

VERTEX OPERATOR SUPERALGEBRAS AND THE
16-FOLD WAY

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ABSTRACT. Let V be a vertex operator superalgebra with the natural order 2 automorphism σ . Under suitable conditions on V , the σ -fixed subspace V_0 is a vertex operator algebra and the V_0 -module category $\mathcal{C}_{V_0}$ is a modular tensor category. In this paper, we prove that $\mathcal{C}_{V_0}$ is a fermionic modular tensor category and the Müger centralizer $\mathcal{C}_{V_0}^0$ of the fermion in $\mathcal{C}_{V_0}$ is generated by the irreducible V_0 -submodules of the V -modules. In particular, $\mathcal{C}_{V_0}^0$ is a super-modular tensor category and $\mathcal{C}_{V_0}$ is a minimal modular extension of $\mathcal{C}_{V_0}^0$. We provide a construction of a vertex operator superalgebra V^l for each positive integer l such that $\mathcal{C}_{(V^l)_0}$ is a minimal modular extension of $\mathcal{C}_{V_0}^0$. We prove that these modular tensor categories $\mathcal{C}_{(V^l)_0}$ are uniquely determined, up to equivalence, by the congruence class of l modulo 16.

1. INTRODUCTION

Modular (tensor) categories are mathematical formalization of topological phases of matters, which are also called topological orders [W]. The 2+1D symmetry protected topological (SPT) orders are recently described by using unitary braided fusion categories $\mathcal{C}$ with the symmetry determined by their Müger centers $\mathcal{E}$, which are symmetric fusion categories (cf. [LKW1, LKW2] and the references therein). It follows from [De, DR] that a Müger center $\mathcal{E}$ is Tannakian or super-Tannakian, i.e. $\mathcal{E}$ is equivalent to the braided fusion category $\text{Rep}(G)$ or $\text{Rep}(G, z)$ where G is a finite group uniquely determined by $\mathcal{E}$ and z is a central order 2 element of G . Modular tensor categories are exactly those braided fusion categories with trivial Müger centers. The category sVec of super vector spaces over $\mathbb{C}$ is the *smallest* super-Tannakian category. By gauging the minimal topological order with the fermionic symmetry [K], Kitaev discovered the 16-fold way: The braided fusion category sVec has 16 exactly inequivalent unitary minimal modular extensions, which are unitary modular tensor categories of dimension 4 containing a full braided fusion subcategory equivalent to sVec .

Throughout this paper, a super-modular category means a braided fusion category over $\mathbb{C}$ whose Müger center is equivalent to sVec as braided fusion categories. Modular or super-modular categories are assumed to be *pseudounitary* and

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equipped with the canonical pivotal structures, i.e. the categorical (or quantum) dimension of each object is a positive number. Motivated by Kitaev's 16-fold way, it is conjectured in [BGH] that every super-modular category $\mathcal{C}$ has exactly 16 minimal modular extensions up to equivalence, i.e. pseudounitary modular categories of dimension $2\dim(\mathcal{C})$ containing a braided fusion full subcategory equivalent to $\mathcal{C}$. If $\mathcal{C}$ admits a minimal modular extension, it has been proved independently in [LKW1] that $\mathcal{C}$ admits 16 minimal modular extensions. However, the existence of a minimal modular extension for any super-modular category is still an open problem.

Rational conformal field theory is closely related to modular categories. According to [Hu1, Hu2], the representation category of a rational C_2 -cofinite vertex operator algebra (VOA) is modular. In fact, it is an open problem whether every modular category over $\mathbb{C}$ can be realized by a VOA. Super-modular categories are not modular, and so they cannot be realized as the module category of any rational VOA. One would ask what kind of rational VOA could realize a minimal modular extension of super-modular category $\mathcal{C}$, and how one can obtain other VOAs whose module categories are minimal modular extensions of $\mathcal{C}$.

A vertex operator superalgebra $V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$ is $\frac{1}{2}\mathbb{Z}$ -graded in this paper. The $\mathbb{Z}_2$ -grading $V = V_0 \oplus V_1$ with $V_0 = \sum_{n \in \mathbb{Z}} V_n$ and $V_1 = \sum_{n \in \frac{1}{2} + \mathbb{Z}} V_n$ determines a natural order 2 automorphism σ of V such that $\sigma|_{V_r} = (-1)^r$ for $r = 0, 1$. The twisted representations and orbifold theory of rational vertex operator superalgebras are well-studied in [DZ1, DZ2]. With suitable assumptions ((A1) and (A2) in Section 4), the V_0 -module category $\mathcal{C}_{V_0}$ is a modular tensor category, and V_1 is a *fermion* in $\mathcal{C}_{V_0}$ (cf. Lemma 9.1). In particular, V_1 is an order 2 simple current of V_0 . We prove in Theorem 8.1 that the full subcategory $\mathcal{C}_{V_0}^0$ of $\mathcal{C}_{V_0}$, generated by the simple V_0 -submodules of V -modules, are closed under the tensor product of $\mathcal{C}_{V_0}$. In particular, $\mathcal{C}_{V_0}^0$ is a braided fusion subcategory of $\mathcal{C}_{V_0}$ with the fermion V_1 . Moreover, $\mathcal{C}_{V_0}^0$ is the Müger centralizer of the fermion V_1 in $\mathcal{C}_{V_0}$. Hence, $\mathcal{C}_{V_0}^0$ is super-modular (cf. Lemma 9.2), and $\mathcal{C}_{V_0}$ is a minimal modular extension $\mathcal{C}_{V_0}^0$. The modular category $\mathcal{C}_{V_0}$ is also $\mathbb{Z}_2$ -graded with $\mathcal{C}_{V_0} = \mathcal{C}_{V_0}^0 \oplus \mathcal{C}_{V_0}^1$ where $\mathcal{C}_{V_0}^1$ is the full subcategory of $\mathcal{C}_{V_0}$ generated by the irreducible V_0 -submodules of the σ -twisted V -modules, and $\dim(\mathcal{C}_{V_0}^1) = \dim(\mathcal{C}_{V_0}^0)$ (cf. Section 8).

Since a *nice* vertex operator superalgebra V (i.e. satisfying (A1) and (A2)) naturally yields a super-modular category $\mathcal{C}_{V_0}^0$ and a minimal modular extension $\mathcal{C}_{V_0}$, one would like to construct other vertex operator superalgebras from V to realize the 16-fold way of the super-modular category $\mathcal{C}_{V_0}^0$. To achieve this goal, we establish in Theorem 9.6 that if U is a holomorphic vertex operator superalgebra, then $V \otimes U$ is a vertex operator superalgebra and $\mathcal{C}_{V_0}^0$ equivalent to $\mathcal{C}_{(V \otimes U)_0}^0$ as braided fusion categories. In particular, $\mathcal{C}_{(V \otimes U)_0}$ is another minimal modular extension of $\mathcal{C}_{V_0}^0$.

For each positive integer l , there is a nice holomorphic vertex operator superalgebra $V(l, \mathbb{Z} + \frac{1}{2})$ (cf. [EFR], [KW], [LT]). For any nice vertex operator superalgebra V , the tensor product vertex operator superalgebra $V^l := V \otimes V(l, \mathbb{Z} + \frac{1}{2})$ yields the super-modular category $\mathcal{C}_{(V^l)_0}^0$ and its minimal modular extension $\mathcal{C}_{(V^l)_0}$. Since $\mathcal{C}_{(V^l)_0}^0$ is equivalent to $\mathcal{C}_{V_0}^0$ as braided fusion categories, $\mathcal{C}_{(V^l)_0}$ is a minimal modular extension of $\mathcal{C}_{V_0}^0$ for each positive integer l . We prove in Theorem 10.3 that $\mathcal{C}_{(V^l)_0}$ and $\mathcal{C}_{(V^m)_0}$ are equivalent modular categories if and only if $m \equiv l \pmod{16}$ by computing their Gauss sums and applying [LKW1, Theorems 4.26 and 5.4].

The results in this paper could hold for $\mathbb{Z}$ -graded vertex operator superalgebras with certain assumptions. This situation will be discussed in another paper. A connection between general orbifold theory for a vertex operator algebra and minimal modular extensions has recently been investigated in [DNR].

The paper is organized as follows: A review of vertex operator superalgebras and some of basic results on their representation theory are presented in Section 2. In Section 3, we discuss the tensor product of two vertex operator superalgebras, and investigate its irreducible representations via the representations of their Zhu's superalgebras. We review the modular invariance of the trace functions in the orbifold theory for the vertex operator superalgebras in Section 4. In Section 5, the irreducible V_0 -modules of a vertex operator superalgebra V are determined. In Section 6, we show that the associated representation of $SL_2(\mathbb{Z})$ on the trace functions in the orbifold theory for the vertex operator superalgebras provided in Section 4 is unitary. Some important relations between the quantum dimensions of the irreducible V -modules and the irreducible V_0 -modules are established in Section 7. In Section 8, we prove that the category $\mathcal{C}_{V_0}$ is $\mathbb{Z}_2$ -graded, where $\mathcal{C}_{V_0}^0$ and $\mathcal{C}_{V_0}^1$ are respectively generated by the irreducible V_0 -submodules of V -modules and σ -twisted V -modules. We further prove that $\mathcal{C}_{V_0}^0$ is a super-modular category and $\mathcal{C}_{V_0}$ is a minimal modular extension of $\mathcal{C}_{V_0}^0$ in Section 9. In Section 10, we construct a sequence of vertex operator superalgebras V^l for each positive integer l such that $\mathcal{C}_{(V^l)_0}$ is a minimal modular extension of $\mathcal{C}_{V_0}^0$ and these modular categories $\mathcal{C}_{(V^l)_0}$ are uniquely determined by the congruence class of l modulo 16.

2. PRELIMINARIES

The various notions of twisted modules for a vertex operator superalgebra following [DZ1], [DZ2] are reviewed in this section. The concepts such as rationality, regularity, and C_2 -cofiniteness from [Z] and [DLM3] are discussed.

A super vector space is a $\mathbb{Z}_2$ -graded vector space $U = U_0 \oplus U_1$. The vectors in U_0 (resp. U_1) are called even (resp. odd). An element u in U_i for some $i = 0, 1$ will be called $\mathbb{Z}_2$ -homogeneous. In this case, we define $\tilde{u} = \bar{i}$. We reserve the notation sVec for the category of finite dimensional super vector spaces over $\mathbb{C}$ with morphisms preserving the $\mathbb{Z}_2$ -gradings, and equipped with the super braiding.

If W is another super vector space, then $\text{Hom}(U, W)$ is also a super vector space in which $\text{Hom}(U, W)_0$ and $\text{Hom}(U, W)_1$ are respectively the $\mathbb{Z}_2$ -graded preserving and reversing linear maps.

A vertex operator superalgebra is a $\frac{1}{2}\mathbb{Z}$ -graded super vector space

$$V = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n = V_0 \oplus V_1$$

with $V_0 = \sum_{n \in \mathbb{Z}} V_n$ and $V_1 = \sum_{n \in \frac{1}{2} + \mathbb{Z}} V_n$ satisfying $\dim V_n < \infty$ for all n , and $V_m = 0$ if m is sufficiently small. V is equipped with a linear map

$$\begin{aligned} V &\rightarrow (\text{End } V)[[z, z^{-1}]], \\ v &\mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in (\text{End } V)_{\bar{v}}) \end{aligned}$$

and with two distinguished vectors $\mathbf{1} \in V_0$, $\omega \in V_2$ satisfying the following conditions for $u, v \in V$, and $m, n \in \mathbb{Z}$:

$$\begin{aligned} u_n v &= 0 \quad \text{for } n \text{ sufficiently large;} \\ Y(\mathbf{1}, z) &= Id_V; \\ Y(v, z)\mathbf{1} &\in V[[z]] \quad \text{and} \quad \lim_{z \rightarrow 0} Y(v, z)\mathbf{1} = v; \\ [L(m), L(n)] &= (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c; \\ \frac{d}{dz}Y(v, z) &= Y(L(-1)v, z); \\ L(0)|_{V_n} &= n \end{aligned}$$

where $L(m) = \omega_{m+1}$, that is,

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},$$

and the *Jacobi identity*

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y(u, z_1) Y(v, z_2) - (-1)^{\tilde{u}\tilde{v}} z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y(v, z_2) Y(u, z_1) \\ = z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2) \end{aligned}$$

holds, where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and $(z_i - z_j)^n$ is expanded as a formal power series in z_j , and $u, v \in V$ are $\mathbb{Z}_2$ -homogeneous. Throughout the paper, z_0, z_1, z_2 , etc. are independent commuting formal variables. A vertex operator superalgebra will be denoted by $V = (V, Y, \mathbf{1}, \omega)$. In the case $V_{\bar{1}} = 0$, V is a vertex operator algebra given in [FLM3].

Let V be a vertex operator superalgebra. There is a canonical order 2 linear automorphism σ of V associated with the structure of super vector space V such that $\sigma|_{V_i} = (-1)^i$ for $i = 0, 1$. It is easy to show that $\sigma\mathbf{1} = \mathbf{1}$, $\sigma\omega = \omega$ and $\sigma Y(v, z)\sigma^{-1} = Y(\sigma v, z)$ for $v \in V$. That means σ is an automorphism of vertex operator superalgebra V , and will be denoted by σ_V when the clarification is necessary.

Let $g = \sigma^i$ for $i = 0, 1$ and $T = o(g)$. Let $V^r = \{v \in V | gv = e^{2\pi i r/T} v\}$ for $r = 0, T - 1$. A weak g -twisted V -module M is a vector space equipped with a linear map

$$\begin{aligned} V &\rightarrow (\text{End } M)[[z^{1/T}, z^{-1/T}]] \\ v &\mapsto Y_M(v, z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } M) \end{aligned}$$

which satisfies that for all $0 \leq r \leq T - 1$, $u \in V^r$, $v \in V$, $w \in M$,

$$\begin{aligned} Y_M(u, z) &= \sum_{n \in \frac{r}{T} + \mathbb{Z}} u_n z^{-n-1}; \\ u_l w &= 0 \quad \text{for } l \gg 0; \\ Y_M(\mathbf{1}, z) &= Id_M; \end{aligned}$$

$$\begin{aligned} z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - (-1)^{\tilde{u}\tilde{v}} z_0^{-1} \delta \left(\frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1) \\ = z_2^{-1} \left(\frac{z_1 - z_0}{z_2} \right)^{-r/T} \delta \left(\frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2) \end{aligned}$$

where we assume that u, v are $\mathbb{Z}_2$ -homogeneous.

Let $o(g\sigma) = T'$. An *admissible* g -twisted V -module is a weak g -twisted V -module M which carries a $\frac{1}{T'}\mathbb{Z}_+$ -grading

$$M = \bigoplus_{n \in \frac{1}{T'}\mathbb{Z}_+} M(n)$$

satisfying

$$v_m M(n) \subseteq M(n + \text{wt } v - m - 1)$$

for homogeneous $v \in V$.

An (ordinary) g -twisted V -module is a weak g -twisted V -module

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

such that $\dim M_\lambda$ is finite, and for fixed λ , $M_{n+\lambda} = 0$ for all small enough integers n where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. We will write $\text{wt } w = \lambda$ if $w \in M_\lambda$.

If $M = \bigoplus_{n \in \frac{1}{T'}\mathbb{Z}_+} M(n)$ is an admissible g -twisted V -module, the contragredient module M' is defined as follows:

$$M' = \bigoplus_{n \in \frac{1}{T'}\mathbb{Z}_+} M(n)^*,$$

where $M(n)^* = \text{Hom}_{\mathbb{C}}(M(n), \mathbb{C})$. The vertex operator $Y_{M'}(a, z)$ is defined for $a \in V$ via

$$\langle Y_{M'}(a, z)f, w \rangle = \langle f, Y_M(e^{zL(1)}(e^{\pi i} z^{-2})^{L(0)} a, z^{-1})w \rangle,$$

where $\langle f, w \rangle = f(w)$ is the natural pairing $M' \times M \rightarrow \mathbb{C}$. It follows from [FHL] and [X] that $(M', Y_{M'})$ is an admissible g -twisted V -module. We can also define the contragredient module M' for a g -twisted V -module M . In this case, M' is also a g -twisted V -module. Moreover, M is irreducible if and only if M' is irreducible.

A vertex operator superalgebra V is called *g -rational*, if the category of its admissible g -twisted modules is semisimple. We simply call V rational if V is 1-rational. V is called holomorphic if V is rational and V is the only irreducible module of itself up to isomorphism.

We also need another important concept called C_2 -cofiniteness [Z]. We say that a vertex operator superalgebra V is C_2 -cofinite if $V/C_2(V)$ is finite dimensional, where $C_2(V) = \langle v_{-2}u \mid v, u \in V \rangle$. A vertex operator superalgebra V is called regular if every weak V -module is a direct sum of ordinary irreducible V -modules.

The following results about σ^i -rationality are given in [DZ1] and [DZ2]. Also see [DLM4] and [DLM7].

Theorem 2.1. *Let V be a g -rational vertex operator superalgebra where $g = \sigma^i$ and $i = 0, 1$. Then:*

(1) *Any irreducible admissible g -twisted V -module M is an ordinary g -twisted V -module. Moreover, there exists a number $\lambda \in \mathbb{C}$ such that $M_\lambda \neq 0$ and $M = \bigoplus_{n \in \frac{1}{T'}\mathbb{Z}_+} M_{\lambda+n}$. The λ is called the conformal weight of M .*

(2) *There are only finitely many irreducible admissible g -twisted V -modules up to isomorphism.*

(3) If V is also C_2 -cofinite and σ^i -rational for $i = 0, 1$ then the central charge c and the conformal weight λ of any irreducible σ^i -twisted V -module M are rational numbers.

A vertex operator superalgebra $V = \oplus_{n \in \frac{1}{2}\mathbb{Z}} V_n$ is said to be of *CFT type* if $V_n = 0$ for negative n and $V_0 = \mathbb{C}\mathbf{1}$. We know from [L3] and [ABD] that if V is a vertex operator algebra of CFT type, then regularity is equivalent to rationality and C_2 -cofiniteness. Moreover, V is regular if and only if the weak module category is semisimple [DYU]. The same results also hold for vertex operator superalgebras with similar proof [HA].

We discuss more on V -modules. Let $M = \oplus_{n \in \frac{1}{2}\mathbb{Z}_+} M(n)$ be an admissible V -module. We set $M_{\bar{0}} = \oplus_{n \in \mathbb{Z}_+} M(n)$ and $M_{\bar{1}} = \oplus_{n \in \mathbb{Z}_+} M(n + \frac{1}{2})$. From now on we assume that V is a simple vertex operator superalgebra and $V_{\bar{1}} \neq 0$. Then $V_{\bar{0}}$ is a simple vertex operator algebra and $V_{\bar{1}}$ is an irreducible $V_{\bar{0}}$ -module.

Lemma 2.2. *Let $M = (M, Y_M)$ be a nonzero admissible V -module. Then $M_{\bar{i}} \neq 0$ for $i = 0, 1$. Moreover, we can define a linear automorphism σ on M such that $\sigma|_{M_{\bar{i}}} = (-1)^i$ and $\sigma Y_M(u, z)\sigma^{-1} = Y_M(\sigma u, z)$ for all $u \in V$.*

Proof. Assume $M_{\bar{i}} \neq 0$. For any nonzero $u \in V_{\bar{1}}$, there exists $n \in \mathbb{Z}$ such that $0 \neq u_n M_{\bar{i}} \in M_{\bar{i}+1}$ by Proposition 11.9 of [DL1]. This implies that $M_{\bar{i}+1} \neq 0$. The rest of the lemma is clear. $\square$

Recall from [DLM7] that M is called σ -stable if $M \circ \sigma$ and M are isomorphic where $M \circ \sigma$ is a V -module such that $M \circ \sigma = M$ as vector spaces and $Y_{M \circ \sigma}(v, z) = Y_M(\sigma v, z)$ for all $v \in V$. Lemma 2.2 asserts that for any admissible V -module M , $M \circ \sigma$ and M are isomorphic, or M is σ -stable.

We now turn our attention to σ -twisted V -module. In this case, an admissible σ -twisted module M has gradation $M = \oplus_{n \in \mathbb{Z}_+} M(n)$. So we cannot use gradation to divide M into even and odd parts. In this case, we have to use $M \circ \sigma$.

Lemma 2.3. *Suppose M is an irreducible admissible σ -twisted V -module. If $M \circ \sigma$ and M are not isomorphic, then M is an irreducible $V_{\bar{0}}$ -module. If $M \circ \sigma$ and M are isomorphic, then M is a direct sum of two inequivalent irreducible $V_{\bar{0}}$ -modules. In this case, there exists an involution $\sigma \in \text{GL}(M)$ such that $\sigma Y_M(v, z)\sigma^{-1} = Y_M(\sigma v, z)$ for $v \in V$ and the two irreducible $V_{\bar{0}}$ -modules are the two different eigenspaces of σ .*

Proof. If $M \circ \sigma$ and M are not isomorphic, it follows from the proof of Theorem 6.1 of [DM] that $M \circ \sigma$ and M are isomorphic irreducible $V_{\bar{0}}$ -modules. If $M \circ \sigma$ is isomorphic to M , we also denote this isomorphism by σ without confusion. Then $\sigma : M \rightarrow M$ is a linear isomorphism such that $\sigma Y_M(v, z)\sigma^{-1} = Y_M(\sigma v, z)$ for v in V by Schur's Lemma. We can choose σ such that $\sigma^2 = 1$. We denote the eigenspace with eigenvalue $(-1)^i$ by $M_{\bar{i}}$. Then $M_{\bar{i}}$ is irreducible $V_{\bar{0}}$ -module. The inequivalence of $M_{\bar{0}}$ and $M_{\bar{1}}$ as $V_{\bar{0}}$ -modules follows the same proof of Theorem 5.4 [DY]. $\square$

We now introduce the notion of an admissible σ -twisted super V -module. An admissible σ -twisted V -module M is called an *admissible σ -twisted super V -module* if M is σ -stable. The ordinary σ -twisted super V -module can be defined similarly.

Lemma 2.4. *If N is an admissible σ -twisted V -module which is not a σ -stable, then $N \oplus N \circ \sigma$ is an admissible σ -twisted super V -module. Moreover, N is irreducible if and only if $N \oplus N \circ \sigma$ is an irreducible admissible σ -twisted super V -module.*

Proof. For short, we set $\overline{N} = N \circ \sigma$ and $M = N \oplus \overline{N}$. Since $N = \overline{N}$ as vector spaces, we can define a linear isomorphism $\sigma : M \rightarrow M$ by $\sigma(w, w') = (w', w)$ for any $w, w' \in N$. Obviously, $\sigma^2 = \text{id}_M$ and one can verify directly that $\sigma Y_M(u, z)\sigma = Y_M(\sigma u, z)$ for $u \in V$. Therefore, $M \circ \sigma \cong M$ and M is an admissible σ -twisted super V -module with

$$M_{\bar{0}} = \{w + \sigma w | w \in M\}, \quad M_{\bar{1}} = \{w - \sigma w | w \in M\}.$$

Note that $M_{\bar{r}}$ and N are isomorphic $V_{\bar{0}}$ -modules for $r = 0, 1$. If N is irreducible, then $M_{\bar{0}}$ and $M_{\bar{1}}$ are irreducible $V_{\bar{0}}$ -modules by Lemma 2.3. Let $X \subset M$ be a nonzero admissible σ -twisted super V -submodule. Then $X = X_{\bar{0}} + X_{\bar{1}}$. Without loss, we can assume that $X_{\bar{0}}$ is nonzero. Then $X_{\bar{0}}$ is a submodule of the irreducible $V_{\bar{0}}$ -module $M_{\bar{0}}$. Thus $X_{\bar{0}} = M_{\bar{0}}$. Since V is simple, for any nonzero $u \in V_{\bar{1}}$ and any nonzero $w \in M_{\bar{0}}$ we know $Y(u, z)w$ is nonzero by Proposition 11.9 of [DL1]. This implies $X_{\bar{1}}$ is nonzero and equal to $M_{\bar{1}}$. So X has to be M and hence M is an irreducible super V -module. Conversely, if M is super irreducible, take a nonzero proper admissible σ -twisted submodule Z of N . It is easy to see that $Z + \sigma(Z)$ is a nonzero proper admissible σ -twisted super module. This is a contradiction. The proof is complete. $\square$

In summary, every admissible σ -twisted super V -module M of a vertex operator superalgebra V admits a $\mathbb{Z}_2$ -grading which is determined by an V -module isomorphism from $M \circ \sigma$ on M of order 2.

3. TENSOR PRODUCTS

For the remaining discussion, we investigate the tensor product $U \otimes V$ of two vertex operator superalgebras U and V , and its twisted modules, which are not well-known in literature. The tensor product of vertex operator algebras and their modules were studied in [FHL], but the *super* case is slightly more complicated. For example, the tensor product $M \otimes N$ of a σ_U -twisted U -module M and a σ_V -twisted V -module N may not be a $\sigma_{U \otimes V}$ -twisted $U \otimes V$ -module. We will use σ for any vertex operator superalgebra if there is no confusion.

Lemma 3.1. *Let U, V be vertex operator superalgebras. Then*

(1) *$U \otimes V$ is also a vertex operator superalgebra with*

$$(U \otimes V)_{\bar{0}} = U_{\bar{0}} \otimes V_{\bar{0}} + U_{\bar{1}} \otimes V_{\bar{1}}, \quad (U \otimes V)_{\bar{1}} = U_{\bar{0}} \otimes V_{\bar{1}} + U_{\bar{1}} \otimes V_{\bar{0}}$$

and

$$Y(u \otimes v, z)(u' \otimes v') = (-1)^{\bar{v}\bar{u}'} Y(u, z)u' \otimes Y(v, z)v'$$

for any $\mathbb{Z}_2$ -homogeneous elements $u, u' \in U$ and $v, v' \in V$.

(2) *The map $f : U \otimes V \rightarrow V \otimes U$ such that $f(u \otimes v) = (-1)^{\bar{u}\bar{v}} v \otimes u$ gives an isomorphism of vertex operator superalgebras.*

(3) *If M is a σ^i -twisted U -module such that $M \circ \sigma^i \cong M$ and N is σ^i -twisted V -module with $i = 0, 1$. Then $M \otimes N$ is a $\sigma^i \otimes \sigma^i$ -twisted $U \otimes V$ -module such that*

$$Y(u \otimes v, z)(x \otimes y) = (-1)^{\bar{v}\bar{x}} Y(u, z)x \otimes Y(v, z)y$$

$u \in U, v \in V$ and $x \in M$ and $y \in N$ where as usual $\bar{x} = r$ if $x \in M_{\bar{r}}$. In particular, the tensor product $M \otimes N$ of an U -module M and an V -module N is a module of $U \otimes V$.

(4) *If both U and V are rational, then any irreducible $U \otimes V$ -module is isomorphic to $M \otimes N$ for some irreducible U -module M and some irreducible V -module N .*

(5) If M is a σ -twisted super U -module and N is a σ -twisted super V -module then $M \otimes N$ is a $\sigma \otimes \sigma$ -twisted super $U \otimes V$ -module with

$$(M \otimes N)_{\bar{0}} = M_{\bar{0}} \otimes N_{\bar{0}} + M_{\bar{1}} \otimes N_{\bar{1}}, \quad (M \otimes N)_{\bar{1}} = M_{\bar{0}} \otimes N_{\bar{1}} + M_{\bar{1}} \otimes N_{\bar{0}}.$$

Proof. The proofs of (1)–(4) are fairly standard [FHL], and (5) follows from (3). $\square$

We deal with the tensor product of σ -twisted modules next. From Lemma 3.1 we need to understand $M \otimes N$ where both M and N are not σ -stable in terms of the tensor product of $A_{\sigma}(U)$ and $A_{\sigma}(V)$ studied in [DZ2]. For this purpose, we need some basic facts on superalgebras and their super modules from [K].

Let $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ be a superalgebra. A super $\mathcal{A}$ -module M is defined as a $\mathbb{Z}_2$ -graded module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ such that $\mathcal{A}_{\bar{r}} M_{\bar{s}} \subset M_{\bar{r}+\bar{s}}$. $\mathcal{A}$ is called semisimple if $\mathcal{A}$ is completely reducible super $\mathcal{A}$ -module. $\mathcal{A}$ is simple if it is semisimple and the only super ideals are 0 and itself.

There are two types of simple superalgebras $\mathcal{Q}_k$ ($\mathcal{Q}$ type) and $\mathcal{M}_{m,n}$ ($\mathcal{M}$ type) for any positive integer k and nonnegative integers m, n with $m+n > 0$. The $\mathcal{Q}_k$ is defined to be the algebra consisting of $2k \times 2k$ matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, where A and B are arbitrary $k \times k$ complex matrices, with $B = 0$ for the even part and $A = 0$ for the odd one. The $\mathcal{M}_{m,n}$ is the complex matrix algebra $M_{(m+n) \times (m+n)}$. Write each matrix as $\begin{pmatrix} A & C \\ D & B \end{pmatrix}$, where A, B, C and D are complex matrices of dimensions $m \times m$, $n \times n$, $m \times n$ and $n \times m$, respectively, with $C = 0, D = 0$ for the even part and $A = 0, B = 0$ for the odd part. Clearly, $\mathcal{Q}_k$ is a direct sum of two copies of a full matrix algebra.

One can find the following results in [K].

Theorem 3.2. *Let $\mathcal{A}$ be a finite dimensional superalgebra.*

(1) *The following are equivalent: (a) $\mathcal{A}$ is a semisimple superalgebra, (b) $\mathcal{A}$ is a semisimple associative algebra, (c) $\mathcal{A}$ is a direct sum of simple superalgebras.*

(2) *Any finite dimensional simple superalgebra over $\mathbb{C}$ is of either $\mathcal{Q}$ type or $\mathcal{M}$ type.*

(3) *For $k > 0$, $\mathcal{Q}_k$ has a unique irreducible super module of dimension $2k$ which is a direct sum of two inequivalent $\mathcal{Q}_k$ -modules of dimension k .*

(4) *For $m, n \geq 0$ with $m+n > 0$, $\mathcal{M}_{m,n}$ has a unique irreducible super module of dimension $m+n$ which is also irreducible $\mathcal{M}_{m,n}$ -module.*

Now we discuss the tensor products of superalgebras and their super modules. For any superalgebra $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ and $a \in \mathcal{A}_{\bar{i}}$, we define $\tilde{a} := \bar{i}$. Superalgebras are algebras in sVec , which is a braided tensor category. Therefore, the tensor product of two superalgebras is a superalgebra. More precisely, if $\mathcal{A}$ and $\mathcal{B}$ are superalgebras, then $\mathcal{A} \otimes \mathcal{B}$ is a superalgebra with

$$(\mathcal{A} \otimes \mathcal{B})_{\bar{0}} = \mathcal{A}_{\bar{0}} \otimes \mathcal{B}_{\bar{0}} + \mathcal{A}_{\bar{1}} \otimes \mathcal{B}_{\bar{1}}, \quad (\mathcal{A} \otimes \mathcal{B})_{\bar{1}} = \mathcal{A}_{\bar{0}} \otimes \mathcal{B}_{\bar{1}} + \mathcal{A}_{\bar{1}} \otimes \mathcal{B}_{\bar{0}}$$

and

$$(a \otimes b)(a' \otimes b') = (-1)^{\tilde{b}\tilde{a}'} aa' \otimes bb'$$

for any homogeneous elements $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. Note that the map $f : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ with $f(a \otimes b) = (-1)^{\tilde{a}\tilde{b}} b \otimes a$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$ is the braiding of $\mathbf{sVec}$. By [K1],

$$\mathcal{Q}_m \otimes \mathcal{Q}_n \cong \mathcal{M}_{mn, mn}, \quad \mathcal{Q}_k \otimes \mathcal{M}_{m, n} \cong \mathcal{Q}_{(m+n)k}, \quad \mathcal{M}_{m, n} \otimes \mathcal{M}_{k, l} \cong \mathcal{M}_{mk+nl, ml+nk}$$

as superalgebras or algebras in $\mathbf{sVec}$.

We now return to vertex operator superalgebra V . Recall the associative algebra $A_\sigma(V)$ from [DZ2]. Let $O_\sigma(V)$ to be the subspace of V spanned by $u \circ_\sigma v$ for $u, v \in V$ where

$$u \circ_\sigma v = \text{Res}_z Y(u, z) v \frac{(1+z)^{\text{wt } u}}{z^2}.$$

Set

$$u *_\sigma v = \text{Res}_z Y(u, z) v \frac{(1+z)^{\text{wt } u}}{z}$$

and $A_\sigma(V) = V/O_\sigma(V)$. Note that the definition of $A_\sigma(V)$ is the same as the Zhu's algebra for a vertex operator algebra.

The statements (1)–(4) of the following proposition are known in [DZ2], and the statements (1')–(4') can be proved similarly with obvious modifications.

Proposition 3.3. *Let V be a vertex operator superalgebra. Then*

(1) $A_\sigma(V)$ is an associative algebra with product induced from $*_\sigma$ on V and identity $\mathbf{1} + O_\sigma(V)$. Moreover, $\omega + O_\sigma(V)$ is a central element.

(1') $A_\sigma(V)$ is a superalgebra with

$$A_\sigma(V)_{\bar{r}} = (V_{\bar{r}} + O_\sigma(V))/O_\sigma(V) \cong V_{\bar{r}}/O_\sigma(V) \cap V_{\bar{r}}.$$

(2) If $M = \bigoplus_{n \geq 0} M(n)$ is an admissible σ -twisted V -module with $M(0) \neq 0$ then $M(0)$ is an $A_\sigma(V)$ -module such that $v + O_\sigma(V)$ acts as $o(v)$ where $o(v) = v_{\text{wt } v} - 1$.

(2') If $M = \bigoplus_{n \geq 0} M(n)$ is an admissible σ -twisted super V -module with $M(0) \neq 0$ then $M(0)$ is a super $A_\sigma(V)$ -module such that $v + O_\sigma(V)$ acts as $o(v)$.

(3) The assignment, $M \rightarrow M(0)$, defines a bijection between inequivalent irreducible admissible σ -twisted V -modules and inequivalent irreducible $A_\sigma(V)$ -modules.

(3') The assignment, $M \rightarrow M(0)$, defines a bijection between inequivalent irreducible admissible σ -twisted super V -modules and inequivalent irreducible super $A_\sigma(V)$ -modules.

(4) If V is σ -rational then $A_\sigma(V)$ is a finite dimensional semisimple associative algebra.

(4') If V is σ -rational then $A_\sigma(V)$ is a finite dimensional semisimple superalgebra.

Now we assume that V is σ -rational. Let

$$\{N^0, N_\sigma^0, \dots, N^q, N_\sigma^q, N^{q+1}, \dots, N^p\}$$

be a complete set of inequivalent irreducible σ -twisted V -modules, where $N^i, N_\sigma^i = N^i \circ \sigma$ are inequivalent for $i = 0, \dots, q$ and $N^j \cong N^j \circ \sigma$ for $j = q+1, \dots, p$. Then

$$A_\sigma(V) = \bigoplus_{i=0}^q (\text{End } N^i(0) \oplus \text{End } N_\sigma^i(0)) \bigoplus_{j=q+1}^p \text{End } N^j(0).$$

For short we denote the $\text{End } N^i(0) \oplus \text{End } N_\sigma^i(0)$ by $A_\sigma(V)^i$ for $i = 0, \dots, q$ and $\text{End } N^j(0)$ by $A_\sigma(V)^j$ for $j = q+1, \dots, p$. Then $A_\sigma(V) = \bigoplus_{i=0}^p A_\sigma(V)^i$.

Lemma 3.4. *Let V be a σ -rational vertex operator superalgebra. For $i = 0, \dots, q$, $A_\sigma(V)^i$ is a $\mathcal{Q}$ type simple superalgebra with the unique irreducible super module $N^i(0) \oplus N_\sigma^i(0)$, a direct sum of the inequivalent irreducible $A_\sigma(V)^i$ -modules $N^i(0)$ and $N_\sigma^i(0)$. If $i = q+1, \dots, p$, $A_\sigma(V)^i$ is an $\mathcal{M}$ type simple superalgebra with the unique irreducible super module $N^i(0)$.*

Proof. By Proposition 3.3, $A_\sigma(V)^i$ is semisimple. Clearly, if $i > q$, $A_\sigma(V)^i$ is an $\mathcal{M}$ type simple superalgebra with the unique irreducible super module $N^i(0)$. If $i \leq q$, note that $N^i(0) \oplus N_\sigma^i(0)$ is a super $A_\sigma(V)^i$ -module with $(N^i(0) \oplus N_\sigma^i(0))_{\bar{r}}$ spanned by $(w, (-1)^r w)$ for $w \in N^i(0)$. Since both $(N^i(0) \oplus N_\sigma^i(0))_{\bar{r}}$ for $r = 0, 1$ are isomorphic irreducible $A_\sigma(V)_0^i$ -modules, we immediately see that $N^i(0) \oplus N_\sigma^i(0)$ is an irreducible super $A_\sigma(V)^i$ -module and $A_\sigma(V)^i$ is a simple superalgebra of $\mathcal{Q}$ type. The proof is complete. $\square$

We now can establish the following results on the tensor product of σ -twisted modules. Let U be another σ -rational vertex operator superalgebra and

$$\{W^{i'}, W_\sigma^{i'}, W^{j'} \mid i' = 0, \dots, q', j' = q' + 1, \dots, p'\}$$

is a complete set of inequivalent irreducible σ -twisted U -modules.

Theorem 3.5. *Let U, V be as above. Then $U \otimes V$ is σ -rational. Moreover, we have*

(1) *For $i' = 0, \dots, q', i = 0, \dots, q$, $(W^{i'} \oplus W_\sigma^{i'}) \otimes (N^i \oplus N_\sigma^i)$ is a sum of two isomorphic irreducible σ -twisted $U \otimes V$ -modules which are σ -stable.*

(2) *For $i' = 0, \dots, q', j = q+1, \dots, p$, $(W^{i'} \oplus W_\sigma^{i'}) \otimes N^j$ is a sum of two inequivalent irreducible σ -twisted $U \otimes V$ -modules $W^{i'} \otimes N^j$ and $W_\sigma^{i'} \otimes N^j$. In particular, $(W^{i'} \otimes N^j) \circ \sigma \cong W_\sigma^{i'} \otimes N^j$.*

(3) *For $j' = q'+1, \dots, p', i = 0, \dots, q$, $W^{j'} \otimes (N^i \oplus N_\sigma^i)$ is a sum of two inequivalent irreducible σ -twisted $U \otimes V$ -modules $W^{j'} \otimes N^i$ and $W^{j'} \otimes N_\sigma^i$ such that $(W^{j'} \otimes N^i) \circ \sigma \cong W^{j'} \otimes N_\sigma^i$.*

(4) *For $j' = q'+1, \dots, p', j = q+1, \dots, p$, $W^{j'} \otimes N^j$ is an irreducible σ -twisted $U \otimes V$ -module which is σ -stable.*

(5) *Every irreducible σ -twisted $U \otimes V$ -module is isomorphic to one of the irreducible σ -twisted modules listed in (1)–(4).*

Proof. The proof of σ -rationality of $U \otimes V$ is similar to that of Proposition 2.7 of [DMZ]. (2)–(4) can be verified directly by Lemma 3.1. For (1), we need $A_\sigma(U \otimes V)$. Using the exact proof of Lemma 2.8 in [DMZ] yields $A_\sigma(U \otimes V) \cong A_\sigma(U) \otimes A_\sigma(V)$. This gives

$$A_\sigma(U \otimes V) = \bigoplus_{0 \leq i' \leq p', 0 \leq i \leq p} A_\sigma(U)^{i'} \otimes A_\sigma(V)^i.$$

Note that these tensor product superalgebras are superalgebras with the multiplication given in the remark after Theorem 3.2. Using Lemma 3.4 and the tensor products of simple superalgebras we can give a different proof of (2)–(4). We now prove (1). In this case, $i' \leq q'$, $i \leq q$ and $A_\sigma(U)^{i'} \otimes A_\sigma(V)^i$ is isomorphic to the simple superalgebra $\mathcal{M}_{mn, mn} = M_{2mn \times 2mn}$, where $m = \dim W^{i'}(0)$ and $n = \dim N^i(0)$. So $A_\sigma(U)^{i'} \otimes A_\sigma(V)^i$ has a unique irreducible module of dimension $2mn$. Since $(W^{i'}(0) \oplus W_\sigma^{i'}(0)) \otimes (N^i(0) \oplus N_\sigma^i(0))$ is an $A_\sigma(U)^{i'} \otimes A_\sigma(V)^i$ -module of dimension

$4mn$, it has to be a sum of two isomorphic irreducible super $A_\sigma(U)^{i'} \otimes A_\sigma(V)^i$ -modules. As a result, $(W^{i'} + W_\sigma^{i'}) \otimes (N^i + N_\sigma^i)$ is a sum of two isomorphic irreducible σ -twisted $U \otimes V$ -modules which are σ -stable. (5) follows from Proposition 3.3(3). $\square$

4. MODULAR INVARIANCE

In this section, we review the modular invariance property of the trace functions in orbifold theory for vertex operator superalgebras from [DZ1], [DLM7] and [Z]. We also correct a mistake on the number of irreducible σ -twisted V -modules in [DZ1].

For the discussion of the modular invariance of trace functions, we recall the vertex operator superalgebra $(V, Y[\cdot], \mathbf{1}, \tilde{\omega})$ associated to a vertex operator superalgebra V defined in [Z]. Here $\tilde{\omega} = \omega - c/24$ and

$$Y[v, z] = Y(v, e^z - 1)e^{z \cdot \text{wt } v} = \sum_{n \in \mathbb{Z}} v[n]z^{-n-1}$$

for homogeneous v . Write

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n]z^{-n-2}.$$

The weight of $v \in V$ in $(V, Y[\cdot], \mathbf{1}, \tilde{\omega})$ is denoted by $\text{wt}[v]$.

In the rest of this paper, we assume that $V = \bigoplus_{n \geq 0} V_n$ is a simple vertex operator superalgebra such that

(A1) V_0 is regular vertex operator algebra of CFT type,

(A2) The weight of any irreducible σ^i -twisted V -module is positive except for V itself with $i = 0, 1$.

We remark that Assumption (A2) is not necessary in the discussions on modular invariance and classification of irreducible V_0 -modules. However this assumption is required for applying some results from [LKW1], where the categories are assumed to be pseudounitary.

Under the assumptions of (A1) and (A2), V is σ^i -rational for $i = 0, 1$ by Theorem 4.1 of [DH] and C_2 -cofinite [ABD]. Using the arguments from [M] and [CM] one can show, in fact, that V is regular if and only if V_0 is regular.

Denote by $\mathcal{M}(g)$ a complete set of inequivalent irreducible g -twisted V -modules for $g = 1, \sigma$ and set $\mathcal{M}(g, h) = \{M \in \mathcal{M}(g) | M \circ h \cong M\}$ for $g, h = 1, \sigma$. Note from Lemma 2.2 that $\mathcal{M}(1, h) = \mathcal{M}(1)$ for $h = 1, \sigma$. Also, $\mathcal{M}(\sigma, 1) = \mathcal{M}(\sigma)$. Then $\mathcal{M}(g)$ and $\mathcal{M}(g, h)$ are finite sets.

Let $M \in \mathcal{M}(\sigma g, \sigma h)$ and $T = o(g)$. For any homogeneous element $v \in V$, we denote $v_{\text{wt } v-1}$ by $o(v)$ as usual and set

$$Z_M(v, (g, h), \tau) = \text{tr}_M o(v) \sigma h q^{L(0)-c/24} = q^{\lambda-c/24} \sum_{n \in \frac{1}{T}\mathbb{Z}_+} \text{tr}_{M_{\lambda+n}} o(v) \sigma h q^n$$

if either $(g, h) \neq (1, \sigma)$ or $(g, h) = (1, \sigma)$ and $M \circ \sigma \cong M$. If $(g, h) = (1, \sigma)$ and $M \circ \sigma \not\cong M$.

$$Z_M(v, (g, h), \tau) = \frac{1}{\sqrt{2}} \text{tr}_M o(v + \sigma v) q^{L(0)-c/24} = \frac{1}{\sqrt{2}} q^{\lambda-c/24} \sum_{n \in \frac{1}{T}\mathbb{Z}_+} \text{tr}_{M_{\lambda+n}} o(v + \sigma v) q^n$$

By convention, $Z_M(v, (g, h), \tau) := 0$ whenever $M \notin \mathcal{M}(\sigma g, \sigma h)$.

Note that if $(g, h) = (1, \sigma)$ and $M \circ \sigma \not\cong M$ then

$$\begin{aligned} Z_M(v, (1, \sigma), \tau) &= \frac{1}{\sqrt{2}} (\text{tr}_M o(v) q^{L(0)-c/24} + \text{tr}_{M \circ \sigma} o(v) q^{L(0)-c/24}) \\ &= \sqrt{2} \text{tr}_M o(v) q^{L(0)-c/24} \\ &= Z_{M \circ \sigma}(v, (1, \sigma), \tau). \end{aligned}$$

The insertion of $\sqrt{2}$ in the definition of $Z_M(v, (1, \sigma), \tau)$ will ensure that the corresponding S -matrix is unitary (see the discussion in Section 6).

From [DZ1] we know that $Z_M(v, (g, h), \tau)$ are holomorphic function on the upper half plane $\mathbb{H}$ with $q = e^{2\pi i \tau}$ [DZ1]. The definition of $Z_M(v, (1, \sigma), \tau)$ given in this paper in the case $M \circ \sigma \not\cong M$ is different from [DZ1] where this case was not considered. This new definition ensures that $Z_M(v, (1, \sigma), \tau)$ is a vector in the conformal block $\mathcal{C}(1, \sigma)$ [DZ1]. According to the definition of the conformal block given in [DZ1], $Z_M(v, (1, \sigma), \tau)$ is required to be zero if $\sigma(v) = -v$. Clearly, $\text{tr}_M o(v) q^{L(0)-c/24}$ is not necessarily zero for such M . But $Z_M(v, (1, \sigma), \tau)$ is zero in our new definition.

Define $Z_M(v, \tau) = \text{tr}_M o(v) q^{L(0)-c/24}$ for σ^s -twisted V -modules M and $s = 0, 1$. Then $Z_M(v, \tau) = Z_M(v, (\sigma, \sigma), \tau)$ if M is a V -modules M , $Z_M(v, \tau) = \frac{1}{\sqrt{2}} Z_M(v, (1, \sigma), \tau)$ if M is an irreducible σ -twisted V -module such that $M \not\cong M \circ \sigma$ and $Z_M(v, \tau) = Z_M(v, (1, \sigma), \tau)$ if M is an irreducible σ -twisted V -module such that $M \cong M \circ \sigma$. We also set $\chi_M(\tau) = \text{tr}_M q^{L(0)-c/24}$ which is called the character of M .

Lemma 4.1. *If $M \in \mathcal{M}(\sigma g, \sigma h)$ and $v \in V_1$ then $Z_M(v, (g, h), \tau) = 0$ for any g, h .*

Proof. If $M \circ \sigma \cong M$ the result was obtained in Lemma 6.3 of [DZ1]. It remains to prove the result if M is an irreducible σ -twisted V -module M with $M \circ \sigma \not\cong M$. However, this follows from the preceding discussion. $\square$

Let W be the vector space spanned by $Z_M(v, (g, h), \tau)$ for $g, h \in \{1, \sigma\}$ and $M \in \mathcal{M}(\sigma g, \sigma h)$. Then, for any given pair (g, h) , Z_M can be regarded as a function on $V \times \mathbb{H}$. Now, we define an action of the modular group $\Gamma = SL_2(\mathbb{Z})$ on W such that

$$Z_M|_{\gamma}(v, (g, h), \tau) = (c\tau + d)^{-\text{wt}[v]} Z_M(v, (g, h), \gamma\tau),$$

where

$$\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Recall that $G = \{1, \sigma\}$ acts on $\mathcal{M}(1)$ and $\mathcal{M}(\sigma)$ such that the action of σ on M is $M \circ \sigma$. We have already known that each G -orbit in $\mathcal{M}(1)$ has exactly one module by Lemma 2.2, and each G -orbit in $\mathcal{M}(\sigma)$ has either one or two σ -twisted modules. Note that if two σ -twisted modules M^1 and M^2 are in the same G -orbit, then $Z_{M^1}(v, (g, h), \tau) = Z_{M^2}(v, (g, h), \tau)$ for all $v \in V$. Let $\mathcal{O}_{\sigma^i}$ be the collection of orbit representatives in $\mathcal{M}(\sigma^i)$.

The following result is essentially obtained in [DZ1] with suitable modification:

Theorem 4.2. *Let V be a vertex operator superalgebra satisfying the assumptions (A1)–(A2).*

(1) $\{Z_M(v, (g, h), \tau) | M \in \mathcal{O}_{\sigma g}\}$ *is linearly independent.*

(2) There is a representation $\rho_V : \Gamma \rightarrow GL(W)$ such that for $g, h \in \{1, \sigma\}$ $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and $M \in \mathcal{O}_{\sigma g}$,

$$Z_M|_{\gamma}(v, (g, h), \tau) = \sum_{N \in \mathcal{O}_{\sigma g^a h^c}} \gamma_{M,N}^{(g,h)} Z_N(v, (g^a h^c, g^b h^d), \tau)$$

where $\rho(\gamma) = (\gamma_{M,N}^{(g,h)})$. That is,

$$Z_M(v, (g, h), \gamma\tau) = (c\tau + d)^{\text{wt}[v]} \sum_{N \in \mathcal{O}_{\sigma g^a h^c}} \gamma_{M,N}^{(g,h)} Z_N(v, (g^a h^c, g^b h^d), \tau).$$

(3) The number of G -orbits in $\mathcal{M}(\sigma)$ or the number of inequivalent irreducible σ -twisted super modules is equal to the number of inequivalent irreducible V -modules.

Theorem 4.2(3) gives a correction of Theorem 8.6(2) in [DZ1]. Let $\mathcal{C}(g, h)$ be the vector spaces spanned by $Z_M(v, (g, h), \tau)$ for $M \in \mathcal{O}_{\sigma g}$. Then by Theorem 4.2(2) we know that $\mathcal{C}(1, \sigma)$ and $\mathcal{C}(\sigma, 1)$ have the same dimension by using the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So $\mathcal{O}_{\sigma^i}$ have the same cardinality for $i = 0, 1$. In particular, the number of inequivalent irreducible σ -twisted modules is always greater than or equal to the number of inequivalent irreducible modules. Two numbers are equal if and only if every irreducible σ -twisted V -module is σ -stable. This result is different from that in [DLM7] when V is a vertex operator algebra and g is an order 2 automorphism. Moreover, if we replace the irreducible σ -twisted modules by the irreducible σ -twisted super modules, the result is the same as in the case of vertex operator algebra.

If $V = V_0$ is a vertex operator algebra then ρ_V is a unitary representation of Γ and the kernel of ρ_V is a congruence subgroup of Γ [Z, DLN].

We use the free fermion as an example to illustrate Theorem 4.2. Let $A(\frac{1}{2} + \mathbb{Z})$ be the associative algebra generated by $a(m)$ with $m \in \frac{1}{2} + \mathbb{Z}$ subject to the relation $a(m)a(n) + a(n)a(m) = 2\delta_{m+n,0}$, and $A(\frac{1}{2} + \mathbb{Z})^+$ the subalgebra generated by $a(m)$ with $m > 0$. Consider $\mathbb{C}$ as an $A(\frac{1}{2} + \mathbb{Z})^+$ -module with the trivial action $a(m) \cdot 1 = 0$ for $m > 0$. Then $V(\frac{1}{2} + \mathbb{Z}) = A(\frac{1}{2} + \mathbb{Z}) \otimes_{A(\frac{1}{2} + \mathbb{Z})^+} \mathbb{C}$ is the unique irreducible highest weight $A(\frac{1}{2} + \mathbb{Z})$ -module. As vector spaces, $V(\frac{1}{2} + \mathbb{Z})$ is isomorphic to the free exterior algebra $\bigwedge[a(m) | m \leq 0]$. It is well known that $V(\frac{1}{2} + \mathbb{Z})$ is a rational, C_2 -cofinite vertex operator superalgebra with only one irreducible module $V(\frac{1}{2} + \mathbb{Z})$ up to isomorphism [KW] and [L1]. Moreover, $V(\frac{1}{2} + \mathbb{Z})$ is generated by $a(-1/2)$ such that $Y(a(-1/2), z) = \sum_{n \in \mathbb{Z}} a(n + 1/2) z^{-n-1}$.

The vertex operator superalgebra $V(\frac{1}{2} + \mathbb{Z})$ has two inequivalent irreducible σ -twisted modules. To construct these two σ -twisted modules we need another associative algebra $A(\mathbb{Z})$ generated by $a(m)$ with $m \in \mathbb{Z}$ satisfying the relation $a(m)a(n) + a(n)a(m) = 2\delta_{m+n,0}$. Let $A(\mathbb{Z})^+$ be the subalgebra of $A(\mathbb{Z})$ generated by $a(m)$ with $m > 0$. Consider the induced $A(\mathbb{Z})$ -module $V(\mathbb{Z}) = A(\mathbb{Z}) \otimes_{A(\mathbb{Z})^+} \mathbb{C}$ where $\mathbb{C}$ is $A(\mathbb{Z})^+$ -module such that $a(m)1 = 0$ for all $m > 0$. It is easy to see that $V(\mathbb{Z})$ is isomorphic to $\bigwedge[a(n) | n \in \mathbb{Z}, n \leq 0]$, in which $a(m)$ acts by multiplication if $m \leq 0$ and $a(m)$ acts as $\pm 2 \frac{\partial}{\partial a(-m)}$ if $m > 0$. Let $W = \bigwedge[a(m) | m \in \mathbb{Z}, m < 0]$, and $W = W_0 \oplus W_1$ the decomposition of W into the sum even and odd subspaces. Then

$$V(\mathbb{Z})_{\pm} = (1 \pm a(0))W_0 \oplus (1 \mp a(0))W_1$$

are irreducible $A(\mathbb{Z})$ -submodules of $V(\mathbb{Z})$ and $V(\mathbb{Z}) = V(\mathbb{Z})_+ \oplus V(\mathbb{Z})_-$. Moreover, $V(\mathbb{Z})_{\pm}$ are the inequivalent irreducible σ -twisted $V(\frac{1}{2} + \mathbb{Z})$ -modules such that $Y(a(-1/2), z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1/2}$ [L2], [DZ2]. It is easy to verify that $V(\mathbb{Z})_+ \circ \sigma$ is isomorphic to $V(\mathbb{Z})_-$. Furthermore, $V(\mathbb{Z})$ is the unique irreducible σ -twisted super $V(\frac{1}{2} + \mathbb{Z})$ -module.

Next we want to discuss more on the trace functions $Z_M(v, (g, h), \tau)$. We know from the Lemma 4.1 that $Z_M(v, (g, h), \tau) = 0$ if $\sigma v = -v$. But we can still consider $\text{tr}_M o(v)q^{L(0)-c/24}$ for $M \in \mathcal{M}(\sigma)$ such that M and $M \circ \sigma$ are not isomorphic, and $v \in V_{\bar{1}}$. In general, $\text{tr}_M o(v)q^{L(0)-c/24}$ does not vanish. But our result does not tell anything about such $\text{tr}_M o(v)q^{L(0)-c/24}$. Now consider the example $V(\frac{1}{2} + \mathbb{Z})$. Let $v = a(-1/2) \in V(\frac{1}{2} + \mathbb{Z})_{\bar{1}}$. Then $\text{wt}[v] = \frac{1}{2}$ and $o(v) = a(0)$ on the twisted module. It is easy to compute that

$$\text{tr}_{V(\mathbb{Z})_{\pm}} o(v)q^{L(0)-c/24} = \pm q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

which is a modular form of weight $\frac{1}{2}$ over Γ . This suggests that for an arbitrary rational vertex operator superalgebra V , an irreducible σ -twisted module M and $v \in V_{\bar{1}}$, $\text{tr}_M o(v)q^{L(0)-c/24}$ is still a modular form of weight $\text{wt}[v]$.

The following corollary is immediate.

Corollary 4.3. *If $\gamma = S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $v \in V_{\bar{0}}$ we have:*

$$Z_M(v, (1, 1), -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{N \in \mathcal{O}_{\sigma}} S_{M,N}^{(1,1)} Z_N(v, (1, 1), \tau),$$

$$Z_M(v, (1, \sigma), -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{N \in \mathcal{M}(1)} S_{M,N}^{(1,\sigma)} Z_N(v, (\sigma, 1), \tau)$$

for $M \in \mathcal{O}_{\sigma}$, and

$$(4.1) \quad Z_N(v, (\sigma, 1), -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{M \in \mathcal{O}_{\sigma}} S_{N,M}^{(\sigma,1)} Z_M(v, (1, \sigma), \tau),$$

$$(4.2) \quad Z_N(v, (\sigma, \sigma), -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_{M \in \mathcal{M}(1)} S_{N,M}^{(\sigma,\sigma)} Z_M(v, (\sigma, \sigma), \tau)$$

for any $N \in \mathcal{M}(1)$. The matrix $\rho(S) = (S_{M,N}^{(g,h)})$ is called S -matrix of V and is independent of the choice of vector $v \in V_{\bar{0}}$.

Remark 4.4. If $V_{\bar{1}} = 0$ then $V = V_{\bar{0}}$ is a vertex operator algebra and $\sigma = 1$. In this case, the representation ρ is unitary and the kernel of ρ is a congruence subgroup [DLN].

5. IRREDUCIBLE $V_{\bar{0}}$ -MODULES

We classify the irreducible $V_{\bar{0}}$ -modules in this section and show that every irreducible $V_{\bar{0}}$ -module occurs in an irreducible V -module or σ -twisted module. The main idea is to use the S -matrix to classify the irreducible $V_{\bar{0}}$ -modules as in [DRX].

By Theorem 4.2(3), the number of inequivalent irreducible V -modules is equal to the number of inequivalent of irreducible σ -twisted super V -modules. Let

$\{M^0, \dots, M^p\}$ be a complete set of inequivalent irreducible V -modules with $M^0 = V$ and

$$\{N^0, N_\sigma^0, \dots, N^q, N_\sigma^q, N^{q+1}, \dots, N^p\}$$

the inequivalent irreducible σ -twisted V -modules such that N^j and $N^j \circ \sigma$ are equivalent for $j > q$. Then $M^i = M_0^i \oplus M_1^i$ and $N^j = N_0^j \oplus N_1^j$ are direct sum of two irreducible V_0 -modules by Lemmas 2.2 and 2.3 for $i = 0, \dots, p$ and $j = q+1, \dots, p$.

Theorem 5.1. *Let V be a vertex operator superalgebra satisfying the assumptions (A1)–(A2). Then*

$$\{M_s^i, N_s^j, N_s^k \mid i = 0, \dots, p, j = 0, \dots, q, k = q+1, \dots, p, s = 0, 1\}$$

is a complete set of inequivalent irreducible V_0 -modules.

Proof. We first prove that $\{M_s^i \mid i = 0, \dots, p, s = 0, 1\}$ is a complete set of inequivalent V_0 -modules. Following [DLM5] we can define associative algebras $A_n(V)$ for $n \in \frac{1}{2}\mathbb{Z}_+$ such that $A_0(V) = A(V)$ as defined in [KW] and both $A_m(V)$ and $A_{m+\frac{1}{2}}(V)$ for any nonnegative integer m are quotient algebras of $A_m(V_0)$. Moreover,

$$A_n(V) = \bigoplus_{i=0}^p \bigoplus_{m \leq n} \text{End } M^i(m)$$

as V is rational. Noting that $M_s^i = \bigoplus_{n \in \mathbb{Z}_+} M^i(\frac{1}{2}s + n)$, we see immediately that M_s^i are inequivalent V_0 -modules.

We prove next that $\{N_s^j, N_s^k \mid j = 0, \dots, q, k = q+1, \dots, p, s = 0, 1\}$ is a complete set of inequivalent V_0 -modules. In this case we need to construct associative algebras $A_{\sigma,n}(V)$ for $n \in \mathbb{Z}_+$ following [DLM6] so that $A_{\sigma,0}(V) = A_\sigma(V)$ as defined in [DZ2]. We can then follow the proof given in [DY] to show that $\{N_s^j, N_s^k \mid j = 0, \dots, q, k = q+1, \dots, p, s = 0, 1\}$ are inequivalent V_0 -modules.

Finally we prove that any M_s^i and N_s^j or M_s^i and N_t^k are not isomorphic. From Proposition 7.2, we see that $\text{qdim}_{V_0} V_1 = \text{qdim}_V V = 1$. Thus V_1 is a simple current [DJX]. This forces $V_1 \boxtimes M_s^i = M_{s+1}^i$ and $V_1 \boxtimes N_t^k = N_{t+1}^k$ and $V_1 \boxtimes N^j = N^j$ as V_0 -modules. Note that the weight difference between M_0^i and M_1^i is a half-integer, and the weight difference between N_0^k and N_1^k is an integer. So any M_s^i and N_t^k or M_s^i and N^j for $i = 0, \dots, p, j = 0, \dots, q, k = q+1, \dots, p$ and $s, t = 0, 1$ are not isomorphic. $\square$

Our next goal is to prove that the irreducible modules given in Theorem 5.1 are complete.

Theorem 5.2. *Let V be a vertex operator superalgebra satisfying the assumptions (A1)–(A2). Then*

$$\{M_s^i, N_s^j, N_s^k \mid i = 0, \dots, p, j = 0, \dots, q, k = q+1, \dots, p, s = 0, 1\}$$

is a complete list of inequivalent irreducible V_0 -modules.

Proof. The main idea in the proof is to use the S -matrix for vertex operator algebra V_0 . Observe that for $v \in V_0$,

$$Z_{V_0}(v, \tau) = \frac{1}{2}(Z_V(v, (\sigma, \sigma), \tau) + Z_V(v, (\sigma, 1), \tau)).$$

Thus

$$Z_{V_0}(v, -\frac{1}{\tau}) = \frac{1}{2}(Z_V(v, (\sigma, \sigma), -\frac{1}{\tau}) + Z_V(v, (\sigma, 1), -\frac{1}{\tau})).$$

Using (4.2) and Theorem 5.1 we know that

$$\begin{aligned} Z_V(v, (\sigma, \sigma), -\frac{1}{\tau}) &= \tau^{\text{wt}[v]} \sum_{i=0}^p S_{V, M^i}^{(\sigma, \sigma)} Z_{M^i}(v, (\sigma, \sigma), \tau) \\ &= \tau^{\text{wt}[v]} \sum_{i=0}^p S_{V, M^i}^{(\sigma, \sigma)} (Z_{M_0^i}(v, \tau) + Z_{M_1^i}(v, \tau)). \end{aligned}$$

By (4.1) and Theorem 5.1

$$\begin{aligned} Z_V(v, (\sigma, 1), -\frac{1}{\tau}) &= \tau^{\text{wt}[v]} \sum_{M \in \mathcal{O}_\sigma} S_{V, M}^{(\sigma, 1)} Z_M(v, (1, \sigma), \tau) \\ &= \sqrt{2} \tau^{\text{wt}[v]} \sum_{j=0}^q S_{V, N^j}^{(\sigma, 1)} Z_{N^j}(v, \tau) + \tau^{\text{wt}[v]} \sum_{j=q+1}^p S_{V, N^j}^{(\sigma, 1)} (Z_{N_0^j}(v, \tau) + Z_{N_1^j}(v, \tau)). \end{aligned}$$

From [2], $Z_{M^i}(v, \tau)$, $Z_{N^j}(v, \tau)$, $Z_{N_s^k}(v, \tau)$ for $i = 0, \dots, p$, $j = 0, \dots, q$, $k = q+1, \dots, p$, $r, s = 0, 1$ are linearly independent vectors in the conformal block of V_0 . From [Hu2], $\tau^{-\text{wt}[v]} Z_{V_0}(v, -\frac{1}{\tau})$ is a linear combination of $Z_W(v, \tau)$ for the irreducible V_0 -modules W and the coefficient of each $Z_W(v, \tau)$ in the linear combination is nonzero. This implies that the list of irreducible V_0 -modules in Theorem 5.1 is complete. $\square$

6. THE UNITARITY OF ρ

The representation ρ given in Section 4 is essentially the representation of $SL_2(\mathbb{Z})$ associated with the modular invariance of the trace functions of V_0 with respect to a new basis. In this section, we show that this representation ρ is unitary, which means this new basis is orthonormal.

Since the modular group is generated by S and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it is good enough to show $\rho_V(S)$ and $\rho_V(T)$ are unitary matrices. Recall that $\rho_{V_0}(S)$ and $\rho_{V_0}(T)$ are the S and T matrices of V_0 . The main idea is to use the unitarity of ρ_{V_0} to establish the unitarity of ρ_V . For this purpose we need to determine the relation between $\rho_V(S)$ and $\rho_{V_0}(S)$, and $\rho_V(T)$ and $\rho_{V_0}(T)$.

Recall that $Z_M(v, \tau) = \text{tr}_M o(v) q^{L(0) - c/24}$ for any irreducible V_0 -module M and $v \in V_0$. The S and T matrices of V_0 are given defined by

$$Z_M(v, -\frac{1}{\tau}) = \tau^{\text{wt}[v]} \sum_N S_{M, N} Z_N(v, \tau).$$

$$Z_M(v, \tau + 1) = e^{2\pi i(-c/24 + \lambda_M)} Z_M(v, \tau)$$

where N runs through the inequivalent irreducible V_0 -modules, c is the central charge of V , λ_M is the lowest weight of M . In particular, the T matrix of V_0 is diagonal with $T_{M, M} = e^{2\pi i(-c/24 + \lambda_M)}$ which is a root of unity as both c and λ_M are rational [DLM7].

According to Theorem 5.2 we have three cases (i) $M = M_s^i$ for $i = 0, \dots, p$ and $s = 0, 1$, (ii) $M = N^j$ for $j = 0, \dots, q$, (iii) $M = N_s^k$ for $k = q+1, \dots, p$ and $s = 0, 1$. We first compute $S_{M_s^i, N}$ for $i = 0, \dots, p$ and $s = 0, 1$. The computation is

similar to those given in the proof of Theorem 5.2 for $v \in V_{\bar{0}}$:

$$\begin{aligned} Z_{M_{\bar{s}}^i}(v, -\frac{1}{\tau}) &= \frac{1}{2}(Z_{M^i}(v, (\sigma, \sigma), -\frac{1}{\tau}) + (-1)^s Z_{M^i}(v, (\sigma, 1), -\frac{1}{\tau})) \\ &= \frac{1}{2}\tau^{\text{wt}[v]} \sum_{j=0}^p S_{M^i, M^j}^{(\sigma, \sigma)} Z_{M^j}(v, (\sigma, \sigma), \tau) \\ &\quad + \frac{(-1)^s}{2}\tau^{\text{wt}[v]} \sum_{N \in \mathcal{O}_{\sigma}} S_{M^i, N}^{(\sigma, 1)} Z_N(v, (1, \sigma), \tau) \\ &= \frac{1}{2}\tau^{\text{wt}[v]} \sum_{j=0}^p S_{M^i, M^j}^{(\sigma, \sigma)} (Z_{M_0^j}(v, \tau) + Z_{M_1^j}(v, \tau)) \\ &\quad + \frac{(-1)^s}{\sqrt{2}}\tau^{\text{wt}[v]} \sum_{j=0}^q S_{M^i, N^j}^{(\sigma, 1)} Z_{N^j}(v, \tau) \\ &\quad + \frac{(-1)^s}{2}\tau^{\text{wt}[v]} \sum_{j=q+1}^p S_{M^i, N^j}^{(\sigma, 1)} (Z_{N_0^j}(v, \tau) + Z_{N_1^j}(v, \tau)). \end{aligned}$$

The following lemma is immediate.

Lemma 6.1. *For $i = 0, \dots, p$ and $s = 0, 1$ we have*

- (1) $S_{M_{\bar{s}}^i, M_{\bar{t}}^j} = \frac{1}{2}S_{M^i, M^j}^{(\sigma, \sigma)}$ for $j = 0, \dots, p$ and $t = 0, 1$,
- (2) $S_{M_{\bar{s}}^i, N^j} = \frac{(-1)^s}{\sqrt{2}}S_{M^i, N^j}^{(\sigma, 1)}$ for $j = 0, \dots, q$,
- (3) $S_{M_{\bar{s}}^i, N_{\bar{t}}^j} = \frac{(-1)^s}{2}S_{M^i, N^j}^{(\sigma, 1)}$ for $j = q+1, \dots, p$ and $t = 0, 1$.

Next we compute $S_{N^i, N}$. Since N^i is an irreducible $V_{\bar{0}}$ -module for $i = 0, \dots, q$, by Corollary 4.3 and the remark preceding Lemma 4.1, we immediately have

$$\begin{aligned} Z_{N^i}(v, -\frac{1}{\tau}) &= \frac{1}{\sqrt{2}}Z_{N^i}(v, (1, \sigma), -\frac{1}{\tau}) \\ &= \frac{1}{\sqrt{2}}\tau^{\text{wt}[v]} \sum_{j=0}^p S_{N^i, M^j}^{(1, \sigma)} Z_{M^j}(v, (\sigma, 1), \tau) \\ &= \frac{1}{\sqrt{2}}\tau^{\text{wt}[v]} \sum_{j=0}^p S_{N^i, M^j}^{(1, \sigma)} (Z_{M_0^j}(v, \tau) - Z_{M_1^j}(v, \tau)). \end{aligned}$$

The discussion above yields

Lemma 6.2. *For $i = 0, \dots, q$, $S_{N^i, M_{\bar{s}}^j} = \frac{(-1)^s}{\sqrt{2}}S_{N^i, M^j}^{(1, \sigma)}$ and $S_{N^i, W} = 0$ for the other irreducible $V_{\bar{0}}$ -modules W .*

Similarly, we have

Lemma 6.3. *For $i = q+1, \dots, p$ and $s, t = 0, 1$, $S_{N_{\bar{s}}^i, M_{\bar{t}}^j} = \frac{(-1)^t}{2}S_{N^i, M^j}^{(1, \sigma)}$ for $j = 0, \dots, p$, $S_{N_{\bar{s}}^i, N_{\bar{t}}^j} = \frac{(-1)^{s+t}}{2}S_{N^i, N^j}^{(1, 1)}$ for $j = q+1, \dots, p$ and $S_{N_{\bar{s}}^i, W} = 0$ for the other irreducible $V_{\bar{0}}$ -modules W .*

Proof. A straightforward calculation using Corollary 4.3 gives

$$\begin{aligned} Z_{N^i_s}(v, -\frac{1}{\tau}) &= \frac{1}{2}(Z_{N^i}(v, (1, \sigma), -\frac{1}{\tau}) + (-1)^s Z_{N^i}(v, (1, 1), -\frac{1}{\tau})) \\ &= \frac{1}{2}\tau^{\text{wt}[v]} \sum_{j=0}^p S_{N^i, M^j}^{(1, \sigma)} Z_{M^j}(v, (\sigma, 1), \tau) + \frac{(-1)^s}{2}\tau^{\text{wt}[v]} \sum_{j=q+1}^p S_{N^i, N^j}^{(1, 1)} Z_{N^j}(v, (1, 1), \tau) \\ &= \frac{1}{2}\tau^{\text{wt}[v]} \sum_{j=0}^p S_{N^i, M^j}^{(1, \sigma)} (Z_{M^j_0}(v, \tau) - Z_{M^j_1}(v, \tau)) \\ &\quad + \frac{(-1)^s}{2}\tau^{\text{wt}[v]} \sum_{j=q+1}^p S_{N^i, N^j}^{(1, 1)} (Z_{N^j_0}(v, \tau) - Z_{N^j_1}(v, \tau)). \end{aligned}$$

The result follows. $\square$

Theorem 6.4. *The representation ρ given in Theorem 4.2 is unitary.*

Proof. The unitarity of $\rho(S)$ follows from Lemmas 6.1–6.3 and the unitarity of S matrix of V_0 . It remains to show that $\rho(T)$ is unitary. We have

$$\begin{aligned} Z_{M^i}(v, (\sigma, \sigma), \tau + 1) &= Z_{M^i_0}(v, \tau + 1) + Z_{M^i_1}(v, \tau + 1) \\ &= e^{2\pi i(-c/24 + \lambda_{M^i_0})} Z_{M^i_0}(v, \tau) - e^{2\pi i(-c/24 + \lambda_{M^i_0})} Z_{M^i_1}(v, \tau) \\ &= e^{2\pi i(-c/24 + \lambda_{M^i_0})} Z_{M^i}(v, (\sigma, 1), \tau) \end{aligned}$$

where we have used the fact that $\lambda_{M^i_0} - \lambda_{M^i_1} + \frac{1}{2}$ is an integer. Similarly,

$$Z_{M^i}(v, (\sigma, 1), \tau + 1) = e^{2\pi i(-c/24 + \lambda_{M^i_0})} Z_{M^i}(v, (\sigma, \sigma), \tau).$$

It is easy to see that for $i = 0, \dots, p$ and $j = q + 1, \dots, p$

$$Z_{N^i}(v, (1, \sigma), \tau + 1) = e^{2\pi i(-c/24 + \lambda_{N^i})} Z_{N^i}(v, (1, \sigma), \tau),$$

$$Z_{N^j}(v, (1, 1), \tau + 1) = e^{2\pi i(-c/24 + \lambda_{N^j})} Z_{N^j}(v, (1, 1), \tau).$$

The unitarity of $\rho(T)$ now follows from that fact that c and λ_{N^i} are rational numbers [DLM7]. $\square$

7. QUANTUM DIMENSIONS

In this section, we compute the quantum dimensions of the irreducible σ^i -twisted V -modules and irreducible V_0 -modules, which are the Frobenius-Perron dimensions of these modules in their underlying fusion categories. The ideas and techniques used here come from [DJX] and [DRX].

Let V be a vertex operator superalgebra as before and M be an irreducible σ^i -twisted module. Recall $\chi_M(\tau)$ from Section 4. The quantum dimension of M over V is defined to be

$$\text{qdim}_V M = \lim_{y \rightarrow 0} \frac{\chi_M(iy)}{\chi_V(iy)}$$

using the relation $q = e^{2\pi i\tau}$ where y is real and positive.

The existence of the quantum dimension for a g -twisted V -module is given below in terms of the S -matrix and the proof is similar to that of Lemma 4.2 of [DJX] by using the S -matrix given in Corollary 4.3.

Proposition 7.1. *Let V be a vertex operator superalgebra satisfying (A1)–(A2), and M an irreducible σ^r -twisted V -module for $r = 0, 1$. Then $\text{qdim}_V M = \frac{S_{M,V}^{(\sigma,\sigma)}}{S_{V,V}^{(\sigma,\sigma)}}$ for any irreducible V -module M . If M is an irreducible σ -twisted V -module, then $\text{qdim}_V M = \frac{S_{M,V}^{(1,\sigma)}}{\sqrt{2}S_{V,V}^{(\sigma,\sigma)}}$ for $M = N^i$ for $i = 0, \dots, q$ and $\text{qdim}_V M = \frac{S_{M,V}^{(1,\sigma)}}{S_{V,V}^{(\sigma,\sigma)}}$ for other M . In particular, $\text{qdim}_V M$ exists.*

We define the global dimension of V as

$$\text{glob}(V) = \sum_{M \in \mathcal{M}(1)} (\text{qdim}_V M)^2.$$

In the case when V is a vertex operator algebra, $\text{glob}(V)$ is exactly the global dimension of V defined in [DJX] and is equal to $\frac{1}{S_{V,V}^2}$.

We now compute the quantum dimensions of irreducible V_0 -modules in terms of quantum dimensions of irreducible V -modules by using Theorem 5.2

Proposition 7.2. *We have*

- (1) $\text{qdim}_{V_0} M_{\bar{r}}^i = \text{qdim}_V M^i$ for $i = 0, \dots, p$ and $r = 0, 1$;
- (2) $\text{qdim}_{V_0} N^j = 2 \text{qdim}_V N^j$ for $j = 0, \dots, q$;
- (3) $\text{qdim}_{V_0} N_s^k = \text{qdim}_V N^k$ for $k = q + 1, \dots, p$ and $s = 0, 1$;
- (4) $\text{glob}(V_0) = 4 \text{glob}(V)$;
- (5) $\sum_{M \in \mathcal{M}(\sigma)} (\text{qdim}_V M)^2 = \text{glob}(V)$;
- (6) $\sum_{X_1} (\text{qdim}_{V_0} X_1)^2 = \sum_{X_2} (\text{qdim}_{V_0} X_2)^2$ where X_i ranges over the inequivalent irreducible V_0 -modules appearing in irreducible σ^i -twisted V -modules.

In particular, $\text{qdim}_{V_0} W = 2 \text{qdim}_V W$ for any irreducible σ^r -twisted module W . Moreover, $\text{qdim}_V M^i$, and $2 \text{qdim}_V N^j$ for $j = 0, \dots, q$, and $\text{qdim}_V N^k$ for $k = q + 1, \dots, p$, take values in $\{2 \cos \frac{\pi}{n} \mid n \geq 3\} \cup [2, \infty)$.

Proof. (1) By Proposition 7.1 and Lemma 6.1 we see that

$$\text{qdim}_{V_0} M_{\bar{r}}^i = \frac{S_{M_{\bar{r}}, V_0}^i}{S_{V_0, V_0}} = \frac{S_{M^i, V}^{(\sigma, \sigma)}}{S_{V, V}^{(\sigma, \sigma)}} = \text{qdim}_V M^i.$$

(2) can be proved similarly by using Lemma 6.2. But we give a different proof here:

$$\begin{aligned} \text{qdim}_{V_0} N^j &= \lim_{y \rightarrow 0} \frac{\chi_{N^j}(iy)}{\chi_{V_0}(iy)} = \lim_{y \rightarrow 0} \frac{\chi_{N^j}(iy)}{\chi_V(iy)} \frac{\chi_V(iy)}{\chi_{V_0}(iy)} \\ &= \lim_{y \rightarrow 0} \frac{\chi_{N^j}(iy)}{\chi_V(iy)} \lim_{y \rightarrow 0} \frac{\chi_V(iy)}{\chi_{V_0}(iy)} = 2 \text{qdim}_V N^j. \end{aligned}$$

(3) The proof is similar.

(4) From [DJX] we know that $\text{glob}(V_0) = \frac{1}{S_{V_0, V_0}^2}$ is positive. This implies that S_{V_0, V_0} is a real number. It follows from Proposition 7.1 that $S_{V, V}^{(\sigma, \sigma)}$ is a real number. Since $\text{qdim}_V M$ is always positive for any irreducible σ^r -twisted V -module M , we see from Proposition 7.1 again that $S_{M, V}^{(\sigma^{1-r}, \sigma)}$ is a real number.

Using Proposition 7.1 and Theorem 6.4 yields

$$\text{glob}(V) = \sum_{i=0}^p (\text{qdim}_V M^i)^2 = \sum_{i=0}^p \left(\frac{S_{M^i, V}^{(\sigma, \sigma)}}{S_{V, V}^{(\sigma, \sigma)}} \right)^2 = \left(\frac{1}{S_{V, V}^{(\sigma, \sigma)}} \right)^2 \sum_{i=0}^p (S_{M^i, V}^{(\sigma, \sigma)})^2 = \left(\frac{1}{S_{V, V}^{(\sigma, \sigma)}} \right)^2.$$

By Lemma 6.1, $S_{V_0 V_0} = \frac{1}{2} S_{V,V}^{(\sigma,\sigma)}$. It follows immediately that $\text{glob}(V_0) = 4 \text{glob}(V)$.

(5) Again by Proposition 7.1 and Theorem 6.4

$$\sum_{M \in \mathcal{M}(\sigma)} (\text{qdim}_V M)^2 = \frac{2}{(S_{V,V}^{(\sigma,\sigma)})^2} \sum_{i=0}^q \left(\frac{S_{N^i,V}^{(1,\sigma)}}{\sqrt{2}} \right)^2 + \left(\frac{1}{S_{V,V}^{(\sigma,\sigma)}} \right)^2 \sum_{i=q+1}^p (S_{N^i,V}^{(1,\sigma)})^2 = \left(\frac{1}{S_{V,V}^{(\sigma,\sigma)}} \right)^2.$$

(6) Note from (1) that

$$\sum_{i=0}^p \sum_{s=0}^1 (\text{qdim}_{V_0} M_s^i)^2 = \sum_{i=0}^p 2(\text{qdim}_V M^i)^2 = 2 \text{glob}(V).$$

The result follows now from (4) and Theorem 5.2.

From [DJX], the quantum dimensions of irreducible V_0 -modules lie in $\{2 \cos \frac{\pi}{n} | n \geq 3\} \cup [2, \infty)$. From (1)–(3) we see immediately that $\text{qdim}_V M^i$, $2 \text{qdim}_V N^j$ for $j = 0, \dots, q$ and $\text{qdim}_V N^k$ for $k = q+1, \dots, p$ take values in $\{2 \cos \frac{\pi}{n} | n \geq 3\} \cup [2, \infty)$. $\square$

8. $\mathbb{Z}_2$ -GRADING ON THE CATEGORY OF V_0 -MODULES

For the discussion below, we introduce several module categories. We use $\mathcal{C}_V$ and $\mathcal{C}_V^\sigma$ to denote the V -module category and the σ -twisted V -module category, respectively. Since $\sigma|_{V_0} = \text{id}_{V_0}$, the objects in $\mathcal{C}_V$ and $\mathcal{C}_V^\sigma$ are V_0 -modules. We denote by $\mathcal{C}_{V_0}^r$ the full abelian subcategory of $\mathcal{C}_{V_0}$ generated by the simple V_0 -submodules of any σ^r -twisted V -modules. Let $\mathcal{C}$ be any of these categories, the dimension of $\mathcal{C}$ is defined as $\dim \mathcal{C} = \sum_M (\text{qdim}_V M)^2$ where M runs over the equivalence classes of simple objects in $\mathcal{C}$. It is clear that $\text{glob}(V) = \dim \mathcal{C}_V$, $\text{glob}(V_0) = \dim \mathcal{C}_{V_0} = \dim \mathcal{C}_{V_0}^0 + \dim \mathcal{C}_{V_0}^1$. From the discussions in Section 7, we know that $\text{glob}(V_0) = 4 \text{glob}(V)$, $\dim \mathcal{C}_V = \dim \mathcal{C}_V^\sigma$, and $\dim \mathcal{C}_{V_0}^0 = \dim \mathcal{C}_{V_0}^1$. By [Hu2], $\mathcal{C}_{V_0}$ is a modular tensor category, and $\dim \mathcal{C}_{V_0}$ coincides with the categorical dimension of the fusion category of $\mathcal{C}_{V_0}$.

Theorem 8.1. *The category $\mathcal{C}_{V_0}^0$ is a fusion subcategory of $\mathcal{C}_{V_0}$ with a complete list of simple objects given by M_r^j , with $j = 0, \dots, p$ and $r = 0, 1$.*

Proof. We need to show that $M_r^i \boxtimes M_s^j$ for $i, j = 0, \dots, p$ and $r, s = 0, 1$ lies in $\mathcal{C}_{V_0}^0$. This is equivalent to that the fusion rules $N_{M_r^i, M_s^j}^W = 0$ for $W = N^k$ for $k = 0, \dots, q$ or $W = N_t^k$ for $k = q+1, \dots, p$ and $t = 0, 1$.

From Proposition 4.9 of [DJX] or the Verlinde formula for modular tensor categories,

$$\text{qdim}_{V_0} M_r^i \text{qdim}_{V_0} M_s^j = \text{qdim}_{V_0} M_r^i \boxtimes M_s^j = \sum_W N_{M_r^i, M_s^j}^W \text{qdim}_{V_0} W$$

where W ranges over the inequivalent irreducible V_0 -modules. By the assumption of V , the quantum dimensions are positive. Our idea is to establish

$$\text{qdim}_{V_0} M_r^i \text{qdim}_{V_0} M_s^j = \sum_{k=0}^p \sum_{t=0,1} N_{M_r^i, M_s^j}^{M_t^k} \text{qdim}_{V_0} M_t^k$$

which implies that $N_{M_r^i, M_s^j}^W = 0$ if W is not any M_t^k .

Recall from [Hu2] the Verlinde formula

$$N_{M_{\bar{r}}, M_{\bar{s}}^j}^{M_{\bar{t}}^k} = \sum_W \frac{S_{M_{\bar{r}}, W} S_{M_{\bar{s}}^j, W} \bar{S}_{W, M_{\bar{t}}^k}}{S_{V_0, W}}$$

where W ranges through the inequivalent irreducible V_0 -modules. Thus

$$\begin{aligned} & \sum_{k=0}^p \sum_{t=0,1} N_{M_{\bar{r}}, M_{\bar{s}}^j}^{M_{\bar{t}}^k} \text{qdim}_{V_0} M_{\bar{t}}^k \\ &= \sum_{k=0}^p \sum_{t=0,1} \sum_W \frac{S_{M_{\bar{r}}, W} S_{M_{\bar{s}}^j, W} \bar{S}_{W, M_{\bar{t}}^k}}{S_{V_0, W}} \frac{S_{M_{\bar{t}}^k, V_0}}{S_{V_0, V_0}} \end{aligned}$$

We claim that $\sum_{k=0}^p \sum_{t=0,1} \bar{S}_{W, M_{\bar{t}}^k} S_{M_{\bar{t}}^k, V_0}$ is 0 if $W \neq V_0, V_1$ and is $\frac{1}{2}$ otherwise.

Note that $\bar{S}_{M_{\bar{t}}^k, W} = S_{W', M_{\bar{t}}^k}$. Let $W = V_a$. By Lemma 6.1 we have $S_{M_{\bar{t}}^k, V_a} = \frac{S_{M^k, V}^{(\sigma, \sigma)}}{2}$ for $a = 0, 1$. Using the unitarity of $\rho(S)$ Theorem 6.4 gives the claim

$$\sum_{k=0}^p \sum_{t=0,1} \bar{S}_{W, M_{\bar{t}}^k} S_{M_{\bar{t}}^k, V_0} = \frac{1}{2} \sum_{k=0}^p (S_{M^k, V}^{(\sigma, \sigma)})^2 = \frac{1}{2}.$$

The proof for $W = M_{\bar{t}}^k$ with $k > 0$ is similar. For $W = N_{\bar{s}}^j$, where $i = 0, \dots, q$, $j = q + 1, \dots, p$ and $s = 0, 1$, the claim follows immediately from Lemmas 6.2 and 6.3.

Finally we have

$$\sum_{k=0}^p \sum_{t=0,1} \sum_W \frac{S_{M_{\bar{r}}, W} S_{M_{\bar{s}}^j, W} \bar{S}_{W, M_{\bar{t}}^k}}{S_{V_0, W}} \frac{S_{M_{\bar{t}}^k, V_0}}{S_{V_0, V_0}} = \frac{1}{2} \frac{S_{M_{\bar{r}}, V_0} S_{M_{\bar{s}}^j, V_0}}{S_{V_0, V_0} S_{V_0, V_0}} + \frac{1}{2} \frac{S_{M_{\bar{r}}, V_1} S_{M_{\bar{s}}^j, V_1}}{S_{V_0, V_1} S_{V_0, V_0}}.$$

Since $S_{M_{\bar{r}}, V_1} = S_{M_{\bar{r}}, V_0}$ we see that

$$\sum_{k=0}^p \sum_{t=0,1} N_{M_{\bar{r}}, M_{\bar{s}}^j}^{M_{\bar{t}}^k} \text{qdim}_{V_0} M_{\bar{t}}^k = \frac{S_{M_{\bar{r}}, V_0} S_{M_{\bar{s}}^j, V_0}}{S_{V_0, V_0} S_{V_0, V_0}} = \text{qdim}_{V_0} M_{\bar{r}}^i \text{qdim}_{V_0} M_{\bar{s}}^j,$$

as desired. $\square$

Remark 8.2. Similarly, one can show that if $M \in \mathcal{C}_{V_0}^r$, $N \in \mathcal{C}_{V_0}^s$ then $M \boxtimes N \in \mathcal{C}_{V_0}^{r+s}$ where $r + s$ is understood to be modulo 2. Therefore, $\mathcal{C}_{V_0}$ is $\mathbb{Z}_2$ -graded.

9. THE 16-FOLD WAY

We discuss in this section on how the representation theory for vertex operator superalgebra is related to the 16-fold way conjecture proposed in [BGH].

Let U be a rational, C_2 -cofinite, simple vertex operator algebra of CFT type such that the weight of any irreducible U -module is positive except U itself. Then the U -module category $\mathcal{C}_U$ is a modular tensor category [Hu2] with positive quantum dimensions. As usual, let $c_{M, N} : M \boxtimes N \rightarrow N \boxtimes M$ be the braiding for U -modules M, N . Let θ denote the ribbon structure on $\mathcal{C}_U$. Then θ_M is a scalar multiple of id_M for any simple U -module M . We use the abuse notation θ_M to denote such scalar. A simple U -module F is called a *fermion* if F is a simple current (or invertible object of $\mathcal{C}_U$) of order 2 and $c_{F, F} = -\text{id}_{F \boxtimes F}$. Since $\text{qdim}_U F = 1$, $\theta_F = -1$.

Lemma 9.1. *Let $V = V_0 \oplus V_1$ be a vertex operator superalgebra satisfying assumptions (A1)–(A2) with $V_1 \neq 0$. Then V_1 is a fermion of $\mathcal{C}_{V_0}$.*

Proof. Clearly, $V_{\bar{1}}$ is a simple current of order 2. Then

$$c_{V_{\bar{1}}, V_{\bar{1}}} = \theta_{V_{\bar{1}}} \text{id}_{V_{\bar{1}} \boxtimes V_{\bar{1}}} = e^{2\pi i L(0)} \text{id}_{V_{\bar{1}} \boxtimes V_{\bar{1}}} = -\text{id}_{V_{\bar{1}} \boxtimes V_{\bar{1}}}$$

from [Hn2] and [BGH] as $V_{\bar{1}} = \bigoplus_{n \in \mathbb{Z}} V_{\frac{1}{2}+n}$. $\square$

Conversely, if U is as before and an U -module F is a fermion, then $V = U \oplus F$ has a structure of a vertex operator superalgebra such that $V_{\bar{0}} = U$ and $V_{\bar{1}} = F$ by Theorem 1.1 of [CKL]. Therefore, the vertex operator superalgebra $V = V_{\bar{0}} \oplus V_{\bar{1}}$ in our sense is completely determined by a fermion $V_{\bar{1}}$ in $\mathcal{C}_{V_{\bar{0}}}$.

Let $\mathcal{B}$ be a braided fusion category. For any family $\mathcal{D}$ of objects in $\mathcal{B}$, the Müger centralizer $C_{\mathcal{B}}(\mathcal{D})$ is the full subcategory of $\mathcal{B}$ consisting of the objects Y in $\mathcal{B}$ such that $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \boxtimes Y}$ for all X in $\mathcal{D}$. The subcategory $C_{\mathcal{B}}(\mathcal{D})$ is closed under the tensor product of $\mathcal{B}$ and hence a braided fusion subcategory of $\mathcal{B}$. The symmetric fusion category $C_{\mathcal{B}}(\mathcal{B})$ is called the Müger center of $\mathcal{B}$ and denoted by $\mathcal{Z}_2(\mathcal{B})$. In this paper, a pseudounitary braided fusion category $\mathcal{B}$ is called *super-modular* if $\mathcal{Z}_2(\mathcal{B})$ equivalent to category sVec , which is equal to $\text{Rep}(\mathbb{Z}_2)$ with the super braiding. In particular, a super-modular category $\mathcal{B}$ admits a fermion F in $C_{\mathcal{B}}(\mathcal{B})$ with $\theta_F = -1$.

Lemma 9.2. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vertex operator superalgebra satisfying assumptions (A1)–(A2). Then $\mathcal{C}_{V_{\bar{0}}}^0 = C_{\mathcal{C}_{V_{\bar{0}}}}(V_{\bar{1}})$ and is super-modular.*

Proof. By Theorem 8.1 that $\mathcal{C}_{V_{\bar{0}}}^0$ is a braided fusion subcategory of $\mathcal{C}_{V_{\bar{0}}}$. We first prove that $V_{\bar{1}}$ lies in $\mathcal{Z}_2(\mathcal{C}_{V_{\bar{0}}}^0)$, and hence $\dim \mathcal{Z}_2(\mathcal{C}_{V_{\bar{0}}}^0) \geq 2$. Equivalently we need to show that

$$c_{V_{\bar{1}}, M_{\bar{r}}^i} \circ c_{M_{\bar{r}}^i, V_{\bar{1}}} = \text{id}_{M_{\bar{r}}^i \boxtimes V_{\bar{1}}}$$

for $i = 0, \dots, p$ and $r = 0, 1$. Since $V_{\bar{1}}$ is a simple current we know that both $V_{\bar{1}} \boxtimes M_{\bar{r}}^i$ and $M_{\bar{r}}^i \boxtimes V_{\bar{1}}$ are isomorphic to $M_{\bar{r}+1}^i$. As usual we will denote the space of intertwining operator of type $\begin{pmatrix} W^3 \\ W^1, W^2 \end{pmatrix}$ by $I\left(\begin{smallmatrix} W^3 \\ W^1, W^2 \end{smallmatrix}\right)$ where W^j are modules of the vertex operator algebra $V_{\bar{0}}$ for $i = 1, 2, 3$. Then $I\left(\begin{smallmatrix} M_{\bar{r}+1}^i \\ M_{\bar{r}}^i, V_{\bar{1}} \end{smallmatrix}\right) = \mathbb{C}\mathcal{Y}$ and $I\left(\begin{smallmatrix} M_{\bar{r}+1}^i \\ V_{\bar{1}}, M_{\bar{r}}^i \end{smallmatrix}\right) = \mathbb{C}Y$ where Y is the restriction of Y defining the V -module structure on $M_{\bar{r}}^i$ to $V_{\bar{1}}$ and $\mathcal{Y}(w, z)u = e^{zL(-1)}Y(u, -z)w$ for $u \in V_{\bar{1}}$ and $w \in M_{\bar{r}}^i$. In this case, $c_{M_{\bar{r}}^i, V_{\bar{1}}}$ is a linear map from $I\left(\begin{smallmatrix} M_{\bar{r}+1}^i \\ M_{\bar{r}}^i, V_{\bar{1}} \end{smallmatrix}\right)$ to $I\left(\begin{smallmatrix} M_{\bar{r}+1}^i \\ V_{\bar{1}}, M_{\bar{r}}^i \end{smallmatrix}\right)$ such that $\mathcal{Y}$ is mapped to $\mathcal{Y}'$ where $\mathcal{Y}'(u, z)w = e^{zL(-1)}\mathcal{Y}(w, -z)u$ for u, w as before. Similarly, $c_{V_{\bar{1}}, M_{\bar{r}}^i}$ is a linear map from $I\left(\begin{smallmatrix} M_{\bar{r}+1}^i \\ V_{\bar{1}}, M_{\bar{r}}^i \end{smallmatrix}\right)$ to $I\left(\begin{smallmatrix} M_{\bar{r}+1}^i \\ M_{\bar{r}}^i, V_{\bar{1}} \end{smallmatrix}\right)$ such that Y is mapped to $\mathcal{Y}$. It is trivial to verify that $\mathcal{Y}' = Y$ and $c_{V_{\bar{1}}, M_{\bar{r}}^i} \circ c_{M_{\bar{r}}^i, V_{\bar{1}}} = \text{id}_{M_{\bar{r}}^i \boxtimes V_{\bar{1}}}$.

It remains to show that $V_{\bar{s}}$ for $s = 0, 1$ are the only simple objects in $\mathcal{Z}_2(\mathcal{C}_{V_{\bar{0}}}^0)$. Since $\mathcal{C}_{V_{\bar{0}}}$ is modular, it follows from Theorem 3.2 of [Mu] that

$$\dim \mathcal{C}_{V_{\bar{0}}} = \dim C_{\mathcal{C}_{V_{\bar{0}}}}(\mathcal{C}_{V_{\bar{0}}}^0) \cdot \dim \mathcal{C}_{V_{\bar{0}}}^0.$$

From the discussion in Section 8 we know that

$$\dim \mathcal{C}_{V_{\bar{0}}} = 2 \dim \mathcal{C}_{V_{\bar{0}}}^0.$$

This forces $\dim C_{\mathcal{C}_{V_0}}(\mathcal{C}_{V_0}^0) = 2$. Clearly, $\mathcal{Z}_2(\mathcal{C}_{V_0}^0) \subset C_{\mathcal{C}_{V_0}}(\mathcal{C}_{V_0}^0)$. This implies that

$$2 \leq \dim \mathcal{Z}_2(\mathcal{C}_{V_0}^0) \leq \dim C_{\mathcal{C}_{V_0}}(\mathcal{C}_{V_0}^0) = 2,$$

and hence $\mathcal{Z}_2(\mathcal{C}_{V_0}^0) = C_{\mathcal{C}_{V_0}}(\mathcal{C}_{V_0}^0)$. $\square$

We can now formulate the 16-fold way conjecture in [BGH] in the context of vertex operator algebra. Let $\mathcal{B}$ be a super-modular category. A modular category $\mathcal{C}$, which contains $\mathcal{B}$ as a full ribbon subcategory, is called a *minimal modular extension* or a *modular closure* of $\mathcal{B}$ if $\dim \mathcal{C} = 2 \dim \mathcal{B}$.

Conjecture 9.3. *If $\mathcal{B}$ is a super-modular category, then $\mathcal{B}$ admits a minimal modular extension. In this case, there are exactly 16 minimal modular extensions of $\mathcal{B}$ up to braided monoidal equivalence.*

Under the assumption of the existence of modular closure of super-modular category, the second part of the conjecture has been proved in [LKW1, Theorems 4.26 and 5.4].

From Lemma 9.2 and its proof, we immediately obtain:

Theorem 9.4. *Let V be a vertex operator superalgebra satisfying (A1)–(A2). Then $\mathcal{C}_{V_0}$ is a minimal modular extension of the super-modular category $\mathcal{C}_{V_0}^0$.*

In view of Theorem 9.4 and the 16-fold way conjecture, the following question arises:

Question 9.5. Does every super-modular category $\mathcal{C}$ equivalent to $\mathcal{C}_{V_0}^0$ for some vertex operator superalgebra V ?

Our next goal is to construct a sequence $\{V^m\}_{m \geq 0}$ of vertex operator superalgebras from V such that $\mathcal{C}_{(V^m)_0}$ are minimal modular extension of $\mathcal{C}_{V_0}^0$ and the equivalence classes of these modular categories $\mathcal{C}_{(V^m)_0}$ are characterized by the congruence class of m modulo 16. We close this section with the following theorem.

Theorem 9.6. *Let V, U be vertex operator superalgebras satisfying (A1)–(A2) and U being holomorphic. Then $\mathcal{C}_{V_0}^0$ and $\mathcal{C}_{(U \otimes V)_0}^0$ are equivalent braided fusion categories. In particular, $\mathcal{C}_{(U \otimes V)_0}$ is a minimal modular extension of $\mathcal{C}_{V_0}^0$.*

Proof. Note that $(U \otimes V)_0$ is an algebra object in $\mathcal{C}_{U_0 \otimes V_0}$. Let $\mathcal{B} = \mathcal{C}_{\mathcal{C}_{U_0 \otimes V_0}}((U \otimes V)_0)$, the Müger centralizer of $(U \otimes V)_0$ in $\mathcal{C}_{U_0 \otimes V_0}$. In view of [CKM, Proposition 2.65], let $F_0 : \mathcal{B} \rightarrow \mathcal{C}_{U \otimes V}$ and $F_1 : \mathcal{B} \rightarrow \mathcal{C}_{(U \otimes V)_0}$ be the induction functors, that means

$$F_0(Y) = (U \otimes V) \boxtimes_{U_0 \otimes V_0} Y, \quad F_1(Y) = (U \otimes V)_0 \boxtimes_{U_0 \otimes V_0} Y$$

for Y in $\mathcal{B}$. By [CKM, Theorem 2.67], F_0, F_1 are braided tensor functors. Since $F_1(Y)$ is a $(U \otimes V)_0$ -submodule of $F_0(Y)$ and $\mathcal{C}_{(U \otimes V)_0}^0$ is generated by the $(U \otimes V)_0$ -submodules of super $U \otimes V$ -modules, $F_1(Y) \in \text{obj}(\mathcal{C}_{(U \otimes V)_0}^0)$ for $Y \in \mathcal{B}$.

Since $(U \otimes V)_0 = U_0 \otimes V_0 \oplus U_1 \otimes V_1$, $U_0 \otimes X \in \mathcal{B}$ for any object X of $\mathcal{C}_{V_0}^0$. Note that the functor $U_0 \otimes -: \mathcal{C}_{V_0}^0 \rightarrow \mathcal{C}_{U_0 \otimes V_0}$ is a faithfully full braided tensor functor, and so is its restriction $F_2 : \mathcal{C}_{V_0}^0 \rightarrow \mathcal{B}$. Therefore, the composite functor $F = F_1 F_2 : \mathcal{C}_{V_0}^0 \rightarrow \mathcal{C}_{(U \otimes V)_0}^0$ is a braided tensor functor. Since $\mathcal{C}_{V_0}^0$ is super-modular, F is faithfully full by [DMNO, Corollary 3.26].

To show that $\mathcal{C}_{V_0}^0$ is braided tensor equivalent to $\mathcal{C}_{(U \otimes V)_0}^0$, it suffices to show that every irreducible $(U \otimes V)_0$ -module is an image of F .

Recall that the inequivalent irreducible super V -modules are M^i with $i = 0, \dots, p$. This implies that $\{M_{\bar{r}}^i \mid i = 0, \dots, p, \text{ and } r = 0, 1\}$ is a complete set of inequivalent simple objects of $\mathcal{C}_{V_0}^0$. Moreover, inequivalent irreducible super $U \otimes V$ -modules are $U \otimes M^i$. Therefore, $(U \otimes M^i)_{\bar{r}} = U_{\bar{0}} \otimes M_{\bar{r}}^i + U_{\bar{1}} \otimes M_{\bar{1}-\bar{r}}^i$ for $i = 0, \dots, p$ and $r = 0, 1$ are all the inequivalent irreducible $(U \otimes V)_{\bar{0}}$ -modules of $\mathcal{C}_{(U \otimes V)_{\bar{0}}}^0$.

For any simple $X \in \text{obj}(\mathcal{C}_{V_0}^0)$,

$$F(X) = (U \otimes V)_{\bar{0}} \boxtimes (U_{\bar{0}} \otimes X)$$

which is isomorphic to $U_{\bar{0}} \otimes X + U_{\bar{1}} \otimes (V_{\bar{1}} \boxtimes X)$ as $V_{\bar{0}} \otimes U_{\bar{0}}$ -modules. Therefore, $F(X)$ is the irreducible $(V \otimes U)_{\bar{0}}$ -module which contains an irreducible $U_{\bar{0}} \otimes V_{\bar{0}}$ -submodule isomorphic to $U_{\bar{0}} \otimes X$. Therefore, by the same reason,

$$F(M_{\bar{r}}^i) \cong (U \otimes M^i)_{\bar{r}}$$

as $(U \otimes V)_{\bar{0}}$ -modules for $i = 1, \dots, p$ and $r = 0, 1$. Thus, $F : \mathcal{C}_{V_0}^0 \rightarrow \mathcal{C}_{(U \otimes V)_{\bar{0}}}^0$ is an equivalence. The last statement follows immediately from Theorem 9.4. $\square$

The Gauss sum $\tau_1(\mathcal{C})$ of a ribbon fusion category $\mathcal{C}$ is defined as

$$\tau_1(\mathcal{C}) = \sum_{X \in \text{Irr}(\mathcal{C})} \text{qdim}(X)^2 \cdot \theta_X$$

where $\text{Irr}(\mathcal{C})$ denotes the set of isomorphism classes of simple objects of $\mathcal{C}$, $\text{qdim}(X)$ is the pivotal (or quantum) dimension of the simple object X , and θ_X denotes the scalar of the twist. The Gauss sums and their higher degree generalizations $\tau_n(\mathcal{C})$ are invariants of ribbon fusion categories (cf. [NSW]). In the case of a fermionic modular category, we follow some idea in [BGN] to prove that the centralizer of the fermion has zero contribution to the Gauss sum.

Lemma 9.7. *Let $\mathcal{C}$ be a pseudounitary modular tensor category over $\mathbb{C}$, f a fermion of $\mathcal{C}$, and $\mathcal{C}^0$ the Müger centralizer of f . Then*

$$\tau_1(\mathcal{C}^0) = 0$$

Proof. Let $X \in \text{Irr}(\mathcal{C}^0)$. Then $S_{X,f} = \text{qdim}(X)$ where $S_{X,Y}$ denotes the categorical trace of $c_{Y,X} \circ c_{X,Y}$. Since f is an invertible object, the tensor product $X \otimes f$ of $\mathcal{C}$ is a simple object of $\mathcal{C}^0$ and $\text{qdim}(X \otimes f) = \text{qdim}(X)$. On the other hand, by the twist equation, we have

$$\theta_{X \otimes f} \text{qdim}(X \otimes f) = \theta_X \theta_f S_{X,f},$$

which implies $\theta_{X \otimes f} = -\theta_X$. In particular, the action of f on $\text{Irr}(\mathcal{C})$ has no fixed point. Therefore, there exists a subset $\mathcal{O}$ of $\text{Irr}(\mathcal{C}^0)$ such that $\bigcup_{X \in \mathcal{O}} \{X, X \otimes f\} = \text{Irr}(\mathcal{C}^0)$. Thus,

$$\tau_1(\mathcal{C}^0) = \sum_{X \in \mathcal{O}} \text{qdim}(X)^2 \cdot \theta_X - \text{qdim}(X \otimes f)^2 \cdot \theta_X = 0. \quad \square$$

10. MINIMAL MODULAR EXTENSIONS OF $\mathcal{C}_{V_0}^0$

In this section we use the holomorphic vertex operator superalgebras $V(l, \mathbb{Z} + \frac{1}{2})$ for $l \geq 1$ and Theorem 9.6 to obtain all the 16 minimal modular extensions of $\mathcal{C}_{V_0}^0$ for any given vertex operator superalgebra V satisfying (A1)–(A2).

The construction of $V(l, \mathbb{Z} + \frac{1}{2})$ is well known (see [FFR], [KW], [L1]). Let $H_l = \bigoplus_{i=1}^l \mathbb{C} a_i$ be a complex vector space equipped with a nondegenerate symmetric

bilinear form $(\cdot, \cdot)$ such that $(a_i, a_j) = 2\delta_{i,j}$. Let $A(l, \mathbb{Z} + \frac{1}{2})$ be the associative algebra generated by $\{a(n) \mid a \in H_l, n \in \mathbb{Z} + \frac{1}{2}\}$ subject to the relation

$$[a(n), b(m)]_+ = (a, b)\delta_{m+n,0}.$$

Let $A(l, \mathbb{Z} + \frac{1}{2})^+$ be the subalgebra generated by $\{a(n) \mid a \in H_l, n \in \mathbb{Z} + \frac{1}{2}, n > 0\}$, and make $\mathbb{C}$ a 1-dimensional $A(l, \mathbb{Z} + \frac{1}{2})^+$ -module so that $a_i(n)1 = 0$ for $n > 0$. The induced module

$$\begin{aligned} V(l, \mathbb{Z} + \frac{1}{2}) &= A(l, \mathbb{Z} + \frac{1}{2}) \otimes_{A(l, \mathbb{Z} + \frac{1}{2})^+} \mathbb{C} \\ &\cong \bigwedge [a_i(-n) \mid n > 0, n \in \mathbb{Z} + \frac{1}{2}, i = 1, 2, \dots, l] \text{ (linearly)} \end{aligned}$$

is a holomorphic vertex operator superalgebra generated by $a_i(-\frac{1}{2})$ for $i = 1, \dots, l$ and $Y(a_i(-\frac{1}{2}), z) = a_i(z) = \sum_{n \in \mathbb{Z}} a_i(-n - \frac{1}{2})z^{-n-1}$. For example, if $l = 1$ then $V(1, \mathbb{Z} + \frac{1}{2})$ is isomorphic to $L(\frac{1}{2}, 0) + L(\frac{1}{2}, \frac{1}{2})$ as a module for the Virasoro vertex operator algebra $L(\frac{1}{2}, 0)$. Moreover, $V(1, \mathbb{Z} + \frac{1}{2})_{\bar{0}} = L(\frac{1}{2}, 0)$ and $V(1, \mathbb{Z} + \frac{1}{2})_{\bar{1}} = L(\frac{1}{2}, \frac{1}{2})$. If $l = 2k$ is even then $V(l, \mathbb{Z} + \frac{1}{2})$ is isomorphic to the lattice vertex operator superalgebra $V_{\mathbb{Z}^k}$ where $\mathbb{Z}^k$ is the lattice in $\mathbb{R}^k$ with the standard inner product.

As usual, we use σ to denote the canonical automorphism of $V(l, \mathbb{Z} + \frac{1}{2})$. To construct σ -twisted $V(l, \mathbb{Z} + \frac{1}{2})$ -modules, we need to consider the cases when l is even or odd. If $l = 2k$ is even, then H_{2k} can be written as

$$H_{2k} = \sum_{i=1}^k \mathbb{C}b_i + \sum_{i=1}^k \mathbb{C}b_i^*$$

with $(b_i, b_j) = (b_i^*, b_j^*) = 0, (b_i, b_j^*) = \delta_{i,j}$. Let $A(2k, \mathbb{Z})$ be the associative algebra generated by $\{b(n) \mid b \in H_{2k}, n \in \mathbb{Z}\}$ subject to the relation

$$[a(m), b(n)]_+ = (a, b)\delta_{m+n,0}.$$

Let $A(2k, \mathbb{Z})^+$ be the subalgebra generated by $\{b_i(n), b_i^*(m) \mid n \geq 0, m > 0, i = 1, \dots, k\}$, and make $\mathbb{C}$ a 1-dimensional $A(2k, \mathbb{Z})^+$ -module with $b_i(n)1 = 0$ and $b_i^*(m)1 = 0$ for $n \geq 0, m > 0, i = 1, \dots, k$. Consider the induced $A(2k, \mathbb{Z})$ -module

$$V(2k, \mathbb{Z}) = A(2k, \mathbb{Z}) \otimes_{A(2k, \mathbb{Z})^+} \mathbb{C} \cong \Lambda[b_i(-n), b_i^*(-m) \mid n, m \in \mathbb{Z}, n > 0, m \geq 0].$$

By Proposition 4.3 in [L2], $V(2k, \mathbb{Z})$ is an irreducible σ -twisted $V(2k, \mathbb{Z} + \frac{1}{2})$ -module such that

$$Y_{V(2k, \mathbb{Z})}(u(-\frac{1}{2}), z) = u(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1/2}$$

for $u \in H_{2k}$. Moreover, $V(2k, \mathbb{Z})$ is the only irreducible σ -twisted $V(2k, \mathbb{Z} + \frac{1}{2})$ -module up to isomorphism [DZ2]. As a result, $V(2k, \mathbb{Z} + \frac{1}{2})_{\bar{0}}$ has 4 inequivalent irreducible modules $V(2k, \mathbb{Z} + \frac{1}{2})_{\bar{r}}$, and $V(2k, \mathbb{Z})_{\bar{r}}$ ($r = 0, 1$) of weights $0, \frac{1}{2}, \frac{k}{8}, \frac{k}{8}$, and quantum dimension 1.

If $l = 2k + 1$ is odd, H_{2k+1} can be decomposed into:

$$H_{2k+1} = \sum_{i=1}^k \mathbb{C}b_i + \sum_{i=1}^k \mathbb{C}b_i^* + \mathbb{C}e$$

with $(b_i, b_j) = (b_i^*, b_j^*) = 0, (b_i, b_j^*) = \delta_{i,j}, (e, b_i) = (e, b_i^*) = 0, (e, e) = 2$. Let $A(2k+1, \mathbb{Z})$ be the associative algebra generated by $a(n)$ for $a \in H_{2k+1}$ and $n \in \mathbb{Z}$ subject to the same relation as before. Let $A(2k+1, \mathbb{Z})^+$ be the subalgebra generated by

$$\{b_i(n), b_i^*(m), e(m) \mid m, n \in \mathbb{Z}, n \geq 0, m > 0, i = 1, \dots, k\}$$

and make $\mathbb{C}$ a 1-dimensional $A(2k+1, \mathbb{Z})^+$ -module with $b_i(n)1 = 0$ for $n \geq 0$ and $b_i^*(m)1 = e(m)1 = 0$ for $m > 0, i = 1, \dots, k$. Set

$$V(2k+1, \mathbb{Z}) = A(2k+1, \mathbb{Z}) \otimes_{A(2k+1, \mathbb{Z})^+} \mathbb{C}.$$

It is easy to see that $V(2k+1, \mathbb{Z})$ is isomorphic to the exterior algebra

$$W(2k+1, \mathbb{Z}) = \Lambda[b_i(-n), b_i^*(-m), e(-m) \mid n, m \in \mathbb{Z}, n > 0, m \geq 0]$$

as vector spaces. Let $W(2k+1, \mathbb{Z}) = W(2k+1, \mathbb{Z})_{\bar{0}} \oplus W(2k+1, \mathbb{Z})_{\bar{1}}$ be the decomposition into the even and odd parity subspaces, and

$$V_{\pm}(2k+1, \mathbb{Z}) = (1 \pm e(0))W(2k+1, \mathbb{Z})_{\bar{0}} \oplus (1 \mp e(0))W(2k+1, \mathbb{Z})_{\bar{1}}.$$

Then

$$V(2k+1, \mathbb{Z}) = V_+(2k+1, \mathbb{Z}) \oplus V_-(2k+1, \mathbb{Z})$$

and $V_{\pm}(2k+1, \mathbb{Z})$ are irreducible $A(2k+1, \mathbb{Z})$ -modules. It follows from Proposition 4.3 in [L2] that $V_{\pm}(2k+1, \mathbb{Z})$ are irreducible σ -twisted modules for $V(2k+1, \mathbb{Z} + \frac{1}{2})$ so that

$$Y_{V(2k+1, \mathbb{Z})}(u(-\frac{1}{2}), z) = u(z) = \sum_{n \in \mathbb{Z}} u(n)z^{-n-1/2}$$

for $u \in H_{2k+1}$. Moreover, $V_{\pm}(2k+1, \mathbb{Z})$ are the only inequivalent irreducible σ -twisted modules and are isomorphic irreducible $V(2k+1, \mathbb{Z} + \frac{1}{2})_{\bar{0}}$ -modules [DZ2]. In this case $V(2k+1, \mathbb{Z} + \frac{1}{2})_{\bar{0}}$ has 3 inequivalent irreducible modules $V(2k+1, \mathbb{Z} + \frac{1}{2})_{\bar{r}}$ for $r = 0, 1$ and $V_+(2k+1, \mathbb{Z})$ of weights $0, \frac{1}{2}$ and $\frac{2k+1}{16}$, and quantum dimensions $1, 1$ and $\sqrt{2}$.

Let V be a vertex operator superalgebra satisfying (A1)–(A2). Set $V^0 = V$ and $V^l = V(l, \mathbb{Z} + \frac{1}{2}) \otimes V$ for $l \geq 1$. According to Theorem 9.6, $\mathcal{C}_{(V^l)_{\bar{0}}}$ is a minimal modular extension of $\mathcal{C}_{V_{\bar{0}}^0}$ for $l \geq 0$. We denote the Virasoro vector of V^l by ω^l for $l \geq 1$ and write $Y(\omega^l, z) = \sum_{n \in \mathbb{Z}} L^l(n)z^{-n-2}$. Let T_l be the corresponding T -matrix associated to $(V^l)_{\bar{0}}$ and set $t_l = e^{2\pi i(c+\frac{1}{2})/24}T_l$ which is the matrix for the operator $e^{2\pi iL^l(0)}$ acting on the inequivalent irreducible $(V^l)_{\bar{0}}$ -modules. Then t_l is the T -matrix of the modular tensor category $\mathcal{C}_{(V^l)_{\bar{0}}}$.

The following result is an immediate consequence of Theorem 3.5

Lemma 10.1. *The inequivalent irreducible σ -twisted V^l -modules are*

$$\{V(l, \mathbb{Z}) \otimes N^j, (V(l, \mathbb{Z}) \otimes N^j) \circ \sigma, V(l, \mathbb{Z}) \otimes N^k \mid j = 0, \dots, q, k = q+1, \dots, p\}$$

if l is even, and

$$\{N^{l,j}, N^{l,k}, N^{l,k} \circ \sigma \mid j = 0, \dots, q, k = q+1, \dots, p\}$$

if l is odd where $V(l, \mathbb{Z}) \otimes (N^j + N^j \circ \sigma) = 2N^{l,j}$ and $V(l, \mathbb{Z}) \otimes N^k = N^{l,k} \oplus N^{l,k} \circ \sigma$.

Corollary 10.2. *The inequivalent simple $(V^l)_{\bar{0}}$ -modules from the σ -twisted V^l -modules are*

$$\{V(l, \mathbb{Z}) \otimes N^j, (V(l, \mathbb{Z}) \otimes N^k)_{\bar{r}} \mid j = 0, \dots, q, k = q + 1, \dots, p, r = 0, 1\}$$

if l is even. In this case,

$$\mathrm{qdim}_{(V^l)_{\bar{0}}}(V(l, \mathbb{Z}) \otimes N^j) = \mathrm{qdim}_{V_0}(N^j), \quad \mathrm{qdim}_{(V^l)_{\bar{0}}}(V(l, \mathbb{Z}) \otimes N^k)_{\bar{r}} = \mathrm{qdim}_{V_0}(N_{\bar{r}}^k)$$

for $j = 0, \dots, q, k = q + 1, \dots, p, r = 0, 1$.

If l is odd, the inequivalent simple $(V^l)_{\bar{0}}$ -modules from the σ -twisted V^l -modules are

$$\{N_{\bar{r}}^{l,j}, N^{l,k} \mid j = 0, \dots, q, k = q + 1, \dots, p, r = 0, 1\}$$

and

$$\mathrm{qdim}_{(V^l)_{\bar{0}}}(N_{\bar{r}}^{l,j}) = \frac{1}{\sqrt{2}} \cdot \mathrm{qdim}_{V_0}(N^j), \quad \mathrm{qdim}_{(V^l)_{\bar{0}}}(N^{l,k}) = \sqrt{2} \cdot \mathrm{qdim}_{V_0}(N_{\bar{r}}^k)$$

for $j = 0, \dots, q, k = q + 1, \dots, p, r = 0, 1$.

Proof. The set of simple $(V^l)_{\bar{0}}$ -modules from the σ -twisted V^l -modules follows immediately from Lemma 10.1 for any nonzero integer l . Let us denote $U^l = V(l, \mathbb{Z} + \frac{1}{2})$. If l is even, then $\mathrm{qdim}_{U^l}(V(l, \mathbb{Z})) = \mathrm{qdim}_{U_0^l}(V(l, \mathbb{Z}))_{\bar{r}} = 1$ for $r = 0, 1$. For $j = 0, \dots, q$, $V(l, \mathbb{Z}) \otimes N^j$ is an unstable σ -twisted V^l -module. It follows from Proposition 7.2 that

$$\mathrm{qdim}_{(V^l)_{\bar{0}}}(V(l, \mathbb{Z}) \otimes N^j) = 2 \mathrm{qdim}_{V^l}(V(l, \mathbb{Z}) \otimes N^j) = 2 \mathrm{qdim}_V(N^j) = \mathrm{qdim}_{V_0}(N^j).$$

For $k = p + 1, \dots, q$, $V(l, \mathbb{Z}) \otimes N^k$ is σ -stable. By Proposition 7.2,

$$\mathrm{qdim}_{(V^l)_{\bar{0}}}(V(l, \mathbb{Z}) \otimes N^k)_{\bar{r}} = \mathrm{qdim}_{V^l}(V(l, \mathbb{Z}) \otimes N^k) = \mathrm{qdim}_V(N^k) = \mathrm{qdim}_{V_0}(N_{\bar{r}}^k)$$

for $r = 0, 1$.

If l is odd, then $\mathrm{qdim}_{U^l}(V(l, \mathbb{Z})) = \mathrm{qdim}_{U_0^l}(V_{\pm}(l, \mathbb{Z})) = \sqrt{2}$. For $j = 0, \dots, p$, $N^{l,j}$ is a σ -stable σ -twisted V^l -module and

$$\mathrm{qdim}_{V^l}(V(l, \mathbb{Z}) \otimes (N^j \oplus N_{\sigma}^j)) = 2 \mathrm{qdim}_{V^l}(N^{l,j}) = 2 \mathrm{qdim}_{(V^l)_{\bar{0}}}(N_{\bar{r}}^{l,j})$$

for any $r = 0, 1$. On the other hand,

$$\mathrm{qdim}_{V^l}(V(l, \mathbb{Z}) \otimes (N^j \oplus N_{\sigma}^j)) = \sqrt{2} \cdot \mathrm{qdim}_V(N^j \oplus N_{\sigma}^j) = \sqrt{2} \mathrm{qdim}_{V_0}(N^j).$$

Thus, we have

$$\mathrm{qdim}_{(V^l)_{\bar{0}}}(N_{\bar{r}}^{l,j}) = \frac{1}{\sqrt{2}} \mathrm{qdim}_{V_0}(N^j)$$

for $r = 0, 1$. Similarly, for $k = p + 1, \dots, q$, $N^{l,k}$ is a σ -unstable σ -twisted V^l -module and

$$\begin{aligned} \mathrm{qdim}_{(V^l)_{\bar{0}}}(N^{l,k}) &= 2 \mathrm{qdim}_{V^l}(N^{l,k}) = \mathrm{qdim}_{V^l}(N^{l,k} \oplus N_{\sigma}^{l,k}) = \mathrm{qdim}_{V^l}(V(l, \mathbb{Z}) \otimes N^k) \\ &= \sqrt{2} \cdot \mathrm{qdim}_V(N^k) = \sqrt{2} \cdot \mathrm{qdim}_{V_0}(N_{\bar{r}}^k) \end{aligned}$$

for $r = 0, 1$. □

Theorem 10.3. *The minimal modular extensions $\mathcal{C}_{(V^l)_0}$, $\mathcal{C}_{(V^m)_0}$ of $\mathcal{C}_{V_0}^0$ are braided equivalent if and only if l and m are congruent modulo 16. In particular, we have constructed 16 minimal modular extensions of $\mathcal{C}_{V_0}^0$.*

Proof. Since $\mathcal{C}_{(V^m)_0}$ has positive quantum dimensions, its spherical pivotal structure is uniquely determined by the fusion category $\mathcal{C}_{(V^m)_0}$. Therefore, $\mathcal{C}_{(V^l)_0}$, $\mathcal{C}_{(V^m)_0}$ are equivalent braided fusion categories if and only if they are equivalent modular categories. The later implies they have the same Gauss sums. Therefore, we proceed to compute the Gauss sum $\tau_1(\mathcal{C}_{(V^l)_0})$. It follows from Lemma 9.7 that

$$\tau_1(\mathcal{C}_{(V^l)_0}) = \sum_{X \in \text{Irr}(\mathcal{C}_{(V^l)_0}^1)} \text{qdim}(X)^2 \cdot \theta_X$$

where $\text{Irr}(\mathcal{C}_{(V^l)_0}^1)$ is the set of inequivalent simple $(V^l)_0$ -modules from the σ -twisted V^l -modules.

By Corollary 10.2, the inequivalent irreducible $(V^l)_0$ -modules from the σ -twisted modules are

$$\{V(l, \mathbb{Z}) \otimes N^j, V(l, \mathbb{Z})_0 \otimes N_{\bar{r}}^k + V(l, \mathbb{Z})_1 \otimes N_{1-r}^k \mid j = 0, \dots, q, k = q+1, \dots, p, r = 0, 1\}$$

if l is even. The actions of $e^{2\pi i L^l(0)}$ on $V(l, \mathbb{Z}) \otimes N^j$ and $V(l, \mathbb{Z})_0 \otimes N_{\bar{r}}^k + V(l, \mathbb{Z})_1 \otimes N_{1-r}^k$ are respectively are $e^{2\pi i(\lambda_{N^j} + \frac{l}{16})}$ and $e^{2\pi i(\lambda_{N^k} + \frac{l}{16})}$ for $r = 0, 1$, where λ_{N^j} is the weight of N^j . Therefore,

$$\begin{aligned} \tau_1(\mathcal{C}_{(V^l)_0}) &= \sum_{j=0}^q \text{qdim}_{(V^l)_0}(V(l, \mathbb{Z}) \otimes N^j)^2 \cdot e^{2\pi i(\lambda_{N^j} + \frac{l}{16})} \\ &\quad + \sum_{r=0}^1 \sum_{k=q+1}^p \text{qdim}_{(V^l)_0}(V(l, \mathbb{Z}) \otimes N_{\bar{r}}^k)^2 \cdot e^{2\pi i(\lambda_{N^k} + \frac{l}{16})} \\ &= e^{\frac{2\pi i l}{16}} \left(\sum_{j=0}^q \text{qdim}_{V_0}(N^j)^2 e^{2\pi i \lambda_{N^j}} + \sum_{r=0}^1 \sum_{k=q+1}^p \text{qdim}_{V_0}(N_{\bar{r}}^k)^2 e^{2\pi i \lambda_{N^k}} \right) \\ &= e^{\frac{2\pi i l}{16}} \tau_1(\mathcal{C}_{V_0}). \end{aligned}$$

Again by Corollary 10.2, the inequivalent irreducible $(V^l)_0$ -modules from the σ -twisted modules are

$$\{N_{\bar{r}}^{l,j}, N^{l,k} \mid j = 0, \dots, q, k = q+1, \dots, p, r = 0, 1\}$$

if l is odd. The actions of $e^{2\pi i L^l(0)}$ on $N_{\bar{r}}^{l,j}$ is $e^{2\pi i(\lambda_{N^j} + \frac{l}{16})}$ and on $N^{l,k}$ is $e^{2\pi i(\lambda_{N^k} + \frac{l}{16})}$. Thus, for $r = 0, 1$,

$$\begin{aligned} \tau_1(\mathcal{C}_{(V^l)_0}) &= \sum_{r=0}^1 \sum_{j=0}^q \text{qdim}_{(V^l)_0}(N_{\bar{r}}^{l,j})^2 e^{2\pi i(\lambda_{N^j} + \frac{l}{16})} \\ &\quad + \sum_{k=q+1}^p \text{qdim}_{(V^l)_0}(N^{l,k})^2 e^{2\pi i(\lambda_{N^k} + \frac{l}{16})} \\ &= e^{\frac{2\pi i l}{16}} \left(\sum_{r=0}^1 \sum_{j=0}^q \frac{1}{2} \text{qdim}_{V_0}(N^j)^2 e^{2\pi i \lambda_{N^j}} + \sum_{k=q+1}^p 2 \text{qdim}_{V_0}(N_{\bar{r}}^k)^2 e^{2\pi i \lambda_{N^k}} \right) \end{aligned}$$

$$\begin{aligned}
 &= e^{\frac{2\pi i l}{16}} \left(\sum_{j=0}^q \text{qdim}_{V_0}(N^j)^2 e^{2\pi i \lambda_{N^j}} + \sum_{r=0}^1 \sum_{k=q+1}^p \text{qdim}_{V_0}(N_{\bar{r}}^k)^2 e^{2\pi i \lambda_{N^k}} \right) \\
 &= e^{\frac{2\pi i l}{16}} \tau_1(\mathcal{C}_{V_0}).
 \end{aligned}$$

Therefore, $\tau_1(\mathcal{C}_{(V^l)_0}) = e^{\frac{2\pi i l}{16}} \tau_1(\mathcal{C}_{V_0})$ for any integer $l \geq 0$. As a result, $\tau_1(\mathcal{C}_{(V^l)_0}) = \tau_1(\mathcal{C}_{(V^m)_0})$ if and only if $l \equiv m$ modulo 16, and there are at least 16 inequivalent modular categories which are minimal extensions of $\mathcal{C}_{V_0}^0$. By [LKW1, Theorems 4.26 and 5.4], $\mathcal{C}_{V_0}^0$ has exactly 16 minimal extensions. Thus, $\mathcal{C}_{(V^l)_0}$ and $\mathcal{C}_{(V^m)_0}$ are equivalent minimal extensions of $\mathcal{C}_{V_0}^0$ if and only if $l \equiv m$ modulo 16. These 16 minimal extensions of are also inequivalent as braided fusion categories as they have distinct Gauss sums. $\square$

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