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A Simple Sufficient Condition for a Unique and Student-Efficient Stable Matching in the College Admissions Problem*

Philip J. Reny

Department of Economics
University of Chicago

Abstract

Consider the college admissions problem. Let us say that (student and college) preferences are *student-oriented* iff whenever two students disagree about the ranking of two colleges, each one of the two students is ranked higher by the college he prefers than the other student. We show that when preferences are oriented there is a unique stable matching, and that no other matching, stable or not, is weakly preferred by every student.

Keywords: school choice, unique stable matching, Pareto efficient matchings.

1 Oriented Preferences

The college admissions model used here is standard. Let I denote the nonempty finite set of students and let C denote the nonempty finite set of colleges. The sets I and C are disjoint. Each college $c \in C$ has a finite number of available seats, or quota, $q_c \in \{1, 2, ...\}$, and has a strict preference ordering P_c over $I \cup \{c\}$. Each student $i \in I$ has a strict preference ordering P_i over $C \cup \{i\}$. For any $i \in I$, we write cR_ic' to mean cP_ic' or $c = c'$. All of these elements are fixed throughout the analysis.

A *matching* is any mapping $\mu : I \to C \cup I$ such that $\mu(i) \in C \cup \{i\}$ for every $i \in I$, and $\#\mu^{-1}(c) \leq q_c$ for every $c \in C$.

For any matching μ and for any student i and college c, say that (i, c) *blocks* μ iff at least one of the following four conditions holds: (i) $iP_i\mu(i)$, (ii) cP_ci, (iii) $cP_i\mu(i)$ and iP_cj for some $j \in \mu^{-1}(c)$, (iv) $cP_i\mu(i)$, iP_cc and $\#\mu^{-1}(c) < q_c$.

Say that a matching μ is *stable* iff there is no student i and college c such that (i, c) blocks μ.

Say that a matching is *student-efficient* if no other matching is weakly preferred by all students and strictly preferred by at least one student.

Say that (student and college) preferences are *student-oriented* iff whenever two students disagree about the ranking of two colleges, each one of the two students is ranked higher by the college he prefers than the other student. One can similarly define *college-oriented* preferences.

Theorem 1.1 *If preferences are student-oriented, then there is a unique stable matching and it is student-efficient.*[1]

Proof. Since a stable matching exists (Gale and Shapley 1962), we need only show that there is at most one and that it is student-efficient. We first show that there is a student and college who are each top-ranked by the other.

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[1] An analogous uniqueness result holds for college-oriented preferences.

Let i be any student and suppose that college c^* is i's most preferred college. Let student i^* be the most preferred student of college c^* among the set of students whose favorite college is c^* (this set is nonempty because it contains student i). For any student j whose favorite college is not c^*, students i^* and j disagree about the ranking of college c^* and j's top-ranked college. Because preferences are student-oriented, college c^* must prefer i^* to j. Hence, student i^* is college c^*'s most preferred student among all students. Thus student i^* and college c^* are each top-ranked by the other and so i^* and c^* must be matched to one another in any stable matching.

Consider the (sub-) market in which student i^* is removed and the quota of college c^* is reduced by one seat. Since the original market has student-oriented preferences, so does this submarket. Hence, there is a student and college in the submarket who are each top-ranked by the other and so must be matched to one another in any stable matching. Repeatedly removing such students and college seats establishes that there is a unique stable matching.

Observe now that the stable matching so obtained can be obtained as a serial dictatorship for students where students can choose their colleges in the order in which students were removed in the above procedure. Being the outcome of a serial dictatorship for students, this stable matching is therefore student-efficient. ■

In the context of one to one matching problems, various conditions are known to yield a unique stable matching, e.g., SPC (Eeckhout 2000),[2] NCC (Clark 2006), and α-reducibility (Alcalde 1995, and Clark 2006). The most permissive of these is Eeckhout's (2000) SPC condition. When specialized to one to one matching settings, oriented preferences must always satisfy each of these conditions except NCC.

Like our oriented-preferences condition, Niederle and Yariv's (2009) alignment condition applies to many to one matching problems, and it can be shown that student-oriented preferences are aligned. (I thank Leeat Yariv for the argument). Aligned preferences, and hence oriented preferences, satisfy natural extensions to many to one matching problems of SPC and α-reducibility.[3]

All of the above conditions imply that there is a pair of agents on opposite sides of the market who are each top-ranked by the other. And all of the above conditions except SPC imply that this holds also in any submarket.

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[2] The name SPC was coined by Clark (2006).

[3] For example, the natural extension of Eeckhout's (2000) SPC condition to the college admissions problem is to require the existence of a sequence of students $i_1, i_2, ..., i_N$ in which each student appears once, and a sequence of colleges $c_1, c_2, ..., c_M$ in which each college appears once for each seat in its quota, such that $c_n P_{i_n} c_k$ for every $k > n$ such that $c_k \neq c_n$, and $i_m P_{c_m} i_k$ for every $k > m$. Our proof here shows that this extension of Eeckhout's condition is implied when preferences are oriented, and it is not difficult to show that this extension is also satisfied when preferences are aligned.