$w_{1+\infty}$ Algebra and the Celestial Sphere: Infinite Towers of Soft Graviton, Photon, and Gluon Symmetries

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It is shown that the infinite tower of tree-level plus-helicity soft graviton symmetries in asymptotically flat 4D quantum gravity can be organized into a single chiral 2D Kac-Moody symmetry based on the wedge algebra of $w_{1+\infty}$, which naturally acts on the celestial sphere at null infinity. The infinite towers of soft photon or gluon symmetries also transform irreducibly under $w_{1+\infty}$.

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Introduction.—A central problem in physics is to find all the fundamental nontrivial symmetries of nature implied by all the experimentally verified physical laws. By this we mean symmetries associated via Noether's theorem to conservation laws with observable consequences. A hundred years ago the answer to this question would have been a short list including Poincare symmetries, (which lead to energy-angular momentum conservation) plus a few more. (We do not consider here the rich variety of emergent symmetries also observable in nature.)

In the 1960s BMS [1,2] showed that the answer cannot be so simple because there is no sense in which the diffeomorphism group of general relativity (GR) in asymptotically flat spacetimes can be reduced to the Poincare group. They did not however (due to then uncertainties about the structure of asymptotic infinity) either identify an alternate larger asymptotic symmetry group of the full past and future spacetime or associate any observable conservation laws. Recently this problem has been translated into the language of quantum field theory and Feynman diagrams where it becomes equivalent to identifying soft theorems. A soft theorem is a linear relation between scattering amplitudes (in asymptotically past and future flat spacetimes) in which one particle becomes "soft" in that its energy is taken to zero. Such theorems can always be recast as a conservation-law-implying symmetry[3–5]. Completing the program of BMS, an infinite number of exact conservation laws were thereby discovered which relate arbitrary moments of ingoing and outgoing energy-momentum fluxes to measurable gravitational memory effects [6].

These developments were satisfying but far from the end of the story. Soft theorems abound in both gauge theory and gravity, with more being discovered only recently, and each associated with an infinite number of symmetries and measurable conservation laws. Moreover, the known symmetries do not close under commutation, implying an infinite tower of soft theorems [7–10].

Hence despite all the progress, finding all the symmetries and conservation laws implied by the standard model plus GR remains an outstanding open problem.

In this Letter we solve the problem in the limited context of the tree-level approximation with vanishing cosmological constant. Moreover we make the significant restriction to symmetries associated with plus-helicity soft particles only. Under these conditions we show that the soft symmetries can be succinctly described by a certain well-known infinite-dimensional w-symmetry group.

We rely heavily on recent progress made in Ref. [11] for tree-level gravity and gauge theory. In this work plus-helicity soft symmetries were compactly represented by 2D (higher-spin) currents in the celestial conformal field theory (CCFT) (This Letter employs a bottom-up approach in which the CCFT is simply defined by a Mellin transform of gauge and gravity scattering amplitudes whose conformal properties are deduced from soft theorems. This differs from the top-down approach usually employed in the AdS/CFT correspondence, where the boundary CFT can be constructed from first principles using string theory. A top down approach is not available here as we strive to describe the real world for which the complete microscopic fundamental laws are unknown.) living on the celestial sphere at null infinity. An infinite tower of such currents and their algebra were derived at tree level using positive-helicity soft theorems and the celestial operator product expansion (OPE).

The somewhat lengthy results of Ref. [11] were displayed in a basis with manifest covariance under the global Lorentz-conformal group of the celestial sphere. The currents include an "$SL(2,\mathbb{R})_{\rm w}$ " Kac-Moody algebra arising from the subleading soft graviton theorem. In this Letter we find dramatic simplifications by reorganizing the generators according to representations of this $SL(2,\mathbb{R})_{\rm w}$, which is accomplished with a version of the light ray transform [12]. The entire tower of currents is assembled into a chiral Kac-Moody symmetry of the wedge algebra of $\mathrm{w}_{1+\infty}$! This same Kac-Moody algebra has appeared previously including in the Penrose twistor construction [13], in discrete states of the $c=1$ string [14] and $\mathrm{w}_{1+\infty}$ gravity [15]. Moreover we find, unlike in Ref. [11], in the $SL(2,\mathbb{R})_{\rm w}$ covariant presentation only positive half-integral weights appear.

In the next section we present explicitly the algebraic transformation from the conventional to the $SL(2,\mathbb{R})_{\rm w}$ covariant basis, and show that the entire algebra is the Kac-Moody symmetry of the wedge algebra of $\mathrm{w}_{1+\infty}$. In section 3 the infinite tower of soft symmetries for gauge theory are written in $SL(2,\mathbb{R})_{\rm w}$ covariant form and again found to dramatically simplify. Again only positive $SL(2,\mathbb{R})_{\rm w}$ weights appear. These results suggest that $\mathrm{w}_{1+\infty}$—or perhaps its quantization $\mathrm{W}_{1+\infty}$ [15]—provides an organizing principle for CCFT. We conclude with speculations on this role.

Gravity.—Let us recap the basic results and notation of Ref. [11] for gravity. Let $G^+_\Delta(z,\bar z)$ denote the positive-helicity conformal-primary graviton operator with 2D conformal weight Δ which crosses the celestial sphere at a point $(z,\bar z)$. Define a discrete family of conformally soft (The conformally soft gravitons defined here differ from the usual energetically soft gravitons in that the conformal weight, rather than the energy, is taken to a limiting value. Nevertheless they obey soft theorems which mirror their energetically defined counterparts [16].) positive-helicity gravitons

$$H^k = \lim_{\varepsilon\to 0}\varepsilon G^+_{k+\varepsilon}, \qquad k = 2,1,0,-1,\ldots, \tag{1}$$

with weights

$$(h,\bar h) = \left(\frac{k+2}{2},\frac{k-2}{2}\right), \tag{2}$$

and a consistently truncated antiholomorphic mode expansion (Outside the specified range of n, the $SL(2,\mathbb{R})_R$-invariant norm vanishes. Such operators may still have contact interactions but in this Letter operators are always at distinct points.)

$$H^k(z,\bar z) = \sum_{n=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_n(z)}{\bar z^{n+\frac{k-2}{2}}}. \tag{3}$$

Each $H^k_n(z)$ is a 2D symmetry-generating conserved current whose Ward identity is a soft theorem. The factor of ε in (1) cancels the soft poles, allowing a finite OPE. Throughout this Letter z and $\bar z$ are treated as independent, which amounts to continuing (3,1) Minkowski space to (2,2) Klein space, the celestial sphere to the celestial torus [17] and Lorentz-$SL(2,\mathbb{C})$ to $SL(2,\mathbb{R})_L\otimes SL(2,\mathbb{R})_R$. The n index in Eq. (3) then transforms in the $3-k$ dimensional representation of the $SL(2,\mathbb{R})_R$. The simplest example of the $k=1$ term generates supertranslations. Expanding $H^1_{\pm\frac12}(z)=\sum_m H^1_{\pm\frac12,m}z^{-m-3/2}$, the four modes $H^1_{\pm\frac12,\pm\frac12}$ generate the four global translations. $k=0$ is related to superrotations and includes Lorentz transformations. Defining the commutator for holomorphic objects (Note that this is a 2D celestial commutator on a 1D circle, not to be mistaken for a 4D commutator on a 3D slice [18].)

$$[A,B](z) = \oint_z \frac{dw}{2\pi i}A(w)B(z), \tag{4}$$

the soft current algebra found in Ref. [11] for gravity is (We note that the (2,0) current $H^2_0(z)$ commutes with all other generators and is a central term in the supertranslation algebra. It has been taken to vanish in most applications but is natural to include here.)

$$[H^k_m, H^l_n] = -\frac{\kappa}{2}[n(2-k)-m(2-l)]\frac{(\frac{2-k}{2}-m+\frac{2-l}{2}-n-1)!}{(\frac{2-k}{2}-m)!(\frac{2-l}{2}-n)!}\frac{(\frac{2-k}{2}+m+\frac{2-l}{2}+n-1)!}{(\frac{2-k}{2}+m)!(\frac{2-l}{2}+n)!}H^{k+l}_{m+n}. \tag{5}$$

To write the algebra (5) in a simpler form we define

$$w^p_n = \frac{1}{\kappa}(p-n-1)!(p+n-1)!H^{-2p+4}_n. \tag{6}$$

This is essentially the light transform or right shadow [12] in one dimension adapted to finite $SL(2,\mathbb{R})$ representations, and p is essentially the right-shadowed weight. Equation (5) then becomes

$$[w^p_m, w^q_n] = [m(q-1)-n(p-1)]w^{p+q-2}_{m+n}. \tag{7}$$

Since the index k on H^k_m runs over $k=2,1,0,\ldots$, p runs over the positive half integral values

$$p = 1,\frac32,2,\frac52,\ldots. \tag{8}$$

The restriction $[(k-2)/2]\le m\le[(2-k)/2]$ becomes [For example, for the $p=2$ Virasoro case this restricts to the $SL(2,\mathbb{R})$ current subalgebra.]

$$1-p \leq m \leq p-1. \tag{9}$$

This is one of our main results.

The commutators (7) were first written down by Bakas in 1989 [19]. In this work m is an arbitrary integer and the resulting algebra is now referred to as $\mathrm{w}_{1+\infty}$. The closed $p=2$ subalgebra of $\mathrm{w}_{1+\infty}$ is the $c=0$ Virasoro algebra. This algebra in Eq. (7) is a subgroup of $\mathrm{w}_{1+\infty}$ with m restricted according to Eq. (9). The resulting restricted algebra is known as the wedge subalgebra of $\mathrm{w}_{1+\infty}$, and can also be rewritten as $GL(\infty,\mathbb{R})$ [15].

Moreover, each $w^p_m(z)$ acts as a current on the celestial sphere. Hence we have a $GL(\infty,\mathbb{R})$ Kac-Moody algebra. The modes of the Kac-Moody currents act on the states of the 2D CCFT associated by the state-operator correspondence to operator insertions in the celestial sphere [18]. In fact, precisely this Kac-Moody algebra has been studied previously both as the symmetry group of $c=1$ string theory [14] and 2D $\mathrm{w}_{1+\infty}$ gravity [20]. (Here we take the classical limit $\mathrm{W}_{1+\infty}\to\mathrm{w}_{1+\infty}$ of the quantum symmetry $\mathrm{W}_{1+\infty}$.) A review of this and other fascinating aspects of W algebras can be found in Ref. [21]. We return to this connection at the end of the Letter.

The closed subgroup generated by

$$L_m \equiv w^2_m, \qquad m=-1,0,1 \tag{10}$$

is an $SL(2,\mathbb{R})_{\mathrm{w}}$ current algebra implied by the subleading soft graviton theorem. It is related to the original generators in Eq. (5) by

$$L_0 = \frac{1}{\kappa}H^0_0, \qquad L_{\pm1} = \frac{2}{\kappa}H^0_{\pm1}. \tag{11}$$

w^q_n transforms under this $SL(2,\mathbb{R})_{\mathrm{w}}$ as

$$[L_m, w^q_n] = [m(q-1)-n]w^q_{m+n}, \tag{12}$$

like the modes of an $SL(2,\mathbb{R})_{\mathrm{w}}$ primary operator of $SL(2,\mathbb{R})_{\mathrm{w}}$ weight q. Since $q\geq 1$ here, these are all positive. However, unlike the H^k_n, they do not transform canonically under the original $SL(2,\mathbb{R})_R$ because of the mode-dependent relation between the two. Indeed the normalization of the H^k_n modes was chosen so that they transform canonically under the original $SL(2,\mathbb{R})_R$ conformal generators with weights $\bar{h}=[(k-2)/2]$.

Evidently $SL(2,\mathbb{R})_R$ and $SL(2,\mathbb{R})_{\mathrm{w}}$ are not quite the same thing. w^p_n on the left-hand side of (6) lies in a positive weight $h_w = p$ representation of $SL(2,\mathbb{R})_{\mathrm{w}}$, while H^{-2p+4}_n on the right-hand side lies in a $2p+1$ dimensional, negative weight $\bar{h}=1-p$ representation of $SL(2,\mathbb{R})_R$. Equation (6) is the relation between them. Both representations are $2p+1$ dimensional, because finite weight h $SL(2,\mathbb{R})$ representations are $2h-1$ dimensional for positive half integral h and $-2h-1$ dimensional for negative half integral h.

Significantly, the $SL(2,\mathbb{R})_{\mathrm{w}}$ representations which appear here are all positive weight.

Gauge theory.—Now we turn to non-Abelian gauge theory. Let $O^{a,+}_k(z,\bar{z})$ denote a positive helicity, conformal weight k gluon operator with adjoint group index a at the point $(z,\bar{z})$ on the celestial sphere. Mode expanding in $\bar{z}$ on the right

$$O^{a,+}_k(z,\bar{z}) = \sum_n \frac{O^{a,+}_{k,n}(z)}{\bar{z}^{n+\frac{k-1}{2}}}, \tag{13}$$

conformally soft currents are defined by

$$R^{k,a}_n(z) \coloneqq \lim_{\varepsilon\to0}\varepsilon O^{a,+}_{k+\varepsilon,n}(z), \qquad k=1,0,-1,-2,\ldots,$$
$$\frac{k-1}{2}\leq n\leq\frac{1-k}{2}. \tag{14}$$

This has $SL(2,\mathbb{R})_L\otimes SL(2,\mathbb{R})_R$ weights

$$(h,\bar{h}) = \left(\frac{k+1}{2},\frac{k-1}{2}\right). \tag{15}$$

These values of conformal weights $\Delta=k$ include all the conformally soft poles encountered in the OPE [22,23]. The factor of ε incorporated in Eq. (14) is needed to cancel these poles, leading to finite OPEs for the rescaled $R^{k,a}$.

The soft current algebra for gauge theory is [11]

$$[R^{k,a}_n, R^{l,b}_{n'}] = -if^{ab}{}_c\frac{(\frac{1-k}{2}-n+\frac{1-l}{2}-n')!}{(\frac{1-k}{2}-n)!(\frac{1-l}{2}-n')!}\frac{(\frac{1-k}{2}+n+\frac{1-l}{2}+n')!}{(\frac{1-k}{2}+n)!(\frac{1-l}{2}+n')!}R^{k+l-1,c}_{n+n'}. \tag{16}$$

Let us define

$$S^{q,a}_m = (q-m-1)!(q+m-1)!R^{3-2q,a}_m, \tag{17}$$

where $q=1,\frac{3}{2},2,\ldots$. One then finds the simple algebra

$$[S^{q,a}_n, S^{p,b}_{n'}] = -if^{ab}{}_cS^{q+p-1,c}_{n+n'}. \tag{18}$$

Moreover using [11]

$$[H^k_m, R^{l,a}_n] = -\frac{\kappa}{2}[n(2-k) - m(1-l)]\frac{(\frac{2-k}{2} - m + \frac{1-l}{2} - n - 1)!}{(\frac{2-k}{2} - m)!(\frac{1-l}{2} - n)!}\frac{(\frac{2-k}{2} + m + \frac{1-l}{2} + n - 1)!}{(\frac{2-k}{2} + m)!(\frac{1-l}{2} + n)!}R^{k+l,a}_{m+n}, \tag{19}$$

one finds the irreducible representation

$$[w^p_m, S^{q,a}_n] = [m(q-1) - n(p-1)]S^{p+q-2,a}_{m+n}. \tag{20}$$

The $SL(2,\mathbb{R})_{\mathrm{w}}$ transformation of S^q_m

$$[L_m, S^{q,a}_n] = [m(q-1) - n]S^{q,a}_{m+n}. \tag{21}$$

is that of a primary of weight q. Again we find only positive weights.

Speculations.—The appearance of W algebras in the 2D celestial symmetry group connects celestial holography to several other research areas. We find it irresistible to speculate on what might lie ahead as these connections unfold.

$\mathrm{w}_{1+\infty}$ has a natural deformation to $W_{1+\infty}$, a significantly more complicated algebra. This deformation can be understood as arising from quantization. In the context of specific classical 2D realizations of $\mathrm{w}_{1+\infty}$, anomalies encountered in quantization deform the classical $\mathrm{w}_{1+\infty}$ algebra to the quantum $W_{1+\infty}$ [24]. While currently not well understood, the tree-level soft algebra of 4D quantum gravity also gets deformed on quantization, as implied among other things by one-loop corrections to the soft theorems. So it is natural to speculate that the action of $\mathrm{w}_{1+\infty}$ on this soft algebra is deformed to $W_{1+\infty}$ in the 4D quantum theory of gravity.

One might have expected a Virasoro algebra rather than an $SL(2,\mathbb{R})_{\mathrm{w}}$ current algebra among the symmetries. However the two are closely related in many similar contexts. For example, the Chern-Simon formulation of AdS_3 gravity, at first sight gives an $SL(2,\mathbb{R})_R \otimes SL(2,\mathbb{R})_L$ current algebra on the boundary. However, the AdS_3 boundary conditions implement a Hamiltonian reduction to $\mathrm{Vir}_R \otimes \mathrm{Vir}_L$ [25,26]. (A similar reduction could perhaps be operative here from constraints related to IR divergences.) The Virasoro generators are field-dependent $SL(2,\mathbb{R})$ transformations which lie in the enveloping algebra of the $SL(2,\mathbb{R})_R \otimes SL(2,\mathbb{R})_L$ current algebra. In another example, 2D gravity in light cone gauge exhibits an $SL(2,\mathbb{R})_R$ current algebra [27], but the same theory in conformal gauge exhibits a Virasoro symmetry [28], again indicating a relation between the two symmetry actions.

This type of relation has an infinite-dimensional uplift to the present context. $SL(2,\mathbb{R})$ is the wedge algebra of Virasoro, just as $GL(\infty,\mathbb{R})$ is the wedge algebra of $\mathrm{w}_{1+\infty}$ [15]. Just as Virasoro lies in the enveloping algebra of $SL(2,\mathbb{R})$ Kac-Moody, $\mathrm{w}_{1+\infty}$ (or $W_{1+\infty}$ at the quantum level) lies in the enveloping algebra of $GL(\infty,\mathbb{R})$ Kac-Moody [20]. So it is natural to speculate that $W_{1+\infty}$ is a celestial symmetry of 4D quantum gravity.

$\mathrm{w}_{1+\infty}$ also appears as the symmetry group of classical self-dual gravity in (2,2) signature Klein space [29], whose sole degree of freedom is given by the Kahler potential. The self-duality suggests that the dual CCFT should be chiral, and acted on (at bulk tree level) by $\mathrm{w}_{1+\infty}$. This is a natural context in which to study celestial holography, because the complicated interactions of opposite chirality sectors and double soft-limit ambiguities are absent. The $N = 2$ string describes the quantization of this theory [30], where $\mathrm{w}_{1+\infty}$ is potentially deformed to $W_{1+\infty}$. Perhaps the holographic CCFT dual of $N = 2$ string theory in 4D Klein space is given by a 2D quantum $W_{1+\infty}$ -gravity theory on the celestial torus.

We leave these thoughts to future explorations.

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