

A STOCHASTIC COLLOCATION METHOD BASED ON SPARSE GRIDS FOR A STOCHASTIC STOKES-DARCY MODEL

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ABSTRACT. In this paper, we develop a sparse grid stochastic collocation method to improve the computational efficiency in handling the steady Stokes-Darcy model with random hydraulic conductivity. To represent the random hydraulic conductivity, the truncated Karhunen-Loève expansion is used. For the discrete form in probability space, we adopt the stochastic collocation method and then use the Smolyak sparse grid method to improve the efficiency. For the uncoupled deterministic subproblems at collocation nodes, we apply the general coupled finite element method. Numerical experiment results are presented to illustrate the features of this method, such as the sample size, convergence, and randomness transmission through the interface.

1. Introduction. The Stokes-Darcy (SD) model is a fundamental model used in many areas of science and engineering, for example, groundwater systems in karst aquifers [25, 49, 60, 61], interaction between surface and subsurface flows [28, 36, 38, 66, 75], oil reservoir in vuggy porous medium [1, 3, 41, 84, 120], and industrial filtrations [44, 62]. The Stokes-Darcy model describes the free flow of a liquid by the Stokes equation and the confined flow in a porous media by the Darcy equation, the two flows are coupled through interface conditions. Due to the fact that the resulting coupled Stokes-Darcy model has higher fidelity than either the Darcy or Stokes systems on their own, it is not surprising that a great deal of effort has been devoted to developing numerical methods for solving the model, such as the coupled finite element methods [21, 72, 85, 99], domain decomposition methods [19, 24, 31, 37, 39, 40, 57, 79, 108], Lagrange multiplier methods [6, 50, 76], multi-grid methods [2, 20, 88],

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discontinuous Galerkin methods [33, 54, 71, 78, 96], mortar discretization [17, 48, 55], and many others [5, 10, 18, 22, 32, 43, 63, 68, 70, 77, 83, 89, 100, 110, 111, 119].

All of the above existing work focused on the deterministic SD model, in which the model parameters, such as the model coefficients, the forcing terms, the domain geometry, the boundary conditions, the initial conditions, etc., are assumed to be perfectly known. Rather surprisingly, there are only limited number of literature studies about the stochastic SD model [74]. In reality, there is a huge amount of uncertainty involved in determining these real-life data due to measurement noise and simplifications. Hence many applications are affected by a relatively large amount of uncertainty in the input data, which cannot be accurately modeled by deterministic partial differential equations. On the other hand, modern engineering relies more and more on computational simulations, and the accuracy requirement in the computational results is growing significantly.

There are many sources that could generate uncertainties in SD model, which may come from either our difficulty in characterizing the investigated complex system or an intrinsic variability of physical quantities. For instance, when we study subsurface flows, the subsurface properties, such as porosity and permeability, are usually extrapolated from measurements taken in a limited number of sampling locations. When enough information is provided for the statistical characterization of the physical system, the input data can be modeled as random fields with a given correlation structure in associated PDEs, see e.g. the Karhunen-Loève [80, 81] and polynomial chaos expansions [116, 117]. Then instead of determining a single solution, we are concerned with the statistical moments of the solutions, such as their mean value, variance, covariance, i.e., the derived statistical quantities of interests.

A number of efficient numerical methods have been developed to solve stochastic PDEs, such as polynomial chaos [69, 116, 117], stochastic Galerkin method [8, 35, 86, 98], stochastic collocation method [7, 59], sparse grid methods [11, 87, 90, 91], multi-level Monte Carlo method [14, 29, 42, 52, 73, 97, 104], and many others [9, 12, 27, 82, 103, 107, 109, 112, 114, 115, 118, 121, 122]. These methods have also been applied to solve the stochastic optimization and control problems [4, 13, 34, 58, 105].

Similar to the Monte Carlo method [46], the stochastic collocation method [7] is characterized by requiring only the solution of uncoupled deterministic problems over the set of collocation points, even in the presence of a hydraulic conductivity which depends nonlinearly on the random variables. That is, the stochastic collocation method is naturally parallel with minimum communication and capable of fully making use of existing packages as black boxes for the corresponding deterministic problems. However, the stochastic collocation method with full tensor product spaces still suffers from the so-called *curse of dimensionality* (COD) when high-dimensional random variables are needed to describe the input data.

Meanwhile, the sparse grid method [15, 51, 102] is an efficient technique to avoid the dilemma caused by COD due to its selection of quadrature points based on their estimated contributions to the numerical quadrature's overall accuracy. When the dimension is moderately large and when the integrand depends smoothly on the underlying random variables, the advantage of sparse grid becomes more clear as the increase of the number of dimensions.

In this article, we will follow the idea in [90, 91] to develop a sparse grid stochastic collocation method for a Stokes-Darcy model with random hydraulic conductivity, which is described by a finite-dimensional random vector. As pointed out in [90, 91], this is often called finite-dimensional noise assumption and may hold either because

the problem itself can be described by a finite number of random variables or because the input coefficients are modeled as truncated random fields.

The rest of paper is organized as follows. In section 2, we introduce the Stokes-Darcy system with random hydraulic conductivity. In section 3, we treat the random hydraulic conductivity by using Karhunen-Loève expansion to obtain the truncated stochastic Stokes-Darcy model. In section 4, a Smolyak based sparse grid method is proposed to solve the truncated stochastic Stokes-Darcy model. Finally, in section 6, we present some numerical results that illustrate the features of the proposed method.

2. Steady Stokes-Darcy model with random hydraulic conductivity. We consider the coupled Stokes-Darcy system on a bounded domain $\Omega = \Omega_D \cup \Omega_S \subset \mathbb{R}^d$, ($d = 2, 3$); see Figure 1. Assume $(\Omega_P, \mathcal{F}, \mathcal{P})$ is a complete probability space.

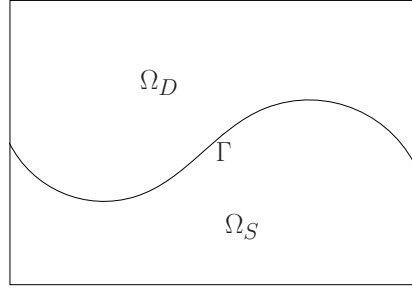


FIGURE 1. A sketch of the porous media domain Ω_D , the free-flow domain Ω_S , and the interface Γ .

Here Ω_P is the set of outcomes, $\mathcal{F}$ is the σ -algebra of events, and $\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$ is a probability measure. In the porous media region Ω_D , the flow is governed by the Darcy system

$$\vec{u}_D(\omega, x) = -K(\omega, x)\nabla\phi_D(\omega, x), \quad (1)$$

$$\nabla \cdot \vec{u}_D(\omega, x) = f_D. \quad (2)$$

where $\omega \in \Omega_P$, $x \in \Omega_D$. Here, $\vec{u}_D$ is the fluid discharge rate in the porous media, K is the random hydraulic conductivity, f_D is a sink/source term, and ϕ_D is the hydraulic head defined as $\phi_D = z + \frac{p_D}{\rho g}$. Here p_D denotes the dynamic pressure, z denotes the height, ρ denotes the density, and g denotes the gravitational acceleration. In this article we will consider the second-order form of the Darcy system

$$-\nabla \cdot (K(\omega, x)\nabla\phi_D(\omega, x)) = f_D. \quad (3)$$

In the fluid region Ω_S , the fluid flow is assumed to satisfy the Stokes system

$$-\nabla \cdot \mathbb{T}(\vec{u}_S, p_S) = \vec{f}_S, \quad (4)$$

$$\nabla \cdot \vec{u}_S = 0, \quad (5)$$

where $\vec{u}_S$ is the fluid velocity, p_S is the kinematic pressure, $\vec{f}_S = (\vec{f}_{S1}, \vec{f}_{S2})$ is the external body force, ν is the kinematic viscosity of the fluid, $\mathbb{T}(\vec{u}_S, p_S) = 2\nu\mathbb{D}(\vec{u}_S) - p_S\mathbb{I}$ is the stress tensor, and $\mathbb{D}(\vec{u}_S) = 1/2(\nabla\vec{u}_S + \nabla^\top\vec{u}_S)$ is the deformation tensor.

Let $\Gamma = \overline{\Omega_D} \cap \overline{\Omega_S}$ denote the interface between the fluid and porous media regions. On Γ , we consider the following three interface conditions which are also affected by the randomness of the hydraulic conductivity K :

$$\vec{u}_S \cdot \vec{n}_S = -\vec{u}_D \cdot \vec{n}_D, \quad (6)$$

$$-\vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = g(\phi_D - z), \quad (7)$$

$$-\tau_j \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = \frac{\alpha \nu \sqrt{d}}{\sqrt{\text{trace}(\Pi)}} \tau_j \cdot (\vec{u}_S - \vec{u}_D), \quad (8)$$

where $\vec{n}_S$ and $\vec{n}_D$ denote the unit outer normal to the fluid and the porous media regions at the interface Γ , respectively; τ_j ($j = 1, \dots, d-1$) denote mutually orthogonal unit tangential vectors to the interface Γ , and $\Pi(\omega, x) = \frac{K(\omega, x)\nu}{g}$. The third condition (8) is referred as the Beavers-Joseph (BJ) interface condition [16, 23, 25, 26, 45, 64, 67, 95, 101]. Compared with the Beavers-Joseph-Saffman-Jones (BJSJ) interface boundary condition [76], the Beavers-Joseph interface condition considers the contribution of the flow in the porous media to the tangential interface condition. More theoretical support for the BJ condition can be found in [30], which demonstrated that the BJ condition is more accurate than the BJSJ condition or its further simplifications.

We also assume that the hydraulic head ϕ_D and the fluid velocity $\vec{u}_S$ satisfy homogeneous Dirichlet boundary condition except on Γ , i.e., $\phi_D = 0$ on the boundary $\partial\Omega_D \setminus \Gamma$ and $\vec{u}_S = 0$ on the boundary $\partial\Omega_S \setminus \Gamma$. We also assume that $K(\omega, \cdot)$ is uniformly bounded below, i.e., there exists a constant $K_0 > 0$ such that $P(\omega \in \Omega_P : K(\omega, x) > K_0, x \in \overline{\Omega_D}) = 1$.

3. Karhunen-Loève expansion and truncated stochastic Stokes-Darcy model. In this section, we will recall the Karhunen-Loève (KL) expansion [80, 81] for a positive random field $r(\omega, x)$, with continuous covariance function $\text{Cov}_r(x, x') : \Omega_D \times \Omega_D \rightarrow \mathbb{R}$, and then utilize the truncated KL expansion to obtain the truncated stochastic Stokes-Darcy model.

We first consider the following eigenvalue problem:

$$\int_{\Omega_D} \text{Cov}_r(x, x') v(x') dx' = \lambda v(x). \quad (9)$$

Due to the symmetry of $\text{Cov}_r(\cdot, \cdot)$ and the positivity of $r(\omega, x)$, the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ are real and nonnegative, and the eigenspace $\text{span}\{v_1, v_2, \dots, v_n, \dots\}$ corresponding to distinct eigenvalues is mutually orthogonal, i.e.,

$$\int_{\Omega_D} v_m(x) v_n(x) dx = \delta_{mn}, \quad \forall m, n \in \mathbb{N}^+. \quad (10)$$

Since the eigenvalues are nonnegative, we may order the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0$. Let $\mu_r(x)$ denote the expected value of $r(\omega, x)$. Define

$$Y_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_{\Omega_D} [r(\omega, x) - \mu_r(x)] v_n(x) dx. \quad (11)$$

Here $\{Y_n(\omega)\}_{n=1}^{\infty}$ are mutually uncorrelated real random variables with zero mean and unit variance, i.e., $\mathbb{E}(Y_n(\omega)) = 0$, $\mathbb{E}(Y_m(\omega)Y_n(\omega)) = \delta_{mn}$. Then the KL expansion is defined by

$$r(\omega, x) = \mu_r(x) + \sum_{n=1}^{\infty} \sqrt{\lambda_n} v_n(x) Y_n(\omega). \quad (12)$$

The validity of the KL expansion results from the fact that the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ decay as n increases. The decay rate depends on the smoothness of the covariance function $\text{Cov}_r(x, x')$ and the correlation length, see [47, 106] and references therein. Based on the desired accuracy, we may retain only the first N terms to obtain the following truncated KL expansion

$$r_N(\omega, x) = \mu_r(x) + \sum_{n=1}^N \sqrt{\lambda_n} v_n(x) Y_n(\omega), \quad \forall N \in \mathbb{N}^+. \quad (13)$$

Using a KL representation of random fields, we consider the following problem instead of (3).

$$-\nabla \cdot (K_N(\omega, x) \nabla \phi_N(\omega, x)) = f_D. \quad (14)$$

One important issue is the elliptic coercivity of $K_N(\omega, x)$. One often performs a truncated Karhunen-Loève expansion for $\log(K - K_{\min})$ instead of K to obtain

$$\log(K_N - K_{\min})(\omega, x) = \mu(x) + \sum_{n=1}^N \sqrt{\lambda_n} v_n(x) Y_n(\omega), \quad \forall N \in \mathbb{N}^+. \quad (15)$$

Here $K(\omega, x) \geq K_{\min}$, $\forall x \in \Omega_D$. Then we have

$$K_N(\omega, x) = K_{\min} + e^{\mu(x) + \sum_{n=1}^N \sqrt{\lambda_n} v_n(x) Y_n(\omega)}, \quad \forall N \in \mathbb{N}^+. \quad (16)$$

In this article, we assume $Y_n(\omega)$ to be bounded, denote the image of Y_n by $I_n = Y_n(\Omega_P)$, and define $\mathbf{I}^N = \prod_{n=1}^N I_n$. Also, we assume that the random variables $\{Y_n\}_{n=1}^N$ have a joint probability density function $\rho_Y : \mathbf{I}^N \rightarrow \mathbb{R}^+$ such that $\rho_Y \in L^\infty(\mathbf{I}^N)$. Hence in the rest of this article, we consider the following problem

$$-\nabla \cdot (K_N(\omega, x) \nabla \phi_N(\omega, x)) = f_D, \quad (17)$$

$$-\nabla \cdot \mathbb{T}(\vec{u}_N(\omega, x), p_N(\omega, x)) = \vec{f}_S, \quad (18)$$

$$\nabla \cdot \vec{u}_N(\omega, x) = 0, \quad (19)$$

$$\vec{u}_N(\omega, x) \cdot \vec{n}_S = K_N(\omega, x) \nabla \phi_N(\omega, x) \cdot \vec{n}_D, \quad (20)$$

$$-\vec{n}_S \cdot (\mathbb{T}(\vec{u}_N(\omega, x), p_N(\omega, x)) \cdot \vec{n}_S) = g(\phi_N(\omega, x) - z), \quad (21)$$

$$\begin{aligned} -\tau_j \cdot (\mathbb{T}(\vec{u}_N(\omega, x), p_N(\omega, x)) \cdot \vec{n}_S) &= \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod_N)}} \tau_j \cdot (\vec{u}_N(\omega, x) \\ &\quad + K_N(\omega, x) \nabla \phi_N(\omega, x)), \end{aligned} \quad (22)$$

where $\prod_N(\omega, x) = \frac{K_N(\omega, x) \nu}{g}$.

4. Sparse grid stochastic collocation method for the Stokes-Darcy model.

In this section, we present the sparse grid stochastic collocation method for the Stokes-Darcy model (17) -(21) with Beavers-Joseph (BJ) interface condition (22), where the random hydraulic conductivity $K(\omega, x)$ is represented by the truncated Karhunen-Loève expansion (16).

We first apply the stochastic collocation method to construct the discrete form in probability, and then use the sparse grid method to reduce the computational cost. To solve the deterministic Stokes-Darcy model at each collocation node (i.e., for each sample), we adopt the regular coupled finite element method (FEM).

4.1. Stochastic collocation method. Define a multi-index $\vec{p} = (p_1, p_2, \dots, p_N)$,

$$|\vec{p}| = \sum_{n=1}^N p_n,$$

$$\mathbf{P}_{p_n}(\mathbf{I}_n) = \text{span}\{y_n^k, k = 0, \dots, p_n\}, \quad n = 1, \dots, N, \quad (23)$$

$$\mathbf{P}_{\vec{p}}(\mathbf{I}^N) = \bigotimes_{n=1}^N \mathbf{P}_{p_n}(\mathbf{I}_n) \subset L_{\rho_Y}^2(\mathbf{I}^N). \quad (24)$$

For stochastic PDEs, one is often interested in determining statistical information about the quantities of interest, given statistical information about the inputs. For the stochastic Stokes-Darcy model, we design a numerical method to calculate the moments for determining statistical information about the quantities of interests. To construct the discrete form in the probability space, we choose a set of collocation points $\{\vec{y}_i\}_{i=1}^M \subset \mathbf{I}^N$ and a set of polynomials $\{l_k^{\vec{p}}(\vec{y})\}_{i=1}^M \subset \mathbf{P}_{\vec{p}}(\mathbf{I}^N)$ correspondingly, such as Lagrange polynomials. Then the discrete solutions in the probability space are

$$\phi_{N,\vec{p}}(\vec{y}, x) = \sum_{i=1}^M \phi_N(\vec{y}_i, x) l_k^{\vec{p}}(\vec{y}), \quad (25)$$

$$\vec{u}_{N,\vec{p}}(\vec{y}_i, x) = \sum_{i=1}^M \vec{u}_N(\vec{y}_i, x) l_k^{\vec{p}}(\vec{y}), \quad (26)$$

$$p_{N,\vec{p}}(\vec{y}_i, x) = \sum_{i=1}^M p_N(\vec{y}_i, x) l_k^{\vec{p}}(\vec{y}). \quad (27)$$

For the discretization in the physical space, the finite element method (FEM) is utilized. In other words, for each $\vec{y}_i (i = 1, \dots, M)$, $\phi_N(\vec{y}_i, x)$, $\vec{u}_N(\vec{y}_i, x)$ and $p_N(\vec{y}_i, x)$ can be approximated numerically by $\phi_{N,h}(\vec{y}_i, x)$, $\vec{u}_{N,h}(\vec{y}_i, x)$ and $p_{N,h}(\vec{y}_i, x)$ by using a deterministic discretization method in Ω . Then the final discrete solutions are

$$\phi_{N,h,\vec{p}}(\vec{y}, x) = \sum_{i=1}^M \phi_{N,h}(\vec{y}_i, x) l_k^{\vec{p}}(\vec{y}), \quad (28)$$

$$\vec{u}_{N,h,\vec{p}}(\vec{y}_i, x) = \sum_{i=1}^M \vec{u}_{N,h}(\vec{y}_i, x) l_k^{\vec{p}}(\vec{y}), \quad (29)$$

$$p_{N,h,\vec{p}}(\vec{y}_i, x) = \sum_{i=1}^M p_{N,h}(\vec{y}_i, x) l_k^{\vec{p}}(\vec{y}). \quad (30)$$

Let $w_k = \mathbb{E}(l_k^{\vec{p}})$. Then the expected value of functions $\gamma(\phi_N)$, $\gamma(\vec{u}_N)$, and $\gamma(p_N)$ can be obtained as follows.

$$\mathbb{E}(\gamma(\phi_N)) \approx \mathbb{E}(\gamma(\phi_{N,h,\vec{p}})) \approx \sum_{i=1}^M w_i \gamma(\phi_{N,h}(\vec{y}_i, x)), \quad (31)$$

$$\mathbb{E}(\gamma(\vec{u}_N)) \approx \mathbb{E}(\gamma(\vec{u}_{N,h,\vec{p}})) \approx \sum_{i=1}^M w_i \gamma(\vec{u}_{N,h}(\vec{y}_i, x)), \quad (32)$$

$$\mathbb{E}(\gamma(p_N)) \approx \mathbb{E}(\gamma(p_{N,h,\vec{p}})) \approx \sum_{i=1}^M w_i \gamma(p_{N,h}(\vec{y}_i, x)). \quad (33)$$

Here $w_i (i = 1, \dots, M)$ and $\vec{y}_i (i = 1, \dots, M)$ are the weights and nodes of some numerical quadratures, such as Gauss quadratures. One regular method for choosing $\vec{y}_i (i = 1, \dots, M)$ is the full tensor-product interpolation. Let $x_n^{j_n}, w_n^{j_n}, j_n = 1, 2, \dots, m_n$ be the quadrature nodes and weights over the interval $I_n, n = 1, 2, \dots, N$. Here $m_n \in \mathbb{N}$ is the number of quadrature nodes over the one dimensional interval $I_n, n = 1, 2, \dots, N$. Then the full tensor-product interpolation to calculate the expected value of function $\gamma(\phi_N)$ is given as

$$\mathbb{E}(\gamma(\phi_N)) \approx \sum_{j_1=1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \left(\prod_{\ell=1}^N w_\ell^{j_\ell} \right) \gamma(\phi_{N,h}(x_1^{j_1}, \dots, x_N^{j_N}, x)). \quad (34)$$

The set of collocation nodes is given as $\{\vec{y}_i\}_{i=1}^M = X_1 \times \cdots \times X_N \subset \mathbf{I}^N$ with $X_n = \{x_n^1, x_n^2, \dots, x_n^{m_n}\}, n = 1, 2, \dots, N$, and the total number of quadrature nodes is $M = \prod_{n=1}^N m_n$. When the dimension N of the truncated stochastic space becomes larger, the number M of quadrature nodes of full tensor product interpolation increases significantly. For each collocation node $\vec{y}_i$, we need to solve the corresponding deterministic Stokes-Darcy model one time to obtain the FEM approximations $\phi_{N,h}(\vec{y}_i, x), \vec{u}_{N,h}(\vec{y}_i, x), p_{N,h}(\vec{y}_i, x)$. Therefore the computational cost increases quickly when the dimension N of the truncated stochastic space becomes larger. This leads to a great need in an efficient numerical method using small number of collocation nodes to reduce the computational cost. One efficient technique to dramatically reduce this curse of dimensionality is sparse grid [15, 51, 102], which will be discussed in the next subsection.

4.2. Algorithm of sparse grid stochastic collocation method. Let $w_j^{i_n}, x_j^{i_n}, j = 1, 2, \dots, m_i \in \mathbb{N}$ be the weights and nodes of numerical quadrature with the accuracy level i over the one dimensional interval $I_n, n = 1, 2, \dots, N$ so that the order of polynomial exactness increases with i_n . For the numerical integral Q^{i_n} with different accuracy level i_n of the function f over the one dimensional interval I_n , the quadrature formulas are given as

$$Q^{i_n}(f) = \sum_{j=1}^{m_{i_n}} f(x_j^{i_n}) w_j^{i_n}, \quad n = 1, 2, \dots, N. \quad (35)$$

For the multivariate case $N > 1$, we define the quadrature formula by the full tensor product interpolation as

$$(Q^{i_1} \otimes \cdots \otimes Q^{i_N})(f) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_N=1}^{m_{i_N}} \left(\prod_{\ell=1}^N w_{\ell}^{j_{\ell}} \right) f(x_{j_1}^{i_1}, \dots, x_{j_N}^{i_N}). \quad (36)$$

The full tensor product interpolation (36) has the accuracy level in each dimension is $i_n, n = 1, 2, \dots, N$.

For the multi-index $\vec{i} = (i_1, i_2, \dots, i_N)$, we define $|\vec{i}| = \sum_{n=1}^N i_n$. For the numerical integral $Q^{\vec{i}}$, we denote

$$Q^0 = 0, \quad \Delta^i = Q^i - Q^{i-1}, \quad i \in \mathbb{N}. \quad (37)$$

Then the quadrature formula in the multivariate case $N > 1$ by the algorithm of Smolyak [15, 65, 90, 91, 102] with accuracy level $s \in \mathbb{N}$ to construct the sparse grid is given by

$$\mathcal{Q}(s, N) = \sum_{\ell=0}^{s-1} \sum_{|\vec{i}|=N+\ell} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_N}). \quad (38)$$

Equivalently, the analogous formula [65, 113] is

$$\mathcal{Q}(s, N) = \sum_{\ell=s-N}^{s-1} (-1)^{s-1-\ell} \binom{N-1}{s-1-\ell} \sum_{|\vec{i}|=N+\ell} (Q^{i_1} \otimes \cdots \otimes Q^{i_N}). \quad (39)$$

Let $X^i = \{x_1^i, x_2^i, \dots, x_{m_i}^i\}$ be the set of quadrature nodes for the numerical integral Q^i . Based on formula (36), the set of quadrature nodes for the full tensor numerical integral $(Q^{i_1} \otimes \cdots \otimes Q^{i_N})$ is $X^{i_1} \times \cdots \times X^{i_N}$. And the set of quadrature nodes for the algorithm of Smolyak to construct the sparse grid in the formula (38) or (39) is

$$\mathcal{H}(s, N) = \bigcup_{\ell=s-N}^{s-1} \bigcup_{|\vec{i}|=N+\ell} (X^{i_1} \times \cdots \times X^{i_N}). \quad (40)$$

There are many numerical integral methods to determine the quadrature weights and nodes over the one dimensional interval I_n for the formula (35). In this paper, we adopt the Gauss-Legendre abscissas (GQU) [15, 65, 90, 91] and Kronrod-Patterson abscissas (KPU) [65, 92–94] to generate the quadrature nodes set $X^i = \{x_1^i, x_2^i, \dots, x_{m_i}^i\}$. The quadrature nodes of the Gauss-Legendre abscissas are the roots of Legendre polynomials with the Gauss-Legendre quadrature rule. The quadrature nodes of Kronrod-Patterson abscissas are generated by the Kronrod-Patterson quadrature formula, which is a variant of Gaussian quadrature formula to derive the nested quadrature nodes. For either the Kronrod-Patterson abscissas or Gauss-Legendre abscissas, the number of the quadrature nodes in sparse grid (40) is much smaller than the number of the quadrature nodes in the full tensor product interpolation (36), since the sparse grid method is capable of attaining a high accuracy when only choosing lower level $\vec{i}$. Hence the computational expense is reduced significantly by the sparse grid method. We now apply the sparse grid using the Gauss-Legendre abscissas (GQU) and Kronrod-Patterson abscissas (KPU) to solve the Stokes-Darcy model (17)–(22). The details of using sparse grid method to solve the Stokes-Darcy model are given in the following Algorithm 1.

Algorithm 1 SG-SC algorithm

Input: The interpolation abscissas and the sparse grid accuracy level s .

- 1: Obtain the collocation points $\{\vec{y}_i\}_{i=1}^M$ and numerical quadrature weights $\{w_i\}_{i=1}^M$ from the Smolyak sparse grid of GQU and KPU.
- 2: For $i = 1, \dots, M$, independently solve the Stokes-Darcy model (17)-(22) on each collocation point $\vec{y}_i$. More precisely, the finite element solutions $\phi_{N,h}(\vec{y}_i, x)$, $\vec{u}_{N,h}(\vec{y}_i, x)$ and $p_{N,h}(\vec{y}_i, x)$ are obtained to approximate the solutions of the model

$$-\nabla \cdot (K_N(\vec{y}_i, x) \nabla \phi_N(\vec{y}_i, x)) = f_D, \quad (41)$$

$$-\nabla \cdot \mathbb{T}(\vec{u}_N(\vec{y}_i, x), p_N(\vec{y}_i, x)) = \vec{f}_S, \quad (42)$$

$$\nabla \cdot \vec{u}_N(\vec{y}_i, x) = 0, \quad (43)$$

$$\vec{u}_N(\vec{y}_i, x) \cdot \vec{n}_S = K_N(\vec{y}_i, x) \nabla \phi_N(\vec{y}_i, x) \cdot \vec{n}_D, \quad (44)$$

$$-\vec{n}_S \cdot (\mathbb{T}(\vec{u}_N(\vec{y}_i, x), p_N(\vec{y}_i, x)) \cdot \vec{n}_S) = g(\phi_N(\vec{y}_i, x) - z), \quad (45)$$

$$\begin{aligned} & -\tau_j \cdot (\mathbb{T}(\vec{u}_N(\vec{y}_i, x), p_N(\vec{y}_i, x)) \cdot \vec{n}_S) - \tau_j \cdot (\mathbb{T}(\vec{u}_N(\vec{y}_i, x), p_N(\vec{y}_i, x)) \cdot \vec{n}_S) \\ & = \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod_N)}} \tau_j \cdot (\vec{u}_N(\vec{y}_i, x) + K_N(\vec{y}_i, x) \nabla \phi_N(\vec{y}_i, x)). \end{aligned} \quad (46)$$

- 3: **Return:** Calculate the expected value:

$$\mathbb{E}[\phi_N](x) = \sum_{i=1}^M w_i \phi_N(\vec{y}_i, x), \quad (47)$$

$$\mathbb{E}[\vec{u}_N](x) = \sum_{i=1}^M w_i \vec{u}_N(\vec{y}_i, x), \quad (48)$$

$$\mathbb{E}[p_N](x) = \sum_{i=1}^M w_i p_N(\vec{y}_i, x). \quad (49)$$

4.3. Numerical algorithm for deterministic Stokes-Darcy model. In order to obtain the finite element solutions $\phi_{N,h}(\vec{y}_i, x)$, $\vec{u}_{N,h}(\vec{y}_i, x)$ and $p_{N,h}(\vec{y}_i, x)$ at the second step in Algorithm 1, we recall the finite element method to solve the deterministic Stokes-Darcy model at each collocation node (i.e., for each sample):

$$-\nabla \cdot (K \nabla \phi_D) = f_D, \text{ in } \Omega_D, \quad (50)$$

$$-\nabla \cdot \mathbb{T}(\vec{u}_S, p_S) = \vec{f}_S, \text{ in } \Omega_S, \quad (51)$$

$$\nabla \cdot \vec{u}_S = 0, \text{ in } \Omega_S, \quad (52)$$

$$\vec{u}_S \cdot \vec{n}_S = -\vec{u}_D \cdot \vec{n}_D, \text{ on } \Gamma, \quad (53)$$

$$-\vec{n}_S \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = g(\phi_D - z), \text{ on } \Gamma, \quad (54)$$

$$-\tau_j \cdot (\mathbb{T}(\vec{u}_S, p_S) \cdot \vec{n}_S) = \frac{\alpha \nu \sqrt{\mathbf{d}}}{\sqrt{\text{trace}(\prod)}} \tau_j \cdot (\vec{u}_S - \vec{u}_D), \text{ on } \Gamma. \quad (55)$$

From the assumptions in Section 2, the hydraulic head ϕ_D and the fluid velocity $\vec{u}_S$ satisfy homogeneous Dirichlet boundary condition except on Γ , i.e., $\phi_D = 0$ on the boundary $\partial\Omega_D \setminus \Gamma$ and $\vec{u}_S = 0$ on the boundary $\partial\Omega_S \setminus \Gamma$.

The spaces that we utilize are

$$\begin{aligned} X_S &= \{\vec{v} \in [H^1(\Omega_S)]^d \mid \vec{v} = 0 \text{ on } \partial\Omega_S \setminus \Gamma\}, \\ Q_S &= L^2(\Omega_S), \\ X_D &= \{\psi \in H^1(\Omega_D) \mid \psi = 0 \text{ on } \partial\Omega_D \setminus \Gamma\}. \end{aligned}$$

For the domain D ($D = \Omega_S$ or Ω_D), $(\cdot, \cdot)_D$ denotes the L^2 inner product on the domain D , and $\langle \cdot, \cdot \rangle$ denotes the L^2 inner product on the interface Γ or the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$. Let P_τ denote the projection onto the tangent space on Γ , i.e.,

$$P_\tau \vec{u} = \sum_{j=1}^{d-1} (\vec{u} \cdot \tau_j) \tau_j.$$

Define the following bilinear terms.

$$\begin{aligned} a_D(\phi_D, \psi) &= (K \nabla \phi_D, \nabla \psi)_{\Omega_D}, \\ a_S(\vec{u}_S, \vec{v}) &= 2\nu(\mathbb{D}(\vec{u}_S), \mathbb{D}(\vec{v}))_{\Omega_S}, \\ b_S(\vec{v}, q) &= -(\nabla \cdot \vec{v}, q)_{\Omega_S}. \end{aligned}$$

We assume that we have in hand regular subdivisions of Ω_D and Ω_S into finite elements. Then one can define finite element spaces $X_{Dh} \subset X_D$, $X_{Sh} \subset X_S$, and $Q_{Sh} \subset Q_S$. We assume that X_{Sh} and Q_{Sh} satisfy the inf-sup condition [53, 56]

$$\inf_{0 \neq q \in Q_{Sh}} \sup_{0 \neq \vec{v} \in X_{Sh}} \frac{b_S(\vec{v}, q)}{\|\vec{v}\|_1 \|q\|_0} > \gamma, \quad (56)$$

where $\gamma > 0$ is a constant independent of h ; this condition is needed to ensure that the spatial discretizations of the Stokes system used here are stable, see [53, 56] for details and many examples of pairs of finite element spaces X_{Dh} , X_{Sh} , and Q_{Sh} that satisfy (56). One typical example is the Taylor-Hood element pair that we use in the numerical experiments; for that pair, X_{Dh} and X_{Sh} consist of continuous piecewise quadratic polynomials and Q_{Sh} consists of continuous piecewise linear polynomials.

Then we recall the finite element formulation of the Stokes-Darcy problem as follows: [25, 26]: find $(\vec{u}_h, p_h) \in X_{Sh} \times Q_{Sh}$ and $\phi_h \in X_{Sh}$ such that

$$\begin{aligned} a_S(\vec{u}_h, \vec{v}_h) + b_S(\vec{v}_h, p_h) + a_D(\phi_h, \psi_h) + \langle g\phi_h, \vec{v}_h \cdot \vec{n}_S \rangle \\ - \langle \vec{u}_h \cdot \vec{n}_S, \psi_h \rangle + \frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}} \langle P_\tau(\vec{u}_h + K\nabla\phi_h), P_\tau\vec{v}_h \rangle \end{aligned} \quad (57)$$

$$\begin{aligned} &= (f_D, \psi_h)_{\Omega_D} + (\vec{f}, \vec{v}_h)_{\Omega_S} + \langle gz, \vec{v}_h \cdot \vec{n}_S \rangle, \quad \forall \vec{v}_h \in X_{Sh}, \psi_h \in X_{Dh}, \\ &b_S(\vec{u}_h, q_h) = 0, \quad \forall q_h \in Q_{Sh}. \end{aligned} \quad (58)$$

5. Computational examples. We consider the model problem (3)-(8) on $\Omega = [0, 1] \times [-0.25, 0.75]$, where $\Omega_D = [0, 1] \times [0, 0.75]$ and $\Omega_S = [0, 1] \times [-0.25, 0]$. For the physical parameters, we choose $\frac{\alpha\nu\sqrt{d}}{\sqrt{\text{trace}(\Pi)}} = 1$, $\nu = 1$, $g = 1$, and $z = 0$.

The boundary condition data functions and the source terms are

$$\begin{aligned} f_D &= -\pi^3 \sin(\pi x)(-y + \cos(\pi(1-y))) - (2 - \pi \sin(\pi x))(-\pi^2 \cos(\pi(1-y))), \\ \vec{f}_{S1} &= -2\nu x^2 - 2\nu y^2 - \nu e^{-y} + \pi^2 \cos(\pi x) \cos(2\pi y), \\ \vec{f}_{S2} &= 4\nu xy - \nu \pi^3 \sin(\pi x) + 2\pi(2 - \pi \sin(\pi x)) \sin(2\pi y), \\ \phi_D &= (2 - \pi \sin(\pi x))(-y + \cos(\pi(1-y))), \quad \text{on } \partial\Omega_D \setminus \Gamma, \\ \vec{u}_S &= (x^2 y^2 + e^{-y}, -2xy^3/3 + 2 - \pi \sin(\pi x)), \quad \text{on } \partial\Omega_S \setminus \Gamma. \end{aligned}$$

For the discrete form in physical space by the finite element method, we use a uniform grid with grid size $h = 1/32$. The Taylor-Hood element pair is used for the Stokes system and the quadratic finite element is used for the primary formulation of the Darcy system.

By the truncated KL expansion, the random hydraulic conductivity $K(\omega, x)$ is given as [91]

$$\ln(K(\omega, x) - 0.5) = 1 + Z_1(\omega) \left(\frac{\sqrt{\pi}L}{2} \right)^{1/2} + \sum_{n=2}^N \zeta_n \phi_n(x) Z_n(\omega), \quad (59)$$

where

$$\zeta_n = (\sqrt{\pi}L)^{1/2} \exp\left(\frac{-(\lfloor n/2 \rfloor \pi L)^2}{8}\right), \quad n > 1, \quad (60)$$

and

$$\phi_n(x) = \begin{cases} \sin\left(\frac{\lfloor n/2 \rfloor \pi x}{L_p}\right), & \text{if } n \text{ is even,} \\ \cos\left(\frac{\lfloor n/2 \rfloor \pi x}{L_p}\right), & \text{if } n \text{ is odd.} \end{cases} \quad (61)$$

Here $\lfloor \cdot \rfloor$ is the floor function giving as output the greatest integer less than or equal to the input. In this example, the random variables $\{Z_n(\omega)\}_{n=1}^N$ are independent, and have zero mean and unit variance, i.e., $\mathbb{E}[Z_n] = 0$ and $\mathbb{E}[Z_n Z_m] = \delta_{nm}$ for $n, m \in N+$. We assume that the random variables $\{Z_n(\omega)\}_{n=1}^N$ are uniformly distributed in the interval $[-\sqrt{3}, \sqrt{3}]$. Let L_c be a desired physical correlation length for the coefficient K . Then the parameter L_p is $L_p = \max\{1, 2L_c\}$ and the parameter L is $L = L_c/L_p$. The decay rate of the eigenvalues λ_n depends on the correlation length L_c .

For evaluating the accuracy of numerical solutions with different accuracy level s in the sparse grid method, we generate the reference solutions with the mesh size $h = 1/32$ and $s = 7$, when we choose $N = 5$ in the truncated Karhunen-Loève expansion. And the reference solutions are constructed with the mesh size $h = 1/32$ and $s = 6$ when we choose $N = 10$. In the Table 1 and Table 2, the number of sparse grid nodes with different accuracy level s are listed for $N = 5$ and $N = 10$, respectively. It is obvious that significant computational cost can be saved if the cases with smaller s can still provide accurate enough solutions.

TABLE 1. Number of sparse grid nodes with different accuracy level s when $N = 5$

s	2	3	4	5	6	7
KPU	11	51	151	391	903	1743
GQU	11	61	241	781	2203	5593

TABLE 2. Number of sparse grid nodes with different accuracy level s when $N = 10$

s	2	3	4	5	6
KPU	21	201	1201	5281	19105
GQU	21	221	1581	8761	40405

We exhibit the convergence of the numerical errors in L^2 norm of the mean value and the variance of the velocity in Figure 2 and Figure 3 with $L_c = 1/64$. It is observed that errors in L^2 norm decrease fast as the parameter s increases. Based on this convergence performance to the reference case ($s = 7$ for $N = 5$ or $s = 6$ for $N = 10$), it is accurate enough to choose smaller s in the sparse grid algorithm to calculate the numerical solutions with much less computational cost.

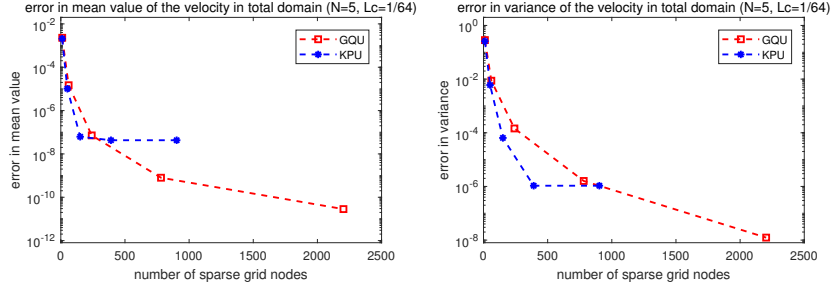


FIGURE 2. The convergence in L^2 norm for the expected value (left) and the variance (right) of velocity with $N = 5$ and $L_c = 1/64$.

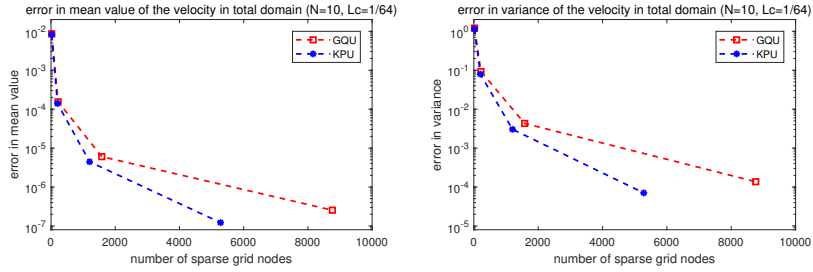


FIGURE 3. The convergence in L^2 norm for the expected value (left) and the variance (right) of velocity with $N = 10$ and $L_c = 1/64$.

To investigate the influence of correlation length L_c , we show the convergence of the numerical errors in L^2 norm of the mean value and the variance of the velocity with different correlation lengths L_c in Figure 4 for $N = 5$ and GQU sparse grid

nodes. As expected, the larger correlation lengths L_c slow down the convergence. But it is still accurate enough to choose smaller s in the sparse grid algorithm to calculate the numerical solutions with much less computational cost.

In Figure 5, we show the convergence of the numerical errors in L^2 norm of the mean value and the variance of the velocity with different correlation lengths L_c for $N = 10$ and GQU sparse grid nodes. Similar to the case of $N = 5$, numerical errors in L^2 norm decrease when s becomes larger.

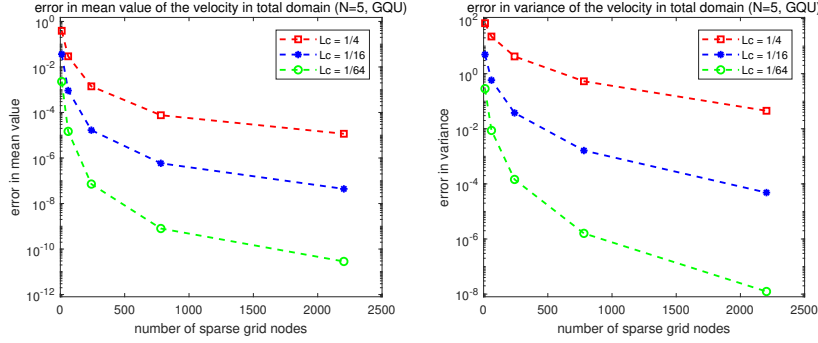


FIGURE 4. The convergence in L^2 norm for the expected value (left) and the variance (right) of velocity with $N = 5$, GQU method, and different correlation length L_c .

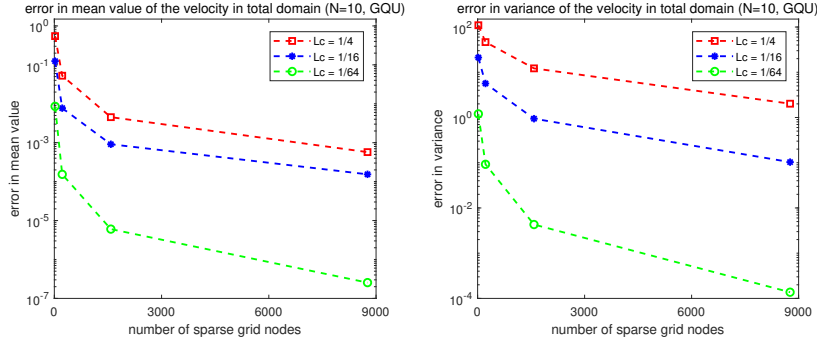


FIGURE 5. The convergence in L^2 norm for the expected value (left) and the variance (right) of velocity with $N = 10$, GQU method, and different correlation length L_c .

In Figure 6, we pick three samples to illustrate the influence of the random hydraulic conductivity K on the flow field. Even though all the set-up inside the free flow subdomain Ω_S is deterministic, it is expected to observe that the randomness is passed from the porous media subdomain Ω_D to the free flow subdomain Ω_S through the interface conditions which are affected by the randomness in Ω_D . To further illustrate the randomness in porous media subdomain Ω_D and free flow

subdomain Ω_S , we show the variance of the speed in total domain in Figure 7. One can see that the variance of speed in porous media subdomain Ω_D is much larger than the variance of speed in free flow subdomain Ω_S .

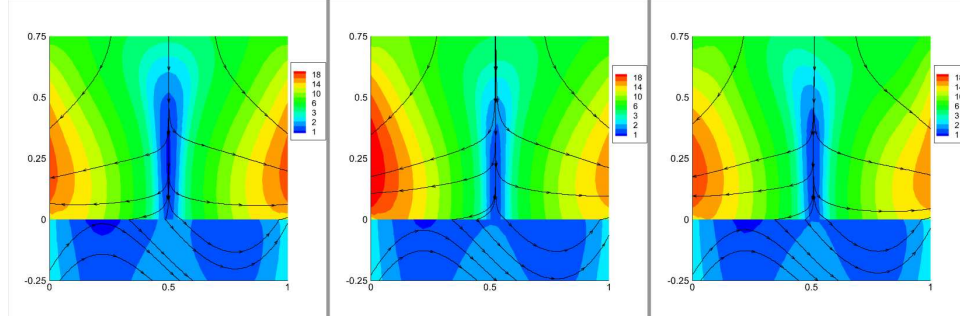


FIGURE 6. Numerical solutions of three samples of GQU with $N = 10$ and $s = 6$. The color represents the speed of flow and the streamlines show the direction of the flow.

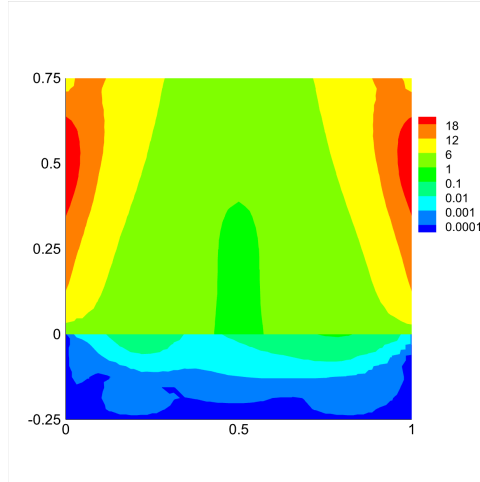


FIGURE 7. Variance of the speed of samples of GQU with $N = 10$ and $s = 6$ in total domain. The color represents the variance of the speed.

6. Conclusions. In this paper we propose a sparse grid stochastic collocation method for a stochastic Stokes-Darcy model with Beavers-Joseph interface condition. Truncated Karhunen-Loève expansion is used to approximate the random field of the hydraulic conductivity. Gauss-Legendre abscissas (GQU) and Kronrod-Patterson abscissas (KPU) are used in the construction of the Smolyak sparse grid formula. The numerical results show that this method is convergent and a small number of sparse grid nodes can be used to calculate the numerical solutions with

enough accuracy and much less computational cost. The influence of the interface conditions on the randomness transmission is also illustrated.

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