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Perfect submonoids of dominant weights

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Let G be a connected semisimple group. Vinberg introduced the notion of perfect submonoids of dominant weights of G in the study of Vinberg monoids. In this paper, we give explicit descriptions of the perfect submonoids.

1. Introduction

1.1. Perfect submonoids

Let K be an algebraic closed field of characteristic 0 and G be a connected reductive group over K. Let T be a maximal torus of G. Denote the weight lattice and the root lattice of G by $X^*(T)$ and Q. Let $X^*_+(T)$ be the set of dominant weights of G. For any λ in $X^*_+(T)$, we let $L(\lambda)$ be the irreducible representation of G with highest weight λ. For any two dominant weights λ, μ, define

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$$X(\lambda,\mu) = \{\nu \in X^*_+(T) \mid L(\nu) \text{ is a direct summand of } L(\lambda) \bigotimes L(\mu)\}.$$

In the study of reductive monoids, Vinberg introduced the following definition.

Definition 1.1. [12, §1] A submonoid L of the additive group of dominant weights is called *perfect* if

$$\lambda, \mu \in L \text{ implies } X(\lambda,\mu) \subset L.$$

In this paper, we give a complete characterization of perfect submonoids of dominant weights for connected semisimple groups. We also discuss the perfect submonoids for reductive groups.

1.2. Main results

The main result of this paper is the following.

Theorem A. *Let G be a connected semisimple algebraic group with a maximal torus T.*

a) The perfect submonoids of $X^*_+(T)$ with full component support are exactly the intersections of $X^*_+(T)$ with sublattices of $X^*(T)$ containing Q.

b) There is a natural bijection between the perfect submonoids of $X^*_+(T)$ with full component support and the subgroups of the center of G.

We refer to Definition 3.6 and Definition 3.11 for the definition of component support. Based on Theorem A, one can deduce the characterization for arbitrary perfect submonoids of dominant weights.

1.3. Strategy of the proof

We first reduce the general case to simply connected case by considering the simply connected cover. By applying *PRV conjecture* [10][7, Theorem 2.10], we show that if L is a nonzero perfect submonoid of dominant weights, then for any dominant weight λ in L, the dominant weights which are also weights of $L(\lambda)$ are all contained in L. We define the component support for each submonoid of dominant weights. Then we relate the perfect submonoids of dominant weights with full component support to the subgroups of the cocenter and prove Theorem A in simply connected case.

Then we prove Theorem A based on simply connected case. The general case for arbitrary perfect submonoids of dominant weights can be deduced from Theorem A.

At the end of the paper, we look at the connected reductive groups. We also compare our results with the classification of reductive monoids in [12].

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2. Preliminaries

2.1. Basic facts about algebraic groups

Recall that K is algebraically closed of characteristic 0 and G is a connected reductive algebraic group over K. Let T be a maximal torus of G. The *root datum* of G is a quadruple $(X^*(T), R, X_*(T), R^\vee)$, where $X^*(T)$ is the weight lattice, $X_*(T)$ is the coweight lattice, R is the set of roots and $R^\vee$ is the corresponding set of coroots.

Let $Q = \mathbb{Z}R$ be the root lattice of G. Let $V = X^*(T) \otimes \mathbb{R}$, there is a natural pairing $\langle , \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ and $P := \{x \in V \mid \langle x, R^\vee\rangle \subset \mathbb{Z}\}$. Then $Q \subset X^*(T) \subset P$. If G is simply connected, then $X^*(T) = P$.

Choose the set of positive roots $R_+ \subset R$. Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset R_+$ be the set of simple roots. The fundamental dominant weights with respect to Δ are $\omega_1, \omega_2, ..., \omega_n$. For any two weights λ, μ in $X^*(T)$, write $\mu \preceq \lambda$ if $\mu = \lambda - \sum_{i=1}^{n} k_i\alpha_i$ where $k_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n$. Let W be the Weyl group of G. Then W is generated by simple reflections $\{s_i\}_{i=1}^n$, where s_i acts on $X^*(T)$ by $s_i(\lambda) = \lambda - \langle\lambda, \alpha_i^\vee\rangle\alpha_i$, for all $1 \leq i \leq n$.

Let $X^*_+(T)$ be the set of dominant weights of G. Recall that for any dominant weight λ in $X^*_+(T)$, $L(\lambda)$ is the irreducible representation of G with highest weight λ. Let $L(\lambda)^*$ be its dual representation, which is irreducible with highest weight λ^*. Denote the set of weights of $L(\lambda)$ by $\Pi(\lambda)$. For any $\mu \in \Pi(\lambda)$, denote the μ-weight space of $L(\lambda)$ by $L(\lambda)_\mu$ and the dimension of $L(\lambda)_\mu$ by $n_\mu(\lambda)$. It is well known that $\mu \in \Pi(\lambda)$ implies $\mu \preceq \lambda$.

We say a subset Π of $X^*(T)$ is *saturated* if for any $\lambda \in \Pi, \alpha \in R$ and $0 \leq i \leq \langle\lambda, \alpha^\vee\rangle$, we have $\lambda - i\alpha \in \Pi$. The following properties are well-known, see e.g. [4, §21].

- For any $\lambda' \in \Pi(\lambda)$ and $w \in W$, we have $w(\lambda') \in \Pi(\lambda)$ and $\dim L(\lambda)_{\lambda'} = \dim L(\lambda)_{w(\lambda')}$;
- $\Pi(\lambda)$ is saturated and if $\mu \in X^*(T)$, then $\mu \in \Pi(\lambda)$ is equivalent to that for any $w \in W, w(\mu) \preceq \lambda$. Therefore, $\Pi(\lambda)$ is a finite set and for any dominant weight $\mu \preceq \lambda$, we have $\mu \in \Pi(\lambda)$.

2.2. Tensor product decomposition

Let λ, μ be two dominant weights of G. We have the tensor product decomposition:

$$L(\lambda) \bigotimes L(\mu) = \bigoplus_{\nu \in X^*_+(T)} L(\nu)^{\oplus m^\nu_{\lambda,\mu}}.$$

Here $m^{\nu}_{\lambda,\mu}$ is the *tensor product multiplicity*. By definition, $m^{\nu}_{\lambda,\mu} > 0$ if and only if $\nu \in X(\lambda, \mu)$. Therefore, a perfect submonoid of dominant weights is closed under taking direct summands of tensor product.

Recall the following classical results describing the possible weights in $X(\lambda, \mu)$.

Lemma 2.1. *[6, Theorem 5.1] Let λ, μ, ν be dominant weights in $X^*_+(T)$. If $\nu \in X(\lambda, \mu)$, then $\nu = \lambda' + \mu$ for some $\lambda' \in \Pi(\lambda)$. In particular, $\nu = \lambda + \mu - \sum_{i=1}^{n} k_i\alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n$.*

Lemma 2.2. *[4, §24] Let λ, μ be dominant weights in $X^*_+(T)$. Suppose that for any $\mu' \in \Pi(\mu)$, $\lambda + \mu'$ is dominant. Then for any $\mu' \in \Pi(\mu)$, $\lambda + \mu' \in X(\lambda, \mu)$ with multiplicity $m^{\lambda+\mu'}_{\lambda,\mu} = n_{\mu'}(\mu)$.*

Another key ingredient in our proof is the *PRV conjecture* conjectured in [10], which was first proved by Kumar.

Theorem 2.3. *[7, Theorem 2.10] (PRV conjecture) Let G be a semisimple group with Weyl group W over K. Let λ, μ be two dominant weights of G. For any $w \in W$, $\overline{\lambda + w\mu} \in X(\lambda, \mu)$, where $\overline{\lambda + w\mu}$ is the only dominant weight in the W-orbit of $\lambda + w\mu$. In particular, if $\lambda + w\mu$ is dominant, then $\lambda + w\mu \in X(\lambda, \mu)$.*

3. Semisimple case

We prove Theorem A in this section.

3.1. Reduction

We first reduce the general case to the case when G is simply connected. Let G be a connected semisimple algebraic group with a maximal torus T and center Z. Let G^{sc} be the simply connected cover of G with a maximal torus T^{sc} and center Z^{sc}. We know $G \simeq G^{sc}/Z'$, where Z' is a subgroup of Z^{sc}. We have the following exact sequence

$$1 \longrightarrow X^*(T) \longrightarrow X^*(T^{sc}) \longrightarrow X^*(Z') \longrightarrow 1,$$

and $X^*_+(T) = \{\lambda \in X^*_+(T^{sc}) \mid \lambda|_{Z'} = 1\}$ is a subset of $X^*_+(T^{sc})$ by natural inclusion.

Recall that the functor between tensor categories $\mathbf{Rep}(G) \to \mathbf{Rep}(G^{sc})$ is fully faithful by [3]. Then the tensor product multiplicities $m^{\nu}_{\lambda,\mu}$ are the same for G and G^{sc} if λ, μ, ν are dominant weights of G. This can also be seen in [8, Corollary 3.6]. Therefore, if L is perfect as a submonoid of $X^*_+(T)$, then it is also perfect as a submonoid of $X^*_+(T^{sc})$. Thus we may focus on the case when G is simply connected.

3.2. Characterization of perfect submonoids of dominant weights

Assume G is semisimple simply connected, then there is a decomposition $G = G_1 \times \cdots \times G_n$, where each G_k is simply connected quasi-simple with a maximal torus T_k, center Z_k and Weyl group W_k. Let $\Xi = \{1, ..., n\}$ be the index set of quasi-simple factors. There are also corresponding decompositions of the weight lattice $X^*(T) = X^*(T_1) \bigoplus \cdots \bigoplus X^*(T_n)$ and the root lattice $Q = Q_1 \bigoplus \cdots \bigoplus Q_n$. We also have

$$X^*(T)/Q \simeq \bigoplus_{k=1}^{n} X^*(T_k)/Q_k.$$

Let the set of simple roots of G be $\{\alpha_i\}_{i\in I}$ and the corresponding simple reflections be $\{s_i\}_{i\in I}$. Write $I = \bigsqcup_{k=1}^{n} I_k$, where I_k is the index set of simple roots of G_k.

First we give some perfect submonoids of dominant weights.

Proposition 3.1. *Suppose that G is a simply connected semisimple group. If $\tilde{L}$ is a sublattice of $X^*(T)$ containing Q, then $\tilde{L} \cap X^*_+(T)$ is a perfect submonoid of $X^*_+(T)$.*

Proof. Let λ, μ be two dominant weights in $\tilde{L}$. For any $\nu \in X(\lambda, \mu)$, by Lemma 2.1, we have $\nu = \lambda + \mu - \sum_{i\in I} k_i\alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$ for $i \in I$. Since $\tilde{L}$ is a lattice containing Q, we have λ, μ and $-\sum_{i\in I} k_i\alpha_i$ are all in $\tilde{L}$. Thus ν is also in $\tilde{L}$. Therefore, we have $\nu \in \tilde{L} \cap X^*_+(T)$ and $\tilde{L} \cap X^*_+(T)$ is perfect. □

Next we focus on the necessary conditions for perfectness of a submonoid $L \subset X^*_+(T)$. By above decomposition of weight lattice, any weight $\lambda \in X^*(T)$ can be denoted by $\big(\pi_1(\lambda), ..., \pi_n(\lambda)\big)$, where $\pi_k : X^*(T) \to X^*(T_k)$ is the canonical projection, for $1 \leq k \leq n$. Suppose that λ is dominant, define the *support* of λ as

$$\mathrm{supp}(\lambda) = \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle > 0\}.$$

For any $1 \leq k \leq n$, say λ is *k-regular* if $\mathrm{supp}(\lambda) \supset I_k$. If λ is *k-regular* for all k, $1 \leq k \leq n$, then λ is a *regular* dominant weight in $X^*(T)$.

Definition 3.2. Let G be a semisimple group. For any dominant weight λ of G, the *component support* of λ is the set $\{1 \leq k \leq n \mid \pi_k(\lambda) \text{ is nontrivial}\}$.

Let L be a perfect submonoid of $X^*_+(T)$. It is clear that for any $1 \leq k \leq n$, $\pi_k(L)$ is a perfect submonoid of $X^*_+(T_k)$. We claim the existence of some certain k-regular dominant weights in a nonzero perfect submonoid L of dominant weights.

Lemma 3.3. *Suppose that G is simply connected semisimple and L is a nonzero perfect submonoid of $X^*_+(T)$. Let λ be a dominant weight in L. Then there exists a dominant* *weight $\omega_\lambda \in L$ (not uniquely determined by λ) such that for any $1 \le k \le n$, ω_λ is k-regular if $\pi_k(\lambda)$ is nontrivial.*

Proof. It suffices to prove the lemma for quasi-simple group G. Indeed, suppose that for any $1 \le k \le n$ such that $\pi_k(\lambda)$ is nontrivial, there is a k-regular dominant weight $\pi_k(\mu_k) \in \pi_k(L)$, where $\mu_k \in L$. Then $\sum\limits_{k, \pi_k(\lambda) \text{ is nontrivial}} \mu_k$ is a desirable dominant weight $\omega_\lambda \in L$.

Assume that G is quasi-simple. It suffices to show that for any dominant weight $\lambda \in L$ with $\mathrm{supp}(\lambda) \subsetneq I$, there is another dominant weight $\mu \in L$ such that $\mathrm{supp}(\mu) \supsetneq \mathrm{supp}(\lambda)$.

Let $\mathcal{D}$ be the Dynkin diagram of G and λ be a dominant weight in L with $\mathrm{supp}(\lambda) \subsetneq I$. There are vertices $j \in \mathrm{supp}(\lambda)$ and $i_1 \notin \mathrm{supp}(\lambda)$ such that j and i_1 are joint with each other in $\mathcal{D}$. Then $\langle \lambda, \alpha_j^\vee \rangle > 0$. Let $I' = \{i_1, i_2, ..., i_m, j\}$ be the subset of I consisting of all vertices joint with j and j itself. Consider the weight $\mu = 2\lambda + s_j(\lambda)$. We show it is dominant.

For any $i \in I$, we have

$$\langle \mu, \alpha_i^\vee \rangle = 3\langle \lambda, \alpha_i^\vee \rangle - \langle \lambda, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle.$$

If $i = j$, then $\langle \mu, \alpha_j^\vee \rangle = \langle \lambda, \alpha_j^\vee \rangle > 0$. If $i \in I' \setminus \{j\}$, then $\langle \mu, \alpha_i^\vee \rangle > 3\langle \lambda, \alpha_i^\vee \rangle \ge 0$ since $\langle \alpha_j, \alpha_i^\vee \rangle < 0$. If $i \in I \setminus I'$, then $\langle \mu, \alpha_i^\vee \rangle = 3\langle \lambda, \alpha_i^\vee \rangle \ge 0$ since $\langle \alpha_j, \alpha_i^\vee \rangle = 0$. By above computations, we have μ is dominant. Then by Theorem 2.3, we have $\mu \in X(2\lambda, \lambda)$ is contained in L.

Now we look at the support. Still by above computations, for $i \in I \setminus I'$, we have $i \in \mathrm{supp}(\mu)$ if and only if $i \in \mathrm{supp}(\lambda)$. We also have $\mathrm{supp}(\mu)$ contains I' while $i_1 \notin \mathrm{supp}(\lambda)$. Therefore, we have $\mathrm{supp}(\mu) \supsetneq \mathrm{supp}(\lambda)$ and the lemma is proved. □

Based on above property of ω_λ and the fact that $\Pi(\lambda)$ is a finite set, we have a direct corollary.

Corollary 3.4. *Suppose that G is simply connected semisimple and L is a nonzero perfect submonoid of $X_+^*(T)$. Let λ be a dominant weight in L. Then there is a positive integer m such that $\mu + m\omega_\lambda \in L$ for any weight $\mu \in \Pi(\lambda)$.*

We also need the following technical proposition, which will be proved in Section 4.

Proposition 3.5. *Suppose that G is simply connected semisimple. If L is a nonzero perfect submonoid of $X_+^*(T)$, then for any $\lambda \in L$, all the dominant weights in $\Pi(\lambda)$ are contained in L.*

For the proof of Proposition 3.5 and our later discussions, we cannot reduce them directly to the case when G is quasi-simple. This is because $\big(\pi_1(\lambda_1), \pi_2(\lambda_2), ..., \pi_n(\lambda_n)\big)$ may not be in L even if $\lambda_1, ..., \lambda_n$ are all in L.

Definition 3.6. Let L be a submonoid of $X_+^*(T)$, the *component support* of L is the set $\{1 \le k \le n \mid \pi_k(L) \neq \{0\}\}$. If the component support of L is equal to $\{1, 2, ..., n\}$, then L is said to have *full component support*. In particular, when G is quasi-simple, every nonzero submonoid of $X_+^*(T)$ has full component support.

We first restrict ourselves to perfect submonoids of $X_+^*(T)$ with full component support.

Lemma 3.7. *Suppose that G is simply connected semisimple. If L is a perfect submonoid of $X_+^*(T)$ with full component support, then for any $1 \le k \le n$, we have $Q_k \cap X_+^*(T)$ is contained in L. In particular, $Q \cap X_+^*(T)$ is contained in L.*

Proof. Let μ be arbitrary in $Q_k \cap X_+^*(T)$. Since L has full component support, there is a dominant weight λ in L with full component support. By Lemma 3.3, there is a regular dominant weight $\omega_\lambda = \big(\pi_1(\omega_\lambda), ..., \pi_n(\omega_\lambda)\big)$ in L. We know that ω_λ is a $\mathbb{Q}_{\ge 0}$-combination of simple roots. Then one can take a positive integer m such that $m\omega_\lambda \in Q \cap X_+^*(T)$. Moreover, since ω_λ is regular, we have $\langle \omega_\lambda, \alpha_i^\vee \rangle > 0$ for any $i \in I_k$. One can take m large enough such that $\langle m\omega_\lambda, \alpha_i^\vee \rangle \ge \langle \mu, \alpha_i^\vee \rangle$ for any $i \in I_k$. Then $m\pi_k(\omega_\lambda) - \mu$ is a $\mathbb{Z}_{\ge 0}$-combination of simple roots in Q_k. Moreover, we have $m\omega_\lambda - \mu := \big(m\pi_1(\omega_\lambda), ..., m\pi_{k-1}(\omega_\lambda), m\pi_k(\omega_\lambda) - \mu, m\pi_{k+1}(\omega_\lambda), ..., m\pi_n(\omega_\lambda)\big)$ is a $\mathbb{Z}_{\ge 0}$-combination of simple roots and thus $\mu \preceq m\omega_\lambda$. Then $\mu \in \Pi(m\omega_\lambda) \cap X_+^*(T)$ is in L by Proposition 3.5. Therefore, $Q_k \cap X_+^*(T)$ is contained in L.

In particular, since L is a submonoid of $X_+^*(T)$, we have $Q \cap X_+^*(T)$ is contained in L by adding $Q_k \cap X_+^*(T)$ for $1 \le k \le n$. $\square$

Based on the above lemma, we use the cocenter to characterize the perfect submonoids of $X_+^*(T)$. Consider the canonical projection map $p : X^*(T) \to X^*(T)/Q$. If L is a perfect submonoid of $X_+^*(T)$ with full component support, then $p(L)$ is a submonoid of $X^*(T)/Q$. Moreover, it is a subgroup of $X^*(T)/Q$ since $X^*(T)/Q$ is finite.

Proposition 3.8. *Suppose that G is simply connected semisimple and L is a perfect submonoid of $X_+^*(T)$ with full component support. Then $L = p^{-1}(\mathcal{L}) \cap X_+^*(T)$ for some subgroup $\mathcal{L}$ of $X^*(T)/Q$.*

Proof. Let $\mathcal{L} = p(L)$ be a subgroup of $X^*(T)/Q$. By definition we have $L \subset p^{-1}(\mathcal{L}) \cap X_+^*(T)$. Then it suffices to show:

$$\text{For any } a \in \mathcal{L},\ p^{-1}(a) \cap X_+^*(T) \text{ is contained in } L.$$

Indeed, by our choice of $\mathcal{L}$, there exists $\lambda \in L$ such that $p(\lambda) = a$. Let μ be an arbitrary dominant weight in $p^{-1}(a)$. Then $\lambda - \mu \in Q$. By same argument as in the proof of Lemma 3.7, there exists a regular dominant weight $\omega \in L$. Then there is a positive integer m such that $\lambda - \mu + m\omega$ is dominant by regularity of ω. Then $\lambda - \mu + m\omega \in Q \cap X_+^*(T)$

is a $\mathbb{Z}_{\geq 0}$-combination of simple roots. By Proposition 3.5, $\mu \in \Pi(\lambda + m\omega) \cap X^*_+(T)$ is in L. Therefore, we have $p^{-1}(a) \cap X^*_+(T) \subset L$ and $L = p^{-1}(\mathcal{L}) \cap X^*_+(T)$. □

Based on above, we can give the characterization of perfect submonoids of dominant weights.

Proposition 3.9. *Let G be a simply connected semisimple group. The perfect submonoids of $X^*_+(T)$ with full component support are exactly $\tilde{L} \cap X^*_+(T)$, where $\tilde{L}$ is any sublattice of $X^*(T)$ containing Q.*

Proof. By Proposition 3.1, the intersection of sublattices of $X^*(T)$ containing Q with $X^*_+(T)$ is perfect. Moreover, these perfect submonoids clearly have full component support since $Q \cap X^*_+(T)$ has full component support.

Let L be a perfect submonoid of $X^*_+(T)$ with full component support. By Proposition 3.8, $L = p^{-1}(\mathcal{L}) \cap X^*_+(T)$ for some subgroup $\mathcal{L}$ of $X^*(T)/Q$. We also have $p^{-1}(\mathcal{L})$ is a subgroup of $X^*(T)$. Moreover, $p^{-1}(\mathcal{L})$ contains $p^{-1}(0) = Q$. Therefore, the perfect submonoid L is the intersection of a sublattice $p^{-1}(\mathcal{L})$ of $X^*(T)$ containing Q with $X^*_+(T)$. □

3.3. Reformulation of the characterization

In Proposition 3.8, we relate our perfect submonoids of $X^*_+(T)$ with the cocenter of G. Now we give a reformulation of perfect submonoids of dominant weights using central characters. Still assume G is simply connected in this subsection. Keep the notations in Subsection 3.2.

Let L be an arbitrary perfect submonoid of $X^*_+(T)$ with full component support. Define a subset Z_L of Z as

$$Z_L = \{z \in Z \mid \lambda(z) = 1, \forall \lambda \in L\}.$$

Since $Z_L = \bigcap_{\lambda \in L} \mathrm{Ker}(\lambda|_Z)$, we have that Z_L is a subgroup of Z.

Conversely, let Z' be an arbitrary subgroup of Z. Define a subset $L_{Z'}$ of $X^*_+(T)$ as

$$L_{Z'} = \{\lambda \in X^*_+(T) \mid \lambda|_{Z'} = 1\}.$$

Then $L_{Z'}$ is a perfect submonoid of $X^*_+(T)$ with full component support. Indeed, there is a unique (up to isomorphism) connected algebraic group G' with simply connected cover G such that $G' \simeq G/Z'$. By [2, §1.2], the maximal torus T' of G' satisfying

$$1 \longrightarrow Z' \longrightarrow T \longrightarrow T' \longrightarrow 1$$

gives rise to

$$1 \longrightarrow X^*(T') \longrightarrow X^*(T) \longrightarrow X^*(Z') \longrightarrow 1,$$

and $X^*_+(T') = \{\lambda \in X^*_+(T) \mid \lambda|_{Z'} = 1\} = L_{Z'}$. Moreover, as weights in $Q \cap X^*_+(T)$ are trivial on $Z \supset Z'$, we have $L_{Z'} \supset Q \cap X^*_+(T)$ and $L_{Z'}$ is a perfect submonoid of $X^*_+(T)$ with full component support.

Proposition 3.10. *Let G be a simply connected semisimple group. The maps $\varphi : L \mapsto Z_L$, $\psi : Z' \mapsto L_{Z'}$ give a natural bijection between the perfect submonoids of $X^*_+(T)$ with full component support and the subgroups of Z.*

Proof. Let L be an arbitrary perfect submonoid of $X^*_+(T)$ with full component support. By Proposition 3.9, we have $L = \tilde{L} \cap X^*_+(T)$ for some sublattice $\tilde{L}$ of $X^*(T)$ containing Q. Then there is a unique (up to isomorphism) connected semisimple group G' with simply connected cover G and a maximal torus T' such that $X^*(T') = \tilde{L}$. Since $G' \simeq G/Z'$ for a unique subgroup Z' of Z, we have $X^*(T') = \{\lambda \in X^*(T) \mid \lambda|_{Z'} = 1\}$. Then we have $L = \tilde{L} \cap X^*_+(T) = \{\lambda \in X^*_+(T) \mid \lambda|_{Z'} = 1\} = \psi(Z')$ and ψ is surjective. Meanwhile, by uniqueness of G', ψ is injective.

Now we show that φ and ψ are inverse to each other. Consider $(\psi \circ \varphi)(L)$ is also a perfect submonoid of $X^*_+(T)$ with full component support. For any $\lambda \in L$ and any $z \in \varphi(L)$, we have $\lambda(z) = 1$. Then by definition, λ is in $(\psi \circ \varphi)(L)$ and $L \subset (\psi \circ \varphi)(L)$. Meanwhile, since ψ is surjective, $L = \psi(Z')$ for some subgroup Z' of Z. Then $\varphi(L)$ contains Z'. Then $(\psi \circ \varphi)(L)$ is a subset of $\psi(Z') = L$. Therefore, we have $(\psi \circ \varphi)(L) = L$. For any $Z' < Z$, we have $(\psi \circ \varphi \circ \psi)(Z') = \psi(Z')$ by above. Since ψ is injective, we have $(\varphi \circ \psi)(Z') = Z'$. Therefore, the pair (φ, ψ) gives a bijection and it is clearly natural by definition. □

3.4. Proof of the main result

Now we return to the setting in Subsection 3.1 and prove Theorem A. Let L be a submonoid of $X^*_+(T)$. We first define the component support of L.

Definition 3.11. Let L be a submonoid of $X^*_+(T)$. The *component support* of L is the component support of L as a submonoid of $X^*_+(T^{sc})$ (see Definition 3.6).

a) Let L be a perfect submonoid of $X^*_+(T)$ with full component support. By our discussion above, L is also a perfect submonoid of $X^*_+(T^{sc})$ with full component support. Therefore, by Proposition 3.9, we have $L = \tilde{L} \cap X^*_+(T^{sc})$ where $\tilde{L}$ is a sublattice of $X^*(T^{sc})$ containing the root lattice Q. Since L is contained in $X^*_+(T)$, we have $L = \tilde{L} \cap X^*_+(T)$. One can also write $L = \big(\tilde{L} \cap X^*(T)\big) \cap X^*_+(T)$. Clearly, $\tilde{L} \cap X^*(T)$ is a sublattice of $X^*(T)$ containing Q.

Conversely, let $L = \tilde{L} \cap X^*_+(T)$ where $\tilde{L}$ is a sublattice of $X^*(T)$ containing Q. Then $\tilde{L}$ is also a sublattice of $X^*(T^{sc})$ containing Q. We also know $\tilde{L} \cap X^*_+(T) = \tilde{L} \cap X^*_+(T^{sc})$ since $\tilde{L} \subset X^*(T)$. Then by Proposition 3.9, L is a perfect submonoid of $X^*_+(T^{sc})$ with full component support and is also a perfect submonoid of $X^*_+(T)$ with full component support.

b) Since $G \simeq G^{sc}/Z'$ and $Z \simeq Z^{sc}/Z'$, then it suffices to show: There is a natural bijection between the perfect submonoids of $X_+^*(T)$ with full component support and the subgroups of Z^{sc}/Z'.

Recall that perfect submonoids of $X_+^*(T)$ with full component support are also perfect submonoids of $X_+^*(T^{sc})$ with full component support. By Proposition 3.10, there is a natural bijection between perfect submonoids of $X_+^*(T^{sc})$ with full component support and subgroups of Z^{sc} given by

$$\varphi : L \longmapsto Z_L^{sc} = \{z \in Z^{sc} \mid \lambda(z) = 1, \forall \lambda \in L\},$$

and its inverse

$$\psi : (Z^{sc})' \longmapsto L_{(Z^{sc})'} = \{\lambda \in X_+^*(T^{sc}) \mid \lambda|_{(Z^{sc})'} = 1\}.$$

Note that

$$X_+^*(T) = \{\lambda \in X_+^*(T^{sc}) \mid \lambda|_{Z'} = 1\}.$$

If L is a perfect submonoid of $X_+^*(T)$ with full component support, then $\varphi(L) = Z_L^{sc}$ contains Z'.

Conversely, for any subgroup $(Z^{sc})'$ of Z^{sc} containing Z', $\psi((Z^{sc})') = L_{(Z^{sc})'}$ is actually a perfect submonoid of $X_+^*(T)$. Then the restrictions of φ and ψ actually give a natural bijection between perfect submonoids of $X_+^*(T)$ with full component support and subgroups of Z^{sc} containing Z'. Since there is a natural bijection between subgroups of Z^{sc} containing Z' and subgroups of Z^{sc}/Z', one can combine two natural bijections together and get the required bijection.

3.5. Characterization for arbitrary perfect submonoids

In this subsection we drop the assumption that L has full component support and deal with arbitrary perfect submonoids. Indeed, we only need to consider the nonzero perfect submonoids.

Recall that $\Xi = \{1, 2, ..., n\}$ is the index set of quasi-simple factors of G^{sc}. Let Ξ_0 be an arbitrary nonempty subset of Ξ and L be a perfect submonoid of $X_+^*(T)$ with component support Ξ_0. Then L is also a perfect submonoid of $X_+^*(T^{sc})$ with component support Ξ_0. Let $X^*(T)_{\Xi_0} = X^*(T) \cap X^*(T^{sc})_{\Xi_0}$ and $Q_{\Xi_0} = Q \cap X^*(T^{sc})_{\Xi_0}$, where

$$X^*(T^{sc})_{\Xi_0} := \{(\pi_1(\lambda), ..., \pi_n(\lambda)) \mid \lambda \in X^*(T^{sc}), \pi_k(\lambda) = 0 \text{ for any } k \notin \Xi_0\} \subset X^*(T^{sc}).$$

It is clear that $X^*(T^{sc})_{\Xi_0}$ and Q_{Ξ_0} are isomorphic to the weight lattice and the root lattice of $G_{\Xi_0}^{sc} = \prod_{k \in \Xi_0} G_k^{sc}$, respectively. Then L is contained in $X_+^*(T)_{\Xi_0} \subset X_+^*(T^{sc})_{\Xi_0}$ and one can view L as a perfect submonoid of dominant weights of $G_{\Xi_0}^{sc}$ with full component support.

Then we can slightly modify the maps φ and ψ. Recall that $G \simeq G^{sc}/Z'$. Let $Z^{sc}_{\Xi_0} = \prod_k Z^{sc}_{k,\Xi_0} < Z^{sc}$, where $Z^{sc}_{k,\Xi_0} = Z^{sc}_k$ for $k \in \Xi_0$ and $Z^{sc}_{k,\Xi_0} = \{1\} < Z^{sc}_k$ for $k \notin \Xi_0$. Let $Z'_{\Xi_0} = Z' \cap Z^{sc}_{\Xi_0}$. Define the map φ_{Ξ_0} from perfect submonoids of $X^*_+(T^{sc})$ with component support Ξ_0 to subgroups of $Z^{sc}_{\Xi_0}$ as $\varphi_{\Xi_0}(L) = (Z^{sc}_{\Xi_0})_L$, where

$$(Z^{sc}_{\Xi_0})_L = \{z \in Z^{sc}_{\Xi_0} \mid \lambda(z) = 1, \forall \lambda \in L\}.$$

For the inverse direction, define the map ψ_{Ξ_0} as $\psi_{\Xi_0}\big((Z^{sc}_{\Xi_0})'\big) = L_{(Z^{sc}_{\Xi_0})'}$, where

$$L_{(Z^{sc}_{\Xi_0})'} = \{\lambda \in X^*_+(T^{sc})_{\Xi_0} \mid \lambda|_{(Z^{sc}_{\Xi_0})'} = 1\}.$$

One can also write

$$L_{(Z^{sc}_{\Xi_0})'} = \{\lambda \in X^*_+(T^{sc}) \mid \lambda|_{(Z^{sc}_{\Xi_0})'} = 1, \lambda|_{T^{sc}_k} = 1, \forall k \notin \Xi_0\}.$$

Then we deduce the characterization for perfect submonoids of $X^*_+(T)$ with component support $\Xi_0 \subset \Xi$ and its reformulation as a corollary of Theorem A.

Corollary 3.12. *Let G be a connected semisimple algebraic group. Then*

a) The perfect submonoids of $X^*_+(T)$ with component support $\Xi_0 \subset \Xi$ are exactly $\tilde{L} \cap X^*_+(T)$, where $\tilde{L}$ is any sublattice of $X^*(T)_{\Xi_0}$ containing Q_{Ξ_0};

b) There is a natural bijection between the perfect submonoids of $X^*_+(T)$ with component support $\Xi_0 \subset \Xi$ and the subgroups of $Z^{sc}_{\Xi_0}/Z'_{\Xi_0}$.

Proof. a) One notices that $X^*(T)_{\Xi_0}$ is a sublattice of $X^*(T^{sc})_{\Xi_0}$ containing Q_{Ξ_0}. Then $X^*(T)_{\Xi_0}$ and Q_{Ξ_0} can be viewed as the weight lattice and the root lattice of a connected semisimple group G'_{Ξ_0} with simply connected cover $G^{sc}_{\Xi_0}$, respectively. Then one can check perfect submonoids of $X^*_+(T)_{\Xi_0}$ with full component support are also perfect submonoids of $X^*_+(T^{sc})$ contained in $X^*_+(T)$ with component support Ξ_0. Then by our discussions above, perfect submonoids of $X^*_+(T)$ with component support Ξ_0 are exactly perfect submonoids of $X^*_+(T)_{\Xi_0}$ with full component support.

We view $X^*_+(T)_{\Xi_0}$ as the set of dominant weights of G'_{Ξ_0}. Then by applying Theorem A to perfect submonoids of $X^*_+(T)_{\Xi_0}$ with full component support, part a) is proved.

b) Identify $X^*_+(T^{sc})_{\Xi_0}$ with the set of dominant weights of $G^{sc}_{\Xi_0}$. Then there is a natural bijection between the perfect submonoids of $X^*_+(T^{sc})$ with component support Ξ_0 and the perfect submonoids of $X^*_+(T^{sc})_{\Xi_0}$ with full component support. Then by applying Proposition 3.10 to $G^{sc}_{\Xi_0}$, we have the maps φ_{Ξ_0} and ψ_{Ξ_0} give a natural bijection between the perfect submonoids of $X^*_+(T^{sc})$ with component support Ξ_0 and the subgroups of $Z^{sc}_{\Xi_0}$.

Moreover, same as the proof of Theorem A, the restrictions of φ_{Ξ_0} and ψ_{Ξ_0} actually give a natural bijection between perfect submonoids of $X^*_+(T)$ with component support Ξ_0 and subgroups of $Z^{sc}_{\Xi_0}$ containing Z'_{Ξ_0}. Since there is a natural bijection between subgroups of $Z^{sc}_{\Xi_0}$ containing Z'_{Ξ_0} and subgroups of $Z^{sc}_{\Xi_0}/Z'_{\Xi_0}$, again we can combine two bijections together and get the required natural bijection. □

4. Proof of Proposition 3.5

In this section, we keep the notations in Subsection 3.2 and prove Proposition 3.5. We first give the idea of the proof. Then we reduce it to the case when G is quasi-simple and finally give the computations in different types.

4.1. Idea

Let λ be a dominant weight in L with component support $\Xi_0 \subset \Xi$. We may assume $\Xi_0 = \{1, 2, ..., n_0\}$. Take the dominant weight $\omega_\lambda \in L$ in Lemma 3.3 such that for any $1 \leq k \leq n_0$, ω_λ is k-regular. By Lemma 3.4, there is a positive integer m such that for any $\mu \in \Pi(\lambda)$, $\mu + m\omega_\lambda$ is in L. Without loss of generality, we assume $m = 1$.

Now let μ be an arbitrary dominant weight in $\Pi(\lambda)$. Then the component support of μ is contained in Ξ_0. Our idea is finding a dominant weight η in L based on ω_λ, such that $\mu + \eta$ is also in L and $w_0(\eta) = -\eta$, where w_0 is the longest element in W. Then we have $\mu = \mu + \eta + w_0(\eta) \in X(\mu + \eta, \eta) \subset L$ by Theorem 2.3. Since μ is arbitrary, all dominant weights in $\Pi(\lambda)$ are contained in L, which proves Proposition 3.5.

4.2. Reduction

As in Subsection 3.2, one can write $\omega_\lambda = \big(\pi_1(\omega_\lambda), ..., \pi_n(\omega_\lambda)\big)$, where $\pi_k(\omega_\lambda) \in X_+^*(T_k)$. For any $k \notin \Xi_0$, we know $\pi_k(\omega_\lambda) = 0$. For any $w \in W$, we write $w = (w^{(1)}, w^{(2)}, ..., w^{(n)})$, where $w^{(k)}$ is in the Weyl group W_k of G_k. In particular, $w_0 = (w_0^{(1)}, w_0^{(2)}, ..., w_0^{(n)})$, where $w_0^{(k)}$ is the longest element in W_k. We construct η by some lemmas.

Lemma 4.1. *Suppose that G is simply connected quasi-simple and $\omega \in L$ is regular. There is a sequence $\{\nu_0 = \omega, \nu_1, ..., \nu_r\}$ of nonzero dominant weights in L such that $w_0(\nu_r) = -\nu_r$ and for any $0 \leq l \leq r-1$, $\nu_{l+1} = \beta_l + \sigma_l(\gamma_l)$ for some $\beta_l, \gamma_l \in \{\nu_0, ..., \nu_l\}$ and $\sigma_l \in W$.*

Proof. We give the precise computations for this lemma in different types in Subsection 4.3. □

Lemma 4.2. *Let $1 \leq k_0 \leq n_0$ and $\omega = \big(\pi_1(\omega), ..., \pi_n(\omega)\big)$ be any dominant weight in L with component support Ξ_0 such that, $\mu + \omega \in L$ and ω is k_0-regular. Then there is a dominant weight $\theta_{k_0} = \big(\pi_1(\theta_{k_0}), ..., \pi_n(\theta_{k_0})\big)$ in L with component support Ξ_0 such that $\mu + \theta_{k_0} \in L$, $w_0^{(k_0)}\big(\pi_{k_0}(\theta_{k_0})\big) = -\pi_{k_0}(\theta_{k_0})$ and $\pi_k(\theta_{k_0})$ is a positive integral multiple of $\pi_k(\omega)$ for $k \in \Xi_0 \setminus \{k_0\}$.*

Proof. Recall that $\pi_{k_0}(L)$ is a perfect submonoid of $X_+^*(T_{k_0})$ with full component support since L is perfect and G_{k_0} is quasi-simple. Then there is a regular dominant weight $\pi_{k_0}(\omega)$

of G_{k_0} in $\pi_{k_0}(L)$. Take the sequence $\{\nu_0 = \pi_{k_0}(\omega), ..., \nu_r\}$ in $\pi_{k_0}(L)$ in Lemma 4.1. For any $0 \le l \le r-1$, one can write $\nu_{l+1} = \beta_l + \sigma_l^{(k_0)}(\gamma_l)$ for some $\beta_l, \gamma_l \in \{\nu_0, ..., \nu_l\}$ and $\sigma_l^{(k_0)} \in W_{k_0}$.

Based on above sequence, we can define a sequence $\{\tilde{\nu}_0, \tilde{\nu}_1, ..., \tilde{\nu}_r\}$ of nonzero weights in $X^*(T)$ as follows:
1) $\tilde{\nu}_0 = \omega$;
2) Suppose we have defined $\{\tilde{\nu}_0, \tilde{\nu}_1, ..., \tilde{\nu}_l\}$ and $\nu_{l+1} = \nu_{a_{l+1}} + \sigma_l^{(k_0)}(\nu_{b_{l+1}})$ for some $a_{l+1}, b_{l+1} \in \{0, 1, ..., l\}$. Then set

$$\tilde{\nu}_{l+1} := \tilde{\nu}_{a_{l+1}} + \tilde{\sigma}_l(\tilde{\nu}_{b_{l+1}}),$$

where $\tilde{\sigma}_l = (\tilde{\sigma}_l^{(1)}, ..., \tilde{\sigma}_l^{(n)})$ is given by $\tilde{\sigma}_l^{(k)} = \begin{cases} \sigma_l^{(k_0)}, & k = k_0 \\ Id^{(k)}, & k \neq k_0 \end{cases}$.

Then we claim that for any $0 \le l \le r$, both $\tilde{\nu}_l$ and $\mu + \tilde{\nu}_l$ are in L with component support Ξ_0. Moreover, $\pi_{k_0}(\tilde{\nu}_l) = \nu_l$ and $\pi_k(\tilde{\nu}_l)$ is a positive integral multiple of $\pi_k(\omega)$ for $k \in \Xi_0 \setminus \{k_0\}$.

Prove the claim by induction on l. For $l = 0$, the claim is clearly true since $\tilde{\nu}_0 = \omega$. Suppose that the claim is true for $\{0, 1, ..., l\}$. Since $\tilde{\nu}_{l+1} = \tilde{\nu}_{a_{l+1}} + \tilde{\sigma}_l(\tilde{\nu}_{b_{l+1}})$, we have

$$\pi_k(\tilde{\nu}_{l+1}) = \begin{cases} \pi_{k_0}(\tilde{\nu}_{a_{l+1}}) + \tilde{\sigma_l}^{(k_0)}\big(\pi_{k_0}(\tilde{\nu}_{b_{l+1}})\big) = \nu_{a_{l+1}} + \sigma_l^{(k_0)}(\nu_{b_{l+1}}) = \nu_{l+1}, & k = k_0, \\ \pi_k(\tilde{\nu}_{a_{l+1}}) + \tilde{\sigma_l}^{(k)}\big(\pi_k(\tilde{\nu}_{b_{l+1}})\big) = \pi_k(\tilde{\nu}_{a_{l+1}}) + \pi_k(\tilde{\nu}_{b_{l+1}}), & k \in \Xi_0 \setminus k_0, \\ \pi_k(\tilde{\nu}_{a_{l+1}}) + \tilde{\sigma_l}^{(k)}\big(\pi_k(\tilde{\nu}_{b_{l+1}})\big) = 0, & k \notin \Xi_0. \end{cases}$$

We have $\pi_k(\tilde{\nu}_{l+1}) = \pi_k(\tilde{\nu}_{a_{l+1}}) + \pi_k(\tilde{\nu}_{b_{l+1}})$ is a positive integral multiple of $\pi_k(\omega)$ for $k \in \Xi_0 \setminus \{k_0\}$ by induction hypothesis. Since $\pi_{k_0}(\tilde{\nu}_{l+1}) = \nu_{l+1}$ is dominant, we have $\tilde{\nu}_{l+1}$ and $\mu + \tilde{\nu}_{l+1}$ are both dominant with component support Ξ_0 by above computation. Then we have $\tilde{\nu}_{l+1} \in X(\tilde{\nu}_{a_{l+1}}, \tilde{\nu}_{b_{l+1}})$ and $\mu + \tilde{\nu}_{l+1} \in X(\mu + \tilde{\nu}_{a_{l+1}}, \tilde{\nu}_{b_{l+1}})$ by Theorem 2.3. Therefore, $\tilde{\nu}_{l+1}$ and $\mu + \tilde{\nu}_{l+1}$ are both in L. Then the claim is true for $(l+1)$-case. By induction, the claim is true.

Consider the dominant weight $\tilde{\nu}_r$ in the sequence. From above claim, We know $\tilde{\nu}_r$ and $\mu + \tilde{\nu}_r$ are both in L with component support Ξ_0 and for $k \in \Xi_0 \setminus \{k_0\}$, $\pi_k(\tilde{\nu}_r)$ is a positive integral multiple of $\pi_k(\omega)$. Since $\pi_{k_0}(\tilde{\nu}_r) = \nu_r$, we also have $w_0^{(k_0)}\big(\pi_{k_0}(\tilde{\nu}_r)\big) = -\pi_{k_0}(\tilde{\nu}_r)$ by Lemma 4.1. Then $\tilde{\nu}_r$ is a desirable dominant weight θ_{k_0}. $\square$

Lemma 4.3. *There is a dominant weight η in L, such that $\mu + \eta \in L$ and $w_0(\eta) = -\eta$.*

Proof. Indeed we show there is a sequence $\{\eta_0 = \omega_\lambda, \eta_1, \eta_2, ..., \eta_{n_0}\}$ of dominant weights with component support Ξ_0 satisfying, for any $1 \le k_1 \le n_0$:
a) $\eta_{k_1} \in L$ and $\mu + \eta_{k_1} \in L$;
b) $w_0^{(k_1)}\big(\pi_{k_1}(\eta_{k_1})\big) = -\pi_{k_1}(\eta_{k_1})$ and $\pi_k(\eta_{k_1})$ is a positive integral multiple of $\pi_k(\eta_{k_1-1})$ for $k \in \Xi_0 \setminus \{k_1\}$;
c) For any $k_1 + 1 \le k \le n_0$, η_{k_1} is k-regular.

Construct the sequence by induction. Since $\eta_0 = \omega_\lambda$, it is dominant in L with component support Ξ_0. We know ω_λ is 1-regular and $\mu + \omega_\lambda \in L$. Then by applying Lemma 4.2 to $k_0 = 1$ and $\omega = \omega_\lambda$, one can obtain a dominant weight η_1 in L with component support Ξ_0. Directly by Lemma 4.2, η_1 satisfies condition a) and b). Moreover, for any $2 \le k \le n_0$, $\pi_k(\eta_1)$ is a positive integral multiple of $\pi_k(\omega_\lambda)$, which is k-regular. Thus η_1 is k-regular for any $2 \le k \le k_0$. Then we have constructed η_1 satisfying all conditions.

Suppose we have constructed $\{\eta_1, ..., \eta_l\}$ satisfying all conditions. By induction hypothesis, we have η_l and $\mu + \eta_l$ are in L with component support Ξ_0. We also have η_l is $(l+1)$-regular. Therefore, we can apply Lemma 4.2 to $k_0 = l + 1$ and $\omega = \eta_l$. Then we obtain a dominant weight η_{l+1} in L with component support Ξ_0. Again directly by Lemma 4.2, η_{l+1} satisfies condition a) and b). Moreover, for any $l+2 \le k \le n_0$, $\pi_k(\eta_{l+1})$ is a positive integral multiple of $\pi_k(\eta_l)$, which is k-regular. Then η_{l+1} also satisfies condition c). Then we have constructed η_{l+1} satisfying all conditions. By induction, we have shown the existence of such sequence.

Now we take $\eta = \eta_{n_0} \in L$. Condition a) tells that $\mu + \eta \in L$. For any $1 \le k \le n_0$, Condition b) tells that $w_0^{(k)}\big(\pi_k(\eta_{n_0})\big) = -\pi_k(\eta_{n_0})$. Then $w_0(\eta) = -\eta$. □

Therefore, by our discussion in Subsection 4.1, the proof of Proposition 3.5 reduces to the proof of Lemma 4.1.

4.3. Proof of Lemma 4.1

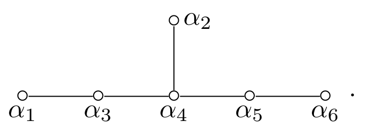
For Lemma 4.1, our computations depend on the type of G. There are four cases in total and the following computations are base on some basic facts of Dynkin diagrams and root data (see e.g. [1, §6.4]).

4.3.1. Type $A_1, B_n, C_n, D_{2n}, E_7, E_8, F_4, G_2$

In these types, we know $w_0 = -1$. Thus the sequence can be chosen as $\{\omega\}$.

4.3.2. Type E_6

Label the vertices of the Dynkin diagram of E_6 as follows:



For convenience, we denote weights in the following way. Let ν be any weight. It is a $\mathbb{Q}$-combination of simple roots, i.e., $\nu = \sum_{i=1}^{6} k_i \alpha_i$. Denote ν by the 6-tuple $\left(\begin{smallmatrix} & & k_2 & & \\ k_1, & k_3, & k_4, & k_5, & k_6 \end{smallmatrix}\right)$. Therefore, one can write down the list of fundamental weights as 6-tuples:

$$\omega_1 = \frac{1}{3}\left(\begin{smallmatrix} & & 3 & & \\ 4, & 5, & 6, & 4, & 2 \end{smallmatrix}\right), \qquad \omega_2 = \left(\begin{smallmatrix} & & 2 & & \\ 1, & 2, & 3, & 2, & 1 \end{smallmatrix}\right), \qquad \omega_3 = \frac{1}{3}\left(\begin{smallmatrix} & & 6 & & \\ 5, & 10, & 12, & 8, & 4 \end{smallmatrix}\right),$$

$$\omega_4 = \left(\begin{smallmatrix} & & 3 & & \\ 2, & 4, & 6, & 4, & 2\end{smallmatrix}\right), \qquad \omega_5 = \frac{1}{3}\left(\begin{smallmatrix} & & 6 & & \\ 4, & 8, & 12, & 10, & 5\end{smallmatrix}\right), \qquad \omega_6 = \frac{1}{3}\left(\begin{smallmatrix} & & 3 & & \\ 2, & 4, & 6, & 5, & 4\end{smallmatrix}\right).$$

In this type, we have w_0 transforms $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ into $-\alpha_6, -\alpha_2, -\alpha_5, -\alpha_4, -\alpha_3, -\alpha_1$, respectively. Then we have $w_0(\omega_2) = -\omega_2$ and $w_0(\omega_4) = -\omega_4$. Therefore, we want the dominant weight ν_r to be a nonnegative linear combination of ω_2 and ω_4.

Now we give the construction of the sequence $\{\nu_1, ..., \nu_r\}$. First we set $\nu_1 := \omega + s_6(\omega)$. For any weight $\nu = \left(\begin{smallmatrix} k_2 \\ k_1, k_3, k_4, k_5, k_6\end{smallmatrix}\right)$, we have $s_6(\nu) = \left(\begin{smallmatrix} k_2 \\ k_1, k_3, k_4, k_5, k_5 - k_6\end{smallmatrix}\right)$. Thus for fundamental weights, we have

$$\begin{cases} \omega_6 + s_6(\omega_6) = \omega_5, \\ \omega_i + s_6(\omega_i) = 2\omega_i, \quad 1 \le i \le 5. \end{cases}$$

Therefore, we have $\nu_1 \in \sum_{i=1}^{5} \mathbb{Z}_{\ge 0}\omega_i$ is dominant. Then $\nu_1 = \omega + s_6(\omega) \in X(\omega, \omega) \subset L$ by Theorem 2.3.

Then we set $\nu_2 := \nu_1 + s_5(\nu_1)$. For any weight $\nu = \left(\begin{smallmatrix} k_2 \\ k_1, k_3, k_4, k_5, k_6\end{smallmatrix}\right)$, we have $s_5(\nu) = \left(\begin{smallmatrix} k_2 \\ k_1, k_3, k_4, k_4 + k_6 - k_5, k_6\end{smallmatrix}\right)$. Thus for fundamental weights, we have

$$\begin{cases} \omega_5 + s_5(\omega_5) = \omega_4 + \omega_6, \\ \omega_i + s_5(\omega_i) = 2\omega_i, \quad i \ne 5. \end{cases}$$

Therefore, ν_2 is dominant by above computation. Then we have $\nu_2 = \nu_1 + s_5(\nu_1) \in X(\nu_1, \nu_1) \subset L$ by Theorem 2.3.

Then set $\nu_3 := \nu_2 + s_6 s_5(\nu_1) = \nu_1 + s_5(\nu_1) + s_6 s_5(\nu_1)$. By above computations, we have

$$s_6 s_5(\omega_i) = \begin{cases} s_6(\omega_4 + \omega_6 - \omega_5) = \omega_4 - \omega_6, & i = 5, \\ s_6(\omega_6) = \omega_5 - \omega_6, & i = 6, \\ s_6(\omega_i) = \omega_i, & i = 1, 2, 3, 4. \end{cases}$$

Then we have

$$\begin{cases} \omega_5 + s_5(\omega_5) + s_6 s_5(\omega_5) = \omega_4 + \omega_6 + \omega_4 - \omega_6 = 2\omega_4, \\ \omega_i + s_5(\omega_i) + s_6 s_5(\omega_i) = 3\omega_i, \quad 1 \le i \le 4. \end{cases}$$

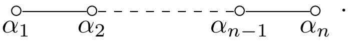
Since $\nu_1 \in \sum_{i=1}^{5} \mathbb{Z}_{\ge 0}\omega_i$, we have $\nu_3 \in \sum_{i=1}^{4} \mathbb{Z}_{\ge 0}\omega_i$ is dominant. Still by Theorem 2.3, we have $\nu_3 = \nu_2 + s_6 s_5(\nu_1) \in X(\nu_2, \nu_1) \subset L$.

Now we have obtained $\nu_3 \in L$ and $\nu_3 \in \sum_{i=1}^{4} \mathbb{Z}_{\ge 0}\omega_i$. Actually, the coefficients of ω_5 and ω_6 vanished in the process of obtaining ν_3. By symmetry of the Dynkin diagram of type

E_6 and the simple roots, we can make the coefficient of ω_1 and ω_3 vanished in a similar way. Set $\nu_4 := \nu_3 + s_1(\nu_3)$, one can check $\nu_4 \in \sum_{i=2}^{4} \mathbb{Z}_{\geq 0}\omega_i$ is dominant and is in L by Theorem 2.3. Then we set $\nu_5 = \nu_4 + s_3(\nu_4)$ and $\nu_6 = \nu_5 + s_1 s_3(\nu_4) = \nu_4 + s_3(\nu_4) + s_1 s_3(\nu_4)$. Similar to ν_2 and ν_3, we have ν_5 and ν_6 are in L and $\nu_6 \in \mathbb{Z}_{\geq 0}\omega_2 + \mathbb{Z}_{\geq 0}\omega_4$. Then we have $w_0(\nu_6) = -\nu_6$ by our computations before. Therefore, $\{\nu_1, ..., \nu_6\}$ is a desirable sequence for Lemma 4.1 in type E_6.

4.3.3. Type $A_n (n \geq 2)$

Label the vertices of the Dynkin diagram of A_n as follows:



For convenience, we still denote each weight ν by a n-tuple $(k_1, k_2, ..., k_n)$, where $\nu = \sum_{i=1}^{n} k_i \alpha_i$. Then for any $1 \leq i \leq n$, the fundamental weight ω_i is denoted by

$$\omega_i = \tfrac{1}{n+1}(n-i+1, 2(n-i+1), ..., i(n-i+1), i(n-i), ..., 2i, i).$$

In this type, w_0 transforms α_i into $-\alpha_{n+1-i}$. Then we have $w_0(\omega_1) = -\omega_n$ and $w_0(\omega_n) = -\omega_1$. Therefore, we want the dominant weight ν_r to be a nonnegative linear combination of ω_1 and ω_n.

Now we give the construction of the sequence $\{\nu_1, ..., \nu_r\}$. First we claim that there is a sequence $\{\zeta_j\}_{i=1}^{n}$ of dominant weights in L such that, for any $1 \leq i \leq n$, $\zeta_i \in \sum_{l=i}^{n} \mathbb{Z}_{>0}\omega_l$. By symmetry, there is also a sequence $\{\theta_i\}_{i=1}^{n}$ of nonzero dominant weights in L such that, for any $1 \leq i \leq n$, $\theta_i \in \sum_{l=1}^{n+1-i} \mathbb{Z}_{>0}\omega_l$.

We proceed to prove the claim. Construct the sequence $\{\zeta_i\}_{i=1}^{n}$ as follows:
(a) $\zeta_1 = \omega$;
(b) $\zeta_{i+1} = \zeta_i + s_i(\zeta_i) + s_{i-1}s_i(\zeta_i) + ... + s_1 s_2 ... s_{i-1} s_i(\zeta_i)$. For any $1 \leq m \leq i+1$, denote the sum of first m terms of right hand side by $\zeta_{i+1,m}$.

By symmetry, we construct the sequence $\{\theta_i\}_{i=1}^{n}$ as follows:
(a) $\theta_1 = \omega$;
(b) $\theta_{i+1} = \theta_i + s_{n+1-i}(\theta_i) + s_{n+1-(i-1)} s_{n+1-i}(\theta_i) + ... + s_n s_{n-1} ... s_{n+1-(i-1)} s_{n+1-i}(\theta_i)$. For any $1 \leq m \leq i+1$, denote the sum of first m terms of right hand side by $\theta_{i+1,m}$.

Then we check these sequences satisfy our requirements by induction on i. By symmetry of the Dynkin diagram of type A_n and the simple roots, we only need to check for $\{\zeta_i\}_{i=1}^{n}$. For any weight $\nu = (k_1, k_2, ..., k_n)$, we have

$$s_i(\nu) = \begin{cases} (k_1, k_2, ..., k_{i-1}, k_{i-1} + k_{i+1} - k_i, k_{i+1}, ..., k_n), & i \neq 1, n, \\ (k_2 - k_1, k_2, ..., k_{i-1}, k_i, k_{i+1}, ..., k_n), & i = 1, \\ (k_1, k_2, ..., k_{i-1}, k_i, k_{i+1}, ..., k_{n-1} - k_n), & i = n. \end{cases}$$

Thus for fundamental weights, we have

$$s_i(\omega_l) = \begin{cases} \omega_{l-1} + \omega_{l+1} - \omega_l, & 2 \leq l = i \leq n-1, \\ \omega_2 - \omega_1, & l = i = 1, \\ \omega_{n-1} - \omega_n, & l = i = n, \\ \omega_l, & l \neq i. \end{cases}$$

For $i = 1$, it is clear that $\zeta_1 = \omega$ is in L and $\omega \in \sum_{l=1}^{n} \mathbb{Z}_{>0}\omega_l$. Suppose that $\{\zeta_1, ..., \zeta_i\}$ satisfies our requirements where $i < n$. Then we look at ζ_{i+1}. We first compute $\omega_i + s_i(\omega_i) + s_{i-1}s_i(\omega_i) + ... + s_1 s_2 ... s_{i-1} s_i(\omega_i)$. For this we need the following lemma. Set $\omega_0 = 0$ for convenience.

Lemma 4.4. *Suppose that* $1 \leq i \leq n-1$. *Then* $s_{i-l+1}...s_{i-1}s_i(\omega_i) = \omega_{i-l} - \omega_{i-l+1} + \omega_{i+1}$ *for* $1 \leq l \leq i$.

Proof. Proceed by induction on l. For $l = 1$, we have $s_i(\omega_i) = \omega_{i-1} - \omega_i + \omega_{i+1}$ by our computations above. Suppose that the equation holds for $l - 1$. Then for l, we have

$$\begin{aligned} s_{i-l+1}...s_{i-1}s_i(\omega_i) &= s_{i-l+1}(\omega_{i-l+1} - \omega_{i-l+2} + \omega_{i+1}) \\ &= \omega_{i-l} - \omega_{i-l+1} + \omega_{i-l+2} - \omega_{i-l+2} + \omega_{i+1} \\ &= \omega_{i-l} - \omega_{i-l+1} + \omega_{i+1}. \end{aligned}$$

Therefore, the equation also holds for l. By induction, the lemma is proved. □

We know $\zeta_i \in \sum_{l=i}^{n} \mathbb{Z}_{>0}\omega_l$ by induction hypothesis. Now by Lemma 4.4, we have $\omega_i + s_i(\omega_i) + s_{i-1}s_i(\omega_i) + ... + s_1 s_2 ... s_{i-1} s_i(\omega_i) = \omega_i + (\omega_{i-1} - \omega_i + \omega_{i+1}) + (\omega_{i-2} - \omega_{i-1} + \omega_{i+1}) + ... + (\omega_0 - \omega_1 + \omega_{i+1}) \in \mathbb{Z}_{>0}\omega_{i+1}$. This equation together with the fact that $s_1, s_2, ..., s_i$ fix ω_l for $i + 1 \leq l \leq n$ show that ζ_{i+1} is contained in $\sum_{l=i+1}^{n} \mathbb{Z}_{>0}\omega_l$.

Then we show that ζ_{i+1} is in L. Recall our construction of $\{\zeta_i\}_{i=1}^{n}$ and $\{\zeta_{i+1,m}\}_{m=1}^{i+1}$. Again by Lemma 4.4, for any $1 \leq m \leq i+1$, the sum of the first m terms of $\omega_i + s_i(\omega_i) + s_{i-1}s_i(\omega_i) + ... + s_1 s_2 ... s_{i-1} s_i(\omega_i)$ is $\omega_i + (\omega_{i-1} - \omega_i + \omega_{i+1}) + ... + (\omega_{i-m+1} - \omega_{i-m+2} + \omega_{i+1}) = (m-1)\omega_{i+1} + \omega_{i-m+1}$, which is dominant. Still together by the fact that $s_1, s_2, ..., s_i$ fix ω_l for $i+1 \leq l \leq n$, $\zeta_{i+1,m}$ is always dominant for $1 \leq m \leq i+1$. Then for any $2 \leq m \leq i+1$, we have $\zeta_{i+1,m} \in X(\zeta_{i+1,m-1}, \zeta_i)$. Since $\zeta_{i+1,1} = \zeta_i$ is in L by induction hypothesis, we have $\zeta_{i+1,m}$ are in L for all $1 \leq m \leq i + 1$ by applying Theorem 2.3 successively. In particular, $\zeta_{i+1} = \zeta_{i+1,i+1}$ is in L. Therefore, the claim is true by induction and symmetry.

Now that the claim is true. Then we have $\zeta_n = a\omega_n \in L$ and $\theta_n = b\omega_1 \in L$ for some positive integers a, b. We set $\zeta = b\zeta_n + a\theta_n \in L$. Notice that $w_0(\zeta) = abw_0(\omega_1 + \omega_n) = -\zeta$.

By above construction, we have two sequences $\{\zeta_{i,m} \mid 2 \leq i \leq n, 2 \leq m \leq i\}$ and $\{\theta_{i,m} \mid 2 \leq i \leq n, 2 \leq m \leq i\}$ in L. We know that

$$\begin{cases} \zeta_{2,2} = \omega + s_1(\omega), \\ \theta_{2,2} = \omega + s_n(\omega), \\ \zeta_{i+1,m} = \zeta_{i+1,m-1} + s_{i-m+2}...s_i(\zeta_{i,i}), \quad i \geq 2, 2 < m \leq i+1, \\ \zeta_{i+1,2} = \zeta_{i,i} + s_i(\zeta_{i,i}), \quad i \geq 2, \\ \theta_{i+1,m} = \theta_{i+1,m-1} + s_{n+1-i+(m-2)}...s_{n+1-i}(\theta_{i,i}), \quad i \geq 2, 2 < m \leq i+1, \\ \theta_{i+1,2} = \theta_{i,i} + s_{n+1-i}(\theta_{i,i}), \quad i \geq 2, \\ l\zeta_{n,n} = (l-1)\zeta_{n,n} + \zeta_{n,n}, \quad l \geq 2, \\ l\theta_{n,n} = (l-1)\theta_{n,n} + \theta_{n,n}, \quad l \geq 2, \\ \zeta = b\zeta_{n,n} + a\theta_{n,n}. \end{cases}$$

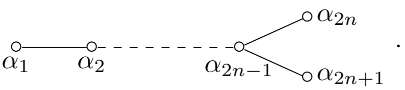
Then consider the following sequence in L,

$$\{\omega, \zeta_{2,2}, \zeta_{3,2}, \zeta_{3,3}, ..., \zeta_{n,n}, 2\zeta_{n,n}, ..., b\zeta_{n,n}, \theta_{2,2}, \theta_{3,2}, \theta_{3,3}, ..., \theta_{n,n}, 2\theta_{n,n}, ..., a\theta_{n,n}, \zeta\}.$$

This is a desirable sequence for Lemma 4.1 in type A_n by equations above.

4.3.4. Type D_{2n+1}

Label the vertices of the Dynkin diagram of D_{2n+1} as follows:



For convenience, we still denote each weight $\nu = \sum_{i=1}^{2n+1} k_i\alpha_i$ by a $(2n+1)$-tuple $(k_1, k_2, ..., k_{2n}, k_{2n+1})$. Then we have the list of fundamental weights as $(2n+1)$-tuples

$$\omega_{2n} = (\frac{1}{2}, \frac{2}{2} = 1, ..., \frac{2n-1}{2}, \frac{2n+1}{4}, \frac{2n-1}{4}),$$
$$\omega_{2n+1} = (\frac{1}{2}, \frac{2}{2} = 1, ..., \frac{2n-1}{2}, \frac{2n-1}{4}, \frac{2n+1}{4}),$$
$$\omega_i = (1, 2, ..., i-1, i, i, ..., i, \frac{i}{2}, \frac{i}{2}), \quad 1 \leq i \leq 2n-1.$$

In this type, we have w_0 transforms $\alpha_{2n}, \alpha_{2n+1}$ into $-\alpha_{2n+1}, -\alpha_{2n}$ and acts as -1 on other simple roots. Then we have $w_0(\omega_i) = -\omega_i$ for $1 \leq i \leq 2n-1$. Therefore, we want the dominant weight ν_r to lie in $\sum_{i=1}^{2n-1} \mathbb{Z}_{\geq 0}\omega_i$.

Now we give the construction of the sequence $\{\nu_1, ..., \nu_r\}$. First set $\nu_1 := \omega + s_{2n+1}(\omega)$. For any weight $\nu = (k_1, k_2, ..., k_{2n+1})$, we have $s_{2n+1}(\nu) = (k_1, k_2, ..., k_{2n-1}, k_{2n}, k_{2n-1} - k_{2n+1})$. Thus for fundamental weights, we have

$$\begin{cases} \omega_{2n+1} + s_{2n+1}(\omega_{2n+1}) = \omega_{2n-1}, \\ \omega_i + s_{2n+1}(\omega_i) = 2\omega_i, \quad i \neq 2n+1. \end{cases}$$

Therefore, we have $\nu_1 \in \sum_{i=1}^{2n} \mathbb{Z}_{\geq 0}\omega_i$ is dominant. By Theorem 2.3, we have $\nu_1 = \omega + s_{2n+1}(\omega) \in X(\omega, \omega) \subset L$.

Then we set $\nu_2 := \nu_1 + s_{2n}(\nu_1)$. For any weight $\nu = (k_1, k_2, ..., k_{2n+1})$, we have $s_{2n}(\nu) = (k_1, k_2, ..., k_{2n-1}, k_{2n-1} - k_{2n}, k_{2n+1})$. Thus for fundamental weights, we have

$$\begin{cases} \omega_{2n} + s_{2n}(\omega_{2n}) = \omega_{2n-1}, \\ \omega_i + s_{2n}(\omega_i) = 2\omega_i, \quad i \neq 2n. \end{cases}$$

Since $\nu_1 \in \sum_{i=1}^{2n} \mathbb{Z}_{\geq 0}\omega_i$, we have $\nu_2 \in \sum_{i=1}^{2n-1} \mathbb{Z}_{\geq 0}\omega_i$ is dominant. By Theorem 2.3 we have $\nu_2 = \nu_1 + s_{2n}(\nu_1) \in X(\nu_1, \nu_1) \subset L$. Since $\nu_2 \in \sum_{i=1}^{2n-1} \mathbb{Z}_{\geq 0}\omega_i$, we have $w_0(\nu_2) = -\nu_2$ by our computations before. Therefore, we have $\{\nu_1, \nu_2\}$ is a desirable sequence for Lemma 4.1 in type D_{2n+1}.

5. Reductive case and comparison

5.1. Comparison with Vinberg's results

The definition of perfect submonoids was given by Vinberg in [12, §1]. Vinberg used this definition to develope his classification of reductive algebraic monoids. All algebraic monoids in this subsection are assumed to be linear and irreducible.

Let G be a connected reductive algebraic group with a maximal torus T. The natural action of $G \times G$ on $K[G]$ induces an isomorphism [9, II.3.1 Satz 3]

$$K[G] \simeq \bigoplus_{\lambda \in X_+^*(T)} K[G]_\lambda,$$

where $K[G]_\lambda \simeq L(\lambda)^* \bigotimes L(\lambda)$ is the linear space spanned by matrix entries of $L(\lambda)$. Every $(G \times G)$-stable subspace of $K[G]$ has the form of $K[G]_L = \bigoplus_{\lambda \in L} K[G]_\lambda$ for some subset L of $X_+^*(T)$. Let M be a reductive monoid with unit group G. Then $K[M]$ is a $G \times G$-stable subalgebra of $K[G]$. By checking the multiplication of $K[M]$, we have $K[M] = K[G]_L$ where L is a perfect submonoid of $X_+^*(T)$. Vinberg gave a description of reductive monoids with unit group G.

Theorem 5.1. *[12, Theorem 1] A submonoid L of $X_+^*(T)$ defines an algebraic monoid M with unit group G, if and only if L is perfect, finitely generated and generating $X^*(T)$ as a group.*

Moreover, based on the fact that every algebraic monoid admits a normalization [11, Proposition 3.15], Vinberg gave a classification of normal reductive monoids in [12, Theorem 2]. This classification is important in Vinberg's construction of Vinberg monoids in [12, Theorem 5].

By [5, Lemma 1.1], if L is perfect, finitely generated and generates $X^*(T)$ as a group, then the reductive monoid M defined by L is normal if and only if $L \subset X^*(T)$ is *saturated* in the following sense.

Definition 5.2. [5, Definition 1.2] Let L be a subset of $X^*(T)$. Suppose that for any $\lambda \in X^*(T)$, if there is an integer $n > 1$ such that $n\lambda \in L$, then $\lambda \in L$. Then L is called *saturated.*

Remark 5.3. Note this definition is different from the definition of *saturated* in Section 2. For example, let λ be a dominant weight. The subset $\Pi(2\lambda) \subset X^*(T)$ is not always saturated in the sense of Definition 5.2 since λ may not be in $\Pi(2\lambda)$. In this section we are always using the definition from [5].

Therefore, Vinberg's results give a characterization for perfect submonoids of $X^*_+(T)$ which are finitely generated, saturated and generates $X^*(T)$ as a group. Then we compare our results with Vinberg's results on perfect submonoids of dominant weights.

5.1.1. Semisimple case

Suppose that G is semisimple. Let G^{sc} be the simply connected cover of G with a maximal torus T^{sc}, and $\Xi = \{1, 2, ..., n\}$ be the index set of the quasi-simple factors of G^{sc}.

Lemma 5.4. *Suppose that G is a connected semisimple group. The perfect submonoids of $X^*_+(T)$ are all finitely generated.*

Proof. Recall the notations in Subsection 3.5. By Subsection 3.5, every perfect submonoid L of $X^*_+(T)$ with component support $\Xi_0 \subset \Xi$ can be viewed as a perfect submonoid of $X^*_+(T^{sc})_{\Xi_0}$ with full component support. Therefore, we may assume that G is simply connected and L has full component support. Then $L = \tilde{L} \cap X^*_+(T)$, where $\tilde{L}$ is a sublattice of $X^*(T)$ containing Q, by Theorem A. Therefore, we have L is the set of dominant weights of a connected semisimple group G' with simply connected cover G. Thus L is clearly finitely generated. □

Therefore, our result differs from Vinberg's theorem in the sense that, we do not assume that L is saturated or generates $X^*(T)$ as a group. Actually, these two conditions do not hold in general. In conclusion, our results give a complete characterization of all perfect submonoids of dominant weights in semisimple case.

5.1.2. Reductive case

Suppose that G is reductive but not necessarily semisimple. Let G_0 be its derived subgroup and Z^0 be its connected center. Fix a maximal unipotent subgroup U and a Borel subgroup $B = TU$ of G. Let $T_0 = T \cap G_0$ be a maximal torus of G_0. Since T is an almost direct product of T_0 and Z^0, there is an natural embedding

$$i : X^*(T) \longrightarrow X^*(T_0) \bigoplus X^*(Z^0)$$

given by restrictions. Therefore, we identify each dominant weight $\lambda \in X^*_+(T)$ with a pair $i(\lambda) = (\mu, \nu)$, where $\mu = \lambda|_{T_0} \in X^*_+(T_0)$ and $\nu = \lambda|_{Z^0} \in X^*(Z^0)$. Then we naturally relate perfect submonoids of $X^*_+(T)$ to perfect submonoids of $X^*_+(T_0) \bigoplus X^*(Z^0)$.

Proposition 5.5. *The perfect submonoids of $X^*_+(T)$ are exactly the perfect submonoids of $X^*_+(T_0) \bigoplus X^*(Z^0)$ contained in the image of $i : X^*(T) \to X^*(T_0) \bigoplus X^*(Z^0)$.*

Proof. It is clear that the embedding i gives a 1-1 correspondence between submonoids of $X^*_+(T)$ with submonoids of $X^*_+(T_0) \bigoplus X^*(Z^0)$ contained in $\mathrm{Im}(i)$. It remains to show that $L \subset X^*_+(T)$ is perfect if and only if $i(L) \subset X^*_+(T_0) \bigoplus X^*(Z^0)$ is perfect.

Let $(\mu_1, \nu_1) = i(\lambda_1)$ and $(\mu_2, \nu_2) = i(\lambda_2)$ be arbitrary. We have

$$X\big((\mu_1, \nu_1), (\mu_2, \nu_2)\big) = \{(\mu, \nu_1 + \nu_2) \mid \mu \in X(\mu_1, \mu_2) \subset X^*_+(T_0)\}.$$

By Lemma 2.1 and the fact that the simple roots of G are exactly the simple roots of G_0, we have

$$i\big(X(\lambda_1, \lambda_2)\big) = \{(\lambda|_{T_0}, \nu_1 + \nu_2) \mid \lambda \in X(\lambda_1, \lambda_2)\}.$$

Now we consider the tensor product decomposition

$$L(\lambda_1) \bigotimes L(\lambda_2) = \bigoplus_{\lambda \in X(\lambda_1, \lambda_2)} L(\lambda)^{\oplus m^{\lambda}_{\lambda_1, \lambda_2}}.$$

We know that $L(\lambda) = \{f \in K[G] \mid f(tg) = \lambda(t) f(g), \forall t \in T, g \in G\}$. By restricting on G_0, we have $L(\lambda)|_{G_0}$ is clearly a nonzero G_0-module. Let $i(\lambda) = (\mu, \nu)$, which means $\lambda|_{T_0} = \mu$. Then by decomposing $L(\lambda)|_{G_0}$ into a direct sum of irreducible G_0-modules, we have $L(\lambda) = L_0(\mu)^{\oplus a_\lambda}$ for some positive integer a_λ, where $L_0(\mu)$ denotes the irreducible G_0-module with highest weight μ. Therefore, the restriction of above equation on G_0 gives

$$L_0(\mu_1)^{\oplus a_{\lambda_1}} \bigotimes L_0(\mu_2)^{\oplus a_{\lambda_2}} = \bigoplus_{\lambda \in X(\lambda_1, \lambda_2)} L_0(\lambda|_{T_0})^{\oplus a_\lambda m^{\lambda}_{\lambda_1, \lambda_2}}.$$

Recall the tensor product decomposition of G_0-modules

$$L_0(\mu_1) \bigotimes L_0(\mu_2) = \bigoplus_{\mu \in X(\mu_1,\mu_2)} L_0(\mu)^{\oplus m^{\mu}_{\mu_1,\mu_2}}.$$

By comparing the direct summands of the right hand side of two equations, we have $\{\lambda|_{T_0} \mid \lambda \in X(\lambda_1,\lambda_2)\} = X(\mu_1,\mu_2)$. Then $X\big((\mu_1,\nu_1),(\mu_2,\nu_2)\big) = i\big(X(\lambda_1,\lambda_2)\big)$. Then $X\big((\mu_1,\nu_1),(\mu_2,\nu_2)\big) \subset i(L)$ is equivalent to $X(\lambda_1,\lambda_2) \subset L$. Therefore, we have $L \subset X^*_+(T)$ is perfect if and only if $i(L) \subset X^*_+(T_0) \bigoplus X^*(Z^0)$ is perfect and the proposition is proved. □

Now Let L be a submonoid of $X^*_+(T_0) \bigoplus X^*(Z^0)$ and $pr_Z(L)$ be the projection of L to $X^*(Z^0)$. One can write L as

$$L = \bigcup_{\nu \in pr_Z(L)} \{(\mu,\nu) \mid \mu \in L_\nu\},$$

where $L_\nu := \{\mu \in X^*_+(T_0) \mid (\mu,\nu) \in L\}$. In general, the perfect submonoids of $X^*_+(T_0) \bigoplus X^*(Z^0)$ could be complicated when G is reductive but not semisimple. To see that, we consider the following example in which $L_0 = \{0\}$.

Example 5.6. Construct the perfect submonoid L of $X^*_+(T_0) \bigoplus X^*(Z^0)$ as follows.

Let $pr_Z(L)$ be $\mathbb{Z}_{\geq 0}\nu$, where ν is a nonzero dominant weight in $X^*(Z^0)$. We construct $L_{i\nu}$ $(i \in \mathbb{Z}_{\geq 0})$ inductively. For $i = 0$, let $L_0 = \{0\}$. Suppose we have already constructed $L_{i\nu}$ for $i < m$. For $i = m$, we take an arbitrary subet X_m of $X^*_+(T)$ and construct $L_{m\nu}$ as

$$L_{m\nu} = \Big(\bigcup_{k=1}^{m-1} X(L_{k\nu}, L_{(m-k)\nu})\Big) \bigcup X_m,$$

where $X(L_{k\nu}, L_{(m-k)\nu})$ denotes the union of sets $X(\lambda,\mu)$ for all $\lambda \in L_{k\nu}$ and $\mu \in L_{(m-k)\nu}$.

We check the perfectness of L. Let $(\mu_1, i\nu), (\mu_2, j\nu)$ be any two dominant weights in L. By our discussion in Proposition 5.5, every dominant weight in $X\big((\mu_1,i\nu),(\mu_2,j\nu)\big)$ has the form of $\big(\mu,(i+j)\nu\big)$ for some $\mu \in X(\mu_1,\mu_2) \subset X(L_{i\nu}, L_{j\nu})$. By our construction above, $\mu \in L_{(i+j)\nu}$ and thus $\big(\mu,(i+j)\nu\big) \in L$. Then L is perfect. Since X_m are all arbitrarily chosen, it is difficult to characterize such L.

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