathcal{G}_{x_{\mathrm{ad}}}$) and, hence, a map on twisted affine Grassmannians $\mathrm{Gr}_{G,x} \to \mathrm{Gr}_{G_{\mathrm{ad}},x_{\mathrm{ad}}}$.

Further, $T \to T_{\mathrm{ad}}$ defines a map $X_*(T)_I \to X_*(T_{\mathrm{ad}})_I$ which sends $X_*(T)_I^+$ to $X_*(T_{\mathrm{ad}})_I^+$. For $\bar{\mu} \in X_*(T)_I^+$, we denote by $\bar{\mu}_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})_I^+$ its image. Since the Schubert varieties are defined as the scheme-theoretic image of the orbit map, we get a natural morphism of k-schemes

$$\mathrm{Gr}_{G,x}^{\le\bar{\mu}} \longrightarrow \mathrm{Gr}_{G_{\mathrm{ad}},x_{\mathrm{ad}}}^{\le\bar{\mu}_{\mathrm{ad}}}. \tag{3.1}$$

PROPOSITION 3.1

The map (3.1) is a finite, birational, universal homeomorphism. In particular, it induces an equivalence of étale sites and an isomorphism on normalizations.

Proof

This is a special case of [16, Proposition 3.5]. The equivalence on étale sites follows from their topological invariance [36, 04DY]. □

Now assume that $G = G_{\mathrm{ad}}$ is adjoint. Then there is a finite index set J, and an isomorphism of F-groups

$$G = \prod_{j \in J} \mathrm{Res}_{F_j/F}(G_j), \tag{3.2}$$

where each F_j/F is a finite separable field extension and G_j is an absolutely simple adjoint F_j-group. The condition on G of being tamely ramified implies that each G_j is tamely ramified (and likewise for F_j/F, but this is not important as we will see). This induces an identification of buildings $\mathscr{B}(G, F) = \prod_{j \in J} \mathscr{B}(G_j, F_j)$ compatible with the simplicial structure (see [14, Proposition 4.6]). Under this identification we get $x = (x_j)_{j \in J}$, where each vertex $x_j \in \mathscr{B}(G_j, F_j)$ is special.

Further, we can write $T = \prod_{j \in J} \mathrm{Res}_{F_j/F}(T_j)$, and likewise for B (see [14, Lemma 4.2]). Note that the splitting field F' of G contains each F_j, and we define $I_j := \mathrm{Gal}(F/F_j)$. By Shapiro's lemma (see [14, Lemma 4.1]), we get $X_*(T)_I = \prod_{j \in J} X_*(T_j)_{I_j}$ compatible with $X_*(T)_I^+ = \prod_{j \in J} X_*(T_j)_{I_j}^+$. For each $\bar{\mu} \in X_*(T)_I^+$, we denote $\bar{\mu} = (\bar{\mu}_j)_{j \in J}$ with $\bar{\mu}_j \in X_*(T_j)_{I_j}^+$.

LEMMA 3.2
Under (3.2) there is an identification of affine Grassmannians

$$\mathrm{Gr}_{G,x} = \prod_{j \in J} \mathrm{Gr}_{G_j, x_j},$$

under which the Schubert varieties (resp., their normalizations) for each $\bar{\mu} = (\bar{\mu}_j)_{j \in J}$ correspond to each other.

Proof
It is enough to treat the following two cases separately.

Products: If $G = G_1 \times G_2$ is a direct product of two F-groups, then we have $\mathrm{Gr}_{G,x} = \mathrm{Gr}_{G_1,x_1} \times \mathrm{Gr}_{G_2,x_2}$, which is obvious. Also the equality $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}} = \mathrm{Gr}_{G_1,x_1}^{\leq \bar{\mu}_1} \times \mathrm{Gr}_{G_2,x_2}^{\leq \bar{\mu}_2}$ is easy to prove using that the product of (geometrically) reduced k-schemes is reduced (see [36, 035Z(2)]). Likewise, the equality holds on normalizations using that the product of (geometrically) normal k-schemes is normal (see [36, 06DG]).

Restriction of scalars: Let $G = \mathrm{Res}_{F'/F}(G')$, where F'/F is a finite separable extension and G' is an F'-group. By [14, Proposition 4.7], we have $\mathscr{G}_x = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F}(\mathscr{G}'_{x'})$, where we use the identification $\mathscr{B}(G, F) = \mathscr{B}(G', F')$. Now choose[2] a uniformizer $u \in \mathcal{O}_{F'}$. Since k is algebraically closed, we have $\mathcal{O}_{F'} = k[\![u]\!]$

[2]The identification of twisted affine Grassmannians is independent of this choice as all loop groups can be defined without choosing uniformizers (see [34, Section 2]).

(resp., $F' = k(\!(u)\!)$). For any k-algebra R, we have $R[\![t]\!] \otimes_{\mathcal{O}_F} \mathcal{O}_{F'} = R[\![u]\!]$ (resp., $R(\!(t)\!) \otimes_F F' = R(\!(u)\!)$). This gives an equality on loop groups $L^+\mathcal{G}_x = L^+\mathcal{G}'_{x'}$ (resp., $LG = LG'$). Hence, there is an equality on twisted affine Grassmannians $\mathrm{Gr}_{G,x} = \mathrm{Gr}_{G',x'}$, and it is obvious that the Schubert varieties (resp., their normalizations) correspond to each other. □

By combining Proposition 3.1 with Lemma 3.2, it is obvious how to extend our classification from the absolutely simple adjoint case to the case of general tamely ramified connected reductive groups. From the discussion, we also see that we can relax the condition on G of being tamely ramified to the condition that each absolutely simple adjoint factor G_i is tamely ramified. In particular, our classification includes all cases where $\mathrm{char}(k) \geq 5$ (see the beginning of the next section).

4. Weight-multiplicity-free representations

We proceed with the notation of Section 2, and assume further that G is adjoint and absolutely simple. Then the splitting field F'/F is of degree $[F' : F] = 1, 2$, or 3 (see [37, Section 4]). So if G is non-split (i.e., $[F' : F] = 2$ or 3), then the assumption of being tame excludes only 2 or 3 from being the residue characteristic $[F' : F]$.

We are interested in classifying all irreducible representations $V_{\bar{\mu}}$ of $(G^\vee)^I$ such that $d_{\bar{\mu}}(\bar{\lambda}) = 1$ for all $\bar{\lambda} \in X_*(T)_I^+$, $\bar{\lambda} \leq \bar{\mu}$. These representations are called *weight-multiplicity-free*.

LEMMA 4.1
The group $(G^\vee)^I$ is a connected reductive $\bar{\mathbb{Q}}_\ell$-group which is simple and semi-simple. Furthermore, it is simply connected except in the case $G^\vee = \mathrm{SL}_{2n+1}$, $n \geq 1$, with a non-trivial I-action, in which case $(G^\vee)^I \cong \mathrm{SO}_{2n+1}$.

Proof
Since the I-action preserves a pinning of $G^\vee$, this follows from Proposition A.1, taking $\kappa = \bar{\mathbb{Q}}_\ell$. □

LEMMA 4.2
The following list gives all possibilities for $(G^\vee)^I$:

(i) $[F' : F] = 1$*: G split; $(G^\vee)^I = G^\vee$.*

(ii) $[F' : F] = 2$*:*

   (a) *$G = \mathrm{PU}_{2n}$, $n \geq 3$, name B-C_n; $(G^\vee)^I = \mathrm{Sp}_{2n}$, type C_n.*

   (b) *$G = \mathrm{PU}_{2n+1}$, $n \geq 1$, name C-BC_n; $(G^\vee)^I = \mathrm{SO}_{2n+1}$, type B_n.*

   (c) *$G = \mathrm{PSO}_{2n+2}$, $n \geq 2$, name C-B_n; $(G^\vee)^I = \mathrm{Spin}_{2n+1}$, type B_n.*

   (d) *$G = {}^2E_{6,4}^2$, ramified E_6, name F_4^I; $(G^\vee)^I = F_4$.*

(iii) $[F':F]=3$: $G={}^3D_{4,2}$, *ramified triality, name* G_2^I; $(G^\vee)^I = G_2$.
Hence, $(G^\vee)^I$ *is simply connected except in case (ii.b) where the connection index is* 2*. In this case,* $(G^\vee)^I = \mathrm{SO}_{2n+1}$ *is adjoint.*

Here, the name refers to the name given by Tits in [37, Table 4.2], and the type refers to the type of the connected reductive group $(G^\vee)^I$. Tables containing essentially this content are contained in [17, Section 5.2], but here we describe the groups in classical terms, including the isogeny type.

Remark 4.3
Case (ii.b) shows that there is no non-zero $\breve{\Sigma}$-minuscule coweight for the non-split group PU_{2n+1}. The fact that SO_{2n+1} is adjoint means that every weight is in the root lattice. This translates to $X_*(T)_I = \mathbb{Z}[\breve{\Sigma}^\vee]$, which in turn implies that the affine Grassmannian for PU_{2n+1} is connected. Similarly, one proves that the affine Grassmannian for a non-split absolutely simple adjoint group G is always connected, except in cases (ii.a) and (ii.c), where it has two connected components. This also shows that only these cases admit minuscule elements: checking the tables in [5, Planche II and III] identifies ω_1 in (ii.a) and ω_n in (ii.c) as the minuscule elements.

Proof
Checking the tables in [37, Section 4] for residually split groups gives the above list. We make the following remarks. Tits's tables list the échelonnage root system attached to G/F. For example, the group named B-C_n is a ramified unitary group PU_{2n} and has échelonnage root system $\breve{\Sigma}$ of type B_n. The group $(G^\vee)^I$ has type dual to $\breve{\Sigma}$ (see [12, Section 5.1]) and, thus, has type C_n, and since it is simply connected (Lemma 4.1), we see that $(G^\vee)^I = \mathrm{Sp}_{2n}$. The other cases are handled similarly. ☐

The following theorem is proven in Howe's article [18, Theorem 4.6.3], and we refer the reader to its introduction for further references on the subject. A classification of multiplicity 1 *primitive* pairs $\bar\lambda < \bar\mu$ is also given in [3]: these are the pairs such that $d_{\bar\mu}(\bar\lambda) = 1$ and every simple root for $(G^\vee)^I$ appears at least once in the difference $\bar\mu - \bar\lambda$; from this one may classify all pairs such that $d_{\bar\mu}(\bar\lambda) = 1$.

THEOREM 4.4
Let $\bar\mu \in X_*(T)_I^+$, *and denote by* X_n *the type of* $(G^\vee)^I$, *where* $n \geq 1$ *is the rank of* $(G^\vee)^I$. *Then the* $(G^\vee)^I$*-representation* $V_{\bar\mu}$ *is weight-multiplicity-free if and only if the pair* $(X_n, \bar\mu)$ *appears in the following list:*
- *any type* X_n, *and* $\bar\mu$ *minuscule;*
- *type* A_1, *and* $\bar\mu$ *arbitrary;*

- *type A_n, $n \geq 2$, and $\bar{\mu} = l \cdot \omega_i$ for $i \in \{1, n\}$ and $l \geq 2$;*
- *type B_n, $n \geq 2$, and $\bar{\mu}$ quasi-minuscule;*
- *type C_3, and $\bar{\mu} = \omega_3$ (not quasi-minuscule);*
- *type G_2, and $\bar{\mu}$ quasi-minuscule.*

Remark 4.5

It is interesting to observe that only in type A_n are there infinitely many weight-multiplicity-free representations. Also, outside of type A cases and the single type C_3 case, the following implication holds: "if $V_{\bar{\mu}}$ is weight-multiplicity-free, then $\bar{\mu}$ is (quasi-)minuscule."

4.1. Proof of Theorem 1.1

This is a combination of Proposition 2.2 and Theorem 4.4 with the list in Lemma 4.2. Indeed, these results make no reference to the choice of special vertex $x \in \mathscr{B}(G, F)$, which we therefore do not specify. Drop it from the notation for the rest of the proof. By Proposition 2.2, the Schubert variety $\mathrm{Gr}_G^{\leq \bar{\mu}}$ is rationally smooth if and only if the $(G^\vee)^I$-representation $V_{\bar{\mu}}$ is weight-multiplicity-free. Theorem 4.4 gives a complete list of all pairs $((G^\vee)^I, \bar{\mu})$ such that $V_{\bar{\mu}}$ is weight-multiplicity-free. Clearly, if $\bar{\mu}$ is minuscule, there are no restrictions on the group. Assume now that $\bar{\mu}$ is not minuscule. If G is split, then Theorem 4.4 directly applies to give the rationally smooth cases listed in our theorem. If G is not split, then we use Lemma 4.2 to translate Theorem 4.4 back in terms of the group G. Note that the group PU_3 appears from Lemma 4.2(ii.b) using the exceptional isomorphism of Lie types $B_1 = A_1$. This proves the theorem.

Remark 4.6

When $\bar{\mu}$ is quasi-minuscule, it is known that the dimension of the zero-weight space $V_{\bar{\mu}}(0)$ is the number of short nodes in the Dynkin diagram for the group $(G^\vee)^I$. From this, Lemma 4.2, and Proposition 2.2, one can easily determine the groups G whose quasi-minuscule Schubert variety is rationally smooth, without invoking Theorem 4.4.

5. Absolutely special vertices

Temporarily we assume G is any connected reductive group over an arbitrary field F endowed with a non-trivial discrete valuation, and F'/F is a finite separable extension splitting G. Following [37] we assume F is complete and its residue field is perfect.

Definition 5.1

A vertex $x \in \mathscr{B}(G, F)$ is called *absolutely special* if its image under the simplicial embedding $\mathscr{B}(G, F) \hookrightarrow \mathscr{B}(G, F')$ is a special vertex.

Note that this notion is independent of the choice of the splitting field F'/F.

LEMMA 5.2
Absolutely special vertices exist in every quasi-split group G and are special.

Proof
This is modeled on Tits's proof of the existence of hyperspecial points for unramified groups (see [37, p. 36]). Since G is quasi-split, there exist $S \subset T \subset B$ defined over F as above. We may assume F'/F is Galois, and we write $\Gamma := \mathrm{Gal}(F'/F)$. Let $a_1, \dots, a_l$ denote a Γ-stable basis of B-simple absolute roots for (G,T). Clearly Γ acts on the apartment $\mathscr{A}(G,T,F') \subset \mathscr{B}(G,F')$ and also on the set of affine roots $\Phi_{\mathrm{aff}} := \Phi_{\mathrm{aff}}(G,T,F')$ by construction (see [37, Section 1.6]). We claim that there is a Γ-stable set $\{\alpha_1, \dots, \alpha_l\} \subset \Phi_{\mathrm{aff}}$ such that the vector part of each α_j is a_j.

Indeed, suppose we are given a Γ-orbit $\{a_{i_1}, \dots, a_{i_r}\}$ of simple roots. We change notation and write these as $a_1, \dots, a_r$. Choose arbitrarily an $\alpha_1 \in \Phi_{\mathrm{aff}}$ whose vector part is a_1. Then for each a_j, $1 \le j \le r$, choose $\gamma_j \in \Gamma$ such that $\gamma_j(a_1) = a_j$, and set $\alpha_j := \gamma_j(\alpha_1)$. This is well-defined because if $\gamma \in \Gamma$ fixes a_1, then it fixes α_1 by the definition of the Γ-action.

Now recall from [7, Section 4.1.2] that any relative root in $X^*(S)$ for G is the restriction of a root in $X^*(T)$ for $G_{F'}$. Hence, any Γ-fixed point in the solution set of $\alpha_1 = \alpha_2 = \dots = \alpha_l = 0$ is the desired absolutely special vertex of $\mathscr{A}(G,T,F')^\Gamma = \mathscr{A}(G,S,F)$. This shows existence and also that any absolutely special vertex is special. □

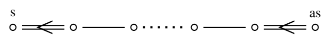
Note that the above result holds even if G is not tamely ramified. Now we continue with the notation and hypotheses of Section 4. In particular, we are again assuming G is adjoint, absolutely simple, and tamely ramified over F.

LEMMA 5.3
Assume that G is not isomorphic to PU_{2n+1} *for any $n \ge 1$. Then all special vertices in $\mathscr{B}(G,F)$ are conjugate under $G(F)$ and, in particular, are absolutely special.*

Proof
The last assertion follows from Lemma 5.2 using that the property of being absolutely special is invariant under $G(F)$-conjugacy. It remains to show that all special vertices are conjugate. This is implicitly contained in [37, Section 2.5], and we add some details. Clearly, it is enough to show that all special points in the apartment $\mathscr{A} := \mathscr{A}(G,T,F)$ are conjugate. Fix a special point $0 \in \mathscr{A}$, and identify $\mathscr{A} = X_*(T)_{I,\mathbb{R}}$. We claim that $X_*(T)_I \subset \mathscr{A}$ is exactly the subset of special points. The claim implies the lemma because the action of $T(F)$ on $\mathscr{A}$ is via translation under

$T(F)/\mathcal{T}^o(\mathcal{O}_F) \simeq X_*(T)_I$, and thus, $T(F)$ permutes all special points. It remains to show the claim. By [5, VI.2.2, Proposition 3], the special points in $\mathscr{A}$ are identified with the weight lattice $P(\check{\Sigma}^\vee)$ for the échelonnage roots. In general, we have an inclusion $X_*(T)_I = X^*((T^\vee)^I) \subset P(\check{\Sigma}^\vee)$, which is an equality if and only if $(G^\vee)^I$ is simply connected. But this holds true by Lemma 4.2 because we excluded case (ii.b) by assumption. This proves the lemma. □

Now assume $G = \mathrm{PU}_{2n+1}$ for $n \geq 1$. Then up to $G(F)$-conjugation there are two kinds of special vertices. If $n \geq 2$, the local Dynkin diagram is of the form

$C\text{-}BC_n$ 

(5.1)

If $n = 1$, then the local Dynkin diagram is drawn in (7.1) below and looks similar. Here F'/F is a ramified quadratic extension, and the vector space $V := (F')^{2n+1}$ is equipped with a non-degenerate split Hermitian form as in [37, Section 3.11] (see also [31, Section 1.2.1]). The vertex labeled "as" is absolutely special and corresponds to a self-dual $\mathcal{O}_{F'}$-lattice in V, whereas the vertex labeled "s" is special, but not absolutely special, and corresponds to an almost modular lattice (see [31, Section 1.2.3(a)]). Here an $\mathcal{O}_{F'}$-lattice $\Lambda \subset V$ is called self-dual if $\Lambda^\perp = \Lambda$, where $(\text{-})^\perp$ denotes the dual lattice with respect to the Hermitian form. The lattice is called almost modular if $u \cdot \Lambda \subset \Lambda^\perp$ with colength 1, where $u \in F'$ is a uniformizer.

5.1. Proof of Theorem 1.2

We start with some preliminary remarks. The normalization

$$\tilde{\mathrm{Gr}}_{G,x}^{\leq \bar{\mu}} \to \mathrm{Gr}_{G,x}^{\leq \bar{\mu}} \tag{5.2}$$

is a finite, birational, universal homeomorphism by [16, Proposition 3.1]. In particular, the source of (5.2) is rationally smooth if and only if its target is rationally smooth. We therefore obtain the same list in Theorem 1.1 for rationally smooth *normalized* Schubert varieties. Also we give references below to articles which include explicit calculations for the special fibers of local models (see [1], [17], [29], [31]). To apply these references we need to often replace the adjoint group G by a suitable central extension $\tilde{G} \to G$ such that $\pi_1(\tilde{G}_{\mathrm{der}}) = 0$ (see [17, (2.11)]). Then the Schubert varieties for $\tilde{G}$ map isomorphically onto the normalized Schubert varieties for G (see Proposition 3.1) using the normality of Schubert varieties for $\tilde{G}$ (see [30, Theorem 6.1]). Hence, by [32, Theorem 9.1] the normalized Schubert variety is isomorphic to the special fiber of a suitable local model for $\tilde{G}$, which allows us to use these references. Also we give references below to articles which contain results about the singularity of Schubert varieties (see [9], [26]) over the complex numbers. Here we refer to Section 6 below for the reduction to $k = \mathbb{C}$, which allows us to use these references.

If $\bar{\mu}$ is minuscule or if G is an odd-dimensional ramified unitary group, x is not absolutely special, and $\bar{\mu}$ is quasi-minuscule, then $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is smooth. Indeed, if $\bar{\mu}$ is minuscule, then $\mathrm{Gr}^{\leq\bar{\mu}}_{G,x} = \mathrm{Gr}^{\bar{\mu}}_{G,x}$ is a single orbit and, hence, is smooth, so that its normalization is smooth as well. The other case was observed by the second named author and follows from an explicit calculation (see [1, Prop. 4.16]).

Conversely assume that $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is smooth. Then it is rationally smooth as well and, hence, appears in the list of Theorem 1.1. We need to exclude from that list all the Schubert varieties which are singular. If $\bar{\mu}$ is minuscule, then $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is smooth as argued above. Therefore, we have reduced to the case where $\bar{\mu}$ is not minuscule. In what follows, we list groups according to the type of the dual group $(G^\vee)^I$.

Type A_1, and $\bar{\mu}$ arbitrary: Note that there is an $l \geq 2$ such that $\bar{\mu} = l \cdot \omega_1$. If $G = \mathrm{PGL}_2$ is split, then $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is singular by [26, Section 5.1] (using Section 6 below to reduce to $k = \mathbb{C}$ here and below; see also [27, Theorem 9.2] for an explicit matrix calculation). These cases are therefore excluded. If G is not split, then according to Lemma 4.2 it is the 3-dimensional quasi-split ramified projective unitary group. Note that only the weights $\bar{\mu} = l \cdot \omega_1$ for *even* $l \geq 2$ appear in this case, because $(G^\vee)^I = \mathrm{SO}_3 \simeq \mathrm{PGL}_2$ is not simply connected. By Section 6 and Proposition 7.1 below, the normalized Schubert variety $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is smooth only in the case where x is special but not absolutely special, and $\bar{\mu} = 2\omega_1$ is quasi-minuscule, and so only this case is not excluded.

Type A_n, $n \geq 2$, and $\bar{\mu} = l \cdot \omega_i$, $i \in \{1, n\}$, $l \geq 2$: By Lemma 4.2, the group G is split, and hence, $G = \mathrm{PGL}_{n+1}$. The singularity of these normalized Schubert varieties is a particular case of [9] and [26] (again reduce to $k = \mathbb{C}$). Let us be more specific. The inverse transpose morphism $G \to G$, $g \mapsto (g^{-1})^t$ induces an isomorphism on affine Grassmannians flipping the connected components and, in particular, restricts to an isomorphism $\tilde{\mathrm{Gr}}^{\leq l\cdot\omega_1}_{G,x} \simeq \tilde{\mathrm{Gr}}^{\leq l\cdot\omega_n}_{G,x}$ for all $l \geq 2$. We are thus reduced to the case where $\bar{\mu} = l \cdot \omega_1$, $l \geq 2$. Also by our general remarks above, we can identify the normalized Schubert variety with an ordinary Schubert variety in GL_{n+1}. We can therefore assume that $G = \mathrm{GL}_{n+1}$ and $\tilde{\mathrm{Gr}}^{\leq l\cdot\omega_1}_{G,x} = \mathrm{Gr}^{\leq l\cdot\omega_1}_{G,x}$.

In this case, we consider the element $\bar{\lambda} = (l-1)\cdot\omega_1 + \omega_2$. Then $\bar{\mu} - \bar{\lambda}$ is a simple coroot. By the Levi lemma of [26, 3.4], the boundary of $\mathrm{Gr}^{\leq\bar{\lambda}}_{G,x} \subset \mathrm{Gr}^{\leq\bar{\mu}}_{G,x}$ is smoothly equivalent to the boundary of a Schubert variety for GL_2 and, hence, is singular by the type A_1 case above. Therefore, all the cases in this paragraph are excluded.

Type B_n, $n \geq 2$, and $\bar{\mu}$ quasi-minuscule: If G is split, then $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is singular (see [26, Sections 2.9, 2.10]). If G is not split, then according to Lemma 4.2 we are left with the cases (ii.b) and (ii.c) for any $n \geq 2$. In case (ii.b), the group G is a quasi-split unitary group on a $2n+1$-dimensional Hermitian space. If x is absolutely special, then $\tilde{\mathrm{Gr}}^{\leq\bar{\mu}}_{G,x}$ is singular at the basepoint by [29, Theorem 4.5, Lemma 4.7] (see also [17, Section 9, 3.b)]). This case is therefore excluded. If x is special, but not absolutely special, then $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}}$ is smooth at the basepoint by [1, Proposition 4.16] (see also [17, Section 9, 3.a)]). This case is therefore not excluded.

In case (ii.c), the group G is a ramified orthogonal group on a $2n+2$-dimensional space. The normalized Schubert variety $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}}$ is singular by [17, Section 9, 3.c)]. Note that by Lemma 5.3 all special vertices are conjugate under $G(F)$ so that we only need to consider the choice of x that is handled in [17, Section 9, 3.c)]. This case is excluded.

Type C_3, and $\bar{\mu} = \omega_3$: If G is split, then $G = \mathrm{PSp}_6$. We have $\bar{\mu} = \omega_3 > \omega_1 =: \bar{\lambda}$, and $\bar{\lambda}$ is minuscule. Checking the tables in [5], we see that $\bar{\lambda}$ is equal to zero on the root subsystem $\mathrm{supp}(\bar{\mu} - \bar{\lambda}) = \{\alpha_2^\vee, \alpha_3^\vee\}$, viewing the latter as simple coroots in $\breve{\Sigma}^\vee$. Also by our general remarks above, we can identify the normalized Schubert variety with an ordinary Schubert variety in GSp_6. We can therefore assume that $G = \mathrm{GSp}_6$ and $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}} = \mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$. By the Levi lemma of [26, 3.4], the boundary of $\mathrm{Gr}_{G,x}^{\leq\bar{\lambda}} \subset \mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ is smoothly equivalent to the quasi-minuscule singularity of type $C_2 = B_2$, which is singular by the previous case. This case is excluded.

If G is not split, then according to Lemma 4.2 we are in case (ii.a) for $n = 3$, that is, G is a quasi-split ramified unitary group on a 6-dimensional Hermitian space. The normalized Schubert variety $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}}$ is isomorphic to the special fiber of the local model of the unitary similitudes group for signature $(3,3)$, and this is singular by [17, Section 9, 2)] (see the equations given in [31, (5.6)]). This case is excluded.

Type G_2, and $\bar{\mu}$ quasi-minuscule: If G is split and μ is quasi-minuscule, then [26, Section 2.9] gives a conceptual proof showing that the basepoint e in $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}}$ is singular. (This does not use the Kumar criterion.) If G is not split, then G is the ramified triality, and the basepoint is again singular. This is the most difficult case in our classification, and it is treated in Section 8 below (see Theorem 8.1). In both the split and non-split cases, we are using Section 6 for the reduction to the case $k = \mathbb{C}$. Thus, the normalized Schubert variety $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}}$ is singular at the basepoint, and hence, it is excluded. This finishes the proof of the classification and, hence, the proof of Theorem 1.2.

5.2. A conjecture on minimal degenerations

Theorem 1.2 classifies the normalized Schubert varieties in twisted affine Grassmannians which are smooth. If the group G is split and $\mathrm{char}(k) = 0$, then the result proven in [9] and [26] is stronger. In this case, every (normalized) Schubert variety is singular along its boundary, that is, $\mathrm{Gr}_{G,x}^{\bar{\mu}} = \tilde{\mathrm{Gr}}_{G,x}^{\bar{\mu}}$ is exactly the smooth locus in $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}} = \tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}}$. As the phenomenon of exotic smoothness shows, this fails in twisted affine Grassmannians for general special vertices.

CONJECTURE 5.4
If x is absolutely special, then the smooth locus of $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar\mu}$ *is precisely* $\tilde{\mathrm{Gr}}_{G,x}^{\bar\mu}$.

In Corollary 6.1 below we give some evidence for this conjecture. The essential difficulty in proving the conjecture consists in handling absolutely simple, non-split groups over $\mathbb{C}(\!(t)\!)$.

Remark 5.5
(i) If Conjecture 5.4 holds for the normalized Schubert varieties, then the same is true for the non-normalized Schubert varieties as well. Indeed, the finite birational universal homeomorphism $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar\mu} \to \mathrm{Gr}_{G,x}^{\leq\bar\mu}$ induces an isomorphism over the smooth locus of the target.
(ii) It would also be interesting to determine the type of singularities which arise. The calculations in Section 8 indicate that these might be different from the minimal degeneration singularities for split groups.

6. Reduction of the remaining cases to $k = \mathbb{C}$

In order to treat the remaining cases in Sections 7 and 8 below, we first reduce the proof that the normalized Schubert varieties in question are singular to the case where $k = \mathbb{C}$. Let $W := W(k)$ be the ring of Witt vectors of k equipped with the natural map $W \to k$. Let $K = \mathrm{Frac}(W)$ be the field of fractions. As the group G is tamely ramified, the twisted affine Grassmannian together with the Schubert varieties lift to W (see [30, Section 7]).

More precisely, there exists a smooth, affine group scheme with connected fibers $\underline{\mathcal{G}} \to W[\![u]\!]$ whose base change $\underline{G}$ to $W(\!(u)\!)$ is reductive and whose base change to $\kappa[\![u]\!]$ for $\kappa = k, K$ is the parahoric group scheme for $\underline{G}_{\kappa(\!(u)\!)}$ attached with the "same" facet (see [32, Corollary 4.2(2)]). We note that $\underline{\mathcal{G}}/W[\![u]\!]$ is a special case of the "parahoric" group schemes constructed in [32, Section 4].

Hence, as in [30, Section 7] there exist a twisted affine Grassmannian $\underline{\mathrm{Gr}}_{G,x}$ defined over W and, for every $\bar\mu \in X_*(T)_I^+$, a normalized Schubert variety $\underline{\tilde{\mathrm{Gr}}}_{G,x}^{\leq\bar\mu}$ such that

$$\underline{\tilde{\mathrm{Gr}}}_{G,x}^{\leq\bar\mu} \otimes k = \tilde{\mathrm{Gr}}_{G,x}^{\leq\bar\mu}. \tag{6.1}$$

Note that we are using the identification $X_*(T)_I = X_*(\underline{T}_K)_{I_K}$ which is compatible with the dominance order, where $\underline{T}$ is the lift of T over W: this is an immediate consequence of the identification of apartments for $\underline{G}_{K(\!(u)\!)}$ and $\underline{G}_{k(\!(u)\!)} = G$ (see [32, (4.2)]). Further, we use the fact that formation of $\underline{\tilde{\mathrm{Gr}}}_{G,x}^{\leq\bar\mu}$ commutes with base change. For simply connected groups this is proved in [30, Proposition 9.11(a) and 9.g], and the case of adjoint groups follows from this by using a standard reduction to affine flag varieties and a translation to the neutral component as, for example, in [16, Proposition 3.1]. The affine Grassmannian in the generic fiber $\underline{\mathrm{Gr}}_{G,x} \otimes K$ is of the same type as the affine Grassmannian in the special fiber $\mathrm{Gr}_{G,x}$, so that $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x} \otimes K$ is the Schubert variety in $\underline{\mathrm{Gr}}^{\le\bar{\mu}}_{G,x} \otimes K$ for the same $\bar{\mu} \in X_*(T)_I$ (using the normality of Schubert varieties in characteristic 0).

As the singular locus in $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x}$ is closed, the generic fiber $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x} \otimes K$ being singular implies that the special fiber $\mathrm{Gr}^{\le\bar{\mu}}_{G,x}$ is singular. We are therefore reduced to the case in which k is an algebraically closed field of characteristic 0. Furthermore, every such group is already defined over $\bar{\mathbb{Q}}(\!(t)\!)$ so that we further reduce to the case $k = \mathbb{C}$. We will assume this whenever convenient in what follows.

With a view toward Conjecture 5.4, we note that if the smooth locus of the generic fiber $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x} \otimes K$ is precisely $\underline{\tilde{\mathrm{Gr}}}^{\bar{\mu}}_{G,x} \otimes K$, then the smooth locus of the special fiber $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x} \otimes k = \tilde{\mathrm{Gr}}^{\le\bar{\mu}}_{G,x}$ is precisely $\tilde{\mathrm{Gr}}^{\bar{\mu}}_{G,x}$. Indeed, the formation of $\underline{\tilde{\mathrm{Gr}}}^{\bar{\lambda}}_{G,x}$ commutes with base change; and hence, for a maximal element $\bar{\lambda} < \bar{\mu}$, any point in the $\bar{\lambda}$-stratum of $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x} \otimes k$ is a specialization of a point in the $\bar{\lambda}$-stratum of $\underline{\tilde{\mathrm{Gr}}}^{\le\bar{\mu}}_{G,x} \otimes K$. Invoking [9] and [26], we thus have proved the following.

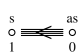
COROLLARY 6.1

(i) *If Conjecture 5.4 holds for $k = \mathbb{C}$, then it holds for general fields k.*

(ii) *If G is split, then Conjecture 5.4 holds.*

7. The 3-dimensional quasi-split ramified unitary groups

Let k be an algebraically closed field with $\mathrm{char}(k) \neq 2$. Let $F = k(\!(t)\!)$, and let F'/F be a quadratic ramified extension. Let $G = \mathrm{PU}_3$, and let $x \in \mathscr{B}(G, F)$ be a special vertex. Up to conjugation by $G_{\mathrm{ad}}(F)$, there are two kinds of vertices: one is absolutely special and one is special, but not absolutely special. The local Dynkin diagram of G is:

$C\text{-}BC_1$  (7.1)

Note that $\mathrm{Gr}_{G,x}$ is connected with a linear order relation on the Schubert varieties, that is, there are no minuscule Schubert varieties. Recall that the pair $(\bar{\mu}, x)$ is said to be of exotic smoothness if $\bar{\mu}$ is quasi-minuscule and x is special, but not absolutely special, that is, the vertex labeled 1 in (7.1). The aim of this section is to prove the following proposition.

PROPOSITION 7.1

Assume that $(\bar{\mu}, x)$ is not of exotic smoothness. Then the smooth locus of $\tilde{\mathrm{Gr}}^{\le\bar{\mu}}_{G,x}$ is exactly $\tilde{\mathrm{Gr}}^{\bar{\mu}}_{G,x}$. In particular, $\tilde{\mathrm{Gr}}^{\le\bar{\mu}}_{G,x}$ is singular if $\bar{\mu}$ is non-trivial.

We may pass to working with $G = \mathrm{SU}_3$ thanks to Proposition 3.1. Also, we use Section 6 to reduce the proof of Proposition 7.1 to the case in which $k = \mathbb{C}$. In this case, $\tilde{\mathrm{Gr}}_{G,x}^{\leq\bar{\mu}} = \mathrm{Gr}_{G,x}^{\leq\bar{\mu}}$ is the ordinary Schubert variety. By [30, Section 9.f] the twisted affine flag varieties for simply connected groups agree with the Kac–Moody affine flag varieties so that Kumar's criterion from [22, Theorem 8.9] is applicable. This is a criterion for the smoothness of Schubert varieties in terms of affine Weyl group combinatorics. We first reduce the proof of Proposition 7.1 to Schubert varieties in the affine flag variety in Section 7.2 and then recall Kumar's criterion in Section 7.3 below. The final verifications are made in Section 7.4 below.

7.1. *Preliminaries on Schubert varieties*

We write $F' = k(\!(u)\!)$ for a choice of uniformizer u with $u^2 = t$, and we fix a basis giving an isomorphism $V = (F')^3$ such that the Hermitian form is given by the anti-diagonal matrix $\mathrm{antidiag}(1,1,1)$. With respect to this basis, we let $T \subset G$ be the diagonal torus, and let $B \subset G$ be the Borel subgroup of upper triangular matrices.

We define the $\mathcal{O}_{F'}$-lattices $\Lambda_0 := \mathcal{O}_{F'}^3$ and $\Lambda_1 := u^{-1}\mathcal{O}_{F'} \oplus \mathcal{O}_{F'}^2$. Up to conjugation by $\mathrm{SU}_3(F)$, the vertex x corresponds either to the absolutely special vertex given by the self-dual lattice Λ_0 or to the special, but not absolutely special vertex given by the almost modular lattice Λ_1 (see (5.1)). We fix the base alcove **a**, which corresponds to the Iwahori subgroup in $\mathrm{SU}_3(F)$ given by the stabilizer of the lattice chain $\Lambda_0 \subset \Lambda_1$.

Observe that $(G^\vee)^I$ is an adjoint group (see Proposition A.1) and so $X_*(T)_I = X^*((T^\vee)^I)$ is generated by $\breve{\Sigma}^\vee$. Write $e_1 = \bar{\mu}_1$ for the simple échelonnage coroot. In this way we identify $X_*(T)_I^+ = \mathbb{Z}_{\geq 0}$, and we denote by $\bar{\mu}_l \in X_*(T)_I^+$ the element which corresponds to $l \in \mathbb{Z}_{\geq 0}$. Explicitly, $\bar{\mu}_l$ is under the Kottwitz map given by the class of the diagonal matrix $\mathrm{diag}(u^l, 1, (-u)^{-l}) \in T(F)$. As closed subschemes in the affine Grassmannian we have

$$\{e\} = \mathrm{Gr}_{G,x}^{\leq\bar{\mu}_0} \subset \mathrm{Gr}_{G,x}^{\leq\bar{\mu}_1} \subset \mathrm{Gr}_{G,x}^{\leq\bar{\mu}_2} \subset \cdots,$$

and $\mathrm{Gr}_{G,x}^{\leq\bar{\mu}_l} \setminus \mathrm{Gr}_{G,x}^{\leq\bar{\mu}_{l-1}} = \mathrm{Gr}_{G,x}^{\bar{\mu}_l}$, which is of dimension $2l$ (because the base alcove corresponds to the interval $(0, \frac{1}{2})$; see below). The element $\bar{\mu}_1$ is the unique quasi-minuscule element.

7.2. *Reduction to the affine flag variety*

We consider the Iwahori $\mathcal{O}_F$-group scheme $\mathcal{G}_{\mathbf{a}}$ given by the automorphisms of the lattice chain $\Lambda_0 \subset \Lambda_1$, and we denote by $\mathcal{F}\ell := LG/L^+\mathcal{G}_{\mathbf{a}}$ the associated twisted affine flag variety in the sense of [30, Section 1.c]. Let $W = W(G,T,F)$ be the affine Weyl group. For each $w \in W$, we denote by $\mathcal{F}\ell^{\leq w} \subset \mathcal{F}\ell$ the $L^+\mathcal{G}_{\mathbf{a}}$-Schubert variety associated with the basepoint $n_w \in \mathcal{F}\ell(k)$ corresponding to w (see [30, Section 8]).

The canonical projection $\pi\colon \mathcal{F}\ell \to \mathrm{Gr}_{G,x}$ is representable by a smooth proper surjective morphism of relative dimension 1 (see [15, Lemma 4.9 i)]). Thus, for each $l \geq 1$ there exists a unique element $w_{l,x} \in W$ such that as subschemes of the affine flag variety

$$\mathcal{F}\ell^{\leq w_{l,x}} = \pi^{-1}(\mathrm{Gr}_{G,x}^{\leq \bar{\mu}_l}).$$

Since the projection $\mathcal{F}\ell^{\leq w_{l,x}} \to \mathrm{Gr}_{G,x}^{\leq \bar{\mu}_l}$ is smooth, to show $\mathrm{Gr}_{G,x}^{\leq \bar{\mu}_l}$ is singular it is enough to show that $\mathcal{F}\ell^{\leq w_{l,x}}$ is singular at a point $v_{l,x}$ lying over $\bar{\mu}_{l-1}$. We need to explicate the elements $w_{l,x}$ in terms of the affine Weyl group W and need to make suitable choices for $v_{l,x}$.

7.2.1. Affine roots

We have the perfect pairing $\langle \text{-},\text{-}\rangle\colon X^*(T)^I_{\mathbb{R}} \times X_*(T)_{I,\mathbb{R}} \to \mathbb{R}$ of 1-dimensional $\mathbb{R}$-vector spaces. Let $\epsilon_1 \in X^*(T)^I_{\mathbb{R}}$ be such that $\langle \epsilon_1, e_1\rangle = 1$. The set of affine roots $\Phi_{\mathrm{aff}} = \Phi_{\mathrm{aff}}(G,T,F)$ is given by

$$\Phi_{\mathrm{aff}} = \{\pm\epsilon_1 + \mathbb{Z}; \pm 2\epsilon_1 + \mathbb{Z}\}.$$

It follows that the simple affine roots are $\alpha_1 = \epsilon_1$, $\alpha_0 = -2\epsilon_1 + 1$, and the simple échelonnage root is $\alpha_{\mathrm{ech}} = 2\epsilon_1$. These roots have coroots $\alpha_1^\vee = 2e_1$, $\alpha_0^\vee = -e_1 + \frac{1}{2}$, and $\alpha_{\mathrm{ech}}^\vee = e_1 = \bar{\mu}_1$. Note this is consistent with our description of $\bar{\mu}_1$ above. The base alcove $\mathbf{a}$ is the open interval $(0, \frac{1}{2}) \subset \mathbb{R}$. In this notation, we have $\bar{\mu}_l = le_1 = l\alpha_{\mathrm{ech}}^\vee$.

7.2.2. Simple reflections

The affine Weyl group W has a Coxeter group structure given by the choice of the base alcove $\mathbf{a}$. We denote by $\leq$ the partial order and by $\ell(w) \in \mathbb{Z}_{\geq 0}$ the length of an element $w \in W$. We let $s_0 := s_{\alpha_0}$ be the simple affine reflection given by α_0, and we let $s_1 := s_{\alpha_1}$ be the simple reflection given by α_1. We have the group presentation

$$W = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1\rangle.$$

The group W acts on Φ_{aff}. We have $s_i(\alpha_i) = -\alpha_i$ for $i = 0, 1$, and

$$s_0(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_0^\vee\rangle \alpha_0 = \alpha_1 + \alpha_0,$$
$$s_1(\alpha_0) = \alpha_0 - \langle \alpha_0, \alpha_1^\vee\rangle \alpha_1 = \alpha_0 + 4\alpha_1.$$

Translation by $\bar{\mu}$ takes the base alcove $\mathbf{a} = (0, \frac{1}{2})$ to the interval $(1, \frac{3}{2})$. Therefore, for $l \geq 1$, translation by $\bar{\mu}_l$ is an element in W with reduced expression $(s_0 s_1)^l$. In what follows we will abbreviate such expressions.

7.2.3. Cases

We extend the definition to every $i \in \mathbb{Z}$ by $s_i := s_0$ (resp., $\alpha_i := \alpha_0$) if i is even, and $s_i := s_1$ (resp., $\alpha_i := \alpha_1$) if i is odd. Further, for every pair $i, j \in \mathbb{Z}$, we define $s_{i,j} := s_i s_{i+1} \cdots s_j$ if $i \leq j$ and $s_{i,j} := 1$ if $i > j$. Fix $l \geq 1$, and consider the following two cases.

Case A. Let x be the absolutely special vertex given by the self-dual lattice Λ_0. Then $\mathcal{G}_x(\mathcal{O}_F)$ contains the affine root groups given by $\pm\alpha_1$. In this case, we have $w_{l,x} = s_{1,2l+1}$, and we fix this reduced expression. We define $v_{l,x} := s_{1,2l-1}$.

Case B. Let x be the special, but not absolutely special vertex given by the almost modular lattice Λ_1. Then $\mathcal{G}_x(\mathcal{O}_F)$ contains the affine root groups given by $\pm\alpha_0$. In this case, we have $w_{l,x} = s_{0,2l}$, and we fix this reduced expression. We define $v_{l,x} := s_{0,2l-2}$, that is, the roles of 0 and 1 are interchanged.

7.3. Kumar's criterion

From now on assume that $k = \mathbb{C}$. Then $\mathcal{F}\ell$ is the Kac–Moody affine flag variety (see [22]) associated with the generalized Cartan matrix of rank 2 given by

$$\begin{pmatrix} \langle \alpha_0, \alpha_0^\vee \rangle & \langle \alpha_0, \alpha_1^\vee \rangle \\ \langle \alpha_1, \alpha_0^\vee \rangle & \langle \alpha_1, \alpha_1^\vee \rangle \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Let Q be the fraction field of the symmetric algebra of the root lattice. Let $w \in W$, and fix a reduced decomposition $w = s_1 \cdot \ldots \cdot s_n$. For $v \leq w$, we define

$$e_v X(w) \stackrel{\text{def}}{=} \sum \prod_{i=1}^{n} \tilde{s}_1 \cdots \tilde{s}_i (\alpha_i)^{-1} \in Q, \tag{7.2}$$

where the sum runs over all sequences $(\tilde{s}_1, \ldots, \tilde{s}_n)$ such that either $\tilde{s}_i = 1$ or $\tilde{s}_i = s_i$ for every i, and $\tilde{s}_1 \cdot \ldots \cdot \tilde{s}_n = v$. The following theorem is [22, Theorems 5.5(b) and 8.9].

THEOREM 7.2

The Schubert variety $\mathcal{F}\ell^{\leq w}$ is smooth at v if and only if

$$e_v X(w) = (-1)^{\ell(v)} \prod_{\alpha \in \Phi_{\text{aff}}^+; s_\alpha v \leq w} \alpha^{-1}, \tag{7.3}$$

where $\Phi_{\text{aff}}^+ \subset \Phi_{\text{aff}}$ is the set of positive affine roots and s_α denotes the associated reflection.

7.4. End of the proof

By Theorem 7.2, we need to calculate the expression $e_{v_{l,x}} X(w_{l,x})$ in both cases A and B. Note that there are $2l$ subexpressions of $v_{l,x}$ in $w_{l,x}$ defined by deleting two neighboring simple reflections and that all subexpressions are of this form. We use the notation introduced in Section 7.2.3.

Case A. An elementary calculation gives

$$e_{v_{l,x}} X(w_{l,x}) = \Big(\prod_{i=1}^{2l-1} s_{1,i}(\alpha_i)^{-1}\Big) \cdot \underbrace{\Big(\sum_{i=0}^{2l-1} s_{1,i}(\alpha_0)^{-1} s_{1,i}(\alpha_1)^{-1}\Big)}_{=:A_l}. \tag{7.4}$$

Case B. The same calculation as before gives

$$e_{v_{l,x}} X(w_{l,x}) = \Big(\prod_{i=0}^{2l-2} s_{0,i}(\alpha_i)^{-1}\Big) \cdot \underbrace{\Big(\sum_{i=0}^{2l-1} s_{0,i-1}(\alpha_0)^{-1} s_{0,i-1}(\alpha_1)^{-1}\Big)}_{=:B_l}. \tag{7.5}$$

COROLLARY 7.3

The Schubert variety $\mathcal{F}\ell^{\leq w_{l,x}}$ is smooth at $v_{l,x}$ if and only if

$$A_l = -\alpha_0^{-1} s_{1,2l}(\alpha_0)^{-1} = \alpha_0^{-1} s_{1,2l-1}(\alpha_0)^{-1}, \qquad \textit{resp.,}$$
$$B_l = -\alpha_1^{-1} s_{0,2l-1}(\alpha_1)^{-1} = \alpha_1^{-1} s_{0,2l-2}(\alpha_1)^{-1}.$$

Proof

The right-hand side in (7.3) takes the form

$$\text{case A:} -\prod_{i=0}^{2l} s_{1,i-1}(\alpha_i)^{-1} = -\prod_{i=0}^{2l} s_{1,i}(\alpha_i)^{-1};$$
$$\text{case B:} -\prod_{i=-1}^{2l-1} s_{0,i-1}(\alpha_i)^{-1} = -\prod_{i=-1}^{2l-1} s_{0,i}(\alpha_i)^{-1}.$$

Here we used $s_i(\alpha_i) = -\alpha_i$ and the conventions $s_{1,-2} = s_{1,-1} = s_{1,0} = 1$ to keep track of the signs. By comparing this with (7.4) (resp., (7.5)), the corollary follows from Theorem 7.2. □

We now need to calculate A_l and B_l for every $l \geq 1$. The following identities are useful.

LEMMA 7.4

For all $i \in \mathbb{Z}_{\geq 0}$, one has

i. $\alpha_0 + s_1(\alpha_0) = 2$, *and* $s_{1,2i}(\alpha_0) = \alpha_0 - 2i$, *and* $s_{1,2i+1}(\alpha_0) = s_1(\alpha_0) + 2i$;

ii. $\alpha_1 + s_0(\alpha_1) = 1$, *and* $s_{0,2i-1}(\alpha_1) = \alpha_1 - i$, *and* $s_{0,2i}(\alpha_1) = s_0(\alpha_1) + i$.

Proof
We have $s_1(\alpha_0) = \alpha_0 + 4\alpha_1 = 2 - \alpha_0$, and $s_0(\alpha_1) = \alpha_0 + \alpha_1 = 1 - \alpha_1$. The remaining identities are proved by an easy induction and are left to the reader. □

Case A. The number of summands in A_l is even, and we add terms in consecutive pairs as follows. For every $0 \le i \le l-1$, one has

$$\begin{aligned} &s_{1,2i}(\alpha_0)^{-1}s_{1,2i}(\alpha_1)^{-1} + s_{1,2i+1}(\alpha_0)^{-1}s_{1,2i+1}(\alpha_1)^{-1} \\ &\quad = 4 \cdot s_{1,2i}(\alpha_0)^{-1}s_{1,2i+1}(\alpha_0)^{-1}, \end{aligned}$$

where we used $s_{1,2i+1}(\alpha_1) = -s_{1,2i}(\alpha_1)$ and $s_{1,2i+1}(\alpha_0) = s_{1,2i}(\alpha_0) + 4s_{1,2i}(\alpha_1)$. This shows

$$A_l = 4 \cdot \sum_{i=0}^{l-1} s_{1,2i}(\alpha_0)^{-1}s_{1,2i+1}(\alpha_0)^{-1} = 4l \cdot \alpha_0^{-1}s_{1,2l-1}(\alpha_0)^{-1}.$$

For the last equality we use Lemma 7.5 below applied to $\alpha_0 + s_1(\alpha_0) = 2$, which is justified by Lemma 7.4(i). Hence, by Corollary 7.3 the Schubert variety $\mathcal{F}\ell^{\le w_{l,x}}$ is singular at $v_{l,x}$ for all $l \ge 1$ in this case.

Case B. Again the number of summands in B_l is even, and we add terms in consecutive pairs. For every $0 \le i \le l-1$, one has

$$s_{0,2i-1}(\alpha_0)^{-1}s_{0,2i-1}(\alpha_1)^{-1} + s_{0,2i}(\alpha_0)^{-1}s_{0,2i}(\alpha_1)^{-1} = s_{0,2i-1}(\alpha_1)^{-1}s_{0,2i}(\alpha_1)^{-1},$$

where we used $s_{0,2i}(\alpha_0) = -s_{0,2i-1}(\alpha_0)$ and $s_{0,2i}(\alpha_1) = s_{0,2i-1}(\alpha_0) + s_{0,2i-1}(\alpha_1)$. This shows

$$B_l = \sum_{i=0}^{l-1} s_{0,2i-1}(\alpha_1)^{-1}s_{0,2i}(\alpha_1)^{-1} = l \cdot \alpha_1^{-1}s_{0,2l-2}(\alpha_1)^{-1}.$$

For the last equality we use Lemma 7.5 below applied to $\alpha_1 + s_0(\alpha_1) = 1$, which is justified by Lemma 7.4(ii). Hence, by Corollary 7.3 the Schubert variety $\mathcal{F}\ell^{\le w_{l,x}}$ is singular (resp., smooth) at $v_{l,x}$ for $l \ge 2$ (resp., $l = 1$). This finishes the proof of Proposition 7.1.

LEMMA 7.5
Let $n \in \mathbb{Z}_{\ge 1}$, and let $\alpha, \beta \in Q \backslash n\mathbb{Z}$ (e.g., α, β are affine roots) with $\alpha + \beta = n$. For any $l \ge 1$, one has

$$\sum_{i=0}^{l-1} \frac{1}{(\alpha - ni)(\beta + ni)} = \frac{l}{\alpha(\beta + n(l-1))}.$$

Proof
This is elementary and left to the reader. □

Remark 7.6
The form of A_l and B_l also shows that the Schubert variety $\mathcal{F}\ell^{\leq w_{l,x}}$ is rationally smooth at $v_{l,x}$ (see [22, Thms. 5.5(a), 8.9]. This is in accordance with Theorem 1.1.

8. The quasi-minuscule Schubert variety for the ramified triality

Let k be an algebraically closed field of characteristic $\neq 3$, and let $F = k(\!(t)\!)$ be the Laurent series local field. Let G be the twisted triality over F, that is, the up to isomorphism unique quasi-split but non-split form of Spin_8 over F. Note that G splits over a totally ramified extension of F of degree 3 and is therefore tamely ramified by the assumption on k. Let $x \in \mathscr{B}(G, F)$ be a special vertex in the Bruhat–Tits building, and denote by $\mathcal{G}_x$ the associated special parahoric group scheme over $\mathcal{O}_F = k[\![t]\!]$. By [37, Section 2.5] the group $G_{\mathrm{ad}}(F)$ acts transitively on the set of special vertices. Therefore, we may justifiably denote by $\mathrm{Gr}_G := LG/L^+\mathcal{G}_x$ the twisted affine Grassmannian in the sense of [30]. We also note that all Schubert varieties inside Gr_G are normal by [30, Theorem 6.1].

THEOREM 8.1
The quasi-minuscule Schubert variety in Gr_G *is a* 6*-dimensional projective* k*-variety which is rationally smooth, but singular at the basepoint.*

Rational smoothness follows from Theorem 1.1 (see also Remark 8.3 below). For the proof that the quasi-minuscule Schubert variety is singular at the basepoint, we use Section 6 to reduce to the case $k = \mathbb{C}$ first, which we assume in Sections 8.6 and 8.7 below. Here our method is similar to the method in [26, Sections 2.9–2.10]: we construct a neighborhood of the basepoint inside the quasi-minuscule Schubert variety in a certain space of nilpotent matrices. We show that the tangent space of our Schubert variety at the basepoint is 7-dimensional, and hence, this point is singular. Note that it is quite different from the quasi-minuscule Schubert variety for a split group of type G_2, in which case the tangent space at the identity is of dimension 14 (= dimension of the Lie algebra of G_2; see [26, Section 2.9]).

Alternatively, one can in principle prove that the quasi-minuscule Schubert variety is singular by using Kumar's criterion as in Section 7 above. However, this is lengthy without computer-aided calculations (see also [26, Section 7.10] for a similar calculation in the split case) and would give less insight as we would not get the dimension of the tangent space at the identity.

8.1. Construction of the twisted triality

Let $F' = k(\!(u)\!)$ be the cubic Galois extension of F defined by $u^3 = t$. The Galois group $I = \mathrm{Gal}(F'/F)$ is cyclic of order 3, and we denote by $\tau \in I$ a generator. Then $\tau u = \zeta \cdot u$, where $\zeta = \zeta_3$ is a primitive third root of unity.

The special orthogonal group in dimension 8 is the functor on k-algebras R given by

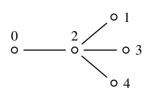
$$\mathrm{SO}_8(R) = \{A \in \mathrm{SL}_8(R) \mid AJA^t J = \mathrm{id}\}, \tag{8.1}$$

where $J := \mathrm{antidiag}(1, \dots, 1) \in \mathrm{GL}_8(R)$. Let $T' \subset B' \subset \mathrm{SO}_8$ be the maximal diagonal torus, contained in the upper triangular Borel. The torus T' is given by

$$T'(R) = \{\mathrm{diag}(a_1, \dots, a_4, a_4^{-1}, \dots, a_1^{-1}) \mid a_1, \dots, a_4 \in R^\times\}. \tag{8.2}$$

The coroot lattice $Q^\vee$ is the index 2 sublattice of $X_*(T') = \mathbb{Z}^4$ with basis $\alpha_1^\vee = \epsilon_1 - \epsilon_2$, $\alpha_2^\vee = \epsilon_2 - \epsilon_3$, $\alpha_3^\vee = \epsilon_3 - \epsilon_4$, and $\alpha_4^\vee = \epsilon_3 + \epsilon_4$. Hence, $\pi_1(\mathrm{SO}_8) = X_*(T')/Q^\vee = \mathbb{Z}/2$.

Define $\pi\colon H := \mathrm{Spin}_8 \to \mathrm{SO}_8$ to be the simply connected degree 2 cover. The preimage $T_H := \pi^{-1}(T')$ is a maximal torus of H contained in the Borel subgroup $B_H := \pi^{-1}(B')$. We have $X_*(T_H) = Q^\vee \subset X_*(T')$. The affine Dynkin diagram of H is

D_4 

(8.3)

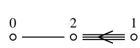
We let $\sigma_0 \in \mathrm{Aut}(D_4)$ denote the automorphism defined by $2 \mapsto 2$ and $1 \mapsto 3 \mapsto 4 \mapsto 1$. We fix a principal nilpotent element in $X_H \in \mathrm{Lie}(B_H)$ and regard σ_0 as an automorphism of H via

$$\mathrm{Aut}(D_4) = \mathrm{Aut}_k(H, B_H, T_H, X_H) \subset \mathrm{Aut}_k(H).$$

Then the twisted triality G is the functor on the category of F-algebras R given by

$$G(R) \overset{\mathrm{def}}{=} \{A \in H(R \otimes_F F') \mid \sigma(A) = A\}, \tag{8.4}$$

where $\sigma := \sigma_0 \otimes \tau$. Likewise, we define the F-subgroups $T \subset B \subset G$ by descent. The affine Dynkin diagram of G has the form (see [37, Section 4.2])

G_2^I 

(8.5)

Here the arrow points to the shorter root and should be regarded as an inequality sign saying that one root is of smaller length than the other.

8.2. The special parahoric

Since the extension F'/F is tamely ramified, we have an identification of buildings $\mathscr{B}(H,F')^\sigma = \mathscr{B}(G,F)$ compatible with the simplicial structure (see [33]). After conjugation by an element in $G_{\mathrm{ad}}(F)$, we reduce to the case where $x=0$ corresponds to the basepoint. In terms of parahoric group schemes, the pair $(H_{F'},0)$ corresponds to the $\mathcal{O}_{F'}$-group $\mathcal{H} := H \otimes_k \mathcal{O}_{F'}$. Hence, the special parahoric group scheme $\mathcal{G}_x = \mathcal{G}$ associated with the pair $(G,0)$ is the $\mathcal{O}_F$-group $\mathcal{G} = \mathrm{Res}_{\mathcal{O}_{F'}/\mathcal{O}_F}(\mathcal{H})^\sigma$.

8.3. Some loop groups

We denote by LG (resp., $L^+\mathcal{G}$) the twisted loop group given on k-algebras R by $LG(R) = G(R(\!(t)\!))$ (resp., $L^+\mathcal{G}(R) = \mathcal{G}(R[\![t]\!])$). Likewise, we denote by LH (resp., $L^+\mathcal{H}$) the loop group given by $LH(R) = H(R(\!(u)\!))$ (resp., $L^+\mathcal{H}(R) = \mathcal{H}(R[\![u]\!])$). Then as k-group functors

$$LG = (LH)^\sigma \qquad (\text{resp., } L^+\mathcal{G} = (L^+\mathcal{H})^\sigma), \tag{8.6}$$

which is an immediate consequence of the definition. The negative loop group $L^-\mathcal{H}$ is defined on k-algebras R by $L^-\mathcal{H}(R) = H(R[u^{-1}])$. Let $L^{--}\mathcal{H} := \ker(L^-\mathcal{H} \to H), u^{-1} \mapsto 0$. Then the morphism given by multiplication

$$L^{--}\mathcal{H} \times L^+\mathcal{H} \to LH, \qquad (h^-,h^+) \mapsto h^- \cdot h^+, \tag{8.7}$$

is relatively representable by an open immersion (see [23, Proposition 4.6]; see also [8, Theorem 2.3.1], [15, Corollary 3.2]). The automorphism $\sigma \in \mathrm{Aut}_k(LH)$ preserves the subgroup $L^{--}\mathcal{H} \subset LH$, and we define the k-group

$$L^{--}\mathcal{G} \stackrel{\text{def}}{=} (L^{--}\mathcal{H})^\sigma.$$

By taking σ-fixed points in (8.7), we see that the multiplication morphism $L^{--}\mathcal{G} \times L^+\mathcal{G} \to LG$ is still an open immersion. Hence, if $e \in \mathrm{Gr}_G = LG/L^+\mathcal{G}$ denotes the basepoint, then the morphism of k-ind-schemes

$$L^{--}\mathcal{G} \hookrightarrow \mathrm{Gr}_G, \qquad g^- \mapsto g^- \cdot e \tag{8.8}$$

is representable by an open immersion.

8.4. The quasi-minuscule Schubert variety

Let $\breve{\Sigma}$ be the échelonnage root system of G which we give explicitly in Section 8.5.2 below. Let $\bar{\mu} \in X_*(T)_I$ be the unique quasi-minuscule cocharacter for this root system. We fix an element $t^{\bar{\mu}} \in T(F)$ mapping to $\bar{\mu}$ under the Kottwitz morphism $T(F) \to X_*(T)_I$. We show in Section 8.5.3 that the element $t^{\bar{\mu}}$ maps under the map $T(F) \to T'(F') \subset \mathrm{SO}_8(F')$ to the diagonal matrix

$$\operatorname{diag}(u^2, u, u, 1, 1, u^{-1}, u^{-1}, u^{-2}) \cdot t_0,$$

for some $t_0 \in T'(\mathcal{O}_{F'})$. Let $C_{\bar{\mu}} \subset \mathrm{Gr}_G$ be the reduced $L^+\mathcal{G}$-orbit of $t^{\bar{\mu}} \cdot e$. The *quasi-minuscule Schubert variety* $S_{\bar{\mu}} \subset \mathrm{Gr}_G$ is the closure of $C_{\bar{\mu}}$ equipped with the reduced scheme structure. Then $S_{\bar{\mu}}$ is a projective k-variety whose smooth locus contains $C_{\bar{\mu}}$. By the Cartan decomposition for twisted affine Grassmannians ([34, Corollary 2.10]), we have

$$S_{\bar{\mu}} = C_{\bar{\mu}} \coprod \{e\}. \tag{8.9}$$

LEMMA 8.2
The Schubert variety $S_{\bar{\mu}}$ is of dimension 6.

Proof
Let $2\rho = 6\alpha_1 + 10\alpha_2 + 6\alpha_3 + 6\alpha_4$ be the sum of the positive roots in the absolute root system Φ_{D_4}. By [34, Corollary 2.10],

$$\dim(S_{\bar{\mu}}) = \langle 2\rho, \mu \rangle,$$

where $\mu = 2\alpha_1^\vee + \alpha_2^\vee$ as in Section 8.5.3 below. A calculation shows $\langle 2\rho, \mu \rangle = 6$. □

Remark 8.3
Note that $(G^\vee)^I = G_2$ by Lemma 4.2(iii). Under the geometric Satake isomorphism for ramified groups (see [35], [38]), the Schubert variety $S_{\bar{\mu}}$ corresponds to the quasi-minuscule fundamental representation $V_{\bar{\mu}}$ of G_2. This is the unique 7-dimensional non-trivial representation, it has 6 extreme weights, and hence, its trivial weight space is 1-dimensional. This shows that $V_{\bar{\mu}}$ is weight-multiplicity-free, and hence, $S_{\bar{\mu}}$ is rationally smooth by Proposition 2.2, without using the classification result in Theorem 4.4.

8.5. Various root systems
We give explicitly the various root systems attached to the twisted triality.

8.5.1. D_4 roots
We use the notation of Bourbaki from [5] for the root system of type D_4. The set of roots Φ_{D_4} carries the automorphism σ_0 of order 3. The simple roots are

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \epsilon_2 - \epsilon_3, \qquad \alpha_3 = \epsilon_3 - \epsilon_4, \qquad \alpha_4 = \epsilon_3 + \epsilon_4.$$

We list the positive roots as σ_0-orbits:

$$\{\alpha_2\}, \qquad \{\alpha_1, \alpha_3, \alpha_4\}, \qquad \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\},$$

$$\{\alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_4\},$$
$$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4\}, \qquad \{\alpha_1+2\alpha_2+\alpha_3+\alpha_4\}.$$

The highest root is $\tilde{\alpha}^{D_4} = \alpha_1+2\alpha_2+\alpha_3+\alpha_4$.

8.5.2. Échelonnage roots

The échelonnage root system $\breve{\Sigma}$ for G can be described explicitly in terms of the absolute roots Φ_{D_4}, by [12, Theorem 6.1]. The simple positive roots in $\breve{\Sigma}$ are the modified norms $N'_I(\alpha)$ of the simple positive roots α for Φ_{D_4} (but here the modified norm coincides with the unmodified norm in [12, Definition 3.1]). Therefore, the simple positive échelonnage roots may be written as

$$\breve{\Delta} = \{\alpha_2, \alpha_1+\alpha_3+\alpha_4\}.$$

In other words, $\alpha := \alpha_2$ is the short simple root, and $\beta := \alpha_1+\alpha_3+\alpha_4$ is the long simple root. It is evident from the angle $\angle(\alpha,\beta)$ that we get a root system of type G_2. Therefore, in these coordinates the highest root is

$$\tilde{\alpha} = 3\alpha+2\beta = 3\alpha_2+2(\alpha_1+\alpha_3+\alpha_4).$$

The coroots are given by

$$\alpha^\vee = \alpha_2^\vee, \qquad \beta^\vee = \frac{\alpha_1^\vee+\alpha_3^\vee+\alpha_4^\vee}{3}$$

and (by using that $\tilde{\alpha}^\vee =: \bar{\mu}$ is quasi-minuscule with respect to $\breve{\Sigma}$ and, hence, is the fundamental coweight $\omega_\beta^\vee$)

$$\tilde{\alpha}^\vee = \alpha^\vee + 2\beta^\vee = \alpha_2^\vee + \frac{2}{3}(\alpha_1^\vee+\alpha_3^\vee+\alpha_4^\vee). \tag{8.10}$$

8.5.3. Quasi-minuscule coweight $\bar{\mu}$ for G

The element (8.10) is the result of applying the σ-averaging map $X_*(T)_I \to X_*(T)^I$ to an element $\mu = \alpha_2^\vee + 2\alpha_1^\vee \in X_*(T)$. By [21, (7.3.2)], under $T(F)/T(\mathcal{O}_F) \to T(F')/T(\mathcal{O}_{F'})$, $t^{\bar{\mu}}$ maps to $u^{\tilde{\mu}} := \tilde{\mu}(u) \in T(F')$, where

$$\tilde{\mu} := N\bar{\mu} = 2(\alpha_1^\vee+\alpha_3^\vee+\alpha_4^\vee)+3\alpha_2^\vee \in X_*(T). \tag{8.11}$$

In terms of the diagonal torus T' in SO_8, the image of $t^{\bar{\mu}}$ takes the form

$$u^{\tilde{\mu}} \cdot t_0 = \mathrm{diag}(u^2, u, u, 1, 1, u^{-1}, u^{-1}, u^{-2}) \cdot t_0,$$

for some $t_0 \in T'(\mathcal{O}_{F'})$.

8.5.4. Fixed-point roots

We can identify the root system of the fixed point group $H^{\sigma_0} = \mathrm{Spin}_8^{\sigma_0}$ using the general procedure of [11, Section 4]. The procedure is to take the non-divisible elements of the set of σ_0-averages of the roots of H. We get the following list, corresponding to the σ_0-orbits listed above:

$$\{\alpha_2\}, \qquad \Big\{\frac{\alpha_1+\alpha_3+\alpha_4}{3}\Big\}, \qquad \Big\{\alpha_2 + \frac{1}{3}(\alpha_1+\alpha_3+\alpha_4)\Big\},$$
$$\Big\{\alpha_2 + \frac{2}{3}(\alpha_1+\alpha_3+\alpha_4)\Big\},$$
$$\{\alpha_1+\alpha_2+\alpha_3+\alpha_4\}, \qquad \{\alpha_1+2\alpha_2+\alpha_3+\alpha_4\}.$$

This is the set of positive roots in a root system of type G_2 (see [5, Pl. IX (II, V)]). The group H^{σ_0} is connected reductive, semi-simple, simple, and simply connected (see Proposition A.1), so that $H^{\sigma_0} = G_2$. (We fix an isomorphism.) Note that we have used the connectedness of T^{σ_0}, which is obvious in this situation.

8.6. Nilpotent orbits with a twist

Now assume that $k = \mathbb{C}$.

8.6.1. The space $(u^{-1}\mathfrak{h})^\sigma$

Write $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{n} = \mathrm{Lie}(H)^{\mathrm{nilp}}$, the set of nilpotent elements in $\mathfrak{h}$. Consider $u^{-1}\mathfrak{h} \subset \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}(\!(u)\!)$. This is a σ-stable $\mathbb{C}$-vector subspace of finite dimension. For a root γ of H, let $u_\gamma : \mathbb{C} \to \mathfrak{h}$ be the corresponding Lie algebra homomorphism. We use the same symbol for $u_\gamma : \mathbb{C}(\!(u)\!) \to \mathfrak{h} \otimes_{\mathbb{C}} \mathbb{C}(\!(u)\!)$.

LEMMA 8.4

(i) *We may identify $(u^{-1}\mathfrak{h})^\sigma$ with the set of vectors*

$$\bigoplus_\gamma u_\gamma(u^{-1}x_\gamma) \oplus (y),$$

where γ ranges over the roots of H, where the $x_\gamma \in \mathbb{C}$ satisfy the condition that $x_{\sigma_0\gamma} = \zeta^{-1}x_\gamma$ for all γ, and where $y \in (u^{-1}\,\mathrm{Lie}(T_H))^\sigma$.

(ii) *The vector space $(u^{-1}\mathfrak{h})^\sigma$ is a 7-dimensional non-trivial representation of $H^\sigma = H^{\sigma_0}$; hence, it is the quasi-minuscule fundamental representation of $G_2 = H^{\sigma_0}$.*

(iii) *Fix $x \in \mathbb{C}\backslash\{0\}$. The variety $(u^{-1}\mathfrak{n})^\sigma$ contains the reduced orbit closure $\overline{G_2 \cdot v_{\max}}$ of*

$$v_{\max} := u_{\alpha_1+\alpha_2+\alpha_3}(u^{-1}x) \oplus u_{\alpha_2+\alpha_3+\alpha_4}(u^{-1}\zeta^{-1}x) \oplus u_{\alpha_1+\alpha_2+\alpha_4}(u^{-1}\zeta^{-2}x).$$

This orbit closure is a 6-dimensional affine $\mathbb{C}$-variety.

Proof
Part (i) is immediate. We see that $x_\gamma = 0$ if γ is σ_0-fixed. Therefore, we are left only with the contributions for γ in

$$\pm\{\alpha_1,\alpha_3,\alpha_4\} \cup \pm\{\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_2+\alpha_4\} \cup \pm\{\alpha_1+\alpha_2+\alpha_3,\alpha_2+\alpha_3+\alpha_4,\alpha_1+\alpha_2+\alpha_4\} \tag{8.12}$$

and the 1-dimensional space $(u^{-1}\operatorname{Lie}(T_H))^\sigma$. Therefore, $\dim_{\mathbb{C}}((u^{-1}\mathfrak{h})^\sigma) = 7$.

Since H and σ act on $u^{-1}\mathfrak{h}$ such that $\sigma(h\cdot v) = \sigma(h)\cdot\sigma(v)$ for $h \in H$ and $v \in u^{-1}\mathfrak{h}$, we see that H^σ acts on $(u^{-1}\mathfrak{h})^\sigma$. The action is visibly non-trivial. Since it is a 7-dimensional semi-simple non-trivial representation, $(u^{-1}\mathfrak{h})^\sigma$ must be the 7-dimensional representation associated to the quasi-minuscule fundamental weight of G_2. This proves (ii).

Finally, let $v_{\max}$ be as in (iii). It is a highest weight vector in the representation $(u^{-1}\mathfrak{h})^\sigma$ of $H^\sigma = G_2$. Therefore, its orbit is at least 6-dimensional, because it contains the non-zero elements of the 6 extreme weight spaces in $(u^{-1}\mathfrak{h})^\sigma$. Since $\mathfrak{n}$ is closed in $\mathfrak{h}$, the orbit closure is contained in $(u^{-1}\mathfrak{n})^\sigma$. As the latter space is 6-dimensional (it does not contain $(u^{-1}\operatorname{Lie}(T_H))^\sigma$), we see that the orbit closure is exactly 6-dimensional. □

8.6.2. The reduced orbit closure is singular
We consider the reduced orbit closure $\overline{G_2\cdot v_{\max}}$. The G_2-orbit is dense in a 6-dimensional vector subspace of $(u^{-1}\mathfrak{h})^\sigma$; hence, its closure contains the origin $0 \in (u^{-1}\mathfrak{h})^\sigma$.

LEMMA 8.5
The 6*-dimensional variety* $\overline{G_2\cdot v_{\max}}$ *has a* 7*-dimensional tangent space*

$$T_0(\overline{G_2\cdot v_{\max}}) = (u^{-1}\mathfrak{h})^\sigma;$$

hence, $\overline{G_2\cdot v_{\max}}$ *is singular at* 0.

Proof
Clearly $T_0(\overline{G_2\cdot v_{\max}}) \subseteq T_0((u^{-1}\mathfrak{h})^\sigma) = (u^{-1}\mathfrak{h})^\sigma$ as a G_2-invariant subspace. Since $(u^{-1}\mathfrak{h})^\sigma$ is an irreducible G_2-representation, the equality holds. The lemma follows from Lemma 8.4(ii). □

8.6.3. The exponential map
Essential to our proof of Theorem 8.1 is the following proposition.

PROPOSITION 8.6
The exponential map

$$\exp\colon u^{-1}\mathfrak{n} \to L^{--}\mathcal{H}, \qquad u^{-1}X \mapsto \sum_{i=0}^{\infty} \frac{(u^{-1}X)^i}{i!} \tag{8.13}$$

is algebraic and equivariant under H. Further, it induces an algebraic map

$$\exp\colon \overline{G_2 \cdot v_{\max}} \longrightarrow (L^{--}\mathcal{H})^{\sigma} \cap S_{\bar{\mu}} = L^{--}\mathcal{G} \cap S_{\bar{\mu}}.$$

Proof
It is clear that (8.13) is algebraic and H-equivariant. Both the scheme-theoretic image Z of $\exp|_{\overline{G_2 \cdot v_{\max}}}$ and $L^{--}\mathcal{G} \cap S_{\bar{\mu}}$ are reduced closed subschemes of $L^{--}\mathcal{H}$. Thus, to show $Z \subset L^{--}\mathcal{G} \cap S_{\bar{\mu}}$ we may argue on $\mathbb{C}$-points. In fact, for the remainder it suffices to show that $v_{\max}$ has σ-fixed image and lands in $C_{\bar{\mu}}$.

Let $U_{\gamma} : \mathbb{C}(\!(u)\!)^{\times} \to \mathrm{SO}_8 \otimes_{\mathbb{C}} \mathbb{C}(\!(u)\!)$ be the root group homomorphism associated to γ. Then, by the definition of $v_{\max}$ in Lemma 8.4(iii), one has the formula

$$\exp(v_{\max}) = U_{\alpha_1+\alpha_2+\alpha_3}(u^{-1}x) \cdot U_{\alpha_2+\alpha_3+\alpha_4}(u^{-1}\zeta^{-1}x) \cdot U_{\alpha_1+\alpha_2+\alpha_4}(u^{-1}\zeta^{-2}x).$$

Note that the three root groups commute with each other, since no pair of the roots in $\{\alpha_1+\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_4\}$ sums to a root of H. Thus, $\exp(v_{\max})$ is evidently fixed by σ, so it lies in $L^{--}\mathcal{G}$.

It remains to show that $\exp(v_{\max})$ lies in $C_{\bar{\mu}}$. A calculation using the root groups shows that the image of $v_{\max}$ in $\mathrm{SO}_8(\mathbb{C}(\!(u)\!))$ is an 8×8 matrix of the form

$$\begin{bmatrix} 1 & & & u^{-1}x & u^{-1}\zeta^{-2}x & 0 & 0 & -u^{-2}\zeta^{-2}x^2 \\ & 1 & & & & u^{-1}\zeta^{-1}x & 0 & 0 \\ & & 1 & & & & -u^{-1}\zeta^{-1}x & 0 \\ & & & 1 & & & & -u^{-1}\zeta^{-2}x \\ & & & & 1 & & & -u^{-1}x \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}, \tag{8.14}$$

where all unlabeled entries are 0.

By using the embedding $\mathrm{SO}_8 \subset \mathrm{GL}_8$ and Section 8.5.3, it suffices to show that

$$\exp(v_{\max}) \in K\,\mathrm{diag}(u^{-2}, u^{-1}, u^{-1}, 1, 1, u, u, u^2)K,$$

where $K = \mathrm{GL}_8(\mathbb{C}[\![u]\!])$. We prove this by using the algorithm for finding the Smith form of a matrix over a principal ideal domain. For the matrix $A = \exp(v_{\max})$ and $1 \le i \le 8$, define the weakly increasing sequence of integers $a_i \in \mathbb{Z}$ by requiring that

$$(u^{\sum_{j\le i} a_j}) = \text{ideal generated by the } i\times i \text{ minors of } A.$$

Then the Smith form of A is the matrix $\operatorname{diag}(u^{a_1}, u^{a_2}, \dots, u^{a_8})$. An inspection of (8.14) shows that $a_1 = -2$, $a_1 + a_2 = -3$, $a_1 + a_2 + a_3 = -4$, and $a_1 + a_2 + a_3 + a_4 = -4$. Thus, the first half of the entries of the Smith form of $\exp(v_{\max})$ are $(u^{-2}, u^{-1}, u^{-1}, 1)$. Because the matrix is in SO_8, this determines the rest of the entries as well. This completes the proof. □

8.7. End of the proof

We now finish the proof of Theorem 8.1. In Proposition 8.6, we have constructed an algebraic morphism of 6-dimensional $\mathbb{C}$-varieties (see Lemmas 8.2 and 8.4(iii) for the dimension)

$$\exp\colon \overline{G_2\cdot v_{\max}} \longrightarrow L^{--}\mathcal{G}\cap S_{\bar\mu}. \tag{8.15}$$

The target is an open neighborhood of the basepoint e in $S_{\bar\mu}$. Note that $0 \mapsto e$ under (8.15). By Lemma 8.5, it suffices to show that (8.15) is an open immersion, that is, an isomorphism onto an open neighborhood of e. For this, we argue as follows.

The map $\exp\colon u^{-1}\mathfrak{h} \to L^{--}\mathcal{H}$ is injective, as can be seen by writing down the formula for the exponential map in SO_8 and by comparing the coefficients of u^{-1}. Hence, the map (8.15) is an injective morphism of irreducible affine $\mathbb{C}$-varieties of the same dimension and, in particular, birational (because the field extension at the generic points is separable, so that it must have degree 1). As the target is normal by [30, Theorem 6.1], the map (8.15) is an open immersion by Zariski's main theorem [10, Corollary 4.4.9]. This completes the proof of Theorem 8.1.

Appendix. A remark on fixed point groups

Let H be a connected reductive group over an algebraically closed field κ, and let (T, B, X) be a pinning which is preserved by the action on H of a finite group I. Recall that H^I is a (possibly disconnected) reductive group, with maximal torus the neutral component $T^{I,\circ}$ of the diagonalizable subgroup $T^I \subset H^I$ (see [11, Proposition 4.1]). In what follows, $Z(A)$ denotes the center of an algebraic group A.

PROPOSITION A.1

Assume $\mathrm{char}(\kappa) \ne 2$. *Then the following statements hold:*

(i) *If T^I is connected (e.g., H is adjoint or simply connected), then H^I is connected.*

(ii) *$Z(H^I) = Z(H)^I$, and this group contains $Z(H^{I,\circ})$.*

(iii) *If H is semi-simple, then H^I is semi-simple.*

(iv) *If H is adjoint, then H^I is adjoint.*

(v) *If H is simple and simply connected, then H^I is simple and simply connected unless H is of type A_{2n} and carries a non-trivial I-action, in which case $H \cong \mathrm{SL}_{2n+1}$ and $H^I \cong \mathrm{SO}_{2n+1}$.*

Proof

By, for example, [11, Proposition 4.1] we know $\pi_0(T^I) \xrightarrow{\sim} \pi_0(H^I)$. If H is adjoint or simply connected, then $X^*(T)$ is an induced I-module, and hence, T^I is connected because $X^*(T^I) = X^*(T)_I$ is $\mathbb{Z}$-free. This proves (i). Note this part holds with no assumption on $\mathrm{char}(\kappa)$.

For (ii), note that $(T_{\mathrm{ad}})^I$ is connected, and thus, by [11, Proposition 4.6] we have $Z(H)^I H^{I,\circ} = H^I$, and hence, $Z(H)^I Z(H^{I,\circ}) = Z(H^I)$. Now by [11, proof of Proposition 4.1], when $\mathrm{char}(\kappa) \neq 2$ the simple roots for $(H^{I,\circ}, T^{I,\circ})$ consist precisely of the restrictions to $T^{I,\circ}$ of the simple roots for (H,T). It follows that $z \in Z(H^{I,\circ})$ is killed by all the roots of (H,T) and, hence, belongs to $Z(H)$. This proves $Z(H^{I,\circ}) \subseteq Z(H)^I$, and (ii) follows. Parts (iii) and (iv) follow from (ii).

For part (v), assume H is simple and simply connected. By (i), H^I is connected. Note that if $\tilde{\alpha}$ is the highest root for (H,T), then the I-average $\tilde{\alpha}^\diamond$ is highest for $(H^{I,\circ}, T^{I,\circ})$, and hence, the root system for the latter is irreducible since it has a unique highest root. This proves H^I is simple. Consider the dual group $H^{I,\vee}$, with dual torus $T^{I,\vee}$. It is enough to determine when $H^{I,\vee}$ is adjoint. We have that $X_*(T^{I,\vee}) = X^*(T^I) = X_*(T^\vee)_I$ is dual to $X^*(T^\vee)^I = \mathrm{Hom}(X_*(T^\vee)_I, \mathbb{Z})$. (Note that $X_*(T^\vee)_I$ is free.) Therefore, $X^*(T^{I,\vee}) \cong X^*(T^\vee)^I = (\mathbb{Z}\Phi^\vee(H))^I$. Therefore, to show $H^{I,\vee}$ is adjoint, it is equivalent to show that

$$\mathbb{Z}\Phi^\vee(H^I) \xhookrightarrow{\sim} \big(\mathbb{Z}\Phi^\vee(H)\big)^I.$$

The right-hand side has as $\mathbb{Z}$-basis the set $N_I(\Delta^\vee(H))$ of unmodified norms of simple coroots of H (see the definition of the operations N_I and N'_I in [12, Def. 3.1][3]). On the other hand, using the notation of [12, Section 3], the proof of [11, Proposition 4.1] shows that $\Phi(H^I) = \mathrm{res}_I\, \Phi(H) \cong (\Phi(H)^\diamond)_{\mathrm{red}}$. By the duality result of [12, Proposition 3.5], $\mathbb{Z}\Phi^\vee(H^I)$ has a $\mathbb{Z}$-basis given by $N'_I(\Delta^\vee(H))$. Therefore, $H^{I,\vee}$ is adjoint if and only if $N'_I(\Delta^\vee(H)) = N_I(\Delta^\vee(H))$, which happens if and only if the I-action is trivial or $\Phi(H)$ is not of type A_{2n}. The rest of the assertions of part (v) are clear. □

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[3]We take this opportunity to point out a typo in the definition of $v^\diamond$ above [12, Definition 3.1]: it should read $v^\diamond := \frac{1}{|I|}\sum_{\sigma \in I} \sigma(v)$.

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